Course notes

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Some notations

$\mathcal{L}$	Laplace transform, §**??**
$\mathcal{L}^{-1}$	inverse Laplace transform, §1.58
$\mathcal{B}$	Borel transform, §**??**
$\mathcal{LB}$	Borel/BE summation operator, §**??** and §**??**
p	usually, Borel plane variable
$\tilde{f}$	formal expansion
$H(p)$	Borel transform of $h(x)$
$\sim$	asymptotic to, §1.1a
$\lesssim$	less than, up to an unimportant constant, §1.1a
$\mathbb{D}_r$	The disk of radius r centered at 0
∂A	The boundary of the set A
$\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ $\mathbb{N}^+, \mathbb{R}^+$	the nonnegative integers, integers, rationals, real numbers, complex numbers, positive integers, and positive real numbers, respectively
$\mathbb{H}$	open right half complex-plane.
$\mathbb{H}_\theta$	half complex-plane centered on $e^{i\theta}$.
$\overline{S}$	closure of the set S.
C_a	absolutely continuous functions, [55]
$f * g$	convolution of f and g, §**??**
$L^1_\nu, \|\cdot\|_\nu$, $\mathcal{A}_{K,\nu}$, etc.	various spaces and norms defined in §**??** and §**??**

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Chapter 1

Introduction

1.1 Expansions and approximations

Classical asymptotic analysis is a set of mathematical results and methods to find the limiting behavior of functions, near a point, most often a singular point. It is particularly efficient in the context of differential or difference equations when the function has no simple representation that immediately conveys the desired limiting behavior.

Asymptotic analysis may involve several variables; however, in this book, we will be mostly concerned with limiting behavior in one scalar variable; in the context of differential or difference equations, this can be the independent variable or a parameter.

1.1a Notation

Let the special point of analysis be $t_0 \in \mathbb{C}$.

Some common notations are: $f = O(1)$ if f is bounded near t_0 and $f = o(1)$ if $f \to 0$ as $t \to t_0$. More generally $f = O(g)$ if $f/g = O(1)$ and similarly $f = o(g)$ if $f/g = o(1)$. We also write $f \ll g$ if $f = o(g)$. It is understood that g cannot vanish close to t_0. The notation $|f| \lesssim |g|$ is used to represent $|f| \le C|g|$ in the domain of interest, where C is a constant whose value is immaterial. Clearly $|f| \lesssim |g|$ in a small neighborhood of t_0 is the same as $f = O(g)$. We write $f = O_s(g)$; when both $f = O(g)$ and $g = O(f)$ near t_0.

The point t_0 may be approached only from one direction, along a curve in $\mathbb{C}$ or even along a given sequence of points tending to t_0 and when such further restrictions are needed, they will be specified. For instance if $t_0 = 0$, then $t = o(1)$ as $t \to 0$ and $e^{-1/t} = o(t^m)$ for any m as $t \downarrow 0$ ($t \in \mathbb{R}^+$ decreases towards 0), but the opposite holds, $t^m = o(e^{-1/t})$, as $t \uparrow 0$.

1.1b Asymptotic expansions

A sequence of functions $\{f_k\}_{k\in\mathbb{N}}$ such that $f_m \ll f_n$ if $m > n$ is called an asymptotic scale at $t = t_0$. In terms of it we can write the leading order behavior of a function, $f = f_0 + o(f_0)$ and also successively higher order corrections: $f = f_0+f_1+o(f_1)$ etc. In a compact form, we write an asymptotic expansion as a formal sum,

$$\sum_{k=0}^{\infty} f_k(t) =: \tilde{f}, \tag{1.1}$$

where no convergence condition is imposed, and define asymptoticity by the following.

Definition 1.2 *A function f is asymptotic to the formal series $\tilde{f}$ as $t \to t_0$ (once more, the approach of t_0 may have to be restricted to a curve) if*

$$f(t) - \sum_{k=0}^{N} \tilde{f}_k(t) = o(\tilde{f}_N(t)) \quad (\forall N \in \mathbb{N}) \tag{1.3}$$

Condition (1.3) can be written in a number of equivalent ways, useful in applications, as the following result shows.

Proposition 1.4 *If $\tilde{f} = \sum_{k=0}^{\infty} \tilde{f}_k(t)$ is an asymptotic series as $t \to t_0$ and f is a function asymptotic to it, then the following characterizations are equivalent to each other and to (1.3).*

(i)

$$f(t) - \sum_{k=0}^{N} \tilde{f}_k(t) = O(\tilde{f}_{N+1}(t)) \quad (\forall N \in \mathbb{N}) \tag{1.5}$$

(ii)

$$f(t) - \sum_{k=0}^{N} \tilde{f}_k(t) = \tilde{f}_{N+1}(t)(1 + o(1)) \quad (\forall N \in \mathbb{N}) \tag{1.6}$$

(iii) There is function $\nu : \mathbb{N} \mapsto \mathbb{N}$, such that $\nu(N) \geq N$ and

$$f(t) - \sum_{k=0}^{\nu(N)} \tilde{f}_k(t) = O(\tilde{f}_{N+1}(t)) \quad (\forall N \in \mathbb{N}) \tag{1.7}$$

Condition (iii) seems strictly weaker, but it is not. It allows us to use less accurate estimates of remainders, provided we can do so to all orders.

PROOF We only show (iii), the others being immediate from the definition. We may assume $\nu(N) > N$, as otherwise there is nothing to prove. Let $N \in \mathbb{N}$. We have

$$f(t) - \sum_{k=0}^{N} \tilde{f}_k(t) \;=\; f(t) - \sum_{k=0}^{\nu(N)} \tilde{f}_k(t) + \sum_{j=N+1}^{\nu(N)} \tilde{f}_j(t) \;=\; O\left(\tilde{f}_{N+1}(t)\right) \quad (1.8)$$

since in the last sum in (1.8) the number of terms is fixed, and thus the sum remains $O\left(\tilde{f}_{N+1}\right)$ as $t \to t_0$. ∎

Whenever possible, the scale is chosen to consist of simple functions, such as powers, logs and exponentials, the behavior of which is manifest. Taylor series are perhaps the simplest nontrivial asymptotic expansions. The following is a way of restating Taylor's theorem with remainder.

Proposition 1.9 *Assume f is C^∞ in an interval containing t_0. Then*

$$f(t) \sim \sum_{k=0}^{\infty} \frac{f^{(k)}(t_0)}{k!}(t - t_0)^k \;\text{ as } t \to t_0 \quad (1.10)$$

Clearly, the asymptotic series of a function f converges to f **iff** f is analytic at t_0. Otherwise, the series is not convergent, or it converges to a function other than f (see Example 1.16).

Note 1.11 In Definition 1.2 none of the f_k is allowed to vanish. For instance, although all right derivatives of $f_1 = e^{-1/t}$ vanish at zero, we cannot write $e^{-1/t} \sim 0$. This is a natural restriction since all the right derivatives vanish at zero for many other functions, for instance $f_2 = \sin(1/t)e^{-1/\sqrt{t}}$, with quite different behavior $t \downarrow 0$. We will however speak of *asymptotic power series*, a weaker notion in which sense f_1 and f_2 above will be represented by the same series.

Example 1.12 (A divergent asymptotic series) A simple example of a divergent asymptotic expansion is obtained by calculating the Taylor series of the function

$$f(z) = \frac{1}{z}e^{-1/z}\mathrm{E}_1\left(\frac{1}{z}\right) = \int_0^\infty \frac{e^{-t}}{1+zt}dt; \;\; z > 0 \quad (1.13)$$

where $E_1(z) = \int_z^\infty \frac{e^{-t}}{t}dt$ is the exponential integral. The exponential decay of the integrand allows for differentiating (1.151) any number of times for $z > 0$,

$$f^{(k)}(z) = k!\int_0^\infty \frac{(-t)^k e^{-t}}{(1+zt)^{k+1}}dt \quad (1.14)$$

Furthermore, $f^{(k)}(z)$ are continuous as $z \to 0^+$ (right limit at zero) for all $k \geq 0$. Elementary analysis tells us that f is C^∞ at zero from the right. The integral representation of the factorial gives $f^{(k)}(0) = (-1)^k(k!)^2$ We have, using Taylor's theorem with one-sided derivatives [54]

$$f(z) \sim \sum_{k=0}^{\infty}(-1)^k k! z^k, \ z \downarrow 0 \tag{1.15}$$

a series with zero radius of convergence, or in short a *divergent series.*

Example 1.16 (A convergent asymptotic series) Since all derivatives of $e^{-1/z}$ vanish as $z \downarrow 0$ we have

$$\frac{1}{1-z} + e^{-1/z} \sim \sum_{k=0}^{\infty} z^k = \frac{1}{1-z}, \ z \to 0^+ \tag{1.17}$$

Convergence of an asymptotic series does not thus imply that the function equals the sum of the series. Note also that here, as it is often done in practice, we have used the *same* notation $\sum_{k=0}^{\infty} z^k$ to mean two different things: an asymptotic series simply displaying the asymptotic scale involved, which is a formal object, and its *sum*, an actual function. We will discuss this ambiguity later.

Example 1.18 (A convergent but *anti*asymptotic series) The following Laurent series converges in $\mathbb{C} \setminus \{0\}$:

$$\sum_{j=0}^{\infty} \frac{(-1)^j}{j! z^j} = e^{-1/z} \tag{1.19}$$

Eq. (1.19) is **not** an asymptotic expansion as $z \to 0$. In (1.19) $f_k \ll f_{k+1}$, the **opposite** of what is required from an asymptotic series. We have $|e^{-1/z} - \sum_{j=0}^{M} \frac{(-1)^j}{j! z^j}| \gtrsim |z^{-M-1}|$ as $z \downarrow 0$ which means the approximations deteriorate the more terms we keep, if $z \downarrow 0$.

In general, for understanding the behavior of a function near a point, an antiasymptotic series, even if convergent, is not very useful. We can see that if we try to determine whether

$$f(z) = \sum_{j=0}^{\infty} \frac{(-1)^j}{(j! + 10^{-j}\sin(j))z^j} \tag{1.20}$$

(the Laurent coefficients are close to those in (1.19)) tends zero or not, as $z \to 0$.

By contrast, although (1.15) is divergent, by the definition of an asymptotic series, in (1.151) we see that $f(z) \to 1$ as $z \downarrow 0$, and that $f(z) - 1 = -z(1 + o(1))$ and so on.

Stirling's formula for $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$, which will be derived later, in §**??**, is an example of a divergent asymptotic expansion, where the scales involve powers of $1/x$ and logs:

$$\ln(\Gamma(x)) \sim (x - 1/2)\ln x - x + \frac{1}{2}\ln(2\pi) + \sum_{j=1}^{\infty} c_j x^{-2j+1},\ x \to +\infty \quad (1.21)$$

where $2j(2j-1)c_j = B_{2j}$ and $\{B_{2j}\}_{j\geq 1} = \{1/6, -1/30, 1/42...\}$ are Bernoulli numbers, [1], eq. 6.140. This expansion is asymptotic as $x \to \infty$: successive terms get smaller and smaller. For $x = 6$, truncating (1.21) at $j = 3$ we get $\Gamma(6) \approx 120.0000002$ (while $\Gamma(6) = 5! = 120$). Stirling's expansion converges for no x, since $\ln(\Gamma(x))$ is singular at all $x \in -\mathbb{N}$ (why is this an obstruction to convergence?).

Remark 1.22 Asymptotic expansions cannot be added, in general. Indeed, we note that $1/(1-z)$ has the same expansion (1.17) as $-e^{-1/z} + 1/(1-z)$, as $z \downarrow 0$. Adding these would give $e^{-1/z} \sim 0$, which is not a valid asymptotic expansion, see Note 1.11. This is one reason for considering, for restricted expansions, a weaker asymptoticity condition; see §1.1c.

Remark 1.23 Sometimes we encounter oscillatory expansions such as $\sin x(1 + a_1x^{-1} + a_2x^{-2} + \cdots)$ $(*)$ for large x, which, while very useful, have to be understood differently. They are not asymptotic expansions, as we saw in Note 1.11. Furthermore, usually the approximation itself is expected to fail near zeros of sin. However, if small neighborhoods of the zeros of sin are excluded, the expansion remains valid in the sense defined. Also, usually there are ways to present the asymptotics in a way that avoids these exclusions,(see §**??**).

1.1c Asymptotic power series

A special role is played by series in *powers* of a small variable, such as

$$\tilde{S} = \sum_{k=0}^{\infty} c_k z^k,\ z \to 0^+ \quad (1.24)$$

With the transformation $z = t - t_0$ (or $z = x^{-1}$, when x is large) the series can be centered at $z = 0$ (or $x = +\infty$, respectively).

Definition 1.25 (Asymptotic power series) *A function is asymptotic to a series as $z \to 0$,* in the sense of power series *if*

$$f(z) - \sum_{k=0}^{N} c_k z^k = O(z^{N+1}) \quad (\forall N \in \mathbb{N}) \quad \textit{as } z \to 0, \quad (1.26)$$

where, as for general asymptotic expansions, it may be necessary to restrict the approach $z \to 0$ to a particular set of curves.

Remark 1.27 If f has an asymptotic expansion (in the sense of Definition 1.2) that happens to be a power series, it is asymptotic to it in the sense of power series as well.

However, the converse is not true, unless all c_k are nonzero, *i.e.* it is possible that $f \sim \tilde{f} \equiv \sum_{k=0}^{\infty} c_k z^k$ in the power series sense, without $\tilde{f}$ being the asymptotic expansion in the sense of Definition 1.2.

For now, whenever confusions are possible, we will use a different symbol, $\sim_p$, for asymptoticity in the sense of power series.

Remark 1.28 Noninteger asymptotic power series, e.g., series of the form

$$z^{\alpha}\sum_{k=0}^{\infty} c_k z^{k\beta}, \quad \mathrm{Re}\,(\beta) > 0 \tag{1.29}$$

as well as asymptoticity of a function to (1.29) can be defined by easily adapting Definition 1.25, and replacing $O(z^N)$ by $O(z^{N\beta+\alpha})$ which is the same as $O(z^{\mathrm{Re}\,\alpha+N\mathrm{Re}\,(\beta)})$. More generally, in (1.29), instead of z^{α}, we could have other simple functions such as exponentials or logs.

The asymptotic power series at zero in $\mathbb{R}$ of e^{-1/z^2} is the zero series, which is not its asymptotic expansion in the sense of Definition 1.2, see again Note 1.11. The advantage of asymptotic power series however is the fact that they form an algebra.

1.1d Operations with asymptotic power series

Addition and multiplication of asymptotic power series are defined as in the convergent case:

$$A\sum_{k=0}^{\infty} c_k z^k + B\sum_{k=0}^{\infty} d_k z^k = \sum_{k=0}^{\infty}(Ac_k + Bd_k)z^k$$

$$\left(\sum_{k=0}^{\infty} c_k z^k\right)\left(\sum_{k=0}^{\infty} d_k z^k\right) = \sum_{k=0}^{\infty}\left(\sum_{j=0}^{k} c_j d_{k-j}\right) z^k$$

Remark 1.30 *If the series $\tilde{f}$ is convergent and f is its sum, $f = \sum_{k=0}^{\infty} c_k z^k$, (note the ambiguity of the sum notation), then $f \sim_p \tilde{f}$.*

The proof follows directly from the definition of convergence

The proof of the following lemma is immediate:

Lemma 1.31 (Algebraic properties of asymptoticity to a power series) *If $f \sim_p \tilde{f} = \sum_{k=0}^{\infty} c_k z^k$ and $g \sim_p \tilde{g} = \sum_{k=0}^{\infty} d_k z^k$, then*

(i) $Af + Bg \sim_p A\tilde{f} + B\tilde{g}$

(ii) $fg \sim_p \tilde{f}\tilde{g}$

Corollary 1.32 (Uniqueness of the asymptotic series to a function) *If $f(z) \sim_p \sum_{k=0}^{\infty} c_k z^k$ as $z \to 0$, then the c_k are unique.*

PROOF Indeed, if $f \sim_p \sum_{k=0}^{\infty} c_k z^k$ and $f \sim_p \sum_{k=0}^{\infty} d_k z^k$, then, by Lemma 1.31 we have $0 \sim_p \sum_{k=0}^{\infty} (c_k - d_k) z^k$ which implies, inductively, that $c_k = d_k$ for all k. ∎

However, division of asymptotic power series is not always possible. For instance, $e^{-1/z^2} \sim_p 0$ for small z in $\mathbb{R}$ while $1/\exp(-1/z^2)$ has no asymptotic power series at zero. Also, classical asymptotics cannot distinguish between functions differing by a quantity which is $o(z^m)$ for all $m > 0$ as $z \to 0$. Indeed, we have the following result (see also Example 1.16)

Proposition 1.33 *Assume f and g have nonzero asymptotic power series as $z \to 0$ and $f - g = h$ where $h = o(z^m)$ for all $m > 0$ as $z \to 0$. Then the asymptotic series of f and g coincide.*

PROOF This follows straightforwardly from Definition 1.26 and the assumption on h. ∎

1.1d.1 Integration and differentiation of asymptotic power series

Asymptotic relations can be integrated termwise as Proposition 1.34 below shows.

Proposition 1.34 *Assume f is integrable near $z = 0$ and that*

$$f(z) \underset{p}{\sim} \tilde{f}(z) = \sum_{k=0}^{\infty} c_k z^k$$

Then

$$\int_0^z f(s)ds \underset{p}{\sim} \int_0^z \tilde{f}(s)ds := \sum_{k=0}^{\infty} \frac{c_k z^{k+1}}{k+1}$$

PROOF This follows from the fact that $\int_0^z o(s^n)ds = o(z^{n+1})$ as it can be seen by straightforward inequalities. ∎

Differentiation is a different issue. Many simple examples show that asymptotic series cannot be unrestrictedly differentiated. For instance $e^{-1/z^2} \sin e^{1/z^4} \sim_p 0$ as $z \to 0$ on $\mathbb{R}$, but the derivative is unbounded and thus it is not asymptotic to zero.

1.1d.2 Asymptotics in regions in $\mathbb{C}$

Asymptotic power series of analytic functions can be differentiated if they hold in a region which is not too rapidly shrinking as $z \to 0$. This is so, since the derivative is expressible as an integral by Cauchy's formula. Such a region is often a sector or strip in $\mathbb{C}$, but can be allowed to be thinner:

Proposition 1.35 *Let $M \geq 0$ and denote*

$$S_a = \{x : \quad |x| \geq R, \quad |x|^M |\mathrm{Im}\,(x)| \leq a\}$$

Assume f is continuous in S_a and analytic in its interior, and

$$f(x) \underset{p}{\sim} \sum_{k=0}^{\infty} c_k x^{-k} \quad \textit{as } x \to \infty \textit{ in } S_a$$

Then, for all $a' \in (0, a)$ we have

$$f'(x) \underset{p}{\sim} \sum_{k=0}^{\infty} (-kc_k) x^{-k-1} \quad \textit{as } x \to \infty \textit{ in } S_{a'}$$

PROOF Here, Proposition 1.4 (iii) will come in handy. Let $\nu(N) = N+M$. By the asymptoticity assumptions, for any N there is some constant $C(N)$ such that $|f(x) - \sum_{k=0}^{\nu(N)} c_k x^{-k}| \leq C(N)|x|^{-\nu(N)-1}$ (*) in S_a.

Let $a' < a$, take x large enough, and let $\rho = \frac{1}{2}(a - a')|x|^{-M}$; then check that $\mathbb{D}_\rho = \{x' : |x - x'| \leq \rho\} \subset S_a$. We have, by Cauchy's formula and (*),

$$\left| f'(x) - \sum_{k=0}^{\nu(N)} (-kc_k) x^{-k-1} \right| = \left| \frac{1}{2\pi i} \oint_{\partial \mathbb{D}_\rho} \left(f(s) - \sum_{k=0}^{\nu(N)} c_k s^{-k} \right) \frac{ds}{(s-x)^2} \right|$$
$$\leq \frac{C(N)}{(|x|-1)^{\nu(N)+1}} \frac{1}{2\pi} \oint_{\partial \mathbb{D}_\rho} \frac{d|s|}{|s-x|^2} \leq \frac{2C(N)}{|x|^{\nu(N)+1}\rho} \leq \frac{4C(N)}{a-a'} |x|^{-N-1} \tag{1.36}$$

and the result follows. ∎

Note 1.37 Usually, we can determine from the context whether $\sim$ or $\sim_p$ should be used. Often in the literature, it is left to the reader to decide which notion is in use. After we have explained the distinction, we will do the same, so as not to complicate notation.

1.2 Asymptotics of integrals

Often when differential equations have closed form solutions, these can be expressed in terms of elementary functions or special functions admitting integral representations. These integral expressions allow for finding the asymptotic behavior of solutions in different regions of the complex domain. Important examples include the equation

$$x^2y'' + xy' + \sigma(x^2 - \sigma\nu^2)y = 0; \quad \sigma = \pm 1 \tag{1.38}$$

For $\sigma = 1$, (1.38) is the Bessel equation [1]; the solution which is regular at the origin is $J_\nu(x)$ – the Bessel function of the first kind and a linearly independent one is $Y_\nu(x)$ – the Bessel function of the second kind. For $\sigma = -1$ (1.38) is the modified Bessel equation; the solution which is regular at the origin is $I_\nu(x)$ – the modified Bessel function of the first kind and a linearly independent one is $K_\nu(x)$ – the modified Bessel function of the second kind. The Airy equation

$$y'' - xy = 0 \tag{1.39}$$

has solutions $\mathrm{Ai}(x)$ and $\mathrm{Bi}(x)$, the Airy functions. The hypergeometric equation

$$x(x-1)y'' + [(a+b+1)x - c]y' + aby = 0 \tag{1.40}$$

has linearly independent solutions ${}_2F_1(a, b; c; x)$ and $x^{1-c}\ {}_2F_1(a-c+1, b-c+1; 2-c; x)$ where ${}_2F_1$ is a hypergeometric function. All these functions have integral representations, in fact a good number of representations suitable for different asymptotic regimes. For instance, see [21] 10.9.17, [6] (Equation 6.6.30, page 298),

$$J_\nu(z) = \frac{1}{2\pi i}\int_{\infty-\pi i}^{\infty+\pi i} \exp(z\sinh t - \nu t)dt; \quad \mathrm{Re}\ \ z\ >\ 0 \tag{1.41}$$

and [21] 9.5.4, and [6] (p. 313, Problem 6.75, with the change of integration variable $t \to -t$).

$$\mathrm{Ai}(z) = \frac{1}{2\pi i}\int_{\infty e^{-\pi i/3}}^{\infty e^{\pi i/3}} \exp\left(t^3/3 - zt\right)\, dt, \tag{1.42}$$

Finally, for $|z| < 1$ [21] 15.1.2 and 15.6.1,

$${}_2F_1(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\,\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\, dt \ \mathrm{Re}(c) > \mathrm{Re}(b) > 0 \tag{1.43}$$

1.2a *The Laplace transform and its properties.

The Laplace transform $\mathcal{L}f$ of a function F is defined by

$$f(x) = \int_0^\infty e^{-xp}F(p)dp, \quad \mathrm{Re}\,(x) > \nu \geq 0 \tag{1.44}$$

Here it is assumed that F is locally integrable in $[0,\infty)$ and does not grow faster than exponentially, for instance

$$\|F\|_{\infty,\nu} = \sup_{p\ge 0}|F(p)|e^{-\nu p} < \infty \text{ or } \|F\|_{L^1_\nu} = \int_0^\infty |F(p)|e^{-\nu p}dp < \infty \quad (1.45)$$

(see §1.8a) for some $\nu \in \mathbb{R}$. Both ensure the existence of $\mathcal{L}f$ if $\mathrm{Re}\, x > \nu$.

As will be seen in the sequel, solutions of linear or nonlinear differential equations, including (1.42) and (1.41) above, can often be written as Laplace transforms of simpler functions. It is then important to understand the asymptotic behavior of Laplace transforms. A general asymptotic result is the following:

Lemma 1.46 *Under the assumption in* (1.45), *we have*

$$\int_0^\infty e^{-xp}F(p)dp \to 0 \ \textit{as} \ \mathrm{Re}\,(x) \to \infty \quad (1.47)$$

PROOF This follows from the dominated convergence theorem, see §1.8a. Indeed, $\int_0^\infty |e^{-xp}F(p)|dp \le \int_0^\infty |e^{-x_0 p}F(p)|dp < \infty$ for $\mathrm{Re}\,(x) \ge x_0 > \nu$, and $e^{-xp}F(p) \to 0$ as $\mathrm{Re}\,(x) \to \infty$ for all $p \in (0,\infty)$. ∎

Furthermore, convergence is exponentially fast iff F is identically zero on some interval $[0,\varepsilon)$, where $\varepsilon > 0$ is independent of x as shown in the following proposition. For the notation, see §1.8a.

Proposition 1.48 *Assume that F is exponentially bounded in the sense of* (1.45)*; let $x_1 = \mathrm{Re}\,(x)$. Then*

$$\int_0^\infty e^{-xp}F(p)dp = o(e^{-x_1\varepsilon}) \ \textit{ as } x_1 \to \infty \ \textbf{iff} \ F = 0 \ \textit{ a.e.}^1 \ \textit{ on } [0,\varepsilon] \ \textit{as } x_1 \to \infty \quad (1.49)$$

Also, $\int_0^\infty e^{-xp}F(p)dp = O(e^{-x_1\varepsilon}) \Leftrightarrow \int_0^\infty e^{-xp}F(p)dp = o(e^{-x_1\varepsilon})$*, implying $F = 0$ a.e. on $[0,\varepsilon]$.*

PROOF (i) Assume that $F = 0$ a.e. on $[0,\varepsilon)$. This implies that

$$\int_0^\infty e^{-xp}F(p)dp = \int_\varepsilon^\infty e^{-xp}F(p)dp = e^{-x\varepsilon}\int_0^\infty e^{-xp}F(p+\varepsilon)dp = e^{-x\varepsilon}o(1) \quad (1.50)$$

as $x_1 \to \infty$ by Lemma 1.46.

(ii) For the converse, assume that $\int_0^\infty e^{-xp}F(p)dp = O(e^{-x_1\varepsilon})$. We write

$$\int_0^\infty e^{-xp}F(p)dp = \int_0^\varepsilon e^{-xp}F(p)dp + \int_\varepsilon^\infty e^{-xp}F(p)dp. \quad (1.51)$$

The rightmost integral in (1.51) is shown to be $o(e^{-x_1\varepsilon})$ by using the change variable $p \to p + \varepsilon$ and using Lemma 1.46. Thus

$$g(x) := e^{x\varepsilon}\int_0^\varepsilon e^{-xp}F(p)dp = O(1) \text{ as } x_1 = \text{Re } x \to +\infty \tag{1.52}$$

It is easy to see that g is entire. Furthermore, it is bounded for $x \in \mathbb{R}^+$ by (1.52) and also manifestly bounded for $x \in i\mathbb{R}$, and $x \in \mathbb{R}^-$. Since g is of exponential order 1, using the Phragmén-Lindelöff theorem in all of the four quadrants (see [17] pp. 19 and 23 for more details) shows g is bounded. From Liouville's theorem, g is a constant. The Riemann-Lebesgue lemma implies that g goes to zero as $x \to \infty$ along the imaginary line. Thus $g = 0$, implying $\int_0^\varepsilon F(p)e^{-px}dp = 0, \forall x \in \mathbb{C}$ implying that the Fourier transform $\int_{-\infty}^\infty e^{-itp}\chi_{[0,\varepsilon]}(p)F(p)dp = 0\ \forall t \in \mathbb{R}$ and thus, by inverse Fourier transform, $F(p) = 0$ *a.e.* on $(0,\varepsilon)$. Now, (i) implies that $\int_0^\infty F(p)e^{-px}dp = o\left(e^{-\varepsilon x_1}\right)$. ∎

Corollary 1.53 *Under the condition* (1.45)*, if* $\mathcal{L}F = 0$ *for all* $x > 0$*, then* $F = 0$ *a.e. on* $\mathbb{R}^+$*.*

PROOF Since, in particular, $\mathcal{L}F = O(e^{-xa})$ for any $a > 0$, from Proposition 1.48, $F = 0$ a.e. on $\mathbb{R}^+$. ∎

First inversion formula

Let $\mathcal{H}$ denote the space of analytic functions in the right half complex plane.

Proposition 1.54 *(i)* $\mathcal{L} : L^1(\mathbb{R}^+) \mapsto \mathcal{H}$ *and* $\|\mathcal{L}F\|_\infty \le \|F\|_1$.

(ii) $\mathcal{L} : L^1(\mathbb{R}^+) \mapsto \mathcal{L}(L^1(\mathbb{R}^+)) \subset \mathcal{H}$ *is invertible, and the inverse is given by*

$$F(x) = \hat{\mathcal{F}}^{-1}\{\mathcal{L}F(it)\}(x) \tag{1.55}$$

for $x \in \mathbb{R}^+$ *where* $\hat{\mathcal{F}}$ *is the Fourier transform (in distributions if* $\mathcal{L}F \notin L^1(i\mathbb{R})$*).*

Second inversion formula

Proposition 1.56 *(i) Assume* f *is analytic in an open sector* $\mathbb{H}_\delta := \{x : |\arg(x)| < \pi/2 + \delta\}$*,* $\delta \ge 0$ *and is continuous on* $\partial\mathbb{H}_\delta$*, and that for some* $K > 0$ *and any* $x \in \overline{\mathbb{H}_\delta}$ *we have*

$$|f(x)| \le K(|x|^2 + 1)^{-1} \tag{1.57}$$

Then $\mathcal{L}^{-1}f$ *is well defined by*

$$F = \mathcal{L}^{-1}f = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} \mathrm{d}t\, e^{pt}f(t) \tag{1.58}$$

and

$$\int_0^\infty \mathrm{d}p\, e^{-px} F(p) = \mathcal{L}\mathcal{L}^{-1} f = f(x) \tag{1.59}$$

We have $\|\mathcal{L}^{-1}\{f\}\|_\infty \le K/2$ *and* $\mathcal{L}^{-1}\{f\} \to 0$ *as* $p \to \infty$.

(ii) If $\delta > 0$*, then* $F = \mathcal{L}^{-1} f$ *is analytic in the sector* $S_\delta = \{p \ne 0 : |\arg(p)| < \delta\}$*. In addition,* $\sup_{S_\delta} |F| \le K/2$ *and* $F(p) \to 0$ *as* $p \to \infty$ *along rays in* S_δ*.*

PROOF Clearly, F in (1.58) is well-defined since $f(is) \in L^1(\mathbb{R})$. (i) We have

$$2\pi i \mathcal{L}\left[\mathcal{L}^{-1} f\right](x) = \int_0^\infty \mathrm{d}p\, e^{-px} \int_{-\infty}^\infty i\mathrm{d}s\, e^{ips} f(is) \tag{1.60}$$

$$= \int_{-\infty}^\infty i\mathrm{d}s\, f(is) \int_0^\infty \mathrm{d}p\, e^{-px} e^{ips} = \int_{-i\infty}^{i\infty} f(z)(x-z)^{-1}\mathrm{d}z = 2\pi i f(x) \tag{1.61}$$

where we applied Fubini's theorem[2] and then pushed the contour of integration past x to infinity. The norm of $\mathcal{L}^{-1}$ is obtained by majorizing $|f(x)e^{px}|$ by $K(|x^2|+1)^{-1}$. The behavior $\left[\mathcal{L}^{-1} f\right](p) \to 0$ as $p \to +\infty$ follows by applying Riemann-Lebesgue Lemma to (1.58).

(ii) For any $\delta' < \delta$ we have, by (1.57),

$$\int_{-i\infty}^{i\infty} \mathrm{d}s\, e^{ps} f(s) = \left(\int_{-i\infty}^0 + \int_0^{i\infty}\right) \mathrm{d}s\, e^{ps} f(s)$$

$$= \left(\int_{-i\infty e^{-i\delta'}}^0 + \int_0^{i\infty e^{i\delta'}}\right) \mathrm{d}s\, e^{ps} f(s) \tag{1.62}$$

Take any $p \in S_\delta$. Choose $\delta' < \delta$ so that $p \in S'_\delta$. Analyticity of (1.62) in $p \in S'_\delta$ is manifest, given the analyticity and exponential decay of the integrand. For the estimates on $F(p)$, we note that (i) applies to $f(xe^{i\phi})$ if $|\phi| < \delta$. □

Many cases can be reduced to (1.57) after transformations. For instance if $f_1 = \sum_{j=1}^N a_j (1+x)^{-k_j} + f(x)$, **where $k_j > 0$ and f satisfies the assumptions above, then (1.58) and (1.59) apply to f_1, since they do apply, by straightforward verification, to the finite sum.

▯

Proposition 1.63 *Let* F *be analytic in the open sector* $S_p = \{e^{i\phi}\mathbb{R}^+ : \phi \in (-\delta, \delta)\}$ *and such that* $|F(|p|e^{i\phi})| \le g(|p|) \in L^1[0,\infty)$*. Then* $f = \mathcal{L}F$ *is analytic in the sector* $S_x = \{x : |\arg(x)| < \pi/2 + \delta\}$ *and* $f(x) \to 0$ *as* $|x| \to \infty, \arg(x) = \theta \in (-\pi/2 - \delta, \pi/2 + \delta)$.

[2]This theorem addresses the permutation of the order of integration; see [55]. Essentially, if $f \in L^1(A \times B)$, then $\int_{A\times B} f = \int_A \int_B f = \int_B \int_A f$.

PROOF Because of the analyticity of F and the decay conditions for large p, the path of Laplace integration can be rotated by any angle ϕ in $(-\delta, \delta)$ without changing[3] $(\mathcal{L}F)(x)$. The fact that$g \in L^1$ also implies that The decay of $(\mathcal{L}F)(x)$ in x follows from Lemma 1.46 with x replaced by $xe^{-i\phi}$ and ϕ chosen $\arg\left(xe^{-i\phi}\right) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ ∎

Note F need not be analytic at $p = 0$ for Proposition 1.63 to apply.

1.2b Watson's Lemma

As will be seen, many integrals after appropriate changes of variable can be cast in a form where Watson's Lemma can be applied. In the following example, we determine the asymptotics of the incomplete Gamma function which we will need later.

Example 1.64 Assume that $\operatorname{Re}\beta > 0$ and $a > 0$. Then as $x \to \infty$ along an arbitrary ray in the open right half plane, $\mathbb{H}$

$$\{x : \arg x = \alpha\}\,; \quad \text{where } \alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \tag{1.65}$$

Then,

$$\int_0^a p^{\beta-1}e^{-px}dp \sim \frac{\Gamma(\beta)}{x^\beta} \tag{1.66}$$

Indeed, changing variable to $t = px$ we get

$$\int_0^\infty p^{\beta-1}e^{-px}dp = \frac{1}{x^\beta}\int_0^{\infty e^{i\arg(x)}} e^{-t}t^{\beta-1}dt = \frac{1}{x^\beta}\int_0^\infty e^{-t}t^{\beta-1}dt = \frac{\Gamma(\beta)}{x^\beta} \tag{1.67}$$

by a homotopic change of contour and the definition of the Gamma function.

Lemma 1.68 *If $x^\alpha \int_0^\infty e^{-xp}F(p)dp$ has an asymptotic power series in $z = x^{-\beta}$ for some β with $\operatorname{Re}\beta > 0$ as $\operatorname{Re}x \to \infty$, then for any fixed $\varepsilon > 0$, $x^\alpha \int_0^\varepsilon e^{-xp}F(p)dp$ has an asymptotic power series as well, and the two power series agree.*

PROOF This is an immediate consequence of Propositions 1.48 and 1.33. ∎

Watson's lemma allows us to integrate power series term by term as stated below.

[3]The fact that $g \in L^1$ implies that $\liminf_{R\to\infty} Rg(R) = 0$; thus there is a subsequence R_n s.t. $R_n g(R_n) \to 0$. By straightforward estimates, or by Jordan's lemma, we see that the integral of Fe^{-px} along an arc of a circle of radius R_n goes to zero with n.

Lemma 1.69 (Watson's lemma) *Assume that* $\|F\|_{L^1_\nu} < \infty$ *(cf.* (1.45)*) and*

$$F(p) = p^{\alpha-1}\sum_{k=0}^{m} c_k p^{k\beta} + o(p^{\alpha-1+m\beta}) \text{ as } p \to 0^+ \text{ for all } m \le m_0 \in \mathbb{N}\cup\infty \tag{1.70}$$

for some α *and* β*, with* $\operatorname{Re}\alpha, \operatorname{Re}\beta > 0$*. Then as* $x \to \infty$ *along an arbitrary ray in* $\mathbb{H}$*, see* (1.65)*, we have* [4]

$$(\mathcal{L}F)(x) = \int_0^\infty e^{-xp}F(p)dp = \sum_{k=0}^{m} c_k\Gamma(k\beta+\alpha)x^{-\alpha-k\beta} + o\left(x^{-\alpha-m\beta}\right). \tag{1.71}$$

for any $m \le m_0$*. The asymptotic expansion (1.71) holds if* $(\mathcal{L}F)(x)$ *is replaced by* $\int_0^a F(p)e^{-px}dp$ *(a independent of x)* $F \in L^1(0,a)$*, and* F *has the same asymptotic series as above as* $p \to 0^+$*.*

PROOF By Lemma 1.68, the conclusion follows if it holds for the integral $\int_0^\varepsilon e^{-xp}F(p)dp$ for some fixed $\varepsilon > 0$. On the other hand, by assumption and the definition of asymptotic power series we have, for any $\delta > 0$, $F(p) = \sum_{k=0}^m c_k p^{k\beta+\alpha-1} + g(p)$ where $|g(p)| \le \delta |p^{\alpha-1+m\beta}|$ for $p \in (0,\varepsilon)$ if $\varepsilon = \varepsilon(\delta, m)$ is small enough. Following the calculations in Example 1.64 we get

$$\left|\int_0^\varepsilon e^{-xp}g(p)dp\right| \le \delta\int_0^\varepsilon e^{-x_1p}p^{\operatorname{Re}\alpha+m\operatorname{Re}\beta-1}dp \le C\delta|x^{-\alpha-m\beta}|,$$

noting that that $\frac{|x|}{x_1}$ is finite along a ray Now, following Example 1.64,

$$\int_0^\varepsilon e^{-px}\sum_{k=0}^m c_k p^{\alpha+k\beta-1}dp \sim \sum_{k=0}^m c_k\Gamma\left(\alpha+k\beta\right)x^{-\alpha-k\beta},$$

finishing the proof for $(\mathcal{L}F)(x)$. For the last part, note that $\int_0^a F(p)e^{-xp}dp = \int_0^\infty F(p)\chi_{[0,a]}(p)e^{-xp}dp$ and $F(p)\chi_{[0,a]}(p)$ satisfies the assumptions in the first part of the lemma. ∎

Note 1.72 Intuitively, we see that, for a fixed F, the larger $\operatorname{Re}x$ is, the more damped is the contribution of any region that is not very close to zero. The behavior of a Laplace transform is gotten from the immediate neighborhood of zero. This will be seen in the next example and is formalized in Watson's lemma following it.

[4] By writing $\int_0^a F(p)e^{-px}dp = \int_0^\infty F(p)e^{-px}\chi_{[0,a]}(p)dp$ where $\chi_{[0,a]}$ is the characteristic function of $[0,a]$, we see that the same result holds for a finite Laplace transform $\int_0^a F(p)e^{-px}dp$.

Note 1.73 Watson's lemma holds for $\int_0^{ae^{i\theta}} F(p)e^{-px}dp$ as $|x| \to \infty$ if the asymptotic behavior (1.70) is valid along a ray $\arg p = \theta$, where $F \in L^1(0, ae^{i\theta})$ $\arg(x)$ satisfies $\theta + \arg x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. The proof is manifest by changing variables $p \to pe^{i\theta}$, $x \to xe^{-i\theta}$ and applying Lemma 1.69.

Exercise 1.74 (A generalization of Watson's Lemma) Assume that for some $\varepsilon > 0$, we have $\sup_{|z|<\varepsilon} \|F(\cdot; z)\|_{L^1_\nu} = C < \infty$ and that

$$F(p; z) = p^{\alpha-1} \sum_{\substack{0\leqslant k\leqslant m\\ 0\leqslant l\leqslant n}} c_{k,l}\, p^{k\beta_1} z^{l\beta_2} + o(p^{\alpha-1+m\beta_1} z^{n\beta_2})$$

$$\text{as } (p, z) \to (0^+, 0) \text{ for all } (m, n) \leqslant (m_0, n_0) \in (\mathbb{N} \cup \infty)^2 \tag{1.75}$$

where $\operatorname{Re}\alpha, \operatorname{Re}\beta_1$ and $\operatorname{Re}\beta_2$ are positive. Then, show that [5]

$$\int_0^\infty e^{-xp} F\left(p, \frac{1}{x}\right) dp = \sum_{\substack{0\leqslant k\leqslant m\\ 0\leqslant l\leqslant n}} \frac{c_{kl}\Gamma(k\beta_1 + \alpha)}{x^{\alpha+k\beta_1+l\beta_2}} + o\left(x^{-\alpha-m\beta_1-n\beta_2}\right). \tag{1.76}$$

Corollary 1.77 (Laplace asymptotics, maximum at an endpoint) *Assume that F is continuously differentiable on $[0, a)$ (as usual when we close the interval we mean right derivative) and $F' > 0$ and g is continuous. Then*

$$\int_0^a e^{-\nu F(x)} g(x)dx \sim e^{-\nu F(0)} \frac{g(0)}{\nu F'(0)} \quad \textit{as } \nu \to \infty \tag{1.78}$$

PROOF By choosing $\tilde{F} = F(x) - F(0)$ we reduce to the case $F(0) = 0$. Since $F' > 0$, F is invertible near zero and, with $h(x) = F^{-1}(x)$, we have

$$\int_0^a e^{-\nu F(x)} g(x)dx = \int_0^{F(a)} e^{-\nu p} g(h(p))h'(p)dp \tag{1.79}$$

By continuity $g(h(p))h'(p) = g(0)h'(0) + o(1)$ as $p \to 0^+$. Noting that $h'(0) = 1/F'(0)$, the rest follows from Watson's lemma. ∎

Corollary 1.80 (Laplace asymptotics, maximum at an inner point) *Assume that $a > 0$, F is twice continuously differentiable on $(-a, a)$ $F'(0) = 0$ and $F''(x) > 0$ on $(-a, a)$, and that g is continuous. Then,*

$$\int_{-a}^a e^{-\nu F(x)} g(x)dx \sim e^{-\nu F(0)} g(0) \sqrt{\frac{2\pi}{\nu F''(0)}} \tag{1.81}$$

[5] It is tacitly assumed m and n are chosen so that no term in the sum is $o\left(x^{-\alpha-m\beta_1-n\beta_2}\right)$.

PROOF As in Corollary 1.77 without loss of generality we may assume $F(0) = 0$. Define $h(x) = \text{signum}(x)\sqrt{F(x)}$ and denote $\frac{1}{2}F''(0) = \lambda^2$. Clearly h is continuously differentiable away from zero. For x close to zero, we have $F(x) = \lambda^2 x^2 + o(x^2)$ and thus $h(x) = \lambda x + o(x)$ for small x. It is then easy to show that h is continuously differentiable on $(-a, a)$ and $h' > 0$. We calculate only the integral from 0 to a: the one from $-a$ to 0 can be computed similarly and has an equal contribution to the final estimate. We make the change of variables $h(x) = \sqrt{u}$ and and note that by continuity $g(\sqrt{u})/h'\left(h^{-1}(\sqrt{u})\right) \sim g(0)/h'(0)$ to obtain

$$\int_0^a e^{-\nu h^2(x)} g(x)dx = \int_0^{F(a)} e^{-\nu u} \frac{g(\sqrt{u})}{h'(h^{-1}(\sqrt{u}))} \frac{1}{2\sqrt{u}} du \sim \frac{g(0)\sqrt{2\pi}}{2\sqrt{\nu F''(0)}} \tag{1.82}$$

by Watson's lemma and the fact that $\Gamma(1/2) = \sqrt{\pi}$. ∎

Note: Only the leading order asymptotic calculations are given in Corollaries 1.77 and 1.80. Watson's Lemma can be used to determine higher order corrections in the asymptotic expansion if F and g are smooth enough near 0.

Exercise 1.83 *Formulate and prove a generalization of Lemma 1.80 for the case when* $F'(0) = \cdots = F^{2m-1}(0) = 0$ *and* $F^{2m}(0) > 0$.

Example: Asymptotics of the Γ function The Gamma function is defined by

$$\Gamma(x+1) \equiv x! = \int_0^\infty e^{-\tau}\tau^x d\tau, = \int_0^\infty e^{x\log\tau} e^{-\tau} d\tau \tag{1.84}$$

for $x > -1$.[6] $x\log\tau - \tau$ is maximal when $\tau = x$. This suggests rescaling $\tau = xs$. This leads to

$$\Gamma(x+1) = x^{x+1}\int_0^\infty e^{-x(s-\log s)}ds = x^{x+1}e^{-x}\int_{-1}^\infty \exp\left[-x\left(t - \log(1+t)\right)\right]dt \tag{1.85}$$

To put it in a form where one of the preceding Lemmas may be used, we introduce

$$q = t\left[\frac{2t - 2\log(1+t)}{t^2}\right]^{1/2} \tag{1.86}$$

Through Taylor series at $t = 0$, it is readily checked that $t \to q$ is an analytic change of variable near $t = 0$, with $q'(0) = 1$. Further, $t \to q$ is mononomic and maps the the real axis interval $(-1, \infty)$ to $q \in (-\infty, \infty)$. We define the unique inverse function to be $t = T(q)$. It follows from (1.85) that

$$\Gamma(x+1) = x^{x+1}e^{-x}\int_{-\infty}^\infty e^{-xq^2/2}\ T'(q)dq \tag{1.87}$$

[6]This representation is valid for complex x as well in the domain $\operatorname{Re} x > -1$.

We decompose the integral in (1.87) as $\int_{-\infty}^{0} + \int_{0}^{\infty}$. We introduce change of variable $q = -\sqrt{2p}$ in the first integral and $q = \sqrt{2p}$ in the second to obtain

$$\Gamma(x+1) = x^{x+1}e^{-x} \int_0^\infty \frac{e^{-px}}{\sqrt{2p}} \left(T'(-\sqrt{2p}) + T'(\sqrt{2p}\right) dp \tag{1.88}$$

Using Taylor series $T(q) = \sum_{j=1}^{\infty} b_j q^j$,

$$\frac{1}{\sqrt{2p}} \left(T'(-\sqrt{2p}) + T'(\sqrt{2p}\right) = \sum_{j=1, j=odd}^{\infty} 2jb_j \quad (2p)^{j/2-1}. \tag{1.89}$$

It follows from Watson's Lemma that

$$\Gamma(x+1) \sim x^{x+1}e^{-x} \sum_{j=1, j=odd}^{\infty} 2^{j/2}\Gamma(j/2)jb_j x^{-j/2} \tag{1.90}$$

The first few b_j is easily computed by substituting a truncation of $t = b_1 q + b_2 q^2 + b_3 q^3 + ..$ into (1.86) and equating like powers of q and solving resulting equations. This gives $b_1 = 1$, $b_3 = \frac{1}{36}$, $b_5 = \frac{1}{4320}$, the even b_j's being inconsequential in (1.90). Using $\Gamma(1/2) = \sqrt{\pi}$, the first few nonzero terms are

$$\Gamma(x+1) = \sqrt{2\pi} x^{x+1/2} e^{-x} \left(1 + \frac{1}{12x} + \frac{1}{288x^2} + O(x^{-3}) \right) \tag{1.91}$$

The three term evaluation at $x = 6$ gives 720.0088692 versus the exact value of 720. If the general term in the asymptotic expansion (1.90) is desired, we can use Lagrange formula for inversion of a series:

$$\begin{aligned} b_j = \frac{1}{2\pi i} \oint \frac{T(q)}{q^{j+1}} dq = \frac{1}{2\pi i} \oint t^2 (1+t)^{-1} \left[2t - 2\log(1+t) \right]^{-j/2-1} dt \\ = \frac{2^{-j/2-1}}{2\pi i} \oint \frac{(e^u - 1)^2}{(e^u - 1 - u)^{j/2+1}} du, \end{aligned} \tag{1.92}$$

where the closed loop contour integrals are assumed to circle the origin in the respective variables in the positive sense.

1.3 Oscillatory integrals and the stationary phase method

In this setting, an integral of a function against a rapidly oscillating exponential becomes small as the frequency of oscillation increases. Again we first look at the case where there is minimal regularity; the following is a version of the Riemann-Lebesgue lemma.

Proposition 1.93 *Assume $f \in L^1[a,b]$. Then $\int_a^b e^{ixt}f(t)dt \to 0$ as $x \to \pm\infty$. The same is true for $\int_{-\infty}^{\infty} e^{ixt}f(t)dt$ for $f \in L^1(\mathbb{R})$.*

It is enough to show the result on a set which is dense[7] in L^1. Since trigonometric polynomials are dense in the continuous functions on a compact set[8], say in $C[a,b]$ in the sup norm, and thus in $L^1[a,b]$, while continuous functions with compact support are dense in $L^1(\mathbb{R})$, it suffices to look at trigonometric polynomials, thus (by linearity), at e^{ikx} for fixed k; for the latter we just calculate explicitly the integral; we have

$$\int_a^b e^{ixs}e^{iks}ds = O(x^{-1}) \text{ for large } x. \qquad \square$$

No rate of decay of the integral in Proposition 1.93 follows without further knowledge about the regularity of f. With some regularity we have the following characterization.

Proposition 1.94 *For $\eta \in (0,1]$ let the $C^\eta[a,b]$ be the Hölder continuous functions of order η on $[a,b]$, i.e., the functions with the property that there is some constant $c > 0$ such that for all $x, x' \in [a,b]$ we have $|f(x) - f(x')| \le c|x - x'|^\eta$.*

(i) We have

$$f \in C^\eta[a,b] \Rightarrow \left|\int_a^b f(s)e^{ixs}ds\right| \le \frac{(b-a)}{2}c\pi^\eta x^{-\eta} + O(x^{-1}) \quad \textit{as} \quad x \to \infty \tag{1.95}$$

(ii) If $f \in L^1(\mathbb{R})$ and $|x|^\eta f(x) \in L^1(\mathbb{R})$ with $\eta \in (0,1]$, then its Fourier transform $\hat{f} = \int_{-\infty}^{\infty} f(s)e^{-ixs}ds$ is in $C^\eta(\mathbb{R})$.

(iii) Let $f \in L^1(\mathbb{R})$. If $x^n f \in L^1(\mathbb{R})$ with $n - 1 \in \mathbb{N}$ then $\hat{f}$ is n times differentiable. If for $A > 0$, $e^{A|x|}f \in L^1(\mathbb{R})$ then $\hat{f}$ extends analytically in a strip of width A centered on $\mathbb{R}$.

PROOF (i) By rescaling, we can choose $[a,b] = [0,1]$. We have as $x \to \infty$ ($\lfloor \cdot \rfloor$ denotes the integer part)

---

[7] A set of functions f_n which, collectively, are arbitrarily close to any function in L^1. Using such a set we can write

$$\int_a^b e^{ixt}f(t)dt = \int_a^b e^{ixt}(f(t) - f_n(t))dt + \int_a^b e^{ixt}f_n(t)dt$$

and the last two integrals can be made arbitrarily small.

[8] One can associate the density of trigonometric polynomials with approximation of functions by Fourier series.

$$
\begin{aligned}
&\left|\int_0^1 f(s)e^{ixs}ds\right| = \\
&\left|\sum_{j=0}^{\lfloor \frac{x}{2\pi}-1\rfloor}\left(\int_{2j\pi x^{-1}}^{(2j+1)\pi x^{-1}} f(s)e^{ixs}ds + \int_{(2j+1)\pi x^{-1}}^{(2j+2)\pi x^{-1}} f(s)e^{ixs}ds\right)\right| + O(x^{-1}) \\
&\qquad = \left|\sum_{j=0}^{\lfloor \frac{x}{2\pi}-1\rfloor}\int_{2j\pi x^{-1}}^{(2j+1)\pi x^{-1}} \big(f(s)-f(s+\pi/x)\big)e^{ixs}ds\right| + O(x^{-1}) \\
&\qquad\qquad \leq \sum_{j=0}^{\lfloor \frac{x}{2\pi}-1\rfloor} c\left(\frac{\pi}{x}\right)^{\eta}\frac{\pi}{x} \leq \frac{1}{2}c\pi^{\eta}x^{-\eta} + O(x^{-1}) \qquad (1.96)
\end{aligned}
$$

(ii) We see that

$$
\left|\frac{\hat f(s)-\hat f(s')}{(s-s')^{\eta}}\right| = \left|\int_{-\infty}^{\infty}\frac{e^{-ixs}-e^{-ixs'}}{x^{\eta}(s-s')^{\eta}}x^{\eta}f(x)dx\right| \leq \int_{-\infty}^{\infty}\left|\frac{e^{-ixs}-e^{-ixs'}}{(xs-xs')^{\eta}}\right|\left|x^{\eta}f(x)\right|dx \qquad (1.97)
$$

is bounded. Indeed, by elementary geometry we see that for $|\phi_1-\phi_2|<1$ we have

$$
|\exp(i\phi_1)-\exp(i\phi_2)| \leq |\phi_1-\phi_2| \leq |\phi_1-\phi_2|^{\eta} \qquad (1.98)
$$

while for $|\phi_1-\phi_2|\geq 1$ we see that

$$
|\exp(i\phi_1)-\exp(i\phi_2)| \leq 2 \leq 2|\phi_1-\phi_2|^{\eta}
$$

(iii) Take any $x \in S_A := \{x\in\mathbb{C} : |\mathrm{Im}\, x| < A\}$. Choose $A' < A$ so that $x\in S_{A'}$. Choose $h\in\mathbb{C}$ so that $|h|\leq\frac{A-A'}{2}$. Then,

$$
[D_h\hat f](x) := \frac{\hat f(x+h)-\hat f(x)}{h} = \int_{\mathbb{R}} f(s)e^{-ixs}\left(\frac{e^{-ihs}-1}{h}\right)ds
$$

and it is readily checked that $|e^{-ixs}\left(\frac{e^{-ihs}-1}{h}\right)| \leq Ce^{A|s|}$ and by the dominating convergence theorem $\hat f'(x) = \lim_{h\to 0}[D_h f](x) = \int_{\mathbb{R}} -isf(s)e^{-ixs}ds$ implying $\hat f$ is analytic in a strip of width A. ∎

Note 1.99 In Laplace type integrals Watson's lemma implies that it suffices for a function to be continuous to ensure an $O(x^{-1})$ decay of the integral, whereas in Fourier-like integrals, the considerably weaker decay (1.95) is optimal as seen in the exercise below.

Exercise 1.100 (*) (a) Consider the function f given by the *lacunary trigonometric series* $f(z)=\sum_{k=2^n,n\in\mathbb{N}} k^{-\eta}e^{ikz}$, $\eta\in(0,1)$. Show that $f\in C^{\eta}[0,2\pi]$.

We want to estimate $f(\phi_1) - f(\phi_2)$ in terms of $|\phi_1 - \phi_2|^\eta$, when $\phi_1 - \phi_2$ is small. We can take $\phi_1 - \phi_2 = 2^{-p}b$ with $|b| < 1$. use the first inequality in (1.98) to estimate the terms in with $n < p$ and the simple bound $2/k^\eta$ for $n \geq p$. Then it is seen that $\int_0^{2\pi} e^{-ijs} f(s)ds = 2\pi j^{-\eta}$ (if $j = 2^m$ and zero otherwise) and the decay of the Fourier transform is exactly given by (1.95).

(b) Use Proposition 1.94 and the result in Exercise 1.100 to show that the function $f(t) = \sum_{k=2^n, n\in\mathbb{N}} k^{-\eta} t^k$, analytic in the open unit disk, has no analytic continuation across the unit circle, that is, the unit circle is a *barrier of singularities for f.*

Note 1.101 *If we are dealing with an analytic function except for isolated singularities (or branch points), then decay is typically better than the one obtained in Proposition 1.94.*

Note. In part (i) of Proposition 1.94, compactness of the interval is crucial. In fact, the Fourier transform of an $L^2(\mathbb{R})$ entire function may not necessarily decrease pointwise. For example, consider $f = \mathcal{F}^{-1}\hat{f}$, where $\hat{f}(x) = 1$ for $x \in [n, n + e^{-n^2}]$ for $n \in \mathbb{N}$ and zero otherwise. Since $\hat{f} \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and for any $A > 0$, $e^{A|x|}\hat{f}(x) \in L^1(\mathbb{R})$, it follows that $f = \mathcal{F}^{-1}\hat{f}$ is entire. Yet $[\mathcal{F}f](x)$, which equals $\hat{f}(x)$ a.e., evidently does not decrease pointwise as $x \to \infty$.

Proposition 1.102 *Assume $f \in C^n[a, b]$. Then we have*

$$\begin{aligned}\int_a^b e^{ixt} f(t)dt &= e^{ixa} \sum_{k=1}^{n} c_k x^{-k} + e^{ixb} \sum_{k=1}^{n} d_k x^{-k} + o(x^{-n}) \\ &= e^{ixt} \left(\frac{f(t)}{ix} - \frac{f'(t)}{(ix)^2} + ... + (-1)^{n-1} \frac{f^{(n-1)}(t)}{(ix)^n} \right) \Bigg|_a^b + o(x^{-n}), \quad (1.103)\end{aligned}$$

where $c_k = -f^{(k-1)}(a)/i^k$ and $d_k = f^{(k-1)}(b)/i^k$.

PROOF This follows by integration by parts and the Riemann-Lebesgue lemma since

$$\begin{aligned}\int_a^b e^{ixt} f(t)dt = e^{ixt} \left(\frac{f(t)}{ix} - \frac{f'(t)}{(ix)^2} + ... + (-1)^{n-1} \frac{f^{(n-1)}(t)}{(ix)^n} \right) \Bigg|_a^b \\ + \frac{(-1)^n}{(ix)^n} \int_a^b f^{(n)}(t) e^{ixt} dt \quad (1.104)\end{aligned}$$

$\square$

Corollary 1.105 *(1) Assume $f \in C^\infty[0, 2\pi]$ is periodic with period 2π. Then $\int_0^{2\pi} f(t)e^{int}dt = o(n^{-m})$ for any $m > 0$ as $n \to +\infty, n \in \mathbb{Z}$.*

(2) Assume $f \in C_0^\infty[a, b]$ vanishes at the endpoints together with all derivatives; then $\hat{f}(x) = \int_a^b f(t)e^{ixt} = o(x^{-m})$ for any $m > 0$ as $x \to \pm\infty$.

Exercise 1.106 Show that if f is analytic in a neighborhood of $[a, b]$ but not entire, then both series in (1.103) have zero radius of convergence.

Exercise 1.107 In Corollary 1.105 (2) show that $\limsup_{x\to\infty} e^{\varepsilon|x|}|\hat{f}(x)| = \infty$ for any $\varepsilon > 0$ unless $f = 0$.

Exercise 1.108 For smooth f, the interior of the interval does not contribute because of cancellations: rework the argument in the proof of Proposition 1.94 under smoothness assumptions. If we write $f(s+\pi/x) = f(s) + f'(s)(\pi/x) + \frac{1}{2}f''(c)(\pi/x)^2$ cancellation is manifest.

Exercise 1.109 Show that if f is piecewise differentiable and the derivative is in L^1, then the Fourier transform is $O(x^{-1})$.

1.4 Steepest descent method

We seek to determine the asymptotic behavior of $I(\nu)$ as $\nu \to +\infty$, where

$$I(\nu) = \int_{\mathcal{C}} g(z)e^{-\nu f(z)}dz \tag{1.110}$$

for f and g that are analytic in some some region of the complex plane[9], and $\mathcal{C}$ is some simple curve that may be finite or infinite. Further, we may assume f is not a constant, as otherwise the asymptotics is trivial. The problem is to determine the asymptotics of I as $\nu \to +\infty$. More generally, if $\nu \to \infty$ along some complex ray $\arg\nu = \phi$, we can replace ν by $|\nu|$ and f by $e^{i\phi}f$ in the ensuing discussion to obtain asymptotics along complex rays.

The idea of the steepest descent method is to use the analyticity of the integrand in (1.110) in z to deform $\mathcal{C}$ homotopically into one or more paths, each of which is characterized by $\operatorname{Im} f = c$, a constant. If $\mathcal{C}$ is homotopically equivalent to just one steepest descent path $\mathcal{C}_s = \{z : z = \gamma(t), a \le t \le b\}$, where γ' exists (and assumed nonzero, without loss of generality) then we may rewrite (1.110)

$$I(\nu) = e^{-i\nu c}\int_a^b g(\gamma(t)) \exp\left[-\nu\left(f(\gamma(t)) - ic\right)\right]\gamma'(t)dt \tag{1.111}$$

[9]The region of analyticity will be dictated by the need to deform C into one or more steepest descent paths and will depend on the specifics of the problem.

Since $f(\gamma(t)) - ic$ is by assumption real valued for $t \in (a, b)$ and $g(\gamma(t))\gamma'(t)$ can decomposed into real and imaginary parts. After breaking up the integral into subintervals where $f - ic$ is monotonic, Watson's lemma can be applied to determine the complete asymptotic expansion of $I(\nu)$. Indeed, for analytic f, the real valued funciton $f(\gamma(t)) - ic$, which is the same as $\operatorname{Re} f$, is monotonically increasing or decreasing in any interval in t that does not contain a singular point of f or a *saddle* point where $f' = 0$ (The term *saddle* refers to the behavior of the harmonic function $\operatorname{Re} f(x + iy)$ for (x, y) near a critical point, where $f' = 0$).

Generally, multiple steepest descent paths, each with a different value of c, are involved in homotopic deformation of $\int_{\mathcal{C}}$; these paths join up at *sinks* where $\operatorname{Re} f \to +\infty$. Multiple descent paths will definitely be needed when $\operatorname{Im} f$ is different at the end points of $\mathcal{C}$, as in the example in §1.4a In such cases, the calculation of $I(\nu)$ generally requires adding up the contributions on each steepest descent path $\int_{\mathcal{C}_s}$ in the manner outlined in the last paragraph. Therefore, the only new element in the steepest descent method is to determine steepest curves which are homotopically equivalent to the original path $\mathcal{C}$. For a point on each such curve, $\operatorname{Re} f$ varies most rapidly relative to all other directions, as may be concluded easily from applying Cauchy-Riemann conditions. This explains the terminology *steepest* descent[10] It should be further noted that without homotopic deformation into descent paths, (1.110) will generally be an oscillatory integral; asymptotics obtained through the stationary phase method leads to substantially weaker results, see note 1.99. The stationary phase method, however, does not require analyticity of f and g.

1.4a Simple illustrative example

Consider

$$I(\nu) = \int_0^1 \frac{e^{i\nu z^2}}{z+1} dz \tag{1.112}$$

In the notation of (1.110), $f(z) = -iz^2$, $g(z) = \frac{1}{z+1}$. Steepest descent paths emanating at $z = 0$ are determined by

$$\operatorname{Im} f = \operatorname{Im} f(0) \quad \text{implying } \operatorname{Re} z^2 = 0, \quad i.e.\ z = re^{\pm i\pi/4} \quad \text{for} \quad r \in (-\infty, \infty) \tag{1.113}$$

However, since $Ref \to +\infty$, when $z = re^{i\pi/4}$ as $r \to \infty$, it follows that $\infty e^{i\pi/4}$ is a sink that is connected to $z = 0$ along the steepest descent path $z = re^{i\pi/4}$. Steepest descent paths from the other end point $z = 1$ in the integral (1.112)

[10] This terminology is confusing, since descent or ascent depends on the direction a path is traversed. Calling it steepest variation path is more appropriate; nonetheless, we will stick to the standard terminology.

is found by setting

$$\operatorname{Im} f = \operatorname{Im} f(1) = -1 \;\; \text{implying} \;\; \operatorname{Re} z^2 = 1, \;\; i.e.\text{hyperbolic path } x^2 - y^2 = 1 \tag{1.114}$$

Since only one branch of this hyperbola passes through (1, 0) and asymptotes to $y = x$, *i.e.* approaches the sink $\infty e^{i\pi/4}$, a homotopic deformation of the $\int_0^1$ may be made to coincide with descent paths $z = re^{i\pi/4}$, $0 \le r < \infty$ followed by integration along steepest descent path C that connects $\infty e^{i\pi/4}$ to 1 along the hyperbola[11] $x^2 - y^2 = 1$. Therefore,

$$I(\nu) = \int_0^{\infty e^{i\pi/4}} \frac{e^{i\nu z^2}}{1+z} dz + \int_C \frac{e^{i\nu z^2}}{1+z} dz \equiv I_1(\nu) + I_2(\nu) \tag{1.115}$$

For $I_1(\nu)$, using $z = re^{i\pi/4}$ for $0 < r < \infty$, we obtain after change of variable and application of Watson's Lemma

$$\begin{aligned} I_1(\nu) = e^{i\pi/4} \int_0^\infty \frac{e^{-\nu r^2}}{1+re^{i\pi/4}} dr &= e^{i\pi/4} \int_0^\infty \frac{e^{-\nu p} dp}{2p^{1/2}[1+p^{1/2}e^{i\pi/4}]} \\ &\sim \frac{1}{2} e^{i\pi/4} \sum_{j=0}^\infty (-1)^j \Gamma\left(\frac{j+1}{2}\right) e^{ij\pi/4} \nu^{-(j+1)/2} \end{aligned} \tag{1.116}$$

As far as $I_2(\nu)$, we know $p = f(z) - f(1) = -iz^2 + i$ is real valued and monotonically increasing on the parabolic path C from $z = 1$ to $z = \infty e^{i\pi/4}$, since $f' \neq 0$ on this path. Therefore, solving for z, inversion leads to

$$z = Z(P) = (1 + ip)^{1/2}, \tag{1.117}$$

where we can readily check that for this branch of square-root, as $p \to +\infty$, $z \to \infty e^{i\pi/4}$ as required. Therefore,

$$I_2(\nu) = -e^{i\nu} \int_0^\infty \frac{e^{-p\nu}}{1 + Z(p)} Z'(p) dp. \tag{1.118}$$

We note that Taylor expansion:

$$\frac{Z'(p)}{1+Z(p)} = \frac{i}{2}(1+ip)^{-1/2} \left[1 + (1+ip)^{1/2}\right]^{-1} = \sum_{j=0}^\infty a_j p^j, \tag{1.119}$$

where the first few coefficents are: $a_0 = \frac{i}{4}$, $a_1 = \frac{3}{16}$, $a_2 = -\frac{5i}{32}$, $a_3 = -\frac{35}{256}$. Applying Watson's Lemma to (1.118), it follows

$$I_2(\nu) \sim -e^{i\nu} \sum_{j=0}^\infty a_j \nu^{-j-1} \Gamma(j+1), \tag{1.120}$$

[11] We do not have the option of going along $re^{-i\pi/4}$, $0 < r < \infty$ since $\operatorname{Re} f \to -\infty$ and so contribution at $\infty e^{-i\pi/4}$ cannot be ignored as it can be for a *sink*.

The full asymptotic expansion of $I(\nu) = I_1(\nu) + I_2(\nu)$ is then obvious from (1.116) and (1.120).

Remark 1.121 A change of variable $\zeta = z^2$ at the outset in (1.112) converts the problem into $I(\nu) = \int_0^1 \frac{e^{i\nu\zeta}d\zeta}{2(\zeta^{1/2}+\zeta)}$, corresponding to which the steepest descent lines connecting each end points are given by $\zeta = ir$ and $\zeta = 1 + ir$ respectively. However, after the change of variables the integrand is generically not explicit. In such cases, finding the steepest descent lines cannot be done explicitly either. Fortunately, descent lines are connected to ODEs amenable to phase plane analysis and we will exploit this connection in the following section.

Note 1.122 If we replace the integrand $\frac{e^{i\nu z^2}}{z+1}$ in (1.112), by $\frac{e^{i\nu z^2}}{z-z_0}$, where z_0 is in the upper-half plane region between $e^{i\pi/4}\mathbb{R}^+$ and steepest descent contour C connecting $\infty e^{i\pi/4}$ to 1, for *e.g.* $z_0 = \frac{1+i}{2}$, then the singulariy at $z = z_0$ interferes with the homotopic deformation into steepest descent paths. Nonetheless, since this singularity is a pole, after collecting residue at $z = z_0$, we can use the same descent paths as in Example 1.4a. Since $\operatorname{Im} z_0^2 > 0$, the residue contribution will be exponentially small in ν relative to (1.120) and (1.116). If this z_0 were a branch point instead, in addition to the steepest descent paths, the homotopically deformed path will include a contour that wraps around z_0. Nonetheless, as in the case of the pole, the contribution of the branch point is exponentially small in ν.

1.4b Finding the steepest variation lines

Prior discussion shows that the main challenge in evaluation of asymptotic behavior of

$$\int_{\mathcal{C}} g(z) \exp(-\nu f(z)) dz \tag{1.123}$$

is the determination of steepest descent path(s) that are homotopically equivalent to $\mathcal{C}$. We now discuss how steepest descent paths may be found when $f(z)$ is not as simple as shown in Example 1.4a.

To simplify the discussion, we will assume that both f and g are entire, and if parts of $\mathcal{C}'$ extend to infinity, the integral along those parts converges. If the functions are not entire, then the contours can be deformed inside the domain of analyticity, and beyond that only in special cases, for instance when the singularities of g are poles. If an integral extends to infinity and the integral would not converge, then we truncate the contour at some large enough z_0 (see Note 1.132) at the price of introducing exponentially small relative errors in the estimates.

When v is very simple, as in 1.4a, one can just plot the curves $v(z) = C$. If not, we can use tools from elementary ODE analysis to find these lines.

If along a curve $z(t) = (x(t), y(t))$ we have $v(z) = C$, then

$$\frac{\partial v}{\partial x}\frac{dx}{dt} + \frac{\partial v}{\partial x}\frac{dy}{dt} = 0 \tag{1.124}$$

which happens along the solution curves of the system

$$\begin{aligned} \frac{dx}{dt} &= \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = \mathrm{Re}\,(f'(z)) \\ \frac{dy}{dt} &= -\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y} = -\mathrm{Im}\,(f'(z) \end{aligned} \tag{1.125}$$

where we used $v = \mathrm{Im}\, f$ to write the system in terms of f'.

Note 1.126 The system (1.125) is autonomous, and the task is to draw the phase portrait. The direction field is parallel with ∇u, that is, it points toward steepest ascent directions of u or steepest descent of $e^{-\nu u}$. To draw the phase portrait more easily we note that:

1. Eq. (1.125) is at the same time a Hamiltonian system as well as a gradient one.
2. There are no closed trajectories since f, thus v, are not identically constant. Indeed, $v = \mathrm{Im}\, f$ is harmonic, and a harmonic function in a domain attains its maximum and minimum value on the boundary; since we are dealing with a level set of v, call it γ, if γ is closed then $\max v = \min v$ in the $\mathrm{int}(\gamma)$ implying that v is constant in an open set, thus constant everywhere.
3. All critical points of the field ($\dot{x} = \dot{y} = 0$) are *saddle points*, the points of interest for our analysis. Indeed, v cannot have, by the maximum modulus principle already used in 2, any interior maxima or minima. (If f is not entire, then of course singularities of f are also singularities of the field.)
4. At a critical point z_0 we have

$$f'(z_0) = 0 \tag{1.127}$$

by (1.125) and (1.127), the local behavior of u near z_0 is

$$u(z) - u(z_0) = \frac{1}{k!}\mathrm{Re}\,\left(f^{(k)}(z_0)(z - z_0)^k\right)(1 + o(1)) \tag{1.128}$$

where k, generically $k = 2$, is the smallest such that $f^{(k)}(z_0) \neq 0$. This is a simple way to plot the directions of steepest ascent of u at z_0. These are the directions

$$f^{(k)}(z_0)(z - z_0)^k \in \mathbb{R}^+ \tag{1.129}$$

5. Trajectories can only intersect at critical points of the field.

6. The properties above, together with the behavior of f at infinity completely determine the topology of the direction field.

7. To find the steepest descent line decomposition of a contour $\mathcal{C}$ we let every point $z_0 = x_0+iy_0 \in \mathcal{C}$ flow with the field: $(x_0, y_0) \mapsto x(t; x_0), (y(t; y_0))$; we denote the set of such points by $\mathcal{C}(t)$. The connected components of the limiting set:
$$\{z : \lim_{t\to\infty} d(z, \mathcal{C}(t)) = 0\}$$
represent the sought-for decomposition.

8. By construction, on each $\mathcal{C}_i$, u is strictly monotonic and v is constant, thus f is one-to-one, and the change of variables $f(z) = f(z_i) + \zeta$ where z_i is an endpoint of $\mathcal{C}_i$, brings the integrals to a Watson's lemma form, see Note 1.134.

9. The asymptotic expansions are collected from the endpoints of the steepest descent lines from which u increases, since $e^{-\nu u}$ decreases rapidly starting from such a point.

We illustrate this on a simple example: we start with the integral

$$\int_{\infty e^{i\pi/4}}^{\infty e^{5\pi i/4}} e^{-\nu(z-z^4/4)}dz \tag{1.130}$$

where $\nu \to +\infty$, and the integral is taken along any curve $\mathcal{C}$ is staring at $+i\infty$ and ending at $-\infty$. Because of the rapid decay in z, the integral converges. We want to find a curve homotopic to $\mathcal{C}$ that consists of paths of steepest descent of e^{-u}. In this example, (1.125) becomes

$$\begin{aligned} \frac{dx}{dt} &= 1 - x^3 + 3xy^2 \\ \frac{dy}{dt} &= 3x^2y - y^3 \end{aligned} \tag{1.131}$$

The equilibria of (1.131) are, by (1.127) the solutions of $1 - z^3 = 0 :\ z_k = e^{2k\pi i/3},\ k = 0, ..., 2$ and near a critical point the directions of descent of e^{-u} are obtained from (1.128), $-3z_k^2(z - z_k)^2 \in \mathbb{R}^+$.

For large $t = |t|e^{i\phi}$, we have $f = -|t|^4e^{4i\phi}(1+o(1))$, and thus asymptotically there are, up to homotopies, four curves of steepest descent of e^{-u}, $\cos(4\phi) = -1$ and four of steepest ascent, $\cos(4\phi) = 1$. All needed qualitative features of the phase portrait, sketched in Fig. 1.1, follow from this information and the fact that trajectories do not intersect except at critical points. In the phase portrait, the arrows point towards steepest descent. We illustrate the detailed arguments that leads one to Fig. 1.1 by showing how we can argue where each

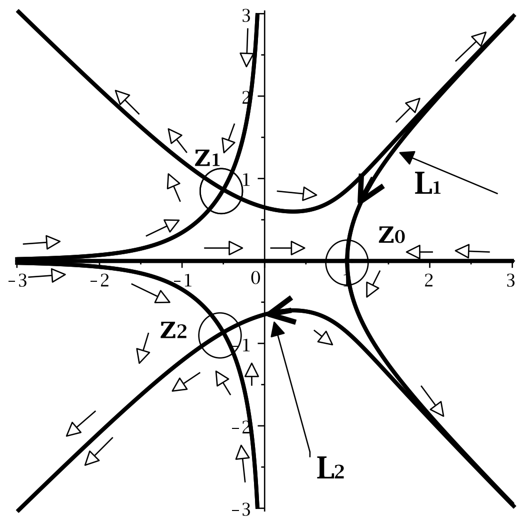

FIGURE 1.1: Phase portrait of (1.131). The white arrows point towards steepest ascent directions of u (steepest descent of $e^{-\nu u}$). The orientations of the paths L_1 and L_2 are shown with dark arrows; $z_k = \exp(2k\pi i/3)$, $k = 0, 1, 2$ are the saddle points.

of the two steepest descent and ascent lines for e^{-u} emanating at the saddle $z_2 = e^{i4\pi/3}$ must end up. First, note that each of the descent paths must end up at sinks $\infty e^{-i\pi/4}$ or $\infty e^{-3i\pi/4}$ since the paths cannot cross the real axis, which is an invariant set of the dynamical system (1.131). Each of the two ascent paths at z_2 must end up at $-\infty$ or $-i\infty$, since they cannot cross the real axis or approach $+\infty$ without crossing the lower-half plane descent path emanating at the sadde $z_0 = 1$. Further, noting that the two ascent or the two descent paths cannot approach the same *sink* or *source* at ∞ without crossing each other, we are qualitatively led to Fig. 1.1.

Note 1.132 Note that if a path of integration starts at ∞ in some direction and ends at ∞ in some other direction, then for large t on the curve the arrows should point towards infinity to ensure convergence of the integral. This is indeed the case for (1.130). The steepest descent line decomposition for (1.130) consists of the curve L_1 joining $\infty e^{i\pi/4}$ to $\infty e^{-i\pi/4}$ passing through the saddle $z_0 = 1$ together with the curve L_2 connecting $\infty e^{-i\pi/4}$ to $\infty e^{i5\pi/4}$ passing through the saddle $z_2 = e^{4i\pi/3}$, as shown in Fig 1.1.

Note 1.133 If the example above were modified to $\int_{\infty e^{i\pi/4}}^{\infty e^{5\pi i/4}} g(z) e^{-\nu(z - z^4/4)} dz$, where $g(z)$ grows too fast along $\infty e^{-i\pi/4}$ to allow meaningful homotopic deformation as shown in Fig 1.1, for *e.g.* $g(z) = \exp\left[e^{-i\pi/6} z^6\right]$, then we truncate the paths L_1 and L_2 at some large enough z_{L_1}, z_{L_2} independent of ν. With such a choice, it is easily seen that the straight line path connecting the two points is exponentially small relative to the saddle point contributions.

Note 1.134 (Connection with Watson's Lemma) For a general entire f, the set of saddle points through which the steepest variation curve passes cannot have accumulation points, because of the assumed analyticity of f. Then along any steepest descent line, the equation $u(x(t), y(t)) = T$ has a unique solution, and $T(u)$ is smooth except at the saddle points where it has algebraic singularities. Furthermore, by construction, $\exp(iv(x(t), y(t)) =$ *const* along such a curve. The change of variables $f(z) = f(z_0) + t$ brings the problem to the Laplace form to which Watson's lemma applies.

Exercise 1.135 *Use the analysis in this section to find the asymptotic behavior of (i)* (1.130)*, and (ii) of* $\int_i^{3+i} e^{-\nu(t-t^4/4)}dt$.

1.4b.1 A singular example

Consider the problem of finding the asymptotic behavior of the Taylor coefficients c_k in the expansion

$$e^{\frac{1}{1-z}} = \sum_{k=0}^{\infty} c_k z^k, \quad |z| < 1 \tag{1.136}$$

We have

$$c_{k-1} = \frac{1}{2\pi i} \oint_{|s|=r<1} \frac{e^{\frac{1}{1-s}}}{s^k} ds = \frac{1}{2\pi i} \oint_{|s|=r<1} e^{\frac{1}{1-s} - k \ln s} ds \tag{1.137}$$

The rightmost integral is of the general form (1.123). What distinguishes this case from the case we considered throughout this section is that $g(z) = e^{\frac{1}{1-z}}$ has an essential singularity at $z = 1$.

The steepest descent lines of f are simply rays towards ∞, but *it is not possible to deform the* $|s| = r$ *path along these lines of steepest descent*, since the singularity at $z = 1$ is not integrable The function g contributes nontrivially to the geometry of the curves of interest. We instead plot the steepest descent lines of $h(s; k) = \frac{1}{1-s} - k \ln s$ for fixed k and let $k \to \infty$; we see that $h(s; k)$ has two saddle points, at $s = 1 \pm k^{-1/2}(1 + o(1))$ We expect that the behavior of c_k relates to the behavior of h on a scale of order $k^{-1/2}$ near $s = 1$. This becomes obvious if we change variable to $s = 1 + k^{-1/2}u$. In anticipation, we deform the contour-clockwise contour $|s| = r$ into the contour shown in Fig 1.2. We note the the cancelation of contributions from coinciding straightline positve real axis segments traversed in opposite directions. Furthermore, the integrand $s^{-k}e^{1/(1-s)}$ vanishes rapidly enough so that the counter-clockwise circular arc of radius R as $R \to \infty$ contributes nothing. We are simply left

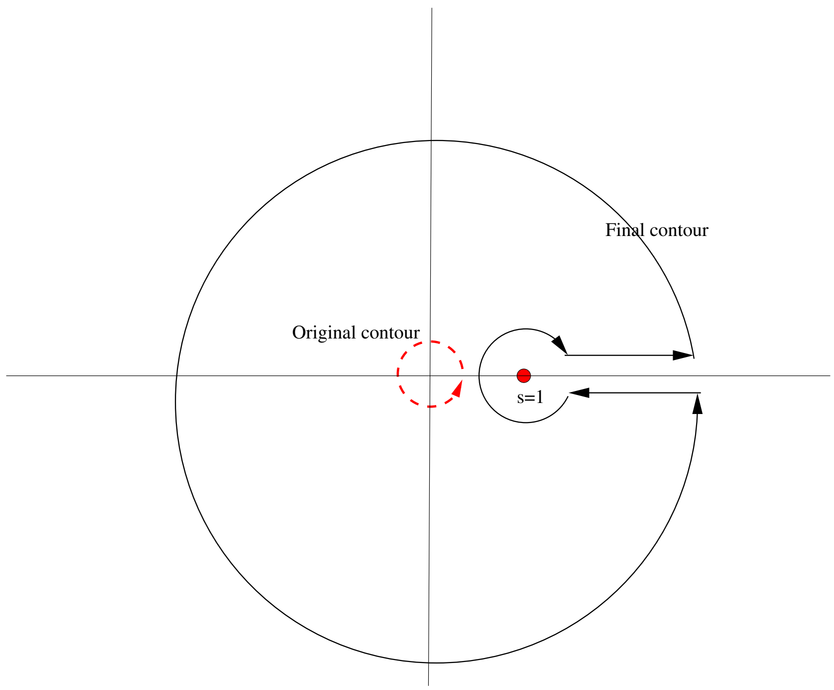

FIGURE 1.2: Deformation of contour for (1.138). The circle around $s = 1$ has radius of $k^{-1/2}$. The large contour can be pushed to infinity and the integrals along the sides of $\mathbb{R}^+$ cancel each-other by single-valuedness.

with a clockwise closed contour C around $s = 1$. We write $s = 1 + u/\nu$, with $\nu = k^{1/2}$ and we have

$$c_{k-1} = \nu^{-1} \frac{1}{2\pi i} \oint_{|u|=1} \exp\left[-\nu(u + u^{-1}) - \nu^2[\ln(1 + u/\nu) - u/\nu]\right] du \quad (1.138)$$

We note that the function

$$z^{-2}[\ln(1 + zu) - zu] = -\tfrac{1}{2}u^2 + \tfrac{1}{3}zu^3 + \cdots \quad (1.139)$$

is analytic at $z = 0$ and we can expand convergently in $z = 1/k$, as $k \to \infty$

$$\exp\left[-\nu^2[\ln(1 + u/\nu) - u/\nu]\right] = e^{u^2/2}\left[1 + \frac{u^3}{3\nu} - \frac{u^4}{4\nu^2} + \frac{u^6}{18\nu^2} + \cdots\right] \quad (1.140)$$

Noting that the saddle point of $e^{-\nu f(u)}$, with $f(u) = u + 1/u$ is at $u = \pm 1$, it is clear that the counter-clockwise C may be chosen to coincide with $|u| = 1$, which is a steepest descent path since $\operatorname{Im} f = 0$ on $u = e^{i\theta}$. We get

$$c_{k-1} = \frac{1}{2\pi i\nu}\oint_{|u|=1} e^{-\nu(u+1/u)+u^2/2}(1+\tfrac{1}{\nu}F_1(u,\tfrac{1}{\nu}))$$
$$= \frac{1}{2\pi i\nu}\oint_{|u|=1} e^{-\nu(u+1/u)+u^2/2}\left(1+\frac{1}{\nu}F_1(u,1/\nu)\right)du \quad (1.141)$$

where $F_1(u,z)$ is analytic in $(z,u)\in\mathbb{D}_{\frac{1}{2}}\times T$ where T is a neighborhood of the circle $\partial\mathbb{D}_1$. Now the substitution $u+1/u=-2+v$ brings the integral to a sum of two integrals, for each of which Exercise 1.74 applies. This gives, to leading order,

$$c_{k-1} = \frac{e^{2\sqrt{k}}}{2\sqrt{\pi e}k^{3/4}}(1+o(1)) \quad (1.142)$$

Alternately, we may use $u = e^{i\theta}$ and use Laplace's method to each of the following integrals to obtain the same result

$$\frac{1}{2\pi\nu}\left(\int_{-\pi/2}^{\pi/2}+\int_{\pi/2}^{3\pi/2}\right)d\theta e^{-2\nu\cos\theta}\exp\left[\frac{1}{2}e^{2i\theta}+i\theta\right] \quad (1.143)$$

It is to be noted that the contribution from the saddle $u=+1$ is exponentially small in k relative to the contribution from $u=-1$.

Higher order corrections are obtained more simply as follows. We note that $f(z)=\exp(1/(1-z))$ satisfies the ODE

$$(1-z)^2f'(z)-f(z)=0 \quad (1.144)$$

The general analytic theory of ODEs implies that there is a on-parameter family of solutions analytic at zero of the form $f(z) = C\sum_{k=0}^{\infty}c_kz^k$. On inserting this power series into (1.144) and collecting like coefficients of powers of z, we obtain recurrence relation for c_k. With normalization $c_0 = 1$, we obtain $C=1$ in order that $f(0)=e$. Recurrence relation shows $c_1=\frac{1}{2}c_0=\frac{1}{2}$, while for $k\geq 2$,

$$c_k = (2-1/k)c_{k-1}-(1-2/k)c_{k-2} \quad (1.145)$$

from which we can get, as we will see in the sequel, the asymptotic behavior of c_k by seeking formal asymptotic solutions of (1.145).

1.5 Formal and actual solutions

Consider the differential equation

$$\frac{df}{dz} = f+f^2+zf^3;\quad f(0)=1 \quad (1.146)$$

which we analyze in a neighborhood of $z = 0$. The general analytic theory of ODEs ensures existence and uniqueness and analyticity of the solution in a neighborhood of $z = 0$. We can calculate the power series solution in a number of ways, for instance by substituting $f(z) = \sum_{k=0}^{\infty} c_k z^k$ into (1.146) and identifying the coefficients c_k. We get

$$f(z) = 1 + 2z + \frac{5}{2}z^2 + \frac{11}{6}z^3 - z^4 \cdots \tag{1.147}$$

If we write the equation in integral form

$$f(z) = 1 + \int_0^z \left[f(s) + f^2(s) + sf^3(s)\right] ds$$

and iterate,

$$f_{n+1}(z) = 1 + \int_0^z (f_n(s) + f_n^2(s) + sf_n^3(s))ds; \quad f_0(z) \equiv 1 \tag{1.148}$$

we can check that, for small z the sequence $\{f_k\}_k$ is uniformly Cauchy, and thus convergent. This can be seen using the fact that if a function h is bounded and integrable, then

$$\left|\int_0^z h(s)ds\right| \le |z| \max_{|s|<|z|} |h(s)| \tag{1.149}$$

The recurrence (1.148) can be used to generate the power series at zero, by inductively replacing f_n by its Maclaurin series truncated to $O(z^n)$ and integrating the resulting series term by term. We will not go over the details here, as we will develop more general tools shortly.

Consider instead the equation

$$\frac{dg}{dz} = z^{-2}g(z) - z^{-1} \tag{1.150}$$

The point $z = 0$ is a singular point of (1.150), in fact an irregular singular point; there are no analytic solutions near zero. An initial value at $z = 0$ is not well defined. Nonetheless, we can find a formal power series formal solutions $\sum_{k=1}^{\infty} c_k z^k$. In this simple example it is easy to insert the power series into (1.150) and identify the coefficients. We get $c_k = \Gamma(k)$, and formally

$$g(z) \quad \text{``}=\text{''} \quad \sum_{k=1}^{\infty} \Gamma(k) z^k, \tag{1.151}$$

where the power series in (1.151) has zero radius of convergence. To generate the power series inductively, we now note that, if we formally differentiate $g(z) = O_s(z^n)$, then $g'(z) = O(z^{n-1}) \ll z^{-2}g(z)$. It then follows that the natural direction of iteration, one in which we place the lower order terms on the right side of the equation is

$$g_{n+1}(z) = z^2 g_n'(z) + z; \quad g_0(z) = z \tag{1.152}$$

The iteration (1.152) is well defined, and it is solved by the polynomial $g_n(z) = \sum_{k=1}^{n+1} \Gamma(k) z^k$. The sequence of polynomials has no limit. Whenever the size of the term containing the highest derivative is formally small with respect to terms involving lower order derivatives, the natural direction of iteration would place the highest derivative on the right side of the equation. However, in general, we cannot bound g' in terms of g, so an iteration of the type (1.152) is not expected to converge. Does the expansion (1.151) relate in any way to the solutions of (1.150)? In this example, we can write down the exact solution of the equation as

$$g(z) = Ce^{-1/z} - e^{-1/z} \int_1^z s^{-1} e^{1/s} ds \tag{1.153}$$

The change of variables $s = 1/t, z = 1/x$ brings (1.153) to the form

$$\begin{aligned} g(1/x) = Ce^{-x} + e^{-x} \int_1^x t^{-1} e^t dt = Ce^{-x} - e^{-x} \int_{-\infty}^1 t^{-1} e^t + e^{-x} \int_{-\infty}^x t^{-1} e^t dt \\ =: C_2 e^{-x} + e^{-x} \int_{-\infty}^x t^{-1} e^t dt = \int_0^\infty \frac{e^{-xu}}{1-u} du + C_2 e^{-x} \end{aligned} \tag{1.154}$$

where the contour of integration avoids $t = 0$ and $u = 1$. Watson's lemma shows that $g(z) \sim \sum_{k=1}^\infty \Gamma(k) z^k$. What we see is that the formal power series solution is, in this case as well as in (1.146), the Maclaurin series as $z \to 0^+$ of some solution (here, in fact, all solutions have the same Maclaurin series). Only now the Maclaurin series diverges. The fact that formal solutions are asymptotic to actual ones is true in much wider generality, as we will see in the sequel.

1.5a An irregular singular point of a nonlinear equation

Consider Abel's equation

$$y' = y^3 + x \tag{1.155}$$

in the limit $x \to +\infty$. We first find the asymptotic behavior of solutions formally, and then justify the argument. We use the method of *dominant balance* that we will discuss in detail later. As x becomes large, y, y', or both need to become large if the equation (1.155) is to hold. Assume first that the balance is between y' and x and that $y \ll x$. If $y' \sim x$ then we have $y \sim x^2/2$ and $y^3 \sim x^6/8$, and this is inconsistent since it would imply $x^8/8 = O(x)$. Now, if we assume $x \ll y^3$ then the balance would be $y' \approx y^3$, implying $y \sim -\frac{1}{2}(x - x_0)^{-2}$; but this is small for $x - x_0 \gg 1$, which conflicts with what we assumed, $x \ll y^3$. We have one possibility left: $y = \alpha x^{1/3}(1+o(1))$, where $\alpha^3 = 1$, which assuming differentiability implies $y' = O(x^{-2/3})$ which is now consistent. We substitute

$$y = \alpha x^{1/3}(1 + v(x)) \tag{1.156}$$

in (1.155); for definiteness, we choose $\alpha = e^{i\pi/3}$, though any cube root of -1 would suffice. We get

$$\alpha x^{1/3}v' + 3xv + 3xv^2 + xv^3 + \frac{\alpha}{3}x^{-2/3} + \frac{\alpha}{3}x^{-2/3}v = 0 \tag{1.157}$$

Now a consistent balance is between $3xv$ and $-\frac{\alpha}{3}x^{-2/3}$ meaning that $v = O(x^{-5/3})$. This makes the nonlinear terms small and, for the purpose of justifying the analysis, we don't need to further expand v. We now aim at writing (1.157) in a suitable integral form. We first place the formally largest term(s) containing v *and* v' on the left side and the smaller terms as well as the terms not depending on v on the right side:

$$\alpha x^{1/3}v' + 3xv = h(x, v(x));\ -h(x, v(x)) := 3xv^2 + xv^3 + \frac{\alpha}{3}x^{-2/3} + \frac{\alpha}{3}x^{-2/3}v \tag{1.158}$$

We treat (1.158) as a linear inhomogeneous equation, and solve it thinking for the moment that h is given.

This leads to

$$v = \mathcal{N}(v);$$
$$\mathcal{N}(v) := Ce^{-\frac{9}{5\alpha}x^{5/3}} + \frac{1}{\alpha}e^{-\frac{9}{5\alpha}x^{5/3}}\int_{x_0}^{x} e^{\frac{9}{5\alpha}s^{5/3}}s^{-1/3}h(s, v(s))ds \tag{1.159}$$

We chose the limits of integration in such a way that the integrand is maximal when $s = x$: if $x \to +\infty$, then $x^{-1/3}e^{\frac{9}{5\alpha}x^{5/3}} \to \infty$, and our choice corresponds indeed to this prescription.

The largest of the terms not containing v on the right side of (1.159) comes from the term $\frac{\alpha}{3}x^{-2/3}$ in h, and is of the order $\frac{1}{3}x^{-5/3}(1 + o(1))$. Indeed, for $\operatorname{Re} b > 0$ by Watson's Lemma or simply by L'Hospital we get

$$\frac{\int_a^x e^{bs^m}/s^n ds}{e^{bx^m}/x^n} \sim b^{-1}m^{-1}x^{1-m};\quad x \to +\infty \tag{1.160}$$

Again by dominant balance, we expect $v = O(x^{-5/3})$. Thus, it is natural to choose x_0 large enough and introduce the Banach space

$$\{f : \|f\| := \sup_{x>x_0} |x^{5/3}f(x)| < \infty\} \tag{1.161}$$

or the region $|x| > x_0$ in a sector $\mathcal{S}$ in the complex domain where Re $\left(\frac{1}{\alpha}x^{5/3}\right) > 0$: $\arg x \in (-\frac{\pi}{10}, \frac{\pi}{2})$:

$$\mathcal{B} = \{f : \|f\| := \sup_{x\in\mathcal{S}} |x^{5/3}f(x)| < \infty\} \tag{1.162}$$

and within this space a ball of size large enough $-\frac{2}{3}-$ to accommodate for the largest term on the right side, $\frac{\alpha}{3}x^{-2/3}$:

$$B_1 := \{f \in \mathcal{B} : \|f\| \leqslant \frac{2}{3}\} \tag{1.163}$$

Lemma 1.164 *For given C, if x_0 is large enough, then the operator $\mathcal{N}$ is contractive in B_1 and thus* (1.159) *(as well as* (1.158)*) has a unique solution there.*

PROOF We first check that $\mathcal{N}(B_1) \subset B_1$, by estimating each term in $\mathcal{N}$. By (1.160) we have for large enough x_0, $|\mathcal{N}x^{-m}| = \frac{1}{3}|x|^{-m-1}(1+o(1))$. In particular, $|\mathcal{N}\frac{\alpha}{3}x^{-2/3}| \leqslant \frac{\alpha}{9}|x|^{-5/3}(1+o(1))$. The contribution of the other terms are much smaller. For instance, $|xv^2| < Cx^{1-5/2}\|v\|$ we have $|\mathcal{N}(xv^2)| = C|x|^{-5/2}(1+o(1))$.

To show contractivity, we note that, for $k > 1$,

$$|\mathcal{N}(v_2^k - v_1^k)| \leqslant k\|v_2 - v_1\||\mathcal{N}\left[x^{-5/3}2(2/3)^{k-1}x^{-5(k-1)/3}\right]$$

$\square$

1.5b The wave equation with potential

The *free* wave equation is $u_{tt} - c^2u_{xx} = 0$; c can be scaled out, by changing variables to $\tilde{x} = x/c$; without loss of generality we can then assume $c = 1$. One common setting is to have the initial position and velocity specified, that is

$$u_{tt} - u_{xx} = 0;\ u(x,0) = f(x), \quad u_t(x,0) = g(x) \tag{1.165}$$

When $f \in C^2(\mathbb{R})$ and $g \in C^1(\mathbb{R})$, the change of variable $\xi = x - t$, $\eta = x + t$ leads to the well-known D'Alembert solution

$$u(x,t) = \tfrac{1}{2}f(x+t) + \tfrac{1}{2}f(x-t) + \tfrac{1}{2}\int_{x-t}^{x+t} g(s)ds \tag{1.166}$$

Without smoothness of f and g (1.166) is interpreted as a weak solution. In the same way we can solve the wave equation with a source,

$$u_{tt} - u_{xx} = S(x,t);\ u(x,0) = f(x), \quad u_t(x,0) = g(x) \tag{1.167}$$

to obtain

$$u(x,t) = \tfrac{1}{2}f(x+t) + \tfrac{1}{2}f(x-t) + \tfrac{1}{2}\int_{x-t}^{x+t} g(s)ds + \frac{1}{2}\int_0^t \int_{x-t+t_1}^{x+t-t_1} S(x_1,t_1)dx_1dt_1 \tag{1.168}$$

The wave equation with potential arises naturally in a number of physical problems, ranging from electrodynamics to the wave evolution in the presence of a black hole. It reads

$$u_{tt} - u_{xx} + V(x)u(x,t) = 0; \; u(x,u) = f(x), \quad u_t(x,0) = g(x) \tag{1.169}$$

Clearly, at least for general V we cannot expect to solve (1.169) in closed form.

Here we assume that $V \in L^\infty(\mathbb{R})$ and f, g are in $L^1(\mathbb{R})$ and show that (1.169) has a global solution $u(\cdot,t) \in L^1(\mathbb{R})$ and $\|u(\cdot,t)\|_{L^1}$ grows at most exponentially in t. That exponential growth is possible for some potentials can be seen in the following way. Looking for solutions in the form $u(x,t) = e^{\lambda t}U(x)$ we obtain

$$-U'' + V(x)U = -\lambda^2 U \tag{1.170}$$

Eq. (1.170) is the time-independent Schrödinger equation; in that setting it is natural to assume that V decays as $x \to \infty$. An L^2 solution of (1.170) for $\lambda \neq 0$ is called a *bound state* of the quantum *Hamiltonian* $-\frac{d^2}{dx^2} + V(x)$, and for many potentials of interest these do exist.

We can use (1.171) to rewrite (1.169) in integral form,

$$\begin{aligned} u(x,t) &= \tfrac{1}{2}f(x+t) + \tfrac{1}{2}f(x-t) \\ &+ \tfrac{1}{2}\int_{x-t}^{x+t} g(s)ds - \frac{1}{2}\int_0^t \int_{x-t+t_1}^{x+t-t_1} V(x_1)u(x_1,t_1)dx_1 dt_1 =: \mathcal{A}[u](x,t) \end{aligned} \tag{1.171}$$

Proposition 1.172 *Assume the initial conditions $f(x) = u(t,0)$ and $g(x) = u_t(x,0)$ are in $L^1(\mathbb{R})$ and $V \in L^\infty(\mathbb{R})$. Then, if $\nu > \sqrt{2}\|V\|_\infty^{\frac{1}{2}}$ we have $\sup_{t>0} e^{-\nu t}\|u(t,\cdot)\|_{L_1} < \infty$.*

PROOF We write the Duhamel formula as

$$\begin{aligned} &u = \mathcal{A}u; \; \mathcal{A}u := \frac{f(x-t) + f(x+t)}{2} \\ &+ \tfrac{1}{2}\int_{-\infty}^{\infty} \chi_t(y-x)g(y)dy + \frac{1}{2}\int_0^t \int_{-\infty}^{\infty} u(y,s)V(y)\chi_{t-s}(y-x))dyds \end{aligned} \tag{1.173}$$

where χ_a is the characteristic function of the interval $[-a,a]$. Consider the Banach space

$$\mathcal{B} = \{u : \|\|u\|_\nu := \sup_{t \in \mathbb{R}^+} e^{-\nu t}\|u(t,\cdot)\|_1 < \infty\}; \quad (\nu > \sqrt{2}\|V\|_\infty^{\frac{1}{2}}) \tag{1.174}$$

Applying Fubini to integrate first in x, we see that $\| \int_{-\infty}^{\infty} \chi_t(y-x)g(y)dy\|_1 \le 2t\|g\|_1$ and (since by definition $\|u(\cdot,s)\|_1 \le \|u\|_\nu e^{\nu s}$)

$$\sup_{t>0} e^{-\nu t} \left\| \int_0^t \int_{-\infty}^{\infty} u(y,s)V(y)\chi_{t-s}(y-x)dydt \right\|_1$$
$$\le \|V\|_\infty \|u\|_\nu \sup_{t>0} e^{-\nu t} \int_0^t 2(t-s)e^{\nu s}ds \le 2\|V\|_\infty \nu^{-2}\|u\|_\nu \quad (1.175)$$

Using (1.175) we see that $\mathcal{A} : \mathcal{B} \to \mathcal{B}$ is contractive. Also, assuming f, g and V are smooth, the solution is seen to be smooth too: since $u \in L^1$, Duhamel's formula shows that it is continuous; then, as usual, using continuity we derive differentiability, and inductively, we see that u is smooth.

$\square$

Exercise 1.176 Complete the details by showing that this result implies global existence of a solution of (1.169).

Exercise 1.177 (i) Assume $V \in L^2(\mathbb{R})$. Prove a similar result with $\|u\|$ given by $\sup_{t\ge 0} e^{-\nu t} \left(\int_{-\infty}^{\infty} |u(x,t)|^2 dx \right)^{1/2}$. Use this result to estimate the largest possible eigenvalue of V.

1.5c Regular versus singular perturbations

Consider first two elementary problems: finding the roots of the polynomials $P_1(x;\varepsilon) = x^5 - x - \varepsilon$ and $P_2(x;\varepsilon) = \varepsilon x^5 - x - \varepsilon$ for small ε.

We see that $P_1(x;0)$ has five roots, $\rho = 0, \pm 1, \pm i$. We choose one of them, say $\rho = 1$ and look for roots of $P_1(x;\varepsilon)$ in the form $\rho(\varepsilon) = 1 + \sum_{k\geqslant 1} c_k \varepsilon^k$. Substituting in the equation $P_1 = 0$ we get $(4c_1 - 1)\varepsilon + (4c_2 + 10c_1^2)\varepsilon^2 + (4c_3 + 20c_1c_2 + 10c_1^3))\varepsilon^3 = 0$, and solving for the coefficients $c_1, \dots, c_3, \dots$ we get

$$c_1 = \frac{1}{4},\ c_2 = -\frac{5}{32},\ c_3 = \frac{5}{32}, \dots \quad (1.178)$$

The series of $\rho(\varepsilon)$ is actually convergent. It would not be very convenient to prove this directly from the recurrence relation, though this is possible. A better way is to substitute $\rho = 1 + \delta$ into the equation, placing the largest term containing δ on the left side, and showing that the equation for δ is contractive for small δ, in a space of functions analytic in ε at $\varepsilon = 0$. We leave the details as an exercise.This is a typical behavior in regularly perturbed problems: the roots of the leading order equation $P_1(x;0)$ give the leading behavior of the actual roots of $P_1(x,\varepsilon)$ as $\varepsilon \to 0$.

By contrast, $P_2(x;0)$ has only one root, $x = 0$. Four solutions of the quintic polynomial $P_2(x,\varepsilon)$ are lost by setting $\varepsilon = 0$ in the equation; this is an example of singular perturtbation since $P_2(x;0)$ does not capture all the behavior of of the five roots of $P_2(x,\varepsilon)$ as $\varepsilon \to 0$. We can find the missing roots by applying a formal dominant balance argument: clearly εx^5 has to be part of the balance. Balancing εx^5 with ε leads to an inconsistency, since $-x$ would turn out to be much larger. We must then have $\varepsilon x^5 \sim x$ or $\varepsilon x^4 \sim 1$. To obtain the higher order corrections, we substitute $x = \varepsilon^{-1/4} y$ and we get

$$y^5 = y + \eta; \quad (\eta = \varepsilon^{5/4}) \tag{1.179}$$

Now the limiting ($\eta \to 0$) equation, $y^5 = y$, has five roots as expected of a quintic polynomial. In fact, the equation (1.179) is $P_1(y;\eta) = 0$ and, if we take $y = 1 + \delta$ we get a convergent expansion $\delta = \frac{1}{4}\eta - \frac{5}{32}\eta^2 + \frac{5}{32}\eta^3 + \ldots$. Substituting to get δ, we see that $\delta(\varepsilon)$ is not analytic; nonetheless it has a convergent expansion in powers of $\varepsilon^{5/4}$. By contrast, we will find that in singular perturbation of differential equations, where a small parameter typically multiples the highest derivative, the asymptotic expansions are generally divergent.

An equation can be regularly perturbed in some regimes and singularly perturbed in some others.

An interesting example is the pendulum of slowly variable length. A model equation is

$$\ddot{q} + \frac{g}{l_0 + \varepsilon t} q = 0 \tag{1.180}$$

where q is the generalized position, g is the gravitational acceleration and l_0 is the initial length. A proper treatment of this problem will have to wait until we study adiabatic invariants.

By changing units and ε we can assume without loss of generality $l_0 = g = 1$. The limiting equation $\ddot{q} + q = 0$ has a two dimensional family of solutions, $y = A\sin t + B\cos t$. Assuming that $y(0) = 0$ and $\dot{y}(0) = 1$ we choose $q_0(t) = \sin t$. We look for solutions y in the form of power series in powers of ε,

$$y(t) = \sin t + \sum_{k=1}^{\infty} \varepsilon^k y_k(t) \tag{1.181}$$

Solving order by order in ε and using the initial condition $y(0) = 0$ and $\dot{y}(0) = 1$, translating to $y_k(0) = 0$, $y_k'(0) = 0$ for $k \geq 1$, we get

$$\begin{aligned} q(t) &= \sin t + \left(\frac{1}{4} t \sin t - \frac{1}{4} t^2 \cos t\right)\varepsilon \\ &\quad + \left[\left(\frac{3}{32} - \frac{3}{32} t^2 - \frac{1}{32} t^4\right)\sin t - \left(\frac{3}{32} t - \frac{1}{16} t^3\right)\cos t\right]\varepsilon^2 + \cdots \end{aligned} \tag{1.182}$$

We see that the validity of the expansion is limited by the condition $t^2 \varepsilon <$ *const.*, where *const.* needs to be relatively small, since otherwise we end up with a series of successively growing terms, and the expansion would be useless: we would have to look at the complete expansion, to all orders in ε, to hope to understand anything about q.

In a region where $t < \delta\varepsilon^{-1/2}$ with δ small enough, we can set up a contractive mapping argument to justify the expansion, which will turn out to be convergent. We leave this as an exercise as well.

Note also that in the time interval $0 < t \ll \sqrt{\varepsilon}$ we have $l = l_0 + O(\sqrt{\varepsilon}$, that is, the length does not change much; this region is not very interesting. A proper treatment of this problem will have to wait until we study adiabatic invariants.

Now, when $t \sim \varepsilon^{-1/2}$ it is natural to take $t\sqrt{\varepsilon} = \tau$ as a new variable, $q(t) = Q(\tau)$ that will not be necessarily small. The equation for Q reads.

$$\varepsilon\ddot{Q} + \frac{Q}{1+\sqrt{\varepsilon\tau}} = 0 \tag{1.183}$$

Now the limit $\varepsilon \to 0$ is singular: in this limit equation (1.183) would become $\frac{Q}{1+\sqrt{\varepsilon\tau}} = 0$; here, as in the case of $P_2(x;\varepsilon)$ we lose most solutions. Furthermore, the surviving solution $Q = 0$ is not very interesting, and it does not satisfy the initial condition. We need to do something else, in this case WKB, which we introduce in §1.5c.1 below.

1.5c.1 Singularly perturbed differential equations

Consider first the very simple equation

$$\varepsilon^2 y'' + y = 0; \quad \varepsilon \ll 1 \tag{1.184}$$

which can be of course solved in closed form, which we will do after we explore some qualitative features. The limit $\varepsilon \to 0$ is singular: taking $\varepsilon = 0$ in (1.184) leaves us with $y = 0$. Most solutions of (1.184) are lost in this limit. This is one of the indications that an equation is singularly perturbed. The other one, that we will return to, is non-analytic behavior in ε.

Similarly, the equation

$$\frac{d^2y}{dx^2} - a^2 y = 0 \tag{1.185}$$

is singularly perturbed as $x \to \infty$, since the change of variable $x = 1/z$ brings it to

$$z^4 \frac{d^2y}{dz^2} + 2z^3 \frac{dy}{dz} - a^2 y(z) = 0 \tag{1.186}$$

and we see that for small z the coefficients of the derivatives on the left side of the equation vanish at $z = 0$, and if we ignored these terms we would be once more left with a scalar equation, $y = 0$.

The eigenvalue problem for the one-dimensional Schrödinger equation

$$-\hbar^2 \psi'' + V(x)\psi = E\psi \tag{1.187}$$

is singularly perturbed in the when the Planck constant is taken to the limit $\hbar \to 0$ (its physical value is $\approx 6.626068 \times 10^{-34} m^2 kg/s$). Here ψ

is the wave function and it has the physical interpretation that $|\psi(x)|^2$ is probability density function for a particle and the total probability is one: $\|\psi\|_2 = \int_{-\infty}^{\infty} |\psi(x)|^2 dx = 1$. For a typical potential V going to zero as $x \to \infty$, Eq. (1.187) is also singularly perturbed when $x \to \infty$. Indeed, scaling out $\hbar$ now and taking $z = 1/x$ we get

$$-z^4 \frac{d^2\psi}{dz^2} - 2z^3 \frac{d\psi}{dz} + V(1/z)\psi = E\psi \tag{1.188}$$

and for $z = 0$ we are left with the scalar equation $E\psi = 0$. The solution $\psi = 0$ is not physically acceptable, as it violates $\|\psi\|_2 = 1$.

We can analyze (1.188) using dominant balance. It is clear that $V(1/z)\psi$ cannot be part of the dominant balance, since it is necessarily much smaller than $E\psi$. We are left with three possible balances, only one of them consistent. If we assume $z^4 \frac{d^2\psi}{dz^2} \sim -2z^3 \frac{d\psi}{dz}$, *i.e.* $V(1/z)\psi, E\psi << z^4 \frac{d^2\psi}{dz^2}$, we get $\psi \sim \text{const}/z$, but then this violates the assumption $E\psi << z^4 \frac{d^2\psi}{dz^2}$, unless $E = 0$. If instead we balance $2z^3 \frac{d\psi}{dz} = E\psi$ we get $\psi \sim e^{-\frac{1}{4}Ez^{-2}}$, assuming other terms in (1.188) to be smaller, then $z^4 \frac{d^2\psi}{dz^2} \gg E\psi$ contrary to the assumption. We are left with $z^4 \frac{d^2\psi}{dz^2} \sim E\psi$.

To analyze the balance $z^4 \frac{d^2\psi}{dz^2} \sim E\psi$ it is useful to note that the same balance is the only consistent one in (1.186) which can be solved exactly, as it is equivalent to (1.185): the solution is $y(x) = \exp(\pm ax) = \exp(\pm a/z)$. The solutions do not have asymptotic power series for small z, but their logs do. An exponential substitution, $y = e^{w(z)}$ is suggested, and this *WKB ansatz* is very helpful in singularly perturbed equations.

We will proceed formally first, and then prove a result for (1.187). So, consider again (1.187) and substitute $\psi(x) = e^{w(x)}$. After dividing by $e^{w(x)}$ we get

$$-\hbar^2(w'' + {w'}^2) = E - V(x) \tag{1.189}$$

or, with $w' = f$, we get the first order nonlinear ODE

$$-\hbar^2(f' + f^2) = E - V(x) \tag{1.190}$$

We analyze (1.189) by dominant balance. We first assume for simplicity that $E > V(x)$ for all x; a similar argument works if $E < V(x)$ for all x, with $\sqrt{V(x) - E}$ replacing $i\sqrt{E - V(x)}$. The situations in which $E = V(x)$ has nontrivial solutions, called *turning points* are important, and we will study them separately.

Note 1.191 In a WKB ansatz, we have $w'' \ll {w'}^2$. Indeed, the balance $-\hbar^2 w'' \sim E - V(x)$ would give $w = O_s(\hbar^{-2})$ and then ${w'}^2 = O_s(\hbar^{-4})$ showing that this choice is inconsistent. The balance $w'' \sim -{w'}^2$ does not work either since then $w, w', w'' = O(1)$ whereas, under our assumptions we have $\hbar^{-2}(E - V(x)) = O_s(\hbar^{-2})$. We are left with the balance ${w'}^2 \sim -\hbar^{-2}(E - V(x))$ with $w'' \ll {w'}^2$ (we have already seen that $w'' \not\sim {w'}^2$).

According to Note 1.191 we place w'' on the right side of the equation, treated as being relatively small. With $f = w'$, (1.190) implies

$$f = \pm\frac{i}{\hbar}\sqrt{E - V(x) + \hbar^2 f'} \tag{1.192}$$

where we choose one sign at a time, say plus for now, and we expand (1.192), by the usual Picard-like asymptotic iterations,

$$f^{[n+1]} = \frac{i}{\hbar}\sqrt{E - V(x) + \hbar^2 f^{[n]\prime}} \tag{1.193}$$

The fact that the highest order derivative is on the right side of the iteration strongly indicates that the expansion thus obtained is divergent.

Expanding in $\hbar$ to three orders we get

$$f^{[n+1]} = \tfrac{i}{\hbar}\sqrt{E - V(x)} + \frac{i\hbar f^{[n]\prime}}{2\sqrt{E - V(x)}} - \frac{i(f^{[n]\prime})^2\hbar^3}{8(E - V(x))^{3/2}} + \cdots \tag{1.194}$$

In this way we get

$$f^{[0]} = \tfrac{i}{\hbar}\sqrt{E - V(x)} \tag{1.195}$$

$$f^{[1]} = \tfrac{i}{\hbar}\sqrt{E - V(x)} + \frac{1}{4}\frac{V'}{E - V} \tag{1.196}$$

$$f^{[2]} = \tfrac{i}{\hbar}\sqrt{E - V(x)} + \frac{1}{4}\frac{V'}{E - V} + \hbar\frac{\frac{5i}{32}V'^2 + \frac{i}{8}V''(E - V))}{(E - V(x))^{5/2}} \tag{1.197}$$

To two orders, this gives

$$w^{[0]} = \tfrac{i}{\hbar}\int_{x_0}^{x}\sqrt{E - V(s)}ds + C \tag{1.198}$$

$$w^{[1]} = \tfrac{i}{\hbar}\int_{x_0}^{x}\sqrt{E - V(s)}ds - \frac{1}{4}\ln(E - V(x)) + C \tag{1.199}$$

or

$$\psi = C_1(E - V(x))^{-1/4}e^{\frac{i}{\hbar}\int_{x_0}^{x}\sqrt{E - V(s)}ds}(1 + o(1)) \tag{1.200}$$

1.5c.2 Proof of existence of a solution of (1.187) in the form (1.200)

One way of proving the expansion is to return to (1.190) where we substitute for $f(x) = f^{[j]}(x) + \delta(x),\ j \geqslant 1$ [12]. Here we choose $j = 1$; assuming that the regularity of V allows for calculating higher order terms which involve higher

[12] The minimum j needed depends on the problem; in some settings, $j = 0$ suffices. As a rule, the more terms we pull out, the more contractive the operator becomes, at the expense of getting a more involved algebra.

derivatives of V as seen in (1.195) taking $j = j_1 > 1$ would allow for proving asymptoticity of the expansion with $j_1 + 1$ terms.

$$f(x) = \tfrac{i}{\hbar}\sqrt{E - V(x)} + \frac{1}{4}\frac{V'}{E - V} + \delta(x) \tag{1.201}$$

The equation for $\delta(x)$ is

$$\hbar\delta' + 2i\sqrt{E - V(x)}\delta + \frac{\hbar V'}{2(E - V(x))}\delta = -\hbar g - \hbar\delta^2$$
$$\text{where} \quad g(x) := \frac{5}{16}\left(\frac{V'}{E - V(x)}\right)^2 + \frac{V''}{4[E - V(x)]} \tag{1.202}$$

We note that we need to keep the highest derivative term, here $\hbar\delta'$ on the left side of (1.202) even though it is multiplied by small $\hbar$. In fact, in general, in a singularly perturbed problem, the singularly perturbed term cannot be discarded.

We then write the equation in integral form. Let $J = \frac{2i}{\hbar}\int_{x_0}^{x} \sqrt{E - V(s)}ds$ and $\mu(x) = (E - V(x))^{-1/2}$. Using the integrating factor for the left side of 1.202, we get

$$\delta(x) = -\frac{e^{-J(x)}}{\mu(x)}\int_{\infty}^{x} \mu(s)e^{J(s)}g(s)ds - \frac{e^{-J(x)}}{\mu(x)}\int_{\infty}^{x} e^{J(s)}\mu(s)\delta^2(s)ds$$
$$:= \delta_0 + \mathcal{N}\delta \tag{1.203}$$

To prove a rigorous result we need some assumptions.

Assumption 1.204 For simplicity. we let $V : \mathbb{R} \to \mathbb{C}$, $V \in C^2(\mathbb{R})$, and V is $O(1/x^{1+\varepsilon})$ for large x and it "acts like a symbol" essentially meaning that we can differentiate the asymptotics: $V' = O(1/x^{2+\varepsilon})$ and $V'' = O(1/x^{3+\varepsilon})$. We work on an interval, say $[x_0, \infty)$, where $E - V(x) > a > 0$. We note that under these assumptions we have $g(x) = O(x^{-3-\varepsilon})$ for large x.

We introduce the Banach space

$$\mathcal{B} = \{\delta : [x_0, \infty) \to \mathbb{C} \mid \|\delta\| := \sup_{x \geqslant x_0} |x^{1+\varepsilon}\delta(x)| < \infty\} \tag{1.205}$$

We first prove that the term

$$\delta_0 := \frac{e^{-J(x)}}{\mu(x)}\int_{x}^{\infty} \mu(s)e^{J(s)}g(s)ds \tag{1.206}$$

in (1.203) decays as $\hbar \to 0$ or as $x_0 \to +\infty$.

Lemma 1.207 *Under Assumption 1.204, we have* $\lim_{\hbar\to 0} \|\delta_0\| = 0$. *Furthermore, for any* $\hbar \neq 0$, $\lim_{x_0\to\infty} \|\delta_0\| \to 0$.

PROOF Let $t = t(x) = \int_{x_0}^{x} \sqrt{E - V(t)}dt$. Note that $t : [x_0, \infty] \to [0, \infty]$ is increasing since $t'(x) = \sqrt{E - V(x)} = 1/\mu(x)$ is bounded away from zero. Let

$$f(x) := x^{1+\varepsilon} \int_x^\infty \mu(s) e^{J(s)} g(s) ds = \int_t^\infty e^{2it'/\hbar} g(x(t')) \left[x(t)\right]^{1+\varepsilon} \mu^2(x(t'))dt' \tag{1.208}$$

Since $|x^{1+\varepsilon} e^{J(s)} g(s)\mu(s)| \le s^{1+\varepsilon}\mu(s)|g(s)|$ is in L^1, the first equality in (1.208) implies

$$\lim_{x\to\infty} f(x) = 0 \tag{1.209}$$

Using the fact that $\frac{1}{\mu(x)} e^{-J(x)}$ is bounded, (1.205), (1.203) and (1.209) imply $\lim_{x_0\to\infty} \|\delta_0\| = 0$ for any $\hbar \ne 0$. Now, consider the case of $\hbar \to 0$ with $x_0 > 0$ fixed.

We claim that for any $\varepsilon > 0$, there exists $\hbar_0$ such that $|f(x)| \le \varepsilon$ for any x if $|\hbar| \le \hbar_0$.

First, from (1.209), it follows that for large enough M and $x > M$ we have $|f(x)| \le \varepsilon$. Then, the Riemann-Lebesgue lemma implies that for $x \in [x_0, M]$ we have $\lim_{\hbar\to 0} f(x) = 0$. Since f is uniformly continuous on compact sets, convergence is uniform in x, *i.e.* there exists $\hbar_0$ so that $|\hbar| \le \hbar_0$ implies $|f(x)| \le \varepsilon$ for any x. Therefore, $\lim_{\hbar\to 0} \|\delta_0\| = 0$ since $e^{-J(x)}/\mu(x)$ is bounded. $\square$

Theorem 1.210 *Under Assumption 1.204, if x_0 is large enough or $\hbar$ is small enough, then two linearly independent solutions of (1.187) for $x \in (x_0, \infty)$, $\psi = \psi_1$ and $\psi = \psi_2$, satisfy*

$$\psi_1(x) = \left[E - V(x)\right]^{-1/4} \exp\left[\frac{i}{\hbar}\int_{x_0}^{x} \left[E - V(t)\right]^{1/2} dt\right] \{1 + o(1)\} \tag{1.211}$$

$$\psi_2(x) = \left[E - V(x)\right]^{-1/4} \exp\left[-\frac{i}{\hbar}\int_{x_0}^{x} \left[E - V(t)\right]^{1/2} dt\right] \{1 + o(1)\} \tag{1.212}$$

PROOF We only prove the result for ψ_1 since the proof for ψ_2 is the same after changing the sign of i. Since $\psi_1 = e^W$, $W' = f^{[1]} + \delta$, it is enough to show that (1.203) has a solution in a ball where $\|\delta\|$ is small:

$$B_\varepsilon = \{\delta \in \mathcal{B} | \|\delta\| \leqslant \varepsilon\} \tag{1.213}$$

Using Lemma 1.207, we see that for any $\varepsilon > 0$, if we x_0 is large enough or $\hbar$ is small enough, then $\|\delta_0\| \le \frac{1}{2}\varepsilon$. Now, for any $\delta \in B_\varepsilon$, we have

$$|x^{1+\varepsilon} \mathcal{N}[\delta]| \le \|\delta\|^2 x^{1+\varepsilon} \int_x^\infty \frac{\mu(s)}{\mu(x)} s^{-2-2\varepsilon} ds \le C\|\delta\|^2 \le C\varepsilon^2,$$

where C is independent of ε. Choosing $\varepsilon < \frac{1}{2C}$, this implies that $\|\delta_0 + \mathcal{N}[\delta]\| < \varepsilon$ and $\|\mathcal{N}[\delta_1] - \mathcal{N}[\delta_2]\| \le 2C\varepsilon\|\delta_1 - \delta_2\| < \|\delta_1 - \delta_2\|$. Therefore, the contraction mapping theorem implies that there exists a unique solution in B_ε. $\square$

If, as mentioned at the beginning of the section we took $j = j_1 > 1$ instead, then the remainder g in the map (1.224) will be of higher order in $\hbar$. With this change, contractivity is proved in the same way, to obtain an asymptotic expansion with $j_1 + 1$ terms.

Remark 1.214 *Replacing i by $-i$ in in (1.200) gives the behavior of a second independent independent solution of (1.187) in (x_0, ∞).*

Remark 1.215 *(i) No decay assumption on V is necessary for Theorem 1.210 to apply for x in a fixed ($\hbar-$independent interval $[a, b]$. (ii) The assumption $x_0 > 0$ in Theorem 1.210 is not needed. To allow for $x_0 < 0$ the proof is largely the same. Assuming $V(x) = O(|x|^{-1-\varepsilon})$ as $x \to -\infty$, we would instead use the norm $\|\delta\| = \sup_{x\in(x_0,\infty)} |1 + |x|)^{1+\varepsilon}|\delta(x)|$.*

1.5d The case $V(x) - E \ge a > 0$

In this case, if $V \in C^2$, the arguments given in §1.5c.1 that lead to WKB solution of (1.187) may be applied in this case again to give the result

$$\psi = C_1 \left[V(x) - E\right]^{-1/4} \exp\left[\pm\frac{1}{\hbar}\int_{x_0}^{x} \sqrt{V(t) - E}dt\right] [1 + o(1)] \quad \text{for } \; x \in (x_0, \infty) \tag{1.216}$$

either for $x_0 \to +\infty$ or $\hbar \to 0$. The precise result is given below:

Theorem 1.217 *For $V \in C^2(x_0, \infty)$, and $V(x) - E \ge a > 0$, as $x_0 \to +\infty$ or $\hbar \to 0^+$, Two independent solutions of (1.187) are given by $\psi = \psi_1$ and $\psi = \psi_2$, where*

$$\psi_1(x) = \left[V(x) - E\right]^{-1/4} \exp\left\{\frac{1}{\hbar}\int_{x_0}^{x} \left[V(t) - E\right]^{1/2} dt\right\} \{1 + o(1)\} \tag{1.218}$$

$$\psi_2(x) = \left[V(x) - E\right]^{-1/4} \exp\left\{-\frac{1}{\hbar}\int_{x_0}^{x} \left[V(t) - E\right]^{1/2} dt\right\} \{1 + o(1)\} \tag{1.219}$$

The proof of this theorem is similar to the proof of Theorem 1.210. The only difference is that in the proof for $\psi \sim \psi_1$, in the equation for δ defined by $W' = \sqrt{V(x) - E} - \frac{V'}{4[V(x)-E]} + \delta$, where $\psi = e^W$, it is necessary to put the integral equation for δ in the following form

$$\delta = -\frac{e^{-J(x)}}{\mu(x)}\int_{x_0}^{x} g(s)\mu(s)e^{J(s)}ds - \frac{e^{-J(x)}}{\mu(x)}\int_{x_0}^{x} e^{J(s)}\mu(s)\delta^2(s)ds =: \delta_0 + \mathcal{N}\delta \tag{1.220}$$

where $J(x) = \frac{2}{\hbar}\sqrt{V(x)-E}$ and $\mu = (V(x)-E)^{-1/2}$, On the other hand, to prove $\psi \sim \psi_2$, On the other hand, when $W' = -\sqrt{V(x)-E} - \frac{V'}{4[V(x)-E]} + \delta$, where $\psi = e^W$, it is necessary we replace the integration limit x_0 in (1.220) by ∞, or the right end of whatever x interval one is concerned with, since in this case, $J(x) = -\frac{2}{\hbar}\sqrt{V(x)-E}$– these choices of limits ensure that $e^{J(x)-J(s)} \le 1$.

Exercise 1.221 Prove Theorem 1.217. You may want to use the fact that for locally integrable q, $\lim_{\hbar\to 0^+}\int_{x_0}^{x} e^{J(x)-J(s)}q(s)ds \to 0$ when $J = \frac{2}{\hbar}\sqrt{V(x)-E}$.

1.5d.1 Turning points

In the previous subsection we assumed that $E - V$ is bounded below. This assumption is in fact necessary, otherwise the asymptotic behavior of the solutions is different. If we examine the procedure used to derive (1.194) from (1.193), we see that the expansion is only valid if $\hbar^2 f^{[n]\prime} \ll E - V(x)$, that is, to have $f \approx f^{[0]}$ we need $\hbar(E-V(x))^{-1/2} \ll E - V(x)$, that is, $E - V(x) \gg \hbar^{2/3}$. Something else must be done when the latter condition fails.

In our assumption V is smooth. Generically, near a zero of $V(x) - E$, also referred to as a turning point, $V(x) = \alpha(x - x_t) + O(x - x_t)^2$, where $\alpha \neq 0$. Without loss of generality we can take $x_t = 0$ and $\alpha = -1$ through translation and scaling. The region where our WKB does not hold is given by $|x| \lesssim \hbar^{2/3}$. It is natural to change variables to $t = x/\hbar^{2/3}$ in (1.187); we get, after dividing by $\hbar^{2/3}$,

$$-\psi''(t) - t\psi(t) = \hbar^{2/3}t^2\varphi_1(x(t))\psi(t) \tag{1.222}$$

where $\varphi_1(x) = x^{-2}[E - V(x) - x]$. To leading order in small $\hbar$, ψ satisfies $-\psi_0''(t) - t\psi_0(t) = 0$ with the general solution

$$\psi_0(t) = C_1\mathrm{Ai}(-t) + C_2\mathrm{Bi}(-t) \tag{1.223}$$

Since the right hand side of (1.223) is a regular perturbation in $\hbar^{2/3}$ for t in any finite interval, we can obtain higher order corrections in $\hbar$ as usual.

1.6 Borderline region: $x \gg \hbar^{2/3}$

Assume a turning point at $x = 0$, *i.e.* , $E = V(0)$ and that $E - V(x) > 0$ for $x > 0$. Then, for $x > x_0 > 0$, independent of $\hbar$, Theorem 1.210 applies. We now write a mapping for an interval (a, x_0) where a is allowed to depend on $\hbar$:

$$\delta(x) = -\frac{e^{-J(x)}}{\mu(x)}\int_a^x e^{J(s)}\mu(s)g(s)ds - \frac{e^{-J(x)}}{\mu(x)}\int_a^x e^{J(s)}\mu(s)\delta(s)^2ds := \delta_0 + \mathcal{N}\delta \tag{1.224}$$

The reasoning is similar to that in §1.5c.2. We choose a as small as possible, while still allowing the right side of (1.224) to be contractive. For this to be the case, we need $|g| \lesssim |x|^{-2}$ and we choose a so that $\delta^2 \ll g$; when this is possible, as shown at the end of the argument, the results of Theorem 1.210 extend to the interval (a, x_0). To determine what this condition entails, we use dominant balance in (1.202): $\delta \ll \hbar|gx^{-1/2}| \ll \hbar|x|^{-5/2}$, and thus $\delta^2 \ll g$ implies $\hbar^2|x|^{-5} \ll \hbar|x|^{-2}$, that is $|x| \gg \hbar^{2/3}$. For contractivity we need, as in §1.5c.2, $|\delta_1 + \delta_1| \ll 1$ which for $\delta_1, \delta_2 = O\left(\hbar x^{-5/2}\right)$ holds if $x \gg \hbar^{2/5}$. This condition is stringent than $|x| \gg \hbar^{2/3}$. We then choose $a = \nu\hbar^{2/3}$ with ν sufficiently large, and with this, the map is contractive on (a, x_0). We leave the details as an exercise.

1.6a Inner region: Rigorous analysis

$$-\psi'' - t\psi = -\hbar^{2/3}t^2\varphi_1(\hbar^{2/3}t)\psi := f(t) \tag{1.225}$$

which can be transformed into an integral equation in the usual way,

$$\psi(t) = \pi\mathrm{Ai}(-t)\int^t f(s)\mathrm{Bi}(-s)\psi(s)ds - \pi\mathrm{Bi}(-t)\int^t f(s)\mathrm{Ai}(-s)\psi(s)ds \\ + C_1\mathrm{Ai}(-t) + C_2\mathrm{Bi}(-t) \tag{1.226}$$

where Ai, Bi are the Airy functions, with the integral representations:

$$\mathrm{Ai}(z) = \frac{1}{2\pi i}\int_{\infty e^{-\pi i/3}}^{\infty e^{\pi i/3}} e^{\frac{1}{3}t^3 - zt}dt \tag{1.227}$$

$$\mathrm{Bi}(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty e^{\pi i/3}} e^{\frac{1}{3}t^3 - zt}dt + \frac{1}{2\pi}\int_{-\infty}^{\infty e^{-\pi i/3}} e^{\frac{1}{3}t^3 - zt}dt \tag{1.228}$$

The integral representations allow us to derive the *global* behavior at ∞, that is, the asymptotic expansion in any direction towards infinity, with explicit constants. With $\zeta = \frac{2}{3}|t|^{\frac{3}{2}}$ we have

$$\mathrm{Ai}(-t) \sim \frac{1}{2\sqrt{\pi}}|t|^{-1/4}e^{-\zeta};\quad \mathrm{Bi}(-t) \sim \frac{1}{\sqrt{\pi}}|t|^{-1/4}e^{\zeta};\quad t \to -\infty \tag{1.229}$$

[1] and

$$\mathrm{Ai}(-t) \sim \frac{1}{\pi^{1/2}t^{1/4}}\sin(\zeta + \tfrac{\pi}{4}),\ \mathrm{Bi}(-t) \sim \frac{1}{\pi^{1/2}t^{1/4}}\cos(\zeta + \tfrac{\pi}{4})\ (t \to \infty) \tag{1.230}$$

as $t \to -\infty$. We have to choose the limits of integration in (1.226) in order for the right side of (1.226) to be a contractive mapping. The general prescription is that the maximum point of the integration contour should be at the variable point of integration, if the integrand behaves exponentially. We note that we cannot quite choose infinity as an upper limit since the Airy-type behavior was derived in the inner region $|x| \ll \hbar^{2/3}$ and in general is not expected to be the same outside. We will choose as large a t-interval $(-M_1, M_2)$, possibly depending on $\hbar$ for which the leading order behavior $\psi \sim C_1 Ai(-t)+C_2 Bi(-t)$ can be shown. We rewrite (1.225) in the integral form

$$\psi(t) = \pi \mathrm{Ai}(-t)\int_0^t f(s)\mathrm{Bi}(-s)\psi(s)ds - \pi \mathrm{Bi}(-t)\int_{-M_1}^t f(s)\mathrm{Ai}(-s)\psi(s)ds \\ + C_1\mathrm{Ai}(-t) + C_2\mathrm{Bi}(-t) = J\psi + \psi_0 \quad (1.231)$$

Next, to control the norm of J, for large M_1 the estimate

$$|t|^{-1/4} e^{-\frac{2}{3}|t|^{3/2}} \hbar^{2/3} \int_0^{|t|} s^2 s^{-1/4} e^{\frac{2}{3}s^{3/2}} ds \lesssim \hbar^{2/3} M_1, \ (t \to -\infty) \qquad (1.232)$$

follows from Watson's Lemma after the change of variable $p = 1 - s^{3/2}/|t|^{3/2}$, and similarly

$$|t|^{-1/4} e^{\frac{2}{3}|t|^{3/2}} \hbar^{3/2} \int_{|t|}^{M_1} s^2 s^{-1/4} e^{-\frac{2}{3}s^{3/2}} ds \lesssim \hbar^{2/3}|t| \lesssim \hbar^{2/3} M_1 \qquad (1.233)$$

The right sides of (1.234) and (1.233) are small if $M_1 \ll \hbar^{-2/3}$. For $t \to +\infty$, estimating crudely $|\sin|, |\cos|$ by one, we get

$$t^{-1/4}\hbar^{2/3} \int_0^t |s^2 s^{-1/4}| ds \lesssim \hbar^{2/3} t^{5/2} \lesssim \hbar^{2/3} M_2^{5/2}, \qquad (1.234)$$

which is small for $M_2 \ll \hbar^{-4/15}$. We now work in the sup norm on $[-M_1, M_2]$ and obtain, in the usual way, the following result

Proposition 1.235 *If $\hbar$ is small enough, then J defined in Eq. (1.231) is contractive in $L^\infty(-M_1, M_2)$ when $\hbar^{2/3} M_1$ and $\hbar^{4/15} M_2$ are small enough.*

We leave the details as an exercise. We see that the region of contractivity for $t < 0$ simply requires $|x| \ll 1$. On the other hand, the same is true for $t > 0$, with the price of making the argument quite a bit more involved.

Note 1.236 The contractivity of the map for $x < 0$ only requires $|x| \ll 1$. However, the norm used, L^∞ does not allow for controlling *the asymptotic behavior of solutions as t becomes large.* In particular, we would like to understand for what range of (large, negative) t does the solution of (1.225) have the behavior described by Airy function asymptotics, (1.229). The behavior (1.229) does not follow from our arguments, and in fact it is not even correct if $|t| \gg \hbar^{-4/15}$ as we will see in §1.6b.

1.6b Matching region

Let's analyze the behavior of solutions in the region $1 \ll |t| \ll \hbar^{-4/15}$. We will only analyze $t < 0$, as for $t > 0$ the analysis is similar (in fact, slightly simpler).

We first write $t = -u$ to make the analysis clearer. We get

$$-\psi'' + u\psi = -\hbar^{2/3}u^2\varphi_1(-\hbar^{2/3}u)\psi \tag{1.237}$$

We next bring (1.225) to a form that is best suited for looking at large t, a process called *normalization.* In the region where solutions have Airy-like asymptotic behavior, roughly $u^{-1/4}e^{\pm\frac{2}{3}u^{3/2}}$, we change variables so that the leading behavior is of the form e^s. A way to do this is simply by rescaling the dependent and independent variables, $\psi(u) = u^{-1/4}g(\frac{2}{3}u^{3/2})$.

With $s = \frac{2}{3}u^{3/2}$, this leads to the equation

$$g'' - g = -\frac{5}{36}s^{-2}g(s) + \frac{18^{1/3}}{2}\hbar\phi_1(s)s^{2/3}g(s) = F(s)g(s) \tag{1.238}$$

where ϕ_1 is bounded. Choosing s_0 large enough, we write (1.238) in the integral form:

$$g = Ae^s + Be^{-s} + \frac{1}{2}\left(e^s\int_M^s F(v)e^{-v}g(v)dv - e^{-s}\int_{s_0}^s F(v)e^v g(v)dv\right) \tag{1.239}$$

where M will be "large but not too large" so that two solutions with asymptotic behavior e^s and e^{-s} respectively exist for $s \in [s_0, M]$.

We now look for a solution with the behavior $g(s) = e^{-s}$ for large s. The adapted norm to measure this type of behavior is $\|g\| = \sup_{s>s_0}|g(s)e^s|$. We should take $A = 0$ in (1.239), since the norm of e^s is very large, of order e^{2M}. To check for the contractivity of the map in this norm, we use the fact that, by the definition of the norm, $|g(v)| \le \|g\|e^{-v}$. For the first integral we have

$$e^s\left|e^s\int_M^s F(v)e^{-v}g(v)dv\right| \lesssim \|g\|e^{2s}\int_M^s(\hbar^{2/3}v^{2/3} + v^{-2})e^{-2v}dv$$
$$\lesssim \|g\|(\hbar^{2/3}s^{2/3} + s^{-2}) \lesssim \|g\|(\hbar^{2/3}M^{2/3} + s_0^{-2}) \tag{1.240}$$

where we used Watson's lemma. In order for the norm of this part of the operator to be less than one, we need s_0 to be large, which we assumed already, and, once more, $|x| \lesssim 1$.

For the second integral, we see that the exponential in the definition of the norm cancels the exponential which was already in the integrand and we get

$$e^s\left|e^{-s}\int_{s_0}^s F(v)e^v g(v)dv\right| \lesssim \|g\|\int_{s_0}^s(\hbar^{2/3}v^{2/3} + v^{-2})ds \lesssim \|g\|\hbar^{2/3}s^{5/3} + s_0^{-1}$$
$$\lesssim x\hbar^{-1}|x|^{5/2} + s_0^{-1} \tag{1.241}$$

which can be made small if s_0 is large, as before, and if $|x| \lesssim \hbar^{2/5}$. The mapping is now contractive in a smaller region– the one that we have obtained before in the oscillatory regime.

Exercise 1.242 *Complete the details of the analysis, and do a similar analysis for the behavior e^s (where now the norm would be $\|g\| = \sup_s |e^{-s} g(s)|$). Show the existence of solutions of* (1.225) *with the behavior of the Airy functions Ai and Bi, cf.* (1.229) *in the region $|x| \lesssim \hbar^{2/5}$.*

Note now that, when approaching $x = 0$ from the outer region, we have $E - V(x) = ax + o(x^2)$ where, by scaling we chose $a = 1$; then $i\hbar^{-1} \int \sqrt{E - V(x)} = i\hbar^{-1} \frac{2}{3}(x^{3/2} + O(x^{5/2}))$ and

$$(E - V(x))^{-1/4} e^{i\hbar^{-1} \int \sqrt{E - V(x)}} = x^{-1/4} e^{i\hbar^{-1} \frac{2}{3}(x^{3/2} + O(x^{5/2}))} \tag{1.243}$$

and by switching to the variable $t = \hbar^{-2/3} x$ we get the behavior of a linear combination of Ai and Bi in the oscillatory region. Similarly, changing i to $-i$ in the analysis above we get a linearly independent solution, with the behavior given by a different combination of Ai and Bi. This was to be expected since we are, after all, dealing with the same equation in the inner and outer region, up to these changes of variables, and the behaviors should correspond to each other.

Matching means simply finding the concrete values of the constants so that an outer solution equals an inner one.

We note that there is a difference between the oscillatory outer region and the one with growing/decaying exponential behavior. If only the decaying exponential is present in the outer solution, the matching is straightforward: it corresponds simply to the solution with the behavior Ai in the inner region (Bi should not be present since it *grows* exponentially). But if the outer solution has both growing and decaying components, matching becomes more delicate since the small exponential is masked by the larger one to all orders of an asymptotic expansion in $\hbar$ and finding the correspondence between constants cannot be done by classical asymptotic means. One has to go to the complex domain if the potential is analytic or use exponential asymptotic tools.

1.7 Recovering actual solutions from formal ones

Consider the simple ODE

$$y' = y + 1/x \tag{1.244}$$

(1.244) has an irregular singularity at infinity. If we look for formal asymptotic series solutions $\tilde{y} = \sum_{k\geqslant 0} c_k x^{-k}$ we get $c_0 = 0$, $c_k = (-1)^k (k-1)!$, that is

$$\tilde{y} = \sum_{k=0}^{\infty} \frac{(-1)^k k!}{x^{k+1}} \tag{1.245}$$

This series has empty domain of convergence. Nonetheless, we can do the following. Writing

$$k! = \int_0^{\infty} e^{-t} t^k dt \Rightarrow \frac{(-1)^k k!}{x^{k+1}} = \int_0^{\infty} e^{-px} p^k dp \tag{1.246}$$

and inserting (1.246) into (1.247), we get

$$\tilde{y} = \sum_{k=0}^{\infty} \int_0^{\infty} e^{-px} p^k dp \tag{1.247}$$

This following step requires serious justification, but for now we formally interchange summation and integration,

$$\tilde{y} = \int_0^{\infty} e^{-px} \sum_{k=0}^{\infty} p^k dp = \int_0^{\infty} \frac{e^{-px}}{1+p} dp = e^x \mathrm{Ei}_1(x) \tag{1.248}$$

If our sole purpose was to solve (1.244) we could bypass the intermediate steps and any need for justification, and simply check that the function we obtained at the end, $e^x \mathrm{Ei}_1(x)$, satisfies the ODE. For the general solution of (1.244), we just add Ce^x, the solution of the associated homogeneous equation, to $e^x \mathrm{Ei}_1(x)$.

Of course however, (1.244) is very simple and we could have solved it by variation of constants or other elementary means. The questions are (1) Can we extend this to a much more general procedure, applicable to generic ODEs near irregular singularities? (the answer is yes) and (2) Can we justify the formal steps that led from (1.247) to the function in (1.248)? (the answer is yes again). We leave these issues for later, now we simply note that there is another way to interpret the operations that led to "summing" the divergent series: (1) we took the formal inverse Laplace transform of the series, that is, term-by-term; (indeed $\mathcal{L}^{-1} x^{-k-1} = p^k/k!$, (2) we summed the geometric series $\sum_{k=0}^{\infty} (-p)^k = (1+p)^{-1}$, and, since the radius of convergence of this geometric series is one, we extended $(1+p)^{-1}$ analytically $(1+p)^{-1}$ on $\mathbb{R}^+$, and (3) we took the Laplace transform of $\mathcal{L}$ the result. Since $\mathcal{L}\mathcal{L}^{-1} = I$ the identity, and at a formal level what we did is just that, $\mathcal{L}\mathcal{L}^{-1}$, we expect that if $\tilde{y}$ satisfied an ODE, so will the $\mathcal{L}\mathcal{L}^{-1}\tilde{y}$. This is the route we will take in justifying this procedure.

We also note that the formal series $\tilde{y}$ is divergent since it is obtained by repeatedly differentiating a function which is not entire: the iterative asymptotic process leading to $\tilde{y}$ is $y^{[n+1]} = -1/x + \partial_x y^{[n]}$. The inverse Laplace transform is a Fourier transform in the imaginary direction, and the Fourier transform is the unitary operator that diagonalizes differentiation. After a form of Fourier transform, repeated differentiation becomes repeated multiplication by the "symbol" of the differential operator, denoted by p here. This can only lead to geometric behavior of the terms of the formal series, something we know much more about: this is dealt with by analytic function theory.

Finally, and this is another important point, in this and many problems, applying the inverse Laplace transform has a regularizing effect. Indeed, the formal solution $\sum_{k=0}^{\infty}(-1)^k k! x^{-k-1}$ becomes, after applying $\mathcal{L}^{-1}$, $\sum_{k=0}^{\infty}(-p)^k$ which is convergent. Whatever problem the new series is a solution of, that new problem is expected to have at mot a regular singularity, given this convergence. Indeed, taking $\mathcal{L}^{-1}$ in (1.244) we get, with $\mathcal{L}^{-1}y = Y$,

$$(p+1)Y = 1 \tag{1.249}$$

an ordinary equation with meromorphic solutions.

The same can be dome in the context of PDEs. Let's take the heat equation,

$$h_t = h_{xx}; \quad \text{with } h(0,x) = \frac{1}{1+x^2} \tag{1.250}$$

Since the equation if parabolic, the Cauchy-Kowalesky does not apply. In fat, looking for power series solutions

$$h = \sum_{k=0}^{\infty} H_k(x) t^k \tag{1.251}$$

we obtain the recurrence

$$H_{k+1}(x) = \frac{H_k''(x)}{k+1}; \quad H_0(x) = \frac{1}{1+x^2} = \mathrm{Re}\left(\frac{1}{1+ix}\right) \tag{1.252}$$

where we wrote the initial condition in a way that facilitates taking high order derivatives. We get for H_k,

$$H_{k+1} = \frac{H_k''}{k+1} \Rightarrow H_k = \frac{H_0^{(2k)}(x)}{k!} = (-1)^k \frac{(2k)!}{k!}\mathrm{Re}\left((1+ix)^{-2k-1}\right) \tag{1.253}$$

and (1.253) shows that, with the given initial condition, (1.254) is divergent.

Denoting $t = 1/T$ we write

$$\tilde{h} = T\sum_{k=0}^{\infty} H_k(x) T^{-k-1} = T\sum_{k=0}^{\infty}(-1)^k \frac{(2k)!}{k!}\mathrm{Re}\left((1+ix)^{-2k-1}\right) T^{-k-1} \tag{1.254}$$

and we apply to the sum in (1.254) the procedure we used in (1.248), (1.247), (1.246), with $x = T$. We get

$$T\sum_{k=0}^{\infty}(-1)^k\frac{(2k)!}{k!}\mathrm{Re}\,\left((1+ix)^{-2k-1}\right)T^{-k-1}$$
$$= t^{-1}\int_0^{\infty} e^{-\frac{p}{t}}\sum_{k=0}^{\infty}\frac{(-1)^k(2k)!(1+ix)^{-2k-1}p^k}{k!^2}dp$$
$$= t^{-1}\int_0^{\infty} e^{-\frac{p}{t}}F(p,x)dp:$$
$$F(p,x)) = -2\mathrm{Re}\left(\frac{4p}{\xi^3+\xi^2\sqrt{\xi^2-4p}-4p\xi}\right)\ \xi=(1+ix) \quad (1.255)$$

1.8 Appendix

In this book we work in $\mathbb{R}^n$ (or $\mathbb{C}$) and we will state the results in this simpler setting. See [55] for general measure spaces. The integrals we use are Lebesgue integrals. A function is in $L^1(S)$ where S is a measurable set if $\int_S |f(x)|dx < \infty$. The Lebesgue measure λ is simply the measure defined first on boxes B by $\lambda(B) = volume(B)$, and then extended to measurable sets by additivity and "continuity" (regularity). A function is measurable if its inverse image of any measurable set is measurable.

1.8a The dominated convergence theorem

Theorem 1.256 (dominated convergence) *Assume $\{f_n\}_{n\in\mathbb{N}}$ is a family of real-valued functions and that $f_n(x) \to f(x)$ for almost all x in S* [13]. *Assume further that for all n $|f_n| \le g$ a.e $[\lambda]$* [13] *, where g is in $L^1(S)$. Then $f \in L^1(S)$ and*

$$\lim_{n\to\infty}\int_S f_n(s)ds \to \int_S f(s)ds \quad (1.257)$$

The Theorem also applies for complex valued functions, when real and imaginary parts have the requisite properties. Furthermore, it is easy to see that a similar statement holds for more general parametric convergence, that is, if n is replaced by a parameter y in a, say, metric space, under similar assumptions: $|f(y,x)| \le g(x)$ for all (x,y) where g is integrable, and $f(y,x) \to f(x)$ as $y \to y_0$ a.e.$[\lambda]$.

Note 1.258

[13] That is, except possibly for a set of measure zero; a set has zero measure if it contained in a union of boxes of arbitrarily small total measure. The notation a.e. $[\lambda]$ simply means for all x except for a zero measure set.

if K is a compact set in $\mathbb{R}$, then $F \in L^1_\nu(K)$, see (1.45), **iff** $F \in L^1(K)$. Indeed, in this case there exist two positive constants $c_1 \le c_2$ such that $c_1 < e^{-\nu p} < c_2$; the rest is straightforward. Nonetheless, if $F \in L^1([a,b])$, it is still useful to work in $L^1_\nu([a,b])$ $0 \le a < b \in \mathbb{R}$, since $\|F\|_{L^1_\nu([a,b])} \to 0$ as $\nu \to \infty$. Indeed, if $\nu > 0$ we have $|F(p)|e^{-\nu p} \le |F(p)|$ and $|F(p)|e^{-\nu p} \to 0$ on $[a,b]$. Thus Theorem 1.8a applies and $\int_a^b F(p)e^{-xp}dp \to 0$.

1.9 Banach spaces and the contractive mapping principle

In rigorously proving asymptotic results about *solutions* of various problems, where a closed form solution does not exist or is awkward, the contractive mapping principle is a handy tool. Once an asymptotic expansion solution has been found, if we use a truncated expansion as a quasi-solution, the remainder should be small. As a result, the complete problem becomes one to which the truncation is an exact solution modulo small errors (usually involving the unknown function). Therefore, most often, asymptoticity to a formal solution can be shown rigorously by rewriting this latter equation as a small perturbation of the identity operator (in a suitable norm) acting on a truncation of the formal solution. Some general guidelines on how to construct this operator are discussed in §1.9b. It is desirable to go through the rigorous proof, whenever possible — this should be straightforward when the asymptotic solution has been correctly found—, one reason being that this quickly signals errors such as omitting important terms, or exiting the region of asymptoticity.

In §1.9.1 we discuss, for completeness, a few basic facts about Banach spaces. There is of course a vast literature on the subject; see e.g. [50].

1.9.1 A brief review of Banach spaces

Familiar examples of Banach spaces are the n-dimensional Euclidian vector spaces $\mathbb{R}^n$. A norm exists in a Banach space, which has the essential properties of a length: scaling, positivity except for the zero vector which has length zero and the triangle inequality (the sum of the lengths of the sides of a triangle is no less than the length of the third one). Once we have a norm, we can define limits, by reducing the notion to that in $\mathbb{R}$: $x_n \to x$ iff $\|x - x_n\| \to 0$. A normed vector space $\mathcal{B}$ is a Banach space if it is complete, that is every sequence with the property $\|x_n - x_m\| \to 0$ uniformly in n, m (a Cauchy sequence) has a limit in $\mathcal{B}$. Note that $\mathbb{R}^n$ can be thought of as the space of functions defined on the set of integers $\{1, 2, ..., n\}$. If we take a space of functions on a domain containing infinitely many points, then the Banach space is usually infinite-dimensional. An example is $L^\infty[0, 1]$, the space of bounded functions on $[0,1]$ with the norm $\|f\| = \sup_{[0,1]} |f|$. A function L between two Banach spaces which is linear, $L(x+y) = Lx + Ly$, is bounded (or continuous) if $\|L\| := \sup_{\|x\|=1} \|Lx\| < \infty$. Assume $\mathcal{B}$ is a Banach space and that S is a closed subset of $\mathcal{B}$. In the *induced topology* (i.e., in the same norm), S is a complete normed space.

1.9.2 Fixed point theorem

Assume $\mathcal{M} : S \mapsto \mathcal{B}$ is a (linear or nonlinear) operator with the property that for any $x, y \in S$ we have

$$\|\mathcal{M}(y) - \mathcal{M}(x)\| \leq \lambda \|y - x\| \tag{1.259}$$

with $\lambda < 1$. Such operators are called **contractive**. Note that if $\mathcal{M}$ is linear, this just means that the norm of $\mathcal{M}$ is less than one.

Theorem 1.260 *Assume $\mathcal{M} : S \mapsto S$, where S is a closed subset of $\mathcal{B}$ is a contractive mapping. Then the equation*

$$x = \mathcal{M}(x) \tag{1.261}$$

has a unique solution in S.

PROOF Consider the sequence $\{x_j\}_j \in \mathbb{N}$ defined recursively by

$$\begin{aligned} x_0 &= x_0 \in S \\ x_1 &= \mathcal{M}(x_0) \\ &\cdots \\ x_{j+1} &= \mathcal{M}(x_j) \\ &\cdots \end{aligned} \tag{1.262}$$

We see that

$$\|x_{j+2} - x_{j+1}\| = \|\mathcal{M}(x_{j+1}) - \mathcal{M}(x_j)\| \leq \lambda \|x_{j+1} - x_j\| \leq \cdots \leq \lambda^j \|x_1 - x_0\| \tag{1.263}$$

Thus,

$$\|x_{j+p+2} - x_{j+2}\| \leq \left(\lambda^{j+p} + \cdots \lambda^j\right) \|x_1 - x_0\| \leq \frac{\lambda^j}{1-\lambda} \|x_1 - x_0\| \tag{1.264}$$

and x_j is a Cauchy sequence, and it thus converges, say to x. Since by (1.259) $\mathcal{M}$ is continuous, passing the equation for x_{j+1} in (1.262) to the limit $j \to \infty$ we get

$$x = \mathcal{M}(x) \tag{1.265}$$

that is existence of a solution of (1.261). For uniqueness, note that if x and x' are two solutions of (1.261), by subtracting their equations we get

$$\|x - x'\| = \|\mathcal{M}(x) - \mathcal{M}(x')\| \leq \lambda \|x - x'\| \tag{1.266}$$

implying $\|x - x'\| = 0$, since $\lambda < 1$. ∎

Note 1.267 *Note that contractivity and therefore existence of a solution of a fixed point problem depends on the norm. An adapted norm needs to be chosen for this approach to give results.*

Definition 1.268 *The norm $\|.\|$ of a linear operator $L : \mathcal{A} \to \mathcal{B}$ is simply defined as*

$$\|L\| = \sup_{\|x\|=1} \|Lx\|$$

Exercise 1.269 *Show that if L is a linear operator from the Banach space $\mathcal{B}$ into itself and $\|L\| < 1$ then $I - L$ is invertible, that is $x - Lx = y$ has always a unique solution $x \in \mathcal{B}$. "Conversely," assuming that $I - L$ is not invertible, then in whatever norm $\|\cdot\|_*$ we choose to make the same $\mathcal{B}$ a Banach space, we must have $\|L\|_* \geq 1$ (why?).*

1.9a Fixed points and vector valued analytic functions

A theory of analytic functions with values in a Banach space can be constructed by almost exactly following the usual construction of analytic functions. For the construction to work, we need the usual vector space operations and a topology in which these operations are continuous. A typical setting is that of a Banach algebra[14]. A detailed presentation is found in [32] and [42], but the basic facts are simple enough for the reader to redo the necessary proofs.

1.9b Choice of the contractive map

An equation can be rewritten in a number of equivalent ways. In solving an asymptotic problem, as a general guideline we mention:

- The operator $\mathcal{N}$ appearing in the final form of the equation, which we want to be contractive, should not contain derivatives of highest order, divided differences with small denominators, or other operations poorly behaved with respect to asymptotics, and it should only depend on the sought-for solution in a formally small way. The latter condition should be, in a first stage, checked for consistency: the discarded terms, calculated using the first order approximation, should indeed turn out to be small.

- To obtain an equation where the discarded part is manifestly small it often helps to write the sought-for solution as the sum of the first few terms of the approximation, plus an exact remainder, say δ. The equation for δ is usually more contractive. It also becomes, up to smaller corrections, linear.

[14]A Banach algebra is a Banach space of functions endowed with multiplication which is distributive, associative and continuous in the Banach norm.

- The norms should reflect as well as possible the expected growth/decay tendency of the solution itself and the spaces chosen should be spaces where this solution lives.

- All freedom in the solution has been accounted for, that is, we should make sure the final equation cannot have more than one solution.

Note 1.270 At the stage where the problem has been brought to a contractive mapping setting, it usually can be recast without conceptual problems, but perhaps complicating the algebra, to a form where the implicit function theorem applies (and vice versa). The contraction mapping principle is often more natural, especially when the topology, suggested by the problem itself, is not one of the common ones. But an implicit function reformulation might bring in more global information.

1.10 Examples

1.10a Linear differential equations in Banach spaces

Consider the equation

$$Y'(t) = L(t)Y(t); \quad Y(0) = Y_0 \tag{1.271}$$

in a Banach space X, where $L(t) : X \to X$ is linear, norm continuous in t and uniformly bounded,

$$\sup_{t\in[0,\infty)} \|L(t)\| < L \tag{1.272}$$

Then the problem (1.271) has a global solution on $[0,\infty)$, and $\|Y(t)\| \leq \|Y_0\|e^{(L+\varepsilon)t}$.

PROOF By comparison with the case when $X = \mathbb{R}$, the natural growth is indeed Ce^{Lt}, so we rewrite (1.271) as an integral equation, in a space where the norm reflects this possible growth. Consider the space of continuous functions $Y : [0,\infty) \mapsto X$ in the norm

$$\|Y\|_{\infty,L} = \sup_{t\in[0,\infty)} e^{-Lt/\lambda}\|Y(t)\| \tag{1.273}$$

with $\lambda < 1$ and the auxiliary equation

$$Y(t) = Y_0 + \int_0^t L(s)Y(s)ds =: \mathcal{A}[Y](t) \tag{1.274}$$

which is well defined on X and is contractive there since

$$e^{-Lt/\lambda}\left|\int_0^t L(s)Y(s)ds\right| \le Le^{-Lt/\lambda}\int_0^t e^{Ls/\lambda}\|Y\|_{\infty,L}ds$$
$$= \lambda(1-e^{-Lt/\lambda})\|Y\|_{\infty,L} \le \lambda\|Y\|_{\infty,L}, \quad (1.275)$$

and therefore in a ball of radius $(1+\gamma)\|Y_0\|$, for large enough γ (in fact, we need $(1+\gamma)(1-\lambda) > 1$),

$$\|\mathcal{A}[Y]\|_{\infty,L} \le \|Y_0\| + \lambda(1+\gamma)\|Y_0\| < (1+\gamma)\|Y_0\|$$

while

$$\|\mathcal{A}[Y_1] - \mathcal{A}[Y_2]\|_{\infty,L} \le \lambda\|Y_1 - Y_1\|_{\infty,L},$$

implying $\mathcal{A}$ to be contraction map; and so a unique solution exists for the initial value problem (1.271) with given exponential bounds for growth as given. We note that in linear problems, we do not need to restrict the analysis to a ball. $\square$

References

[1] M. Abramowitz and I.A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, 9th printing. New York: Dover, pp. 804–806, (1972).

[2] L. Ahlfors, *Complex Analysis*, McGraw-Hill, New York, (1979).

[3] M. Aschenbrenner and L. van den Dries, Asymptotic differential algebra, in *Analyzable Functions and Applications, pp. 49–85*, Contemp. Math., 373, Amer. Math. Soc., Providence, RI, (2005).

[4] W. Balser, B.L.J. Braaksma, J-P. Ramis and Y. Sibuya, *Asymptotic Anal. 5, no. 1, pp. 27–45*, (1991).

[5] W. Balser, *From Divergent Power Series to Analytic Functions*, Springer-Verlag, (1994).

[6] C. Bender and S. Orszag, *Advanced Mathematical Methods for Scientists and Engineers*, McGraw-Hill, (1978), Springer-Verlag, (1999).

[7] M.V. Berry and C.J. Howls, *Proceedings: Mathematical and Physical Sciences*, vol. 430, No. 1880, pp. 653–668, (1990).

[8] M.V. Berry and C.J. Howls, *Proc. Roy. Soc. London* Ser. A 443, no. 1917, pp. 107–126, (1993).

[9] M.V. Berry, *Proc. R. Soc. Lond.* A 422, pp. 7–21, (1989).

[10] M.V. Berry, *Proc. R. Soc. Lond.* A 430, pp. 653–668, (1990).

[11] M V. Berry, *Proc. Roy. Soc. London* Ser. A 434 no. 1891, pp. 465–472, (1991).

[12] E. Borel, *Leçons sur les Series Divérgentes*, Gauthier-Villars, Paris, (1901).

[13] B.L.J. Braaksma, *Ann. Inst. Fourier*, Grenoble, 42, 3, pp. 517–540, (1992).

[14] A. Cauchy, *Oeuvres Completes d'Augustin Cauchy*, publiées sous la direction scientifique de l'Academie de sciences et sous les auspices du m. le ministre de l'instruction publique. Gauthier-Villars, Paris, (1882–90).

[15] M.D. Kruskal and P.A. Clarkson, *Studies in Applied Mathematics*, 86, pp 87–165, (1992).

[16] E.A. Coddington and N. Levinson, *Theory of Ordinary Differential Equations*, McGraw-Hill, New York, (1955).

[17] Costin O., Asymptotics and Borel summability, Chapmann & Hall, New York (2009).

[18] O. Costin and M.D. Kruskal, *Proc. R. Soc. Lond. A*, 452, pp. 1057–1085, (1996).

[19] O. Costin and M.D. Kruskal, *Comm. Pure Appl. Math.*, 58, no. 6, pp. 723–749, (2005).

[20] O. Costin and M. D. Kruskal, *Proc. R. Soc. Lond. A*, 455, pp. 1931–1956, (1999).

[21] NIST Digital Library of Mathematical Functions http://dlmf.nist.gov/

[22] O. Costin and R.D. Costin, SIAM J. Math. Anal. 27, no. 1, pp. 110–134, (1996).

[23] O. Costin and S. Tanveer, *Ann. Inst. H. Poincaré Anal. Non Linéaire* 24, no. 5, pp. 795–823, (2007).

[24] O. Costin, *Analyzable functions and applications*, pp. 137–175, Contemp. Math., 373, Amer. Math. Soc., Providence, RI, (2005).

[25] O. Costin, *Duke Math. J.*, 93, No. 2, (1998).

[26] O. Costin and R.D. Costin, *Inventiones Mathematicae*, 145, 3, pp. 425–485, (2001).

[27] O. Costin, *IMRN* 8, pp. 377–417, (1995).

[28] O. Costin, G. Luo and S. Tanveer, *Integral Equation representation and longer time existence solutions of 3-D Navier-Stokes, submitted.* http://www.math.ohio-state.edu/∼tanveer/ictproc.ns.5.pdf (last accessed: August 2008).

[29] O. Costin and S. Garoufalidis *Annales de L'institut Fourier*, vol. 58 no. 3, pp. 893–914, (2008).

[30] E. Delabaere and C.J. Howls, *Duke Math. J.* 112, no. 2, 199–264, (2002).

[31] R.B. Dingle, *Asymptotic Expansions: Their Derivation and Interpretation*, Academic Press, London-New York, (1973).

[32] N. Dunford and J.T. Schwartz, *Linear Operators, Part I: General Theory*, Interscience, New York, (1960).

[33] J. Écalle and F. Menous, *Publicacions Matemàtiques*, vol. 41, pp. 209–222, (1997).

[34] J. Écalle, Preprint 90-36 of Université de Paris-Sud, (1990).

[35] J. Écalle, *Fonctions Resurgentes*, Publications Mathématiques D'Orsay, (1981).

[36] J. Écalle, in *Bifurcations and periodic orbits of vector fields*, NATO ASI Series, vol. 408, (1993).

[37] G.A. Edgar, *Transseries for Beginners*, preprint, (2008). http://www.math.ohio-state.edu/∼ edgar/WG_W08/edgar/transseries.pdf (last accessed: August 2008).

[38] L. Euler, De seriebus divergentibus, *Novi Commentarii academiae scientiarum Petropolitanae* (1754/55) 1760, pp. 205–237, reprinted in *Opera Omnia Series*, I vol. 14, pp. 585–617. Available through The Euler Archive at www.EulerArchive.org (last accessed, August 2008).

[39] E. Goursat, *A Course in Mathematical Analysis, vol. 2: Functions of a Complex Variable & Differential Equations*, Dover, New York, (1959).

[40] C.G. Hardy, *Divergent Series*, Oxford, (1949).

[41] E. Hille *Ordinary Differential Equations in the Complex Domain*, Dover Publications (1997).

[42] E. Hille and R.S. Phillips, *Functional Analysis and Semigroups*, AMS Providence, R.I. (1957).

[43] J. van der Hoeven, *Transseries and Real Differential Algebra*, Lecture Notes in Mathematics 1888, Springer, New York, (2006).

[44] A.S.B. Holland, *Introduction to the Theory of Entire Functions*, Academic Press, US, (1973).

[45] I. Kaplansky, *Differential Algebra*, Hermann, Paris, (1957).

[46] T. Kawai and Y. Takei, *Advances in Mathematics*, 203, 2,pp. 636–672 (2006).

[47] D.A. Lutz, M. Miyake and R. Schäfke, *Nagoya Math. J.* 154, 1, (1999).

[48] J. Martinet and J-P. Ramis, *Annales de l'Institut Henri Poincaré (A) Physique théorique*, 54 no. 4, pp. 331–401, (1991).

[49] A.B. Olde Daalhuis, *R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci.* 454, no. 1968, pp. 1–29, (1998).

[50] M. Reed and B. Simon, *Methods of Modern Mathematical Physics*, Academic Press, New York, (1972).

[51] J-P. Ramis, *Asterisque*, V. 59-60, pp. 173–204, (1978).

[52] J-P. Ramis, *C.R. Acad. Sci. Paris*, tome 301, pp. 99–102, (1985).

[53] J.F. Ritt *Differential Algebra*, American Mathematical Society, New York, (1950).

[54] W. Rudin, *Principles of Mathematical Analysis* p. 110, McGraw-Hill, New York, (1974).

[55] W. Rudin, *Real and Complex Analysis*, McGraw-Hill, New York, (1987).

[56] D. Sauzin, *RIMS Kokyuroku* 1493, pp. 48–117, (31/05/2006). http://arxiv.org/abs/0706.0137v1. (last accessed: August 2008).

[57] Y. Sibuya, *Global Theory of a Second Order Linear Ordinary Differential Equation with a Polynomial Coefficient*, North-Holland, Amsterdam-New York, (1975).

[58] A. Sokal, *J. Math. Phys.*, 21 pp. 261–263, (1980).

[59] G.G. Stokes *Trans. Camb. Phil. Soc.*, 10, pp. 106–128. Reprinted in *Mathematical and Physical papers by late sir George Gabriel Stokes,* Cambridge University Press, vol. IV, pp. 77–109, (1904).

[60] P. Suppes, *Axiomatic Set Theory*, Dover, (1972).

[61] E.C. Titchmarsh, *The Theory of Functions*, Oxford University Press, USA; 2nd edition, (1976).

[62] W. Wasow *Asymptotic Expansions for Ordinary Differential Equations*, Interscience Publishers, New York, (1968).

[63] E. Zermelo, *Mathematische Annalen*, 65, pp. 261–281, (1908).