# *Course notes*

*CRC PRESS*
*Boca Raton London New York Washington, D.C.*

## Some notations

| | |
|---|---|
| $\mathcal{L}$ | Laplace transform, §?? |
| $\mathcal{L}^{-1}$ | inverse Laplace transform, §1.58 |
| $\mathcal{B}$ | Borel transform, §2.1 |
| $\mathcal{LB}$ | Borel/BE summation operator, §?? and §2.2b |
| $p$ | usually, Borel plane variable |
| $\tilde{f}$ | formal expansion |
| $H(p)$ | Borel transform of $h(x)$ |
| $\sim$ | asymptotic to, §1.1a |
| $\lesssim$ | less than, up to an unimportant constant, §1.1a |
| $\mathbb{D}_r$ | The disk of radius $r$ centered at 0 |
| $\partial A$ | The boundary of the set $A$ |
| $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ $\mathbb{N}^+, \mathbb{R}^+$ | the nonnegative integers, integers, rationals, real numbers, complex numbers, positive integers, and positive real numbers, respectively |
| $\mathbb{H}$ | open right half complex-plane. |
| $\mathbb{H}_\theta$ | half complex-plane centered on $e^{i\theta}$. |
| $\overline{S}$ | closure of the set $S$. |
| $C_a$ | absolutely continuous functions, [74] |
| $f * g$ | convolution of $f$ and $g$, §?? |
| $L^1_\nu, \|\cdot\|_\nu$, $\mathcal{A}_{K,\nu}$, etc. | various spaces and norms defined in §2.6 and §2.7 |

# Contents


**1 Introduction** **1**

1.1 Expansions and approximations . . . . . 1

1.1a Notation . . . . . 1

1.1b Asymptotic expansions . . . . . 2

1.1c Asymptotic power series . . . . . 5

1.1d Operations with asymptotic power series . . . . . 6

1.2 Asymptotics of integrals . . . . . 8

1.2a *The Laplace transform and its properties. . . . . . 9

1.2b Watson's Lemma . . . . . 13

1.3 Oscillatory integrals and the stationary phase method . . . . 18

1.4 Steepest descent method . . . . . 21

1.4a Simple illustrative example . . . . . 23

1.4b Finding the steepest variation lines . . . . . 24

1.5 Formal and actual solutions . . . . . 31

1.5a An irregular singular point of a nonlinear equation . . 32

1.5b The wave equation with potential . . . . . 34

1.5c Regular versus singular perturbations . . . . . 36

1.5d The case $V(x) - E \geq a > 0$ . . . . . 43

1.6 Borderline region: $x \gg \hbar^{2/3}$ . . . . . 44

1.6a Inner region: Rigorous analysis . . . . . 45

1.6b Matching region . . . . . 47

1.7 Recovering actual solutions from formal ones . . . . . 48

**2 Borel summation: an introduction** **53**

2.1 The Borel transform $\mathcal{B}$ . . . . . 53

2.2 Definition of Borel summation and basic properties . . . . . 54

2.2a Borel summation along other directions . . . . . 55

2.2b Limitations of classical Borel summation; BE summation . . . . . 56

2.3 Borel summation as an isomorphism . . . . . 58

2.4 Some examples . . . . . 62

2.4a Whittaker functions . . . . . 64

2.4b Hypergeometric functions . . . . . 65

2.4c The Gamma function . . . . . 66

2.5 Stokes phenomena . . . . . 67

2.5a The Airy equation . . . . . 67

2.5b Nonlinear Stokes phenomena . . . . . 70

2.5c The Gamma function . . . . . . . . . . . . . . . . . . . 71
2.6 Analysis of convolution equations . . . . . . . . . . . . . . . . 72
2.6a Properties of the $p$ plane convolution, Definition 2.24 . 72
2.6b Banach convolution algebras . . . . . . . . . . . . . . 72
2.6c Spaces of sequences of functions . . . . . . . . . . . . 74
2.7 Focusing spaces and algebras . . . . . . . . . . . . . . . . . . 75
2.8 Borel summation analysis of nonlinear ODEs . . . . . . . . . 76
2.8a Borel summation of the transseries solution . . . . . . 78
2.8b Analytic structure along $\mathbb{R}^+$ . . . . . . . . . . . . . . . 81
2.8c The analytic continuations of $Y_0$ . . . . . . . . . . . . 82
2.8d Local analytic structure of $Y_0, \Phi_k$ along $\mathbb{R}^+$ . . . . . . 83
2.8e Determining singularities in Borel plane from asymptotics of Laplace integrals . . . . . . . . . . . . . . . . 86
2.9 Spontaneous singularities and the Painlevé property . . . . . 87
2.9a The Painlevé property . . . . . . . . . . . . . . . . . . 88
2.9b Analysis of a modified $\mathrm{P_I}$ equation . . . . . . . . . . . 88
2.9c Rigorous analysis of the meromorphic expansion for $\mathrm{P_I}$ 92
2.9d The Painlevé property for PDEs . . . . . . . . . . . . 94
2.10 Gevrey classes, least term truncation and Borel summation . . . . . . . . . . . . . . . . . . . . . . . . . . . 97
2.10a Connection between Gevrey asymptotics and Borel summation . . . . . . . . . . . . . . . . . . . . . . . . 98
2.11 Multiple scales; Adiabatic invariants . . . . . . . . . . . . . . 102
2.11a The problem as a regularly perturbed equation; secular terms . . . . . . . . . . . . . . . . . . . . . . . . . . 102
2.11b The Poincaré-Lindstedt method . . . . . . . . . . . . . 103
2.11c Multi-scale analysis . . . . . . . . . . . . . . . . . . . 104
2.11d How do the parameters of the motion change when $L$ is doubled? . . . . . . . . . . . . . . . . . . . . . . . . . 105
2.11e WKB . . . . . . . . . . . . . . . . . . . . . . . . . . 105
2.11f The adiabatic invariant . . . . . . . . . . . . . . . . . 106
2.11g Solution for more general $L$ . . . . . . . . . . . . . . . 107
2.11h Working with action-angle variables . . . . . . . . . . 108
2.11i The Painlevé equations $\mathrm{P_I}$ . . . . . . . . . . . . . . . 110
2.12 Appendix . . . . . . . . . . . . . . . . . . . . . . . . . . . . 117
2.12a The dominated convergence theorem . . . . . . . . . . 118
2.13 Banach spaces and the contractive mapping principle . . . . 119
2.13a Fixed points and vector valued analytic functions . . . 122
2.13b Choice of the contractive map . . . . . . . . . . . . . . 122
2.14 Examples . . . . . . . . . . . . . . . . . . . . . . . . . . . . 123
2.14a Linear differential equations in Banach spaces . . . . . 123

**References** **125**

# Chapter 1

# Introduction

## 1.1 Expansions and approximations

Classical asymptotic analysis is a set of mathematical results and methods to find the limiting behavior of functions, near a point, most often a singular point. It is particularly efficient in the context of differential or difference equations when the function has no simple representation that immediately conveys the desired limiting behavior.

Asymptotic analysis may involve several variables; however, in this book, we will be mostly concerned with limiting behavior in one scalar variable; in the context of differential or difference equations, this can be the independent variable or a parameter.

### 1.1a Notation

Let the special point of analysis be $t_0 \in \mathbb{C}$.

Some common notations are: $f = O(1)$ if $f$ is bounded near $t_0$ and $f = o(1)$ if $f \to 0$ as $t \to t_0$. More generally $f = O(g)$ if $f/g = O(1)$ and similarly $f = o(g)$ if $f/g = o(1)$. We also write $f \ll g$ if $f = o(g)$. It is understood that $g$ cannot vanish close to $t_0$. The notation $|f| \lesssim |g|$ is used to represent $|f| \le C|g|$ in the domain of interest, where $C$ is a constant whose value is immaterial. Clearly $|f| \lesssim |g|$ in a small neighborhood of $t_0$ is the same as $f = O(g)$. We write $f = O_s(g)$; when both $f = O(g)$ and $g = O(f)$ near $t_0$.

The point $t_0$ may be approached only from one direction, along a curve in $\mathbb{C}$ or even along a given sequence of points tending to $t_0$ and when such further restrictions are needed, they will be specified. For instance if $t_0 = 0$, then $t = o(1)$ as $t \to 0$ and $e^{-1/t} = o(t^m)$ for any $m$ as $t \downarrow 0$ ($t \in \mathbb{R}^+$ decreases towards 0), but the opposite holds, $t^m = o(e^{-1/t})$, as $t \uparrow 0$.

### 1.1b Asymptotic expansions

A sequence of functions $\{f_k\}_{k\in\mathbb{N}}$ such that $f_m \ll f_n$ if $m > n$ is called an asymptotic scale at $t = t_0$. In terms of it we can write the leading order behavior of a function, $f = f_0 + o(f_0)$ and also successively higher order corrections: $f = f_0+f_1+o(f_1)$ etc. In a compact form, we write an asymptotic expansion as a formal sum,

$$\sum_{k=0}^{\infty} f_k(t) =: \tilde{f}, \tag{1.1}$$

where no convergence condition is imposed, and define asymptoticity by the following.

**Definition 1.2** *A function $f$ is asymptotic to the formal series $\tilde{f}$ as $t \to t_0$ (once more, the approach of $t_0$ may have to be restricted to a curve) if*

$$f(t) - \sum_{k=0}^{N} \tilde{f}_k(t) = o(\tilde{f}_N(t)) \quad (\forall N \in \mathbb{N}) \tag{1.3}$$

Condition (1.3) can be written in a number of equivalent ways, useful in applications, as the following result shows.

**Proposition 1.4** *If $\tilde{f} = \sum_{k=0}^{\infty} \tilde{f}_k(t)$ is an asymptotic series as $t \to t_0$ and $f$ is a function asymptotic to it, then the following characterizations are equivalent to each other and to (1.3).*

*(i)*

$$f(t) - \sum_{k=0}^{N} \tilde{f}_k(t) = O(\tilde{f}_{N+1}(t)) \quad (\forall N \in \mathbb{N}) \tag{1.5}$$

*(ii)*

$$f(t) - \sum_{k=0}^{N} \tilde{f}_k(t) = \tilde{f}_{N+1}(t)(1 + o(1)) \quad (\forall N \in \mathbb{N}) \tag{1.6}$$

*(iii) There is function $\nu : \mathbb{N} \mapsto \mathbb{N}$, such that $\nu(N) \geq N$ and*

$$f(t) - \sum_{k=0}^{\nu(N)} \tilde{f}_k(t) = O(\tilde{f}_{N+1}(t)) \quad (\forall N \in \mathbb{N}) \tag{1.7}$$

*Condition (iii) seems strictly weaker, but it is not. It allows us to use less accurate estimates of remainders, provided we can do so to all orders.*

**PROOF** We only show (iii), the others being immediate from the definition. We may assume $\nu(N) > N$, as otherwise there is nothing to prove. Let $N \in \mathbb{N}$. We have

$$f(t) - \sum_{k=0}^{N} \tilde{f}_k(t) \; = \; f(t) - \sum_{k=0}^{\nu(N)} \tilde{f}_k(t) + \sum_{j=N+1}^{\nu(N)} \tilde{f}_j(t) \; = \; O\left(\tilde{f}_{N+1}(t)\right) \quad (1.8)$$

since in the last sum in (1.8) the number of terms is fixed, and thus the sum remains $O\left(\tilde{f}_{N+1}\right)$ as $t \to t_0$. ∎

Whenever possible, the scale is chosen to consist of simple functions, such as powers, logs and exponentials, the behavior of which is manifest. Taylor series are perhaps the simplest nontrivial asymptotic expansions. The following is a way of restating Taylor's theorem with remainder.

**Proposition 1.9** *Assume $f$ is $C^\infty$ in an interval containing $t_0$. Then*

$$f(t) \sim \sum_{k=0}^{\infty} \frac{f^{(k)}(t_0)}{k!}(t - t_0)^k \text{ as } t \to t_0 \quad (1.10)$$

Clearly, the asymptotic series of a function $f$ converges to $f$ **iff** $f$ is analytic at $t_0$. Otherwise, the series is not convergent, or it converges to a function other than $f$ (see Example 1.16).

**Note 1.11** In Definition 1.2 none of the $f_k$ is allowed to vanish. For instance, although all right derivatives of $f_1 = e^{-1/t}$ vanish at zero, we cannot write $e^{-1/t} \sim 0$. This is a natural restriction since all the right derivatives vanish at zero for many other functions, for instance $f_2 = \sin(1/t)e^{-1/\sqrt{t}}$, with quite different behavior $t \downarrow 0$. We will however speak of *asymptotic power series*, a weaker notion in which sense $f_1$ and $f_2$ above will be represented by the same series.

**Example 1.12 (A divergent asymptotic series)** A simple example of a divergent asymptotic expansion is obtained by calculating the Taylor series of the function

$$f(z) = \frac{1}{z} e^{-1/z} \mathrm{E}_1\left(\frac{1}{z}\right) = \int_0^\infty \frac{e^{-t}}{1 + zt} dt; \; z > 0 \quad (1.13)$$

where $E_1(z) = \int_z^\infty \frac{e^{-t}}{t} dt$ is the exponential integral. The exponential decay of the integrand allows for differentiating (1.151) any number of times for $z > 0$,

$$f^{(k)}(z) = k! \int_0^\infty \frac{(-t)^k e^{-t}}{(1 + zt)^{k+1}} dt \quad (1.14)$$

Furthermore, $f^{(k)}(z)$ are continuous as $z \to 0^+$ (right limit at zero) for all $k \geq 0$. Elementary analysis tells us that $f$ is $C^\infty$ at zero from the right. The integral representation of the factorial gives $f^{(k)}(0) = (-1)^k (k!)^2$ We have, using Taylor's theorem with one-sided derivatives [73]

$$f(z) \sim \sum_{k=0}^{\infty} (-1)^k k! z^k, \; z \downarrow 0 \tag{1.15}$$

a series with zero radius of convergence, or in short a *divergent series.*

**Example 1.16 (A convergent asymptotic series)** Since all derivatives of $e^{-1/z}$ vanish as $z \downarrow 0$ we have

$$\frac{1}{1-z} + e^{-1/z} \sim \sum_{k=0}^{\infty} z^k = \frac{1}{1-z}, \; z \to 0^+ \tag{1.17}$$

Convergence of an asymptotic series does not thus imply that the function equals the sum of the series. Note also that here, as it is often done in practice, we have used the *same* notation $\sum_{k=0}^{\infty} z^k$ to mean two different things: an asymptotic series simply displaying the asymptotic scale involved, which is a formal object, and its *sum*, an actual function. We will discuss this ambiguity later.

**Example 1.18 (A convergent but *anti*asymptotic series)** The following Laurent series converges in $\mathbb{C} \setminus \{0\}$:

$$\sum_{j=0}^{\infty} \frac{(-1)^j}{j! z^j} = e^{-1/z} \tag{1.19}$$

Eq. (1.19) is **not** an asymptotic expansion as $z \to 0$. In (1.19) $f_k \ll f_{k+1}$, the **opposite** of what is required from an asymptotic series. We have $|e^{-1/z} - \sum_{j=0}^{M} \frac{(-1)^j}{j! z^j}| \gtrsim |z^{-M-1}|$ as $z \downarrow 0$ which means the approximations deteriorate the more terms we keep, if $z \downarrow 0$.

In general, for understanding the behavior of a function near a point, an antiasymptotic series, even if convergent, is not very useful. We can see that if we try to determine whether

$$f(z) = \sum_{j=0}^{\infty} \frac{(-1)^j}{(j! + 10^{-j} \sin(j)) z^j} \tag{1.20}$$

(the Laurent coefficients are close to those in (1.19)) tends zero or not, as $z \to 0$.

By contrast, although (1.15) is divergent, by the definition of an asymptotic series, in (1.151) we see that $f(z) \to 1$ as $z \downarrow 0$, and that $f(z) - 1 = -z(1 + o(1))$ and so on.

Stirling's formula for $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$, which will be derived later, in §**??**, is an example of a divergent asymptotic expansion, where the scales involve powers of $1/x$ and logs:

$$\ln(\Gamma(x)) \sim (x-1/2)\ln x - x + \frac{1}{2}\ln(2\pi) + \sum_{j=1}^{\infty} c_j x^{-2j+1}, \ x \to +\infty \quad (1.21)$$

where $2j(2j-1)c_j = B_{2j}$ and $\{B_{2j}\}_{j\ge1} = \{1/6, -1/30, 1/42...\}$ are Bernoulli numbers, [1], eq. 6.140. This expansion is asymptotic as $x \to \infty$: successive terms get smaller and smaller. For $x = 6$, truncating (1.21) at $j = 3$ we get $\Gamma(6) \approx 120.0000002$ (while $\Gamma(6) = 5! = 120$). Stirling's expansion converges for no $x$, since $\ln(\Gamma(x))$ is singular at all $x \in -\mathbb{N}$ (why is this an obstruction to convergence?).

**Remark 1.22** Asymptotic expansions cannot be added, in general. Indeed, we note that $1/(1-z)$ has the same expansion (1.17) as $-e^{-1/z} + 1/(1-z)$, as $z \downarrow 0$. Adding these would give $e^{-1/z} \sim 0$, which is not a valid asymptotic expansion, see Note 1.11. This is one reason for considering, for restricted expansions, a weaker asymptoticity condition; see §1.1c.

**Remark 1.23** Sometimes we encounter oscillatory expansions such as $\sin x(1 + a_1x^{-1} + a_2x^{-2} + \cdots)$ $(*)$ for large $x$, which, while very useful, have to be understood differently. They are not asymptotic expansions, as we saw in Note 1.11. Furthermore, usually the approximation itself is expected to fail near zeros of sin. However, if small neighborhoods of the zeros of sin are excluded, the expansion remains valid in the sense defined. Also, usually there are ways to present the asymptotics in a way that avoids these exclusions,(see §**??**).

### 1.1c Asymptotic power series

A special role is played by series in *powers* of a small variable, such as

$$\tilde{S} = \sum_{k=0}^{\infty} c_k z^k, \ z \to 0^+ \quad (1.24)$$

With the transformation $z = t - t_0$ (or $z = x^{-1}$, when $x$ is large) the series can be centered at $z = 0$ (or $x = +\infty$, respectively).

**Definition 1.25 (Asymptotic power series)** *A function is asymptotic to a series as $z \to 0$,* in the sense of power series *if*

$$f(z) - \sum_{k=0}^{N} c_k z^k = O(z^{N+1}) \quad (\forall N \in \mathbb{N}) \quad \text{as } z \to 0, \quad (1.26)$$

*where, as for general asymptotic expansions, it may be necessary to restrict the approach $z \to 0$ to a particular set of curves.*

**Remark 1.27** If $f$ has an asymptotic expansion ( in the sense of Definition 1.2) that happens to be a power series, it is asymptotic to it in the sense of power series as well.

However, the converse is not true, unless all $c_k$ are nonzero, *i.e.* it is possible that $f \sim \tilde{f} \equiv \sum_{k=0}^{\infty} c_k z^k$ in the power series sense, without $\tilde{f}$ being the asymptotic expansion in the sense of Definition 1.2.

For now, whenever confusions are possible, we will use a different symbol, $\sim_p$ , for asymptoticity in the sense of power series.

**Remark 1.28** Noninteger asymptotic power series, e.g., series of the form

$$z^{\alpha} \sum_{k=0}^{\infty} c_k z^{k\beta}, \quad \mathrm{Re}\,(\beta) > 0 \tag{1.29}$$

as well as asymptoticity of a function to (1.29) can be defined by easily adapting Definition 1.25, and replacing $O(z^N)$ by $O(z^{N\beta+\alpha})$ which is the same as $O(z^{\mathrm{Re}\,\alpha+N\mathrm{Re}\,(\beta)})$. More generally, in (1.29), instead of $z^{\alpha}$, we could have other simple functions such as exponentials or logs.

The asymptotic power series at zero in $\mathbb{R}$ of $e^{-1/z^2}$ is the zero series, which is not its asymptotic expansion in the sense of Definition 1.2, see again Note 1.11. The advantage of asymptotic power series however is the fact that they form an algebra.

### 1.1d Operations with asymptotic power series

Addition and multiplication of asymptotic power series are defined as in the convergent case:

$$A\sum_{k=0}^{\infty} c_k z^k + B\sum_{k=0}^{\infty} d_k z^k = \sum_{k=0}^{\infty} (Ac_k + Bd_k) z^k$$

$$\left(\sum_{k=0}^{\infty} c_k z^k\right)\left(\sum_{k=0}^{\infty} d_k z^k\right) = \sum_{k=0}^{\infty}\left(\sum_{j=0}^{k} c_j d_{k-j}\right) z^k$$

**Remark 1.30** *If the series $\tilde{f}$ is convergent and $f$ is its sum, $f = \sum_{k=0}^{\infty} c_k z^k$, (note the ambiguity of the sum notation), then $f \sim_p \tilde{f}$.*

The proof follows directly from the definition of convergence

The proof of the following lemma is immediate:

**Lemma 1.31 (Algebraic properties of asymptoticity to a power series)** *If $f \sim_p \tilde{f} = \sum_{k=0}^{\infty} c_k z^k$ and $g \sim_p \tilde{g} = \sum_{k=0}^{\infty} d_k z^k$, then*

*(i) $Af + Bg \sim_p A\tilde{f} + B\tilde{g}$*

*(ii) $fg \sim_p \tilde{f}\tilde{g}$*

**Corollary 1.32 (Uniqueness of the asymptotic series to a function)** *If $f(z) \sim_p \sum_{k=0}^{\infty} c_k z^k$ as $z \to 0$, then the $c_k$ are unique.*

**PROOF** Indeed, if $f \sim_p \sum_{k=0}^{\infty} c_k z^k$ and $f \sim_p \sum_{k=0}^{\infty} d_k z^k$, then, by Lemma 1.31 we have $0 \sim_p \sum_{k=0}^{\infty}(c_k - d_k) z^k$ which implies, inductively, that $c_k = d_k$ for all $k$. ∎

However, division of asymptotic power series is not always possible. For instance, $e^{-1/z^2} \sim_p 0$ for small $z$ in $\mathbb{R}$ while $1/\exp(-1/z^2)$ has no asymptotic power series at zero. Also, classical asymptotics cannot distinguish between functions differing by a quantity which is $o(z^m)$ for all $m > 0$ as $z \to 0$. Indeed, we have the following result (see also Example 1.16)

**Proposition 1.33** *Assume $f$ and $g$ have nonzero asymptotic power series as $z \to 0$ and $f - g = h$ where $h = o(z^m)$ for all $m > 0$ as $z \to 0$. Then the asymptotic series of $f$ and $g$ coincide.*

**PROOF** This follows straightforwardly from Definition 1.26 and the assumption on $h$. ∎

### 1.1d.1 Integration and differentiation of asymptotic power series

Asymptotic relations can be integrated termwise as Proposition 1.34 below shows.

**Proposition 1.34** *Assume $f$ is integrable near $z = 0$ and that*

$$f(z) \underset{p}{\sim} \tilde{f}(z) = \sum_{k=0}^{\infty} c_k z^k$$

*Then*

$$\int_0^z f(s)ds \underset{p}{\sim} \int_0^z \tilde{f}(s)ds := \sum_{k=0}^{\infty} \frac{c_k z^{k+1}}{k+1}$$

**PROOF** This follows from the fact that $\int_0^z o(s^n)ds = o(z^{n+1})$ as it can be seen by straightforward inequalities. ∎

Differentiation is a different issue. Many simple examples show that asymptotic series cannot be unrestrictedly differentiated. For instance $e^{-1/z^2} \sin e^{1/z^4} \sim_p 0$ as $z \to 0$ on $\mathbb{R}$, but the derivative is unbounded and thus it is not asymptotic to zero.

### 1.1d.2 Asymptotics in regions in $\mathbb{C}$

Asymptotic power series of analytic functions can be differentiated if they hold in a region which is not too rapidly shrinking as $z \to 0$. This is so, since the derivative is expressible as an integral by Cauchy's formula. Such a region is often a sector or strip in $\mathbb{C}$, but can be allowed to be thinner:

**Proposition 1.35** *Let $M \geq 0$ and denote*

$$S_a = \{x : \quad |x| \geq R, \quad |x|^M |\mathrm{Im}\,(x)| \leq a\}$$

*Assume $f$ is continuous in $S_a$ and analytic in its interior, and*

$$f(x) \underset{p}{\sim} \sum_{k=0}^{\infty} c_k x^{-k} \quad \text{as } x \to \infty \text{ in } S_a$$

*Then, for all $a' \in (0, a)$ we have*

$$f'(x) \underset{p}{\sim} \sum_{k=0}^{\infty} (-kc_k) x^{-k-1} \quad \text{as } x \to \infty \text{ in } S_{a'}$$

**PROOF** Here, Proposition 1.4 (iii) will come in handy. Let $\nu(N) = N + M$. By the asymptoticity assumptions, for any $N$ there is some constant $C(N)$ such that $|f(x) - \sum_{k=0}^{\nu(N)} c_k x^{-k}| \leq C(N)|x|^{-\nu(N)-1}$ (*) in $S_a$.

Let $a' < a$, take $x$ large enough, and let $\rho = \frac{1}{2}(a - a')|x|^{-M}$; then check that $\mathbb{D}_\rho = \{x' : |x - x'| \leq \rho\} \subset S_a$. We have, by Cauchy's formula and (*),

$$\left| f'(x) - \sum_{k=0}^{\nu(N)} (-kc_k) x^{-k-1} \right| = \left| \frac{1}{2\pi i} \oint_{\partial \mathbb{D}_\rho} \left( f(s) - \sum_{k=0}^{\nu(N)} c_k s^{-k} \right) \frac{ds}{(s-x)^2} \right|$$
$$\leq \frac{C(N)}{(|x| - 1)^{\nu(N)+1}} \frac{1}{2\pi} \oint_{\partial \mathbb{D}_\rho} \frac{d|s|}{|s - x|^2} \leq \frac{2C(N)}{|x|^{\nu(N)+1} \rho} \leq \frac{4C(N)}{a - a'} |x|^{-N-1} \quad (1.36)$$

and the result follows. ∎

**Note 1.37** Usually, we can determine from the context whether $\sim$ or $\sim_p$ should be used. Often in the literature, it is left to the reader to decide which notion is in use. After we have explained the distinction, we will do the same, so as not to complicate notation.

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## 1.2 Asymptotics of integrals

Often when differential equations have closed form solutions, these can be expressed in terms of elementary functions or special functions admitting integral representations. These integral expressions allow for finding the asymptotic behavior of solutions in different regions of the complex domain. Important examples include the equation

$$x^2y'' + xy' + \sigma(x^2 - \sigma\nu^2)y = 0; \quad \sigma = \pm 1 \tag{1.38}$$

For $\sigma = 1$, (1.38) is the Bessel equation [1]; the solution which is regular at the origin is $J_\nu(x)$ – the Bessel function of the first kind and a linearly independent one is $Y_\nu(x)$ – the Bessel function of the second kind. For $\sigma = -1$ (1.38) is the modified Bessel equation; the solution which is regular at the origin is $I_\nu(x)$ – the modified Bessel function of the first kind and a linearly independent one is $K_\nu(x)$ – the modified Bessel function of the second kind. The Airy equation

$$y'' - xy = 0 \tag{1.39}$$

has solutions $\mathrm{Ai}(x)$ and $\mathrm{Bi}(x)$, the Airy functions. The hypergeometric equation

$$x(x-1)y'' + [(a+b+1)x - c]y' + aby = 0 \tag{1.40}$$

has linearly independent solutions ${}_2F_1(a,b;c;x)$ and $x^{1-c}\ {}_2F_1(a-c+1, b-c+1; 2-c; x)$ where ${}_2F_1$ is a hypergeometric function. All these functions have integral representations, in fact a good number of representations suitable for different asymptotic regimes. For instance, see [27] 10.9.17, [8] (Equation 6.6.30, page 298),

$$J_\nu(z) = \frac{1}{2\pi i}\int_{\infty-\pi i}^{\infty+\pi i} \exp(z\sinh t - \nu t)dt; \quad \mathrm{Re}\ z\ >\ 0 \tag{1.41}$$

and [27] 9.5.4, and [8] (p. 313, Problem 6.75, with the change of integration variable $t \to -t$).

$$\mathrm{Ai}(z) = \frac{1}{2\pi i}\int_{\infty e^{-\pi i/3}}^{\infty e^{\pi i/3}} \exp\left(t^3/3 - zt\right)\, dt, \tag{1.42}$$

Finally, for $|z| < 1$ [27] 15.1.2 and 15.6.1,

$${}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\,\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\, dt\ \ \mathrm{Re}(c) > \mathrm{Re}(b) > 0 \tag{1.43}$$

### 1.2a *The Laplace transform and its properties.

The Laplace transform $\mathcal{L}f$ of a function $F$ is defined by

$$f(x) = \int_0^\infty e^{-xp}F(p)dp, \quad \mathrm{Re}\,(x) > \nu \geq 0 \tag{1.44}$$

Here it is assumed that $F$ is locally integrable in $[0,\infty)$ and does not grow faster than exponentially, for instance

$$\|F\|_{\infty,\nu} = \sup_{p\ge 0} |F(p)|e^{-\nu p} < \infty \text{ or } \|F\|_{L^1_\nu} = \int_0^\infty |F(p)|e^{-\nu p}dp < \infty \quad (1.45)$$

(see §2.12a) for some $\nu \in \mathbb{R}$. Both ensure the existence of $\mathcal{L}f$ if $\mathrm{Re}\, x > \nu$.

As will be seen in the sequel, solutions of linear or nonlinear differential equations, including (1.42) and (1.41) above, can often be written as Laplace transforms of simpler functions. It is then important to understand the asymptotic behavior of Laplace transforms. A general asymptotic result is the following:

**Lemma 1.46** *Under the assumption in* (1.45), *we have*

$$\int_0^\infty e^{-xp}F(p)dp \to 0 \text{ as } \mathrm{Re}\,(x) \to \infty \quad (1.47)$$

**PROOF** This follows from the dominated convergence theorem, see §2.12a. Indeed, $\int_0^\infty |e^{-xp}F(p)|dp \le \int_0^\infty |e^{-x_0 p}F(p)|dp < \infty$ for $\mathrm{Re}\,(x) \ge x_0 > \nu$, and $e^{-xp}F(p) \to 0$ as $\mathrm{Re}\,(x) \to \infty$ for all $p \in (0,\infty)$. ◻

Furthermore, convergence is exponentially fast iff $F$ is identically zero on some interval $[0,\varepsilon)$, where $\varepsilon > 0$ is independent of $x$ as shown in the following proposition. For the notation, see §2.12a.

**Proposition 1.48** *Assume that $F$ is exponentially bounded in the sense of* (1.45)*; let $x_1 = \mathrm{Re}\,(x)$. Then*

$$\int_0^\infty e^{-xp}F(p)dp = o(e^{-x_1\varepsilon}) \text{ as } x_1 \to \infty \textbf{ iff } F = 0 \text{ a.e.}^1 \text{ on } [0,\varepsilon] \text{ as } x_1 \to \infty \quad (1.49)$$

*Also,* $\int_0^\infty e^{-xp}F(p)dp = O(e^{-x_1\varepsilon}) \Leftrightarrow \int_0^\infty e^{-xp}F(p)dp = o(e^{-x_1\varepsilon})$, *implying* $F = 0$ *a.e. on* $[0,\varepsilon]$.

**PROOF** (i) Assume that $F = 0$ a.e. on $[0,\varepsilon)$. This implies that

$$\int_0^\infty e^{-xp}F(p)dp = \int_\varepsilon^\infty e^{-xp}F(p)dp = e^{-x\varepsilon}\int_0^\infty e^{-xp}F(p+\varepsilon)dp = e^{-x\varepsilon}o(1) \quad (1.50)$$

as $x_1 \to \infty$ by Lemma 1.46.

(ii) For the converse, assume that $\int_0^\infty e^{-xp}F(p)dp = O(e^{-x_1\varepsilon})$. We write

$$\int_0^\infty e^{-xp}F(p)dp = \int_0^\varepsilon e^{-xp}F(p)dp + \int_\varepsilon^\infty e^{-xp}F(p)dp. \quad (1.51)$$

The rightmost integral in (1.51) is shown to be $o(e^{-x_1\varepsilon})$ by using the change variable $p \to p+\varepsilon$ and using Lemma 1.46. Thus

$$g(x) := e^{x\varepsilon}\int_0^{\varepsilon} e^{-xp}F(p)dp = O(1) \text{ as } x_1 = \text{Re } x \to +\infty \tag{1.52}$$

It is easy to see that $g$ is entire. Furthermore, it is bounded for $x \in \mathbb{R}^+$ by (1.52) and also manifestly bounded for $x \in i\mathbb{R}$, and $x \in \mathbb{R}^-$. Since $g$ is of exponential order 1, using the Phragmén-Lindelöff theorem in all of the four quadrants (see [21] pp. 19 and 23 for more details) shows $g$ is bounded. From Liouville's theorem, $g$ is a constant. The Riemann-Lebesgue lemma implies that $g$ goes to zero as $x \to \infty$ along the imaginary line. Thus $g = 0$, implying $\int_0^{\varepsilon} F(p)e^{-px}dp = 0$, $\forall x \in \mathbb{C}$ implying that the Fourier transform $\int_{-\infty}^{\infty} e^{-itp}\chi_{[0,\varepsilon]}(p)F(p)dp = 0\ \forall t \in \mathbb{R}$ and thus, by inverse Fourier transform, $F(p) = 0$ *a.e.* on $(0,\varepsilon)$. Now, (i) implies that $\int_0^{\infty} F(p)e^{-px}dp = o\left(e^{-\varepsilon x_1}\right)$. ∎

**Corollary 1.53 (Injectivity of the Laplace transform)** *Under the condition* (1.45)*, if $\mathcal{L}F = 0$ for all $x > 0$, then $F = 0$ a.e. on $\mathbb{R}^+$.*

**PROOF** Since, in particular, $\mathcal{L}F = O(e^{-xa})$ for any $a > 0$, from Proposition 1.48, $F = 0$ a.e. on $\mathbb{R}^+$. ∎

### First inversion formula

Let $\mathcal{H}$ denote the space of analytic functions in the right half complex plane.

**Proposition 1.54** *(i) $\mathcal{L} : L^1(\mathbb{R}^+) \mapsto \mathcal{H}$ and $\|\mathcal{L}F\|_\infty \le \|F\|_1$.*

*(ii) $\mathcal{L} : L^1(\mathbb{R}^+) \mapsto \mathcal{L}(L^1(\mathbb{R}^+)) \subset \mathcal{H}$ is invertible, and the inverse is given by*

$$F(x) = \hat{\mathcal{F}}^{-1}\{\mathcal{L}F(it)\}(x) \tag{1.55}$$

*for $x \in \mathbb{R}^+$ where $\hat{\mathcal{F}}$ is the Fourier transform (in distributions if $\mathcal{L}F \notin L^1(i\mathbb{R})$).*

**PROOF** (i) The fact that $\mathcal{L}F$ is analytic in $\mathbb{H}$ follows from the exponential decay of the integrand: by dominated convergence we can differentiate in $x$ under the integral sign. The estimate follows simply from the fact that $|e^{-xp}| < 1$.

(ii) We note that $(\mathcal{L}F)(it)$ exists since $F \in L^1$, and it is, by definition the Fourier transform of $F$ extended by $F(p) = 0$ for $p < 0$. The rest is just Fourier inversion, in the in a generalized sense –in distributions–if $\mathcal{L}F \notin L^1(i\mathbb{R})$. ∎

### Second inversion formula

**Proposition 1.56** *(i) Assume $f$ is analytic in an open sector $\mathbb{H}_\delta := \{x : |\arg(x)| < \pi/2+\delta\}$, $\delta \ge 0$ and is continuous on $\partial\mathbb{H}_\delta$, and that for some $K > 0$ and any $x \in \overline{\mathbb{H}_\delta}$ we have*

$$|f(x)| \le K(|x|^2+1)^{-1} \tag{1.57}$$

*Then $\mathcal{L}^{-1}f$ is well defined by*

$$F = \mathcal{L}^{-1}f = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} \mathrm{d}t\, e^{pt} f(t) \tag{1.58}$$

*and*

$$\int_0^\infty \mathrm{d}p\, e^{-px} F(p) = \mathcal{L}\mathcal{L}^{-1}f = f(x) \tag{1.59}$$

*We have $\|\mathcal{L}^{-1}\{f\}\|_\infty \le K/2$ and $\mathcal{L}^{-1}\{f\} \to 0$ as $p \to \infty$.*

*(ii) If $\delta > 0$, then $F = \mathcal{L}^{-1}f$ is analytic in the sector $S_\delta = \{p \neq 0 : |\arg(p)| < \delta\}$. In addition, $\sup_{S_\delta}|F| \le K/2$ and $F(p) \to 0$ as $p \to \infty$ along rays in $S_\delta$.*

**PROOF** Clearly, $F$ in (1.58) is well-defined since $f(is) \in L^1(\mathbb{R})$. (i) We have

$$2\pi i\mathcal{L}\left[\mathcal{L}^{-1}f\right](x) = \int_0^\infty \mathrm{d}p\, e^{-px}\int_{-\infty}^\infty i\mathrm{d}s\, e^{ips} f(is) \tag{1.60}$$

$$= \int_{-\infty}^\infty i\mathrm{d}s\, f(is)\int_0^\infty \mathrm{d}p\, e^{-px}e^{ips} = \int_{-i\infty}^{i\infty} f(z)(x-z)^{-1}\mathrm{d}z = 2\pi i f(x) \tag{1.61}$$

where we applied Fubini's theorem[2] and then pushed the contour of integration past $x$ to infinity. The norm of $\mathcal{L}^{-1}$ is obtained by majorizing $|f(x)e^{px}|$ by $K(|x^2|+1)^{-1}$. The behavior $\left[\mathcal{L}^{-1}f\right](p) \to 0$ as $p \to +\infty$ follows by applying Riemann-Lebesgue Lemma to (1.58).

(ii) For any $\delta' < \delta$ we have, by (1.57),

$$\int_{-i\infty}^{i\infty} \mathrm{d}s\, e^{ps} f(s) = \left(\int_{-i\infty}^0 + \int_0^{i\infty}\right)\mathrm{d}s\, e^{ps} f(s)$$
$$= \left(\int_{-i\infty e^{-i\delta'}}^0 + \int_0^{i\infty e^{i\delta'}}\right)\mathrm{d}s\, e^{ps} f(s) \tag{1.62}$$

Take any $p \in S_\delta$. Choose $\delta' < \delta$ so that $p \in S'_\delta$. Analyticity of (1.62) in $p \in S'_\delta$ is manifest, given the analyticity and exponential decay of the integrand. For the estimates on $F(p)$, we note that (i) applies to $f(xe^{i\phi})$ if $|\phi| < \delta$. □

[2]This theorem addresses the permutation of the order of integration; see [74]. Essentially, if $f \in L^1(A\times B)$, then $\int_{A\times B} f = \int_A\int_B f = \int_B\int_A f$.

Many cases can be reduced to (1.57) after transformations. For instance if $f_1 = \sum_{j=1}^{N} a_j(1+x)^{-k_j} + f(x)$, **where $k_j > 0$ and $f$ satisfies the assumptions above, then (1.58) and (1.59) apply to $f_1$, since they do apply, by straightforward verification, to the finite sum.

∎

**Proposition 1.63** *Let $F$ be analytic in the open sector $S_p = \{e^{i\phi}\mathbb{R}^+ : \phi \in (-\delta,\delta)\}$ and such that $|F(|p|e^{i\phi})| \le g(|p|) \in L^1[0,\infty)$. Then $f = \mathcal{L}F$ is analytic in the sector $S_x = \{x : |\arg(x)| < \pi/2+\delta\}$ and $f(x) \to 0$ as $|x| \to \infty, \arg(x) = \theta \in (-\pi/2-\delta, \pi/2+\delta)$.*

**PROOF** Because of the analyticity of $F$ and the decay conditions for large $p$, the path of Laplace integration can be rotated by any angle $\phi$ in $(-\delta,\delta)$ without changing[3] $(\mathcal{L}F)(x)$. The fact that$g \in L^1$ also implies that The decay of $(\mathcal{L}F)(x)$ in $x$ follows from Lemma 1.46 with $x$ replaced by $xe^{-i\phi}$ and $\phi$ chosen $\arg\left(xe^{-i\phi}\right) \in \left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ ∎

**Note** $F$ need not be analytic at $p=0$ for Proposition 1.63 to apply.

### 1.2b Watson's Lemma

As will be seen, many integrals after appropriate changes of variable can be cast in a form where Watson's Lemma can be applied. In the following example, we determine the asymptotics of the incomplete Gamma function which we will need later.

**Example 1.64** Assume that $\operatorname{Re}\beta > 0$ and $a > 0$. Then as $x \to \infty$ along an arbitrary ray in the open right half plane, $\mathbb{H}$

$$\{x : \arg x = \alpha\}\,; \quad \text{where } \alpha \in \left(-\frac{\pi}{2},\frac{\pi}{2}\right) \tag{1.65}$$

Then,

$$\int_0^a p^{\beta-1}e^{-px}dp \sim \frac{\Gamma(\beta)}{x^\beta} \tag{1.66}$$

Indeed, changing variable to $t = px$ we get

$$\int_0^\infty p^{\beta-1}e^{-px}dp = \frac{1}{x^\beta}\int_0^{\infty e^{i\arg(x)}} e^{-t}t^{\beta-1}dt = \frac{1}{x^\beta}\int_0^\infty e^{-t}t^{\beta-1}dt = \frac{\Gamma(\beta)}{x^\beta} \tag{1.67}$$

by a homotopic change of contour and the definition of the Gamma function.

[3]The fact that $g \in L^1$ implies that $\liminf_{R\to\infty} Rg(R) = 0$; thus there is a subsequence $R_n$ s.t. $R_n g(R_n) \to 0$. By straightforward estimates, or by Jordan's lemma, we see that the integral of $Fe^{-px}$ along an arc of a circle of radius $R_n$ goes to zero with $n$.

**Lemma 1.68** *If $x^\alpha \int_0^\infty e^{-xp} F(p)dp$ has an asymptotic power series in $z = x^{-\beta}$ for some $\beta$ with $\mathrm{Re}\,\beta > 0$ as $\mathrm{Re}\,x \to \infty$, then for any fixed $\varepsilon > 0$, $x^\alpha \int_0^\varepsilon e^{-xp} F(p)dp$ has an asymptotic power series as well, and the two power series agree.*

**PROOF** This is an immediate consequence of Propositions 1.48 and 1.33. □

Watson's lemma allows us to integrate power series term by term as stated below.

**Lemma 1.69 (Watson's lemma)** *Assume that $\|F\|_{L^1_\nu} < \infty$ (cf. (1.45)) and*

$$F(p) = p^{\alpha-1} \sum_{k=0}^{m} c_k p^{k\beta} + o(p^{\alpha-1+m\beta}) \text{ as } p \to 0^+ \text{ for all } m \le m_0 \in \mathbb{N} \cup \infty \tag{1.70}$$

*for some $\alpha$ and $\beta$, with $\mathrm{Re}\,\alpha, \mathrm{Re}\,\beta > 0$. Then as $x \to \infty$ along an arbitrary ray in $\mathbb{H}$, see (1.65), we have* [4]

$$(\mathcal{L}F)(x) = \int_0^\infty e^{-xp} F(p)dp = \sum_{k=0}^{m} c_k \Gamma(k\beta+\alpha) x^{-\alpha-k\beta} + o\left(x^{-\alpha-m\beta}\right). \tag{1.71}$$

*for any $m \le m_0$. The asymptotic expansion (1.71) holds if $(\mathcal{L}F)(x)$ is replaced by $\int_0^a F(p)e^{-px}dp$ ($a$ independent of $x$) $F \in L^1(0,a)$, and $F$ has the same asymptotic series as above as $p \to 0^+$.*

**PROOF** By Lemma 1.68, the conclusion follows if it holds for the integral $\int_0^\varepsilon e^{-xp} F(p)dp$ for some fixed $\varepsilon > 0$. On the other hand, by assumption and the definition of asymptotic power series we have, for any $\delta > 0$, $F(p) = \sum_{k=0}^m c_k p^{k\beta+\alpha-1} + g(p)$ where $|g(p)| \le \delta |p^{\alpha-1+m\beta}|$ for $p \in (0,\varepsilon)$ if $\varepsilon = \varepsilon(\delta, m)$ is small enough. Following the calculations in Example 1.64 we get

$$\left| \int_0^\varepsilon e^{-xp} g(p)dp \right| \le \delta \int_0^\varepsilon e^{-x_1 p} p^{\mathrm{Re}\,\alpha + m\mathrm{Re}\,\beta - 1} dp \le C\delta |x^{-\alpha-m\beta}|,$$

noting that that $\frac{|x|}{x_1}$ is finite along a ray Now, following Example 1.64,

$$\int_0^\varepsilon e^{-px} \sum_{k=0}^{m} c_k p^{\alpha+k\beta-1} dp \sim \sum_{k=0}^{m} c_k \Gamma\left(\alpha + k\beta\right) x^{-\alpha-k\beta},$$

[4] By writing $\int_0^a F(p)e^{-px}dp = \int_0^\infty F(p)e^{-px}\chi_{[0,a]}(p)dp$ where $\chi_{[0,a]}$ is the characteristic function of $[0,a]$, we see that the same result holds for a finite Laplace transform $\int_0^a F(p)e^{-px}dp$.

finishing the proof for $(\mathcal{L}F)(x)$. For the last part, note that $\int_0^a F(p)e^{-xp}dp = \int_0^\infty F(p)\chi_{[0,a]}(p)e^{-xp}dp$ and $F(p)\chi_{[0,a]}(p)$ satisfies the assumptions in the first part of the lemma. ∎

**Note 1.72** Intuitively, we see that, for a fixed $F$, the larger $\operatorname{Re} x$ is, the more damped is the contribution of any region that is not very close to zero. The behavior of a Laplace transform is gotten from the immediate neighborhood of zero. This will be seen in the next example and is formalized in Watson's lemma following it.

**Note 1.73** Watson's lemma holds for $\int_0^{ae^{i\theta}} F(p)e^{-px}dp$ as $|x| \to \infty$ if the asymptotic behavior (1.70) is valid along a ray $\arg p = \theta$, where $F \in L^1(0, ae^{i\theta})$ $\arg(x)$ satisfies $\theta + \arg x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. The proof is manifest by changing variables $p \to pe^{i\theta}$, $x \to xe^{-i\theta}$ and applying Lemma 1.69.

**Exercise 1.74 (A generalization of Watson's Lemma)** Assume that for some $\varepsilon > 0$, we have $\sup_{|z|<\varepsilon} \|F(\cdot; z)\|_{L^1_\nu} = C < \infty$ and that

$$F(p;z) = p^{\alpha-1} \sum_{\substack{0\leqslant k\leqslant m \\ 0\leqslant l\leqslant n}} c_{k,l}\, p^{k\beta_1} z^{l\beta_2} + o(p^{\alpha-1+m\beta_1} z^{n\beta_2})$$
$$\text{as } (p,z) \to (0^+, 0) \text{ for all } (m,n) \leqslant (m_0, n_0) \in (\mathbb{N} \cup \infty)^2 \qquad (1.75)$$

where $\operatorname{Re}\alpha, \operatorname{Re}\beta_1$ and $\operatorname{Re}\beta_2$ are positive. Then, show that [5]

$$\int_0^\infty e^{-xp} F\left(p, \frac{1}{x}\right) dp = \sum_{\substack{0\leqslant k\leqslant m \\ 0\leqslant l\leqslant n}} \frac{c_{kl}\Gamma(k\beta_1 + \alpha)}{x^{\alpha+k\beta_1+l\beta_2}} + o\left(x^{-\alpha-m\beta_1-n\beta_2}\right). \qquad (1.76)$$

**Corollary 1.77 (Laplace asymptotics, maximum at an endpoint)** *Assume that $F$ is continuously differentiable on $[0,a)$ (as usual when we close the interval we mean right derivative) and $F' > 0$ and $g$ is continuous. Then*

$$\int_0^a e^{-\nu F(x)} g(x)dx \sim e^{-\nu F(0)} \frac{g(0)}{\nu F'(0)} \quad \textit{as } \nu \to \infty \qquad (1.78)$$

**PROOF** By choosing $\tilde{F} = F(x) - F(0)$ we reduce to the case $F(0) = 0$. Since $F' > 0$, $F$ is invertible near zero and, with $h(x) = F^{-1}(x)$, we have

$$\int_0^a e^{-\nu F(x)} g(x)dx = \int_0^{F(a)} e^{-\nu p} g(h(p))h'(p)dp \qquad (1.79)$$

By continuity $g(h(p))h'(p) = g(0)h'(0) + o(1)$ as $p \to 0^+$. Noting that $h'(0) = 1/F'(0)$, the rest follows from Watson's lemma. ∎

[5] It is tacitly assumed $m$ and $n$ are chosen so that no term in the sum is $o\left(x^{-\alpha-m\beta_1-n\beta_2}\right)$.

**Corollary 1.80 (Laplace asymptotics, maximum at an inner point)** *Assume that $a > 0$, $F$ is twice continuously differentiable on $(-a, a)$ $F'(0) = 0$ and $F''(x) > 0$ on $(-a, a)$, and that $g$ is continuous. Then,*

$$\int_{-a}^{a} e^{-\nu F(x)} g(x) dx \sim e^{-\nu F(0)} g(0) \sqrt{\frac{2\pi}{\nu F''(0)}} \tag{1.81}$$

**PROOF** As in Corollary 1.77 without loss of generality we may assume $F(0) = 0$. Define $h(x) = \text{signum}(x)\sqrt{F(x)}$ and denote $\frac{1}{2}F''(0) = \lambda^2$. Clearly $h$ is continuously differentiable away from zero. For $x$ close to zero, we have $F(x) = \lambda^2 x^2 + o(x^2)$ and thus $h(x) = \lambda x + o(x)$ for small $x$. It is then easy to show that $h$ is continuously differentiable on $(-a, a)$ and $h' > 0$. We calculate only the integral from 0 to $a$: the one from $-a$ to 0 can be computed similarly and has an equal contribution to the final estimate. We make the change of variables $h(x) = \sqrt{u}$ and and note that by continuity $g(\sqrt{u})/h'\left(h^{-1}(\sqrt{u})\right) \sim g(0)/h'(0)$ to obtain

$$\int_{0}^{a} e^{-\nu h^2(x)} g(x) dx = \int_{0}^{F(a)} e^{-\nu u} \frac{g(\sqrt{u})}{h'(h^{-1}(\sqrt{u}))} \frac{1}{2\sqrt{u}} du \sim \frac{g(0)\sqrt{2\pi}}{2\sqrt{\nu F''(0)}} \tag{1.82}$$

by Watson's lemma and the fact that $\Gamma(1/2) = \sqrt{\pi}$. ∎

**Note:** Only the leading order asymptotic calculations are given in Corollaries 1.77 and 1.80. Watson's Lemma can be used to determine higher order corrections in the asymptotic expansion if $F$ and $g$ are smooth enough near 0.

**Exercise 1.83** *Formulate and prove a generalization of Lemma 1.80 for the case when $F'(0) = \cdots = F^{2m-1}(0) = 0$ and $F^{2m}(0) > 0$.*

**Example: Asymptotics of the $\Gamma$ function** The Gamma function is defined by

$$\Gamma(x+1) \equiv x! = \int_{0}^{\infty} e^{-\tau} \tau^x d\tau, = \int_{0}^{\infty} e^{x \log \tau} e^{-\tau} d\tau \tag{1.84}$$

for $x > -1$.[6] $x \log \tau - \tau$ is maximal when $\tau = x$. This suggests rescaling $\tau = xs$. This leads to

$$\Gamma(x+1) = x^{x+1} \int_{0}^{\infty} e^{-x(s - \log s)} ds = x^{x+1} e^{-x} \int_{-1}^{\infty} \exp\left[-x\left(t - \log(1+t)\right)\right] dt \tag{1.85}$$

[6]This representation is valid for complex $x$ as well in the domain $\operatorname{Re} x > -1$.

To put it in a form where one of the preceding Lemmas may be used, we introduce

$$q = t\left[\frac{2t - 2\log(1+t)}{t^2}\right]^{1/2} \tag{1.86}$$

Through Taylor series at $t = 0$, it is readily checked that $t \to q$ is an analytic change of variable near $t = 0$, with $q'(0) = 1$. Further, $t \to q$ is mononpremonic and maps the the real axis interval $(-1, \infty)$ to $q \in (-\infty, \infty)$. We define the unique inverse function to be $t = T(q)$. It follows from (1.85) that

$$\Gamma(x+1) = x^{x+1}e^{-x}\int_{-\infty}^{\infty} e^{-xq^2/2} \;\; T'(q)dq \tag{1.87}$$

We decompose the integral in (1.87) as $\int_{-\infty}^{0} + \int_{0}^{\infty}$. We introduce change of variable $q = -\sqrt{2p}$ in the first integral and $q = \sqrt{2p}$ in the second to obtain

$$\Gamma(x+1) = x^{x+1}e^{-x}\int_0^{\infty} \frac{e^{-px}}{\sqrt{2p}}\left(T'(-\sqrt{2p}) + T'(\sqrt{2p})\right)dp \tag{1.88}$$

Using Taylor series $T(q) = \sum_{j=1}^{\infty} b_j q^j$,

$$\frac{1}{\sqrt{2p}}\left(T'(-\sqrt{2p}) + T'(\sqrt{2p})\right) = \sum_{j=1, j=odd}^{\infty} 2jb_j \;\; (2p)^{j/2-1}. \tag{1.89}$$

It follows from Watson's Lemma that

$$\Gamma(x+1) \sim x^{x+1}e^{-x}\sum_{j=1, j=odd}^{\infty} 2^{j/2}\Gamma(j/2)jb_j x^{-j/2} \tag{1.90}$$

The first few $b_j$ is easily computed by substituting a truncation of $t = b_1 q + b_2 q^2 + b_3 q^3 + ..$ into (1.86) and equating like powers of $q$ and solving resulting equations. This gives $b_1 = 1$, $b_3 = \frac{1}{36}$, $b_5 = \frac{1}{4320}$, the even $b_j$'s being inconsequential in (1.90). Using $\Gamma(1/2) = \sqrt{\pi}$, the first few nonzero terms are

$$\Gamma(x+1) = \sqrt{2\pi}x^{x+1/2}e^{-x}\left(1 + \frac{1}{12x} + \frac{1}{288x^2} + O(x^{-3})\right) \tag{1.91}$$

The three term evaluation at $x = 6$ gives 720.0088692 versus the exact value of 720. If the general term in the asymptotic expansion (1.90) is desired, we can use Lagrange formula for inversion of a series:

$$\begin{aligned} b_j = \frac{1}{2\pi i}\oint \frac{T(q)}{q^{j+1}}dq = \frac{1}{2\pi i}\oint t^2(1+t)^{-1}\left[2t - 2\log(1+t)\right]^{-j/2-1}dt \\ = \frac{2^{-j/2-1}}{2\pi i}\oint \frac{(e^u-1)^2}{(e^u-1-u)^{j/2+1}}du, \end{aligned} \tag{1.92}$$

where the closed loop contour integrals are assumed to circle the origin in the respective variables in the positive sense.

## 1.3 Oscillatory integrals and the stationary phase method

In this setting, an integral of a function against a rapidly oscillating exponential becomes small as the frequency of oscillation increases. Again we first look at the case where there is minimal regularity; the following is a version of the Riemann-Lebesgue lemma.

**Proposition 1.93** *Assume $f \in L^1[a,b]$. Then $\int_a^b e^{ixt} f(t)dt \to 0$ as $x \to \pm\infty$. The same is true for $\int_{-\infty}^{\infty} e^{ixt} f(t)dt$ for $f \in L^1(\mathbb{R})$.*

It is enough to show the result on a set which is dense[7] in $L^1$. Since trigonometric polynomials are dense in the continuous functions on a compact set[8], say in $C[a,b]$ in the sup norm, and thus in $L^1[a,b]$, while continuous functions with compact support are dense in $L^1(\mathbb{R})$, it suffices to look at trigonometric polynomials, thus (by linearity), at $e^{ikx}$ for fixed $k$; for the latter we just calculate explicitly the integral; we have

$$\int_a^b e^{ixs} e^{iks} ds = O(x^{-1}) \text{ for large } x. \qquad \square$$

No rate of decay of the integral in Proposition 1.93 follows without further knowledge about the regularity of $f$. With some regularity we have the following characterization.

**Proposition 1.94** *For $\eta \in (0,1]$ let the $C^\eta[a,b]$ be the Hölder continuous functions of order $\eta$ on $[a,b]$, i.e., the functions with the property that there is some constant $c > 0$ such that for all $x, x' \in [a,b]$ we have $|f(x) - f(x')| \leq c|x - x'|^\eta$.*

*(i) We have*

$$f \in C^\eta[a,b] \Rightarrow \left| \int_a^b f(s) e^{ixs} ds \right| \leq \frac{(b-a)}{2} c\pi^\eta x^{-\eta} + O(x^{-1}) \quad \text{as} \quad x \to \infty \tag{1.95}$$

*(ii) If $f \in L^1(\mathbb{R})$ and $|x|^\eta f(x) \in L^1(\mathbb{R})$ with $\eta \in (0,1]$, then its Fourier transform $\hat{f} = \int_{-\infty}^{\infty} f(s) e^{-ixs} ds$ is in $C^\eta(\mathbb{R})$.*

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[7] A set of functions $f_n$ which, collectively, are arbitrarily close to any function in $L^1$. Using such a set we can write

$$\int_a^b e^{ixt} f(t)dt = \int_a^b e^{ixt}(f(t) - f_n(t))dt + \int_a^b e^{ixt} f_n(t)dt$$

and the last two integrals can be made arbitrarily small.

[8] One can associate the density of trigonometric polynomials with approximation of functions by Fourier series.

*(iii) Let $f \in L^1(\mathbb{R})$. If $x^n f \in L^1(\mathbb{R})$ with $n-1 \in \mathbb{N}$ then $\hat{f}$ is $n$ times differentiable. If for $A>0$, $e^{A|x|}f \in L^1(\mathbb{R})$ then $\hat{f}$ extends analytically in a strip of width $A$ centered on $\mathbb{R}$.*

**PROOF** (i) By rescaling, we can choose $[a,b]=[0,1]$. We have as $x\to\infty$ ($\lfloor\cdot\rfloor$ denotes the integer part)

$$\left|\int_0^1 f(s)e^{ixs}ds\right| =$$
$$\left|\sum_{j=0}^{\lfloor\frac{x}{2\pi}-1\rfloor}\left(\int_{2j\pi x^{-1}}^{(2j+1)\pi x^{-1}} f(s)e^{ixs}ds+\int_{(2j+1)\pi x^{-1}}^{(2j+2)\pi x^{-1}} f(s)e^{ixs}ds\right)\right| + O(x^{-1})$$
$$=\left|\sum_{j=0}^{\lfloor\frac{x}{2\pi}-1\rfloor}\int_{2j\pi x^{-1}}^{(2j+1)\pi x^{-1}}\big(f(s)-f(s+\pi/x)\big)e^{ixs}ds\right|+O(x^{-1})$$
$$\leq \sum_{j=0}^{\lfloor\frac{x}{2\pi}-1\rfloor} c\left(\frac{\pi}{x}\right)^{\eta}\frac{\pi}{x}\leq\frac{1}{2}c\pi^{\eta}x^{-\eta}+O(x^{-1}) \quad (1.96)$$

(ii) We see that

$$\left|\frac{\hat{f}(s)-\hat{f}(s')}{(s-s')^{\eta}}\right| = \left|\int_{-\infty}^{\infty}\frac{e^{-ixs}-e^{-ixs'}}{x^{\eta}(s-s')^{\eta}}x^{\eta}f(x)dx\right|\leq\int_{-\infty}^{\infty}\left|\frac{e^{-ixs}-e^{-ixs'}}{(xs-xs')^{\eta}}\right|\left|x^{\eta}f(x)\right|dx \quad (1.97)$$

is bounded. Indeed, by elementary geometry we see that for $|\phi_1-\phi_2|<1$ we have

$$|\exp(i\phi_1)-\exp(i\phi_2)|\leq|\phi_1-\phi_2|\leq|\phi_1-\phi_2|^{\eta} \quad (1.98)$$

while for $|\phi_1-\phi_2|\geq 1$ we see that

$$|\exp(i\phi_1)-\exp(i\phi_2)|\leq 2\leq 2|\phi_1-\phi_2|^{\eta}$$

(iii) Take any $x\in S_A:=\{x\in\mathbb{C}:|\operatorname{Im}x|<A\}$. Choose $A'<A$ so that $x\in S_{A'}$. Choose $h\in\mathbb{C}$ so that $|h|\leq\frac{A-A'}{2}$. Then,

$$[D_h\hat{f}](x):=\frac{\hat{f}(x+h)-\hat{f}(x)}{h}=\int_{\mathbb{R}}f(s)e^{-ixs}\left(\frac{e^{-ihs}-1}{h}\right)ds$$

and it is readily checked that $|e^{-ixs}\left(\frac{e^{-ihs}-1}{h}\right)|\leq Ce^{A|s|}$ and by the dominating convergence theorem $\hat{f}'(x)=\lim_{h\to 0}[D_hf](x)=\int_{\mathbb{R}}-isf(s)e^{-ixs}ds$ implying $\hat{f}$ is analytic in a strip of width $A$. ∎

**Note 1.99** In Laplace type integrals Watson's lemma implies that it suffices for a function to be continuous to ensure an $O(x^{-1})$ decay of the integral, whereas in Fourier-like integrals, the considerably weaker decay (1.95) is optimal as seen in the exercise below.

**Exercise 1.100 (*)** (a) Consider the function $f$ given by the *lacunary trigonometric series* $f(z) = \sum_{k=2^n, n\in\mathbb{N}} k^{-\eta} e^{ikz}$, $\eta \in (0,1)$. Show that $f \in C^\eta[0, 2\pi]$. We want to estimate $f(\phi_1) - f(\phi_2)$ in terms of $|\phi_1 - \phi_2|^\eta$, when $\phi_1 - \phi_2$ is small. We can take $\phi_1 - \phi_2 = 2^{-p}b$ with $|b| < 1$. use the first inequality in (1.98) to estimate the terms in with $n < p$ and the simple bound $2/k^\eta$ for $n \geq p$. Then it is seen that $\int_0^{2\pi} e^{-ijs} f(s)ds = 2\pi j^{-\eta}$ (if $j = 2^m$ and zero otherwise) and the decay of the Fourier transform is exactly given by (1.95).

(b) Use Proposition 1.94 and the result in Exercise 1.100 to show that the function $f(t) = \sum_{k=2^n, n\in\mathbb{N}} k^{-\eta} t^k$, analytic in the open unit disk, has no analytic continuation across the unit circle, that is, the unit circle is a *barrier of singularities for $f$.*

**Note 1.101** *If we are dealing with an analytic function except for isolated singularities (or branch points), then decay is typically better than the one obtained in Proposition 1.94.*

**Note.** In part (i) of Proposition 1.94, compactness of the interval is crucial. In fact, the Fourier transform of an $L^2(\mathbb{R})$ entire function may not necessarily decrease pointwise. For example, consider $f = \mathcal{F}^{-1}\hat{f}$, where $\hat{f}(x) = 1$ for $x \in [n, n + e^{-n^2}]$ for $n \in \mathbb{N}$ and zero otherwise. Since $\hat{f} \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and for any $A > 0$, $e^{A|x|}\hat{f}(x) \in L^1(\mathbb{R})$, it follows that $f = \mathcal{F}^{-1}\hat{f}$ is entire. Yet $[\mathcal{F}f](x)$, which equals $\hat{f}(x)$ a.e., evidently does not decrease pointwise as $x \to \infty$.

**Proposition 1.102** *Assume $f \in C^n[a, b]$. Then we have*

$$\int_a^b e^{ixt} f(t)dt = e^{ixa} \sum_{k=1}^{n} c_k x^{-k} + e^{ixb} \sum_{k=1}^{n} d_k x^{-k} + o(x^{-n})$$

$$= e^{ixt} \left( \frac{f(t)}{ix} - \frac{f'(t)}{(ix)^2} + ... + (-1)^{n-1} \frac{f^{(n-1)}(t)}{(ix)^n} \right) \Bigg|_a^b + o(x^{-n}), \quad (1.103)$$

*where $c_k = -f^{(k-1)}(a)/i^k$ and $d_k = f^{(k-1)}(b)/i^k$.*

**PROOF** This follows by integration by parts and the Riemann-Lebesgue lemma since

$$\int_a^b e^{ixt} f(t)dt = e^{ixt} \left( \frac{f(t)}{ix} - \frac{f'(t)}{(ix)^2} + ... + (-1)^{n-1} \frac{f^{(n-1)}(t)}{(ix)^n} \right) \Bigg|_a^b + \frac{(-1)^n}{(ix)^n} \int_a^b f^{(n)}(t) e^{ixt} dt \quad (1.104)$$

∎

**Corollary 1.105** *(1) Assume $f \in C^\infty[0, 2\pi]$ is periodic with period $2\pi$. Then $\int_0^{2\pi} f(t)e^{int}dt = o(n^{-m})$ for any $m > 0$ as $n \to +\infty, n \in \mathbb{Z}$.*

*(2) Assume $f \in C_0^\infty[a, b]$ vanishes at the endpoints together with all derivatives; then $\hat{f}(x) = \int_a^b f(t)e^{ixt} = o(x^{-m})$ for any $m > 0$ as $x \to \pm\infty$.*

**Exercise 1.106** Show that if $f$ is analytic in a neighborhood of $[a, b]$ but not entire, then both series in (1.103) have zero radius of convergence.

**Exercise 1.107** In Corollary 1.105 (2) show that $\limsup_{x\to\infty} e^{\varepsilon|x|}|\hat{f}(x)| = \infty$ for any $\varepsilon > 0$ unless $f = 0$.

**Exercise 1.108** For smooth $f$, the interior of the interval does not contribute because of cancellations: rework the argument in the proof of Proposition 1.94 under smoothness assumptions. If we write $f(s + \pi/x) = f(s) + f'(s)(\pi/x) + \frac{1}{2}f''(c)(\pi/x)^2$ cancellation is manifest.

**Exercise 1.109** Show that if $f$ is piecewise differentiable and the derivative is in $L^1$, then the Fourier transform is $O(x^{-1})$.

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## 1.4 Steepest descent method

We seek to determine the asymptotic behavior of $I(\nu)$ as $\nu \to +\infty$, where

$$I(\nu) = \int_{\mathcal{C}} g(z)e^{-\nu f(z)}dz \quad (1.110)$$

for $f$ and $g$ that are analytic in some some region of the complex plane[9], and $\mathcal{C}$ is some simple curve that may be finite or infinite. Further, we may assume $f$ is not a constant, as otherwise the asymptotics is trivial. The problem is to determine the asymptotics of $I$ as $\nu \to +\infty$. More generally, if $\nu \to \infty$ along

[9]The region of analyticity will be dictated by the need to deform $C$ into one or more steepest descent paths and will depend on the specifics of the problem.

some complex ray $\arg \nu = \phi$, we can replace $\nu$ by $|\nu|$ and $f$ by $e^{i\phi} f$ in the ensuing discussion to obtain asymptotics along complex rays.

The idea of the steepest descent method is to use the analyticity of the integrand in (1.110) in $z$ to deform $\mathcal{C}$ homotopically into one or more paths, each of which is characterized by $\operatorname{Im} f = c$, a constant. If $\mathcal{C}$ is homotopically equivalent to just one steepest descent path $\mathcal{C}_s = \{z : z = \gamma(t), a \le t \le b\}$, where $\gamma'$ exists (and assumed nonzero, without loss of generality) then we may rewrite (1.110)

$$I(\nu) = e^{-i\nu c} \int_a^b g(\gamma(t)) \exp\left[-\nu \left(f(\gamma(t)) - ic\right)\right] \gamma'(t) dt \qquad (1.111)$$

Since $f(\gamma(t)) - ic$ is by assumption real valued for $t \in (a, b)$ and $g(\gamma(t))\gamma'(t)$ can decomposed into real and imaginary parts. After breaking up the integral into subintervals where $f - ic$ is monotonic, Watson's lemma can be applied to determine the complete asymptotic expansion of $I(\nu)$. Indeed, for analytic $f$, the real valued funciton $f(\gamma(t)) - ic$, which is the same as $\operatorname{Re} f$, is monotonically increasing or decreasing in any interval in $t$ that does not contain a singular point of $f$ or a *saddle* point where $f' = 0$ (The term *saddle* refers to the behavior of the harmonic function $\operatorname{Re} f(x + iy)$ for $(x, y)$ near a critical point, where $f' = 0$).

Generally, multiple steepest descent paths, each with a different value of $c$, are involved in homotopic deformation of $\int_{\mathcal{C}}$; these paths join up at *sinks* where $\operatorname{Re} f \to +\infty$. Multiple descent paths will definitely be needed when $\operatorname{Im} f$ is different at the end points of $\mathcal{C}$, as in the example in §1.4a In such cases, the calculation of $I(\nu)$ generally requires adding up the contributions on each steepest descent path $\int_{\mathcal{C}_s}$ in the manner outlined in the last paragraph. Therefore, the only new element in the steepest descent method is to determine steepest curves which are homotopically equivalent to the original path $\mathcal{C}$. For a point on each such curve, $\operatorname{Re} f$ varies most rapidly relative to all other directions, as may be concluded easily from applying Cauchy-Riemann conditions. This explains the terminology *steepest* descent[10] It should be further noted that without homotopic deformation into descent paths, (1.110) will generally be an oscillatory integral; asymptotics obtained through the stationary phase method leads to substantially weaker results, see note 1.99. The stationary phase method, however, does not require analyticity of $f$ and $g$.

[10]This terminology is confusing, since descent or ascent depends on the direction a path is traversed. Calling it steepest variation path is more appropriate; nonetheless, we will stick to the standard terminology.

### 1.4a Simple illustrative example

Consider

$$I(\nu) = \int_0^1 \frac{e^{i\nu z^2}}{z+1} dz \tag{1.112}$$

In the notation of (1.110), $f(z) = -iz^2$, $g(z) = \frac{1}{z+1}$. Steepest descent paths emanating at $z = 0$ are determined by

$$\operatorname{Im} f = \operatorname{Im} f(0) \quad \text{implying } \operatorname{Re} z^2 = 0, \quad i.e.\ z = re^{\pm i\pi/4} \quad \text{for} \quad r \in (-\infty, \infty) \tag{1.113}$$

However, since $Ref \to +\infty$, when $z = re^{i\pi/4}$ as $r \to \infty$, it follows that $\infty e^{i\pi/4}$ is a sink that is connected to $z = 0$ along the steepest descent path $z = re^{i\pi/4}$. Steepest descent paths from the other end point $z = 1$ in the integral (1.112) is found by setting

$$\operatorname{Im} f = \operatorname{Im} f(1) = -1 \quad \text{implying} \quad \operatorname{Re} z^2 = 1, \quad i.e.\text{hyperbolic path } x^2 - y^2 = 1 \tag{1.114}$$

Since only one branch of this hyperbola passes through (1, 0) and asymptotes to $y = x$, *i.e.* approaches the sink $\infty e^{i\pi/4}$, a homotopic deformation of the $\int_0^1$ may be made to coincide with descent paths $z = re^{i\pi/4}$, $0 \le r < \infty$ followed by integration along steepest descent path $C$ that connects $\infty e^{i\pi/4}$ to 1 along the hyperbola[11] $x^2 - y^2 = 1$. Therefore,

$$I(\nu) = \int_0^{\infty e^{i\pi/4}} \frac{e^{i\nu z^2}}{1+z} dz + \int_C \frac{e^{i\nu z^2}}{1+z} dz \equiv I_1(\nu) + I_2(\nu) \tag{1.115}$$

For $I_1(\nu)$, using $z = re^{i\pi/4}$ for $0 < r < \infty$, we obtain after change of variable and application of Watson's Lemma

$$\begin{aligned} I_1(\nu) = e^{i\pi/4} \int_0^\infty \frac{e^{-\nu r^2}}{1 + re^{i\pi/4}} dr &= e^{i\pi/4} \int_0^\infty \frac{e^{-\nu p} dp}{2p^{1/2}[1 + p^{1/2} e^{i\pi/4}]} \\ &\sim \frac{1}{2} e^{i\pi/4} \sum_{j=0}^\infty (-1)^j \Gamma\left(\frac{j+1}{2}\right) e^{ij\pi/4} \nu^{-(j+1)/2} \end{aligned} \tag{1.116}$$

As far as $I_2(\nu)$, we know $p = f(z) - f(1) = -iz^2 + i$ is real valued and monotonically increasing on the parabolic path $C$ from $z = 1$ to $z = \infty e^{i\pi/4}$, since $f' \neq 0$ on this path. Therefore, solving for $z$, inversion leads to

$$z = Z(P) = (1 + ip)^{1/2}, \tag{1.117}$$

[11]We do not have the option of going along $re^{-i\pi/4}$ , $0 < r < \infty$ since $\operatorname{Re} f \to -\infty$ and so contribution at $\infty e^{-i\pi/4}$ cannot be ignored as it can be for a *sink*.

where we can readily check that for this branch of square-root, as $p \to +\infty$, $z \to \infty e^{i\pi/4}$ as required. Therefore,

$$I_2(\nu) = -e^{i\nu} \int_0^\infty \frac{e^{-p\nu}}{1+Z(p)} Z'(p) dp. \tag{1.118}$$

We note that Taylor expansion:

$$\frac{Z'(p)}{1+Z(p)} = \frac{i}{2}(1+ip)^{-1/2} \left[1 + (1+ip)^{1/2}\right]^{-1} = \sum_{j=0}^\infty a_j p^j, \tag{1.119}$$

where the first few coefficents are: $a_0 = \frac{i}{4}$, $a_1 = \frac{3}{16}$, $a_2 = -\frac{5i}{32}$, $a_3 = -\frac{35}{256}$. Applying Watson's Lemma to (1.118), it follows

$$I_2(\nu) \sim -e^{i\nu} \sum_{j=0}^\infty a_j \nu^{-j-1} \Gamma(j+1), \tag{1.120}$$

The full asymptotic expansion of $I(\nu) = I_1(\nu) + I_2(\nu)$ is then obvious from (1.116) and (1.120).

**Remark 1.121** A change of variable $\zeta = z^2$ at the outset in (1.112) converts the problem into $I(\nu) = \int_0^1 \frac{e^{i\nu\zeta} d\zeta}{2(\zeta^{1/2}+\zeta)}$, corresponding to which the steepest descent lines connecting each end points are given by $\zeta = ir$ and $\zeta = 1 + ir$ respectively. However, after the change of variables the integrand is generically not explicit. In such cases, finding the steepest descent lines cannot be done explicitly either. Fortunately, descent lines are connected to ODEs amenable to phase plane analysis and we will exploit this connection in the following section.

**Note 1.122** If we replace the integrand $\frac{e^{i\nu z^2}}{z+1}$ in (1.112), by $\frac{e^{i\nu z^2}}{z-z_0}$, where $z_0$ is in the upper-half plane region between $e^{i\pi/4}\mathbb{R}^+$ and steepest descent contour $C$ connecting $\infty e^{i\pi/4}$ to 1, for *e.g.* $z_0 = \frac{1+i}{2}$, then the singulariy at $z = z_0$ interferes with the homotopic deformation into steepest descent paths. Nonetheless, since this singularity is a pole, after collecting residue at $z = z_0$, we can use the same descent paths as in Example 1.4a. Since $\operatorname{Im} z_0^2 > 0$, the residue contribution will be exponentially small in $\nu$ relative to (1.120) and (1.116). If this $z_0$ were a branch point instead, in addition to the steepest descent paths, the homotopically deformed path will include a contour that wraps around $z_0$. Nonetheless, as in the case of the pole, the contribution of the branch point is exponentially small in $\nu$.

### 1.4b Finding the steepest variation lines

Prior discussion shows that the main challenge in evaluation of asymptotic behavior of

$$\int_{\mathcal{C}} g(z) \exp(-\nu f(z)) dz \tag{1.123}$$

is the determination of steepest descent path(s) that are homotopically equivalent to $\mathcal{C}$. We now discuss how steepest descent paths may be found when $f(z)$ is not as simple as shown in Example 1.4a.

To simplify the discussion, we will assume that both $f$ and $g$ are entire, and if parts of $\mathcal{C}'$ extend to infinity, the integral along those parts converges. If the functions are not entire, then the contours can be deformed inside the domain of analyticity, and beyond that only in special cases, for instance when the singularities of $g$ are poles. If an integral extends to infinity and the integral would not converge, then we truncate the contour at some large enough $z_0$ (see Note 1.132) at the price of introducing exponentially small relative errors in the estimates.

When $v$ is very simple, as in 1.4a, one can just plot the curves $v(z) = C$. If not, we can use tools from elementary ODE analysis to find these lines.

If along a curve $z(t) = (x(t), y(t))$ we have $v(z) = C$, then

$$\frac{\partial v}{\partial x}\frac{dx}{dt} + \frac{\partial v}{\partial x}\frac{dy}{dt} = 0 \tag{1.124}$$

which happens along the solution curves of the system

$$\begin{aligned} \frac{dx}{dt} &= \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = \mathrm{Re}\,(f'(z)) \\ \frac{dy}{dt} &= -\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y} = -\mathrm{Im}\,(f'(z) \end{aligned} \tag{1.125}$$

where we used $v = \mathrm{Im}\, f$ to write the system in terms of $f'$.

**Note 1.126** The system (1.125) is autonomous, and the task is to draw the phase portrait. The direction field is parallel with $\nabla u$, that is, it points toward steepest ascent directions of $u$ or steepest descent of $e^{-\nu u}$. To draw the phase portrait more easily we note that:

1. Eq. (1.125) is at the same time a Hamiltonian system as well as a gradient one.

2. There are no closed trajectories since $f$, thus $v$, are not identically constant. Indeed, $v = \mathrm{Im}\, f$ is harmonic, and a harmonic function in a domain attains its maximum and minimum value on the boundary; since we are dealing with a level set of $v$, call it $\gamma$, if $\gamma$ is closed then $\max v = \min v$ in the $\mathrm{int}(\gamma)$ implying that $v$ is constant in an open set, thus constant everywhere.

3. All critical points of the field ($\dot{x} = \dot{y} = 0$) are *saddle points*, the points of interest for our analysis. Indeed, $v$ cannot have, by the maximum modulus principle already used in 2, any interior maxima or minima. (If $f$ is not entire, then of course singularities of $f$ are also singularities of the field.)

4. At a critical point $z_0$ we have
$$f'(z_0) = 0 \tag{1.127}$$
by (1.125) and (1.127), the local behavior of $u$ near $z_0$ is
$$u(z) - u(z_0) = \frac{1}{k!}\mathrm{Re}\left(f^{(k)}(z_0)(z - z_0)^k\right)(1 + o(1)) \tag{1.128}$$
where $k$, generically $k = 2$, is the smallest such that $f^{(k)}(z_0) \neq 0$. This is a simple way to plot the directions of steepest ascent of $u$ at $z_0$. These are the directions
$$f^{(k)}(z_0)(z - z_0)^k \in \mathbb{R}^+ \tag{1.129}$$

5. Trajectories can only intersect at critical points of the field.

6. The properties above, together with the behavior of $f$ at infinity completely determine the topology of the direction field.

7. To find the steepest descent line decomposition of a contour $\mathcal{C}$ we let every point $z_0 = x_0 + iy_0 \in \mathcal{C}$ flow with the field: $(x_0, y_0) \mapsto x(t; x_0), (y(t; y_0))$; we denote the set of such points by $\mathcal{C}(t)$. The connected components of the limiting set:
$$\{z : \lim_{t\to\infty} d(z, \mathcal{C}(t)) = 0\}$$
represent the sought-for decomposition.

8. By construction, on each $\mathcal{C}_i$, $u$ is strictly monotonic and $v$ is constant, thus $f$ is one-to-one, and the change of variables $f(z) = f(z_i) + \zeta$ where $z_i$ is an endpoint of $\mathcal{C}_i$, brings the integrals to a Watson's lemma form, see Note 1.134.

9. The asymptotic expansions are collected from the endpoints of the steepest descent lines from which $u$ increases, since $e^{-\nu u}$ decreases rapidly starting from such a point.

We illustrate this on a simple example: we start with the integral
$$\int_{\infty e^{i\pi/4}}^{\infty e^{5\pi i/4}} e^{-\nu(z - z^4/4)} dz \tag{1.130}$$
where $\nu \to +\infty$, and the integral is taken along any curve $\mathcal{C}$ is staring at $+i\infty$ and ending at $-\infty$. Because of the rapid decay in $z$, the integral converges. We want to find a curve homotopic to $\mathcal{C}$ that consists of paths of steepest descent of $e^{-u}$. In this example, (1.125) becomes
$$\frac{dx}{dt} = 1 - x^3 + 3xy^2 \tag{1.131}$$
$$\frac{dy}{dt} = 3x^2y - y^3$$

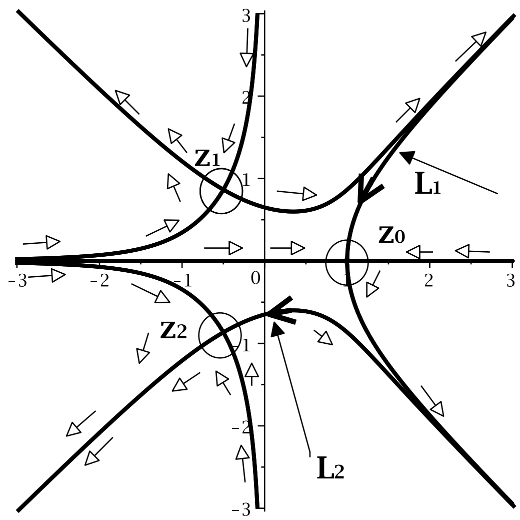


**FIGURE 1.1**: Phase portrait of (1.131). The white arrows point towards steepest ascent directions of $u$ (steepest descent of $e^{-\nu u}$). The orientations of the paths $L_1$ and $L_2$ are shown with dark arrows; $z_k = \exp(2k\pi i/3)$ , $k = 0, 1, 2$ are the saddle points.

The equilibria of (1.131) are, by (1.127) the solutions of $1 - z^3 = 0$ : $z_k = e^{2k\pi i/3}$, $k = 0, ..., 2$ and near a critical point the directions of descent of $e^{-u}$ are obtained from (1.128), $-3z_k^2(z - z_k)^2 \in \mathbb{R}^+$.

For large $t = |t|e^{i\phi}$, we have $f = -|t|^4 e^{4i\phi}(1+o(1))$, and thus asymptotically there are, up to homotopies, four curves of steepest descent of $e^{-u}$, $\cos(4\phi) = -1$ and four of steepest ascent, $\cos(4\phi) = 1$. All needed qualitative features of the phase portrait, sketched in Fig. 1.1, follow from this information and the fact that trajectories do not intersect except at critical points. In the phase portrait, the arrows point towards steepest descent. We illustrate the detailed arguments that leads one to Fig. 1.1 by showing how we can argue where each of the two steepest descent and ascent lines for $e^{-u}$ emanating at the saddle $z_2 = e^{i4\pi/3}$ must end up. First, note that each of the descent paths must end up at sinks $\infty e^{-i\pi/4}$ or $\infty e^{-3i\pi/4}$ since the paths cannot cross the real axis, which is an invariant set of the dynamical system (1.131). Each of the two ascent paths at $z_2$ must end up at $-\infty$ or $-i\infty$, since they cannot cross the real axis or approach $+\infty$ without crossing the lower-half plane descent path emanating at the sadde $z_0 = 1$. Further, noting that the two ascent or the two descent paths cannot approach the same *sink* or *source* at $\infty$ without crossing each other, we are qualitatively led to Fig. 1.1.

**Note 1.132** Note that if a path of integration starts at $\infty$ in some direction and ends at $\infty$ in some other direction, then for large $t$ on the curve the arrows should point towards infinity to ensure convergence of the integral. This is indeed the case for (1.130). The steepest descent line decomposition for (1.130) consists of the curve $L_1$ joining $\infty e^{i\pi/4}$ to $\infty e^{-i\pi/4}$ passing through the saddle $z_0 = 1$ together with the curve $L_2$ connecting $\infty e^{-i\pi/4}$ to $\infty e^{i5\pi/4}$ passing through the saddle $z_2 = e^{4i\pi/3}$, as shown in Fig 1.1.

**Note 1.133** If the example above were modified to $\int_{\infty e^{i\pi/4}}^{\infty e^{5\pi i/4}} g(z) e^{-\nu(z-z^4/4)} dz$, where $g(z)$ grows too fast along $\infty e^{-i\pi/4}$ to allow meaningful homotopic deformation as shown in Fig 1.1, for *e.g.* $g(z) = \exp\left[e^{-i\pi/6} z^6\right]$, then we truncate the paths $L_1$ and $L_2$ at some large enough $z_{L_1}$, $z_{L_2}$ independent of $\nu$. With such a choice, it is easily seen that the straight line path connecting the two points is exponentially small relative to the saddle point contributions.

**Note 1.134 (Connection with Watson's Lemma)** For a general entire $f$, the set of saddle points through which the steepest variation curve passes cannot have accumulation points, because of the assumed analyticity of $f$. Then along any steepest descent line, the equation $u(x(t), y(t)) = T$ has a unique solution, and $T(u)$ is smooth except at the saddle points where it has algebraic singularities. Furthermore, by construction, $\exp(iv(x(t), y(t))) =$ *const* along such a curve. The change of variables $f(z) = f(z_0) + t$ brings the problem to the Laplace form to which Watson's lemma applies.

**Exercise 1.135** *Use the analysis in this section to find the asymptotic behavior of (i)* (1.130)*, and (ii) of* $\int_i^{3+i} e^{-\nu(t-t^4/4)} dt$*.*

### 1.4b.1 A singular example

Consider the problem of finding the asymptotic behavior of the Taylor coefficients $c_k$ in the expansion

$$e^{\frac{1}{1-z}} = \sum_{k=0}^{\infty} c_k z^k, \quad |z| < 1 \tag{1.136}$$

We have

$$c_{k-1} = \frac{1}{2\pi i} \oint_{|s|=r<1} \frac{e^{\frac{1}{1-s}}}{s^k} ds = \frac{1}{2\pi i} \oint_{|s|=r<1} e^{\frac{1}{1-s} - k \ln s} ds \tag{1.137}$$

The rightmost integral is of the general form (1.123). What distinguishes this case from the case we considered throughout this section is that $g(z) = e^{\frac{1}{1-z}}$ has an essential singularity at $z = 1$.

The steepest descent lines of $f$ are simply rays towards $\infty$, but *it is not possible to deform the* $|s| = r$ *path along these lines of steepest descent*, since the singularity at $z = 1$ is not integrable The function $g$ contributes nontrivially to the geometry of the curves of interest. We instead plot the steepest descent lines of $h(s;k) = \frac{1}{1-s} - k\ln s$ for fixed $k$ and let $k \to \infty$; we see that $h(s;k)$ has two saddle points, at $s = 1 \pm k^{-1/2}(1+o(1))$ We expect that the behavior

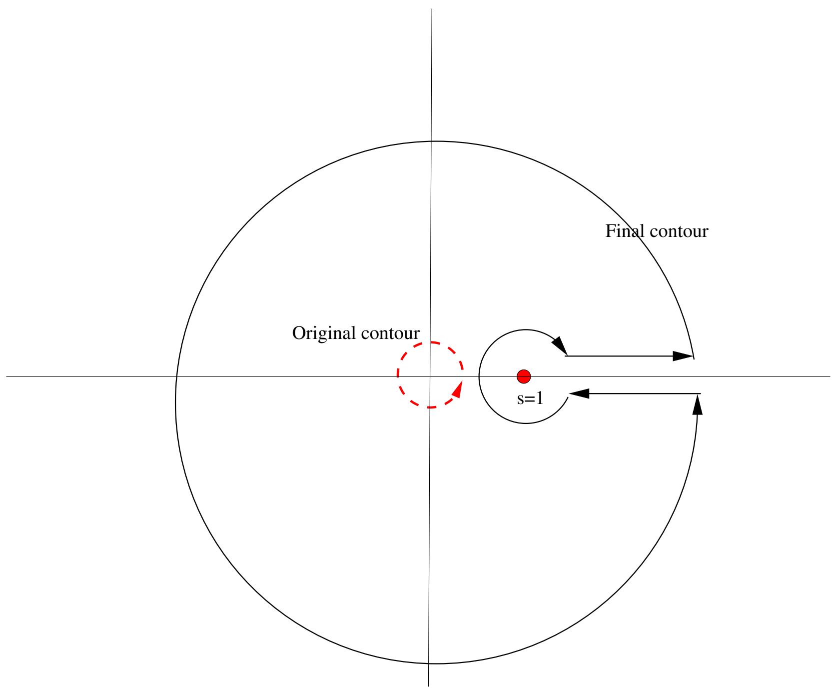


**FIGURE 1.2**: Deformation of contour for (1.138). The circle around $s = 1$ has radius of $k^{-1/2}$. The large contour can be pushed to infinity and the integrals along the sides of $\mathbb{R}^+$ cancel each-other by single-valuedness.

of $c_k$ relates to the behavior of $h$ on a scale of order $k^{-1/2}$ near $s = 1$. This becomes obvious if we change variable to $s = 1 + k^{-1/2}u$. In anticipation, we deform the contour-clockwise contour $|s| = r$ into the contour shown in Fig 1.2. We note the the cancelation of contributions from coinciding straightline positve real axis segments traversed in opposite directions. Furthermore, the integrand $s^{-k}e^{1/(1-s)}$ vanishes rapidly enough so that the counter-clockwise circular arc of radius $R$ as $R \to \infty$ contributes nothing. We are simply left with a clockwise closed contour $C$ around $s = 1$. We write $s = 1 + u/\nu$, with $\nu = k^{1/2}$ and we have

$$c_{k-1} = \nu^{-1}\frac{1}{2\pi i}\oint_{|u|=1} \exp\left[-\nu(u+u^{-1}) - \nu^2[\ln(1+u/\nu) - u/\nu]\right] du \quad (1.138)$$

We note that the function

$$z^{-2}[\ln(1+zu)-zu] = -\tfrac{1}{2}u^2 + \tfrac{1}{3}zu^3 + \cdots \tag{1.139}$$

is analytic at $z=0$ and we can expand convergently in $z=1/k$, as $k\to\infty$

$$\exp\left[-\nu^2[\ln(1+u/\nu)-u/\nu]\right] = e^{u^2/2}\left[1+\frac{u^3}{3\nu}-\frac{u^4}{4\nu^2}+\frac{u^6}{18\nu^2}+\cdots\right] \tag{1.140}$$

Noting that the saddle point of $e^{-\nu f(u)}$, with $f(u)=u+1/u$ is at $u=\pm1$, it is clear that the counter-clockwise $C$ may be chosen to coincide with $|u|=1$, which is a steepest descent path since $\operatorname{Im} f=0$ on $u=e^{i\theta}$. We get

$$\begin{aligned} c_{k-1} &= \frac{1}{2\pi i\nu}\oint_{|u|=1} e^{-\nu(u+1/u)+u^2/2}(1+\tfrac{1}{\nu}F_1(u,\tfrac{1}{\nu})) \\ &= \frac{1}{2\pi i\nu}\oint_{|u|=1} e^{-\nu(u+1/u)+u^2/2}\left(1+\frac{1}{\nu}F_1(u,1/\nu)\right)du \end{aligned} \tag{1.141}$$

where $F_1(u,z)$ is analytic in $(z,u)\in\mathbb{D}_{\frac{1}{2}}\times T$ where $T$ is a neighborhood of the circle $\partial\mathbb{D}_1$. Now the substitution $u+1/u=-2+v$ brings the integral to a sum of two integrals, for each of which Exercise 1.74 applies. This gives, to leading order,

$$c_{k-1} = \frac{e^{2\sqrt{k}}}{2\sqrt{\pi e}k^{3/4}}(1+o(1)) \tag{1.142}$$

Alternately, we may use $u=e^{i\theta}$ and use Laplace's method to each of the following integrals to obtain the same result

$$\frac{1}{2\pi\nu}\left(\int_{-\pi/2}^{\pi/2}+\int_{\pi/2}^{3\pi/2}\right)d\theta e^{-2\nu\cos\theta}\exp\left[\frac{1}{2}e^{2i\theta}+i\theta\right] \tag{1.143}$$

It is to be noted that the contribution from the saddle $u=+1$ is exponentially small in $k$ relative to the contribution from $u=-1$.

Higher order corrections are obtained more simply as follows. We note that $f(z)=\exp(1/(1-z))$ satisfies the ODE

$$(1-z)^2 f'(z) - f(z) = 0 \tag{1.144}$$

The general analytic theory of ODEs implies that there is a on-parameter family of solutions analytic at zero of the form $f(z)=C\sum_{k=0}^{\infty}c_k z^k$. On inserting this power series into (1.144) and collecting like coefficients of powers of $z$, we obtain recurrence relation for $c_k$. With normalization $c_0=1$, we obtain $C=1$ in order that $f(0)=e$. Recurrence relation shows $c_1=\frac{1}{2}c_0=\frac{1}{2}$, while for $k\ge2$,

$$c_k = (2-1/k)c_{k-1} - (1-2/k)c_{k-2} \tag{1.145}$$

from which we can get, as we will see in the sequel, the asymptotic behavior of $c_k$ by seeking formal asymptotic solutions of (1.145).

## 1.5 Formal and actual solutions

Consider the differential equation

$$\frac{df}{dz} = f + f^2 + zf^3; \quad f(0) = 1 \tag{1.146}$$

which we analyze in a neighborhood of $z = 0$. The general analytic theory of ODEs ensures existence and uniqueness and analyticity of the solution in a neighborhood of $z = 0$. We can calculate the power series solution in a number of ways, for instance by substituting $f(z) = \sum_{k=0}^{\infty} c_k z^k$ into (1.146) and identifying the coefficients $c_k$. We get

$$f(z) = 1 + 2z + \frac{5}{2}z^2 + \frac{11}{6}z^3 - z^4 \cdots \tag{1.147}$$

If we write the equation in integral form

$$f(z) = 1 + \int_0^z \left[f(s) + f^2(s) + sf^3(s)\right] ds$$

and iterate,

$$f_{n+1}(z) = 1 + \int_0^z (f_n(s) + f_n^2(s) + sf_n^3(s))ds; \quad f_0(z) \equiv 1 \tag{1.148}$$

we can check that, for small $z$ the sequence $\{f_k\}_k$ is uniformly Cauchy, and thus convergent. This can be seen using the fact that if a function $h$ is bounded and integrable, then

$$\left| \int_0^z h(s)ds \right| \le |z| \max_{|s|<|z|} |h(s)| \tag{1.149}$$

The recurrence (1.148) can be used to generate the power series at zero, by inductively replacing $f_n$ by its Maclaurin series truncated to $O(z^n)$ and integrating the resulting series term by term. We will not go over the details here, as we will develop more general tools shortly.

Consider instead the equation

$$\frac{dg}{dz} = z^{-2}g(z) - z^{-1} \tag{1.150}$$

The point $z = 0$ is a singular point of (1.150), in fact an irregular singular point; there are no analytic solutions near zero. An initial value at $z = 0$ is not well defined. Nonetheless, we can find a formal power series formal solutions $\sum_{k=1}^{\infty} c_k z^k$. In this simple example it is easy to insert the power series into (1.150) and identify the coefficients. We get $c_k = \Gamma(k)$, and formally

$$g(z) \quad \text{“} = \text{”} \quad \sum_{k=1}^{\infty} \Gamma(k) z^k, \tag{1.151}$$

where the power series in (1.151) has zero radius of convergence. To generate the power series inductively, we now note that, if we formally differentiate $g(z) = O_s(z^n)$, then $g'(z) = O(z^{n-1}) \ll z^{-2}g(z)$. It then follows that the natural direction of iteration, one in which we place the lower order terms on the right side of the equation is

$$g_{n+1}(z) = z^2 g_n'(z) + z; \quad g_0(z) = z \tag{1.152}$$

The iteration (1.152) is well defined, and it is solved by the polynomial $g_n(z) = \sum_{k=1}^{n+1} \Gamma(k) z^k$. The sequence of polynomials has no limit. Whenever the size of the term containing the highest derivative is formally small with respect to terms involving lower order derivatives, the natural direction of iteration would place the highest derivative on the right side of the equation. However, in general, we cannot bound $g'$ in terms of $g$, so an iteration of the type (1.152) is not expected to converge. Does the expansion (1.151) relate in any way to the solutions of (1.150)? In this example, we can write down the exact solution of the equation as

$$g(z) = Ce^{-1/z} - e^{-1/z}\int_1^z s^{-1}e^{1/s}ds \tag{1.153}$$

The change of variables $s = 1/t, z = 1/x$ brings (1.153) to the form

$$g(1/x) = Ce^{-x} + e^{-x}\int_1^x t^{-1}e^t dt = Ce^{-x} - e^{-x}\int_{-\infty}^1 t^{-1}e^t + e^{-x}\int_{-\infty}^x t^{-1}e^t dt$$
$$=: C_2 e^{-x} + e^{-x}\int_{-\infty}^x t^{-1}e^t dt = \int_0^\infty \frac{e^{-xu}}{1-u}du + C_2 e^{-x} \tag{1.154}$$

where the contour of integration avoids $t = 0$ and $u = 1$. Watson's lemma shows that $g(z) \sim \sum_{k=1}^\infty \Gamma(k) z^k$. What we see is that the formal power series solution is, in this case as well as in (1.146), the Maclaurin series as $z \to 0^+$ of some solution (here, in fact, all solutions have the same Maclaurin series). Only now the Maclaurin series diverges. The fact that formal solutions are asymptotic to actual ones is true in much wider generality, as we will see in the sequel.

### 1.5a An irregular singular point of a nonlinear equation

Consider Abel's equation

$$y' = y^3 + x \tag{1.155}$$

in the limit $x \to +\infty$. We first find the asymptotic behavior of solutions formally, and then justify the argument. We use the method of *dominant balance* that we will discuss in detail later. As $x$ becomes large, $y$, $y'$, or both need to become large if the equation (1.155) is to hold. Assume first that the balance is between $y'$ and $x$ and that $y \ll x$. If $y' \sim x$ then we have $y \sim x^2/2$

and $y^3 \sim x^6/8$, and this is inconsistent since it would imply $x^8/8 = O(x)$. Now, if we assume $x \ll y^3$ then the balance would be $y' \approx y^3$, implying $y \sim -\frac{1}{2}(x - x_0)^{-2}$; but this is small for $x - x_0 \gg 1$, which conflicts with what we assumed, $x \ll y^3$. We have one possibility left: $y = \alpha x^{1/3}(1+o(1))$, where $\alpha^3 = 1$, which assuming differentiability implies $y' = O(x^{-2/3})$ which is now consistent. We substitute

$$y = \alpha x^{1/3}(1 + v(x)) \tag{1.156}$$

in (1.155); for definiteness, we choose $\alpha = e^{i\pi/3}$, though any cube root of $-1$ would suffice. We get

$$\alpha x^{1/3} v' + 3xv + 3xv^2 + xv^3 + \frac{\alpha}{3} x^{-2/3} + \frac{\alpha}{3} x^{-2/3} v = 0 \tag{1.157}$$

Now a consistent balance is between $3xv$ and $-\frac{\alpha}{3}x^{-2/3}$ meaning that $v = O(x^{-5/3})$. This makes the nonlinear terms small and, for the purpose of justifying the analysis, we don't need to further expand $v$. We now aim at writing (1.157) in a suitable integral form. We first place the formally largest term(s) containing $v$ *and* $v'$ on the left side and the smaller terms as well as the terms not depending on $v$ on the right side:

$$\alpha x^{1/3} v' + 3xv = h(x, v(x)); \ -h(x, v(x)) := 3xv^2 + xv^3 + \frac{\alpha}{3} x^{-2/3} + \frac{\alpha}{3} x^{-2/3} v \tag{1.158}$$

We treat (1.158) as a linear inhomogeneous equation, and solve it thinking for the moment that $h$ is given.

This leads to

$$v = \mathcal{N}(v);$$
$$\mathcal{N}(v) := Ce^{-\frac{9}{5\alpha}x^{5/3}} + \frac{1}{\alpha} e^{-\frac{9}{5\alpha}x^{5/3}} \int_{x_0}^{x} e^{\frac{9}{5\alpha}s^{5/3}} s^{-1/3} h(s, v(s)) ds \tag{1.159}$$

We chose the limits of integration in such a way that the integrand is maximal when $s = x$: if $x \to +\infty$, then $x^{-1/3} e^{\frac{9}{5\alpha}x^{5/3}} \to \infty$, and our choice corresponds indeed to this prescription.

The largest of the terms not containing $v$ on the right side of (2.46) comes from the term $\frac{\alpha}{3}x^{-2/3}$ in $h$, and is of the order $\frac{1}{3}x^{-5/3}(1 + o(1))$. Indeed, for $\operatorname{Re} b > 0$ by Watson's Lemma or simply by L'Hospital we get

$$\frac{\int_a^x e^{bs^m}/s^n ds}{e^{bx^m}/x^n} \sim b^{-1} m^{-1} x^{1-m}; \quad x \to +\infty \tag{1.160}$$

Again by dominant balance, we expect $v = O(x^{-5/3})$. Thus, it is natural to choose $x_0$ large enough and introduce the Banach space

$$\{f : \|f\| := \sup_{x > x_0} |x^{5/3} f(x)| < \infty\} \tag{1.161}$$

or the region $|x| > x_0$ in a sector $\mathcal{S}$ in the complex domain where Re $\left(\frac{1}{\alpha}x^{5/3}\right) > 0$ : $\arg x \in (-\frac{\pi}{10}, \frac{\pi}{2})$:

$$\mathcal{B} = \{f : \|f\| := \sup_{x\in\mathcal{S}} |x^{5/3}f(x)| < \infty\} \tag{1.162}$$

and within this space a ball of size large enough $-\frac{2}{3}-$ to accommodate for the largest term on the right side, $\frac{\alpha}{3}x^{-2/3}$:

$$B_1 := \{f \in \mathcal{B} : \|f\| \leqslant \frac{2}{3}\} \tag{1.163}$$

**Lemma 1.164** *For given $C$, if $x_0$ is large enough, then the operator $\mathcal{N}$ is contractive in $B_1$ and thus* (2.46) *(as well as* (1.158)*) has a unique solution there.*

**PROOF** We first check that $\mathcal{N}(B_1) \subset B_1$, by estimating each term in $\mathcal{N}$. By (1.160) we have for large enough $x_0$, $|\mathcal{N}x^{-m}| = \frac{1}{3}|x|^{-m-1}(1+o(1))$. In particular, $|\mathcal{N}\frac{\alpha}{3}x^{-2/3}| \leqslant \frac{\alpha}{9}|x|^{-5/3}(1+o(1))$. The contribution of the other terms are much smaller. For instance, $|xv^2| < Cx^{1-5/2}\|v\|$ we have $|\mathcal{N}(xv^2)| = C|x|^{-5/2}(1+o(1))$.

To show contractivity, we note that, for $k > 1$,

$$|\mathcal{N}(v_2^k - v_1^k)| \leqslant k\|v_2 - v_1\||\mathcal{N}\left[x^{-5/3}2(2/3)^{k-1}x^{-5(k-1)/3}\right]$$

$\square$

## 1.5b The wave equation with potential

The *free* wave equation is $u_{tt} - c^2u_{xx} = 0$; $c$ can be scaled out, by changing variables to $\tilde{x} = x/c$; without loss of generality we can then assume $c = 1$. One common setting is to have the initial position and velocity specified, that is

$$u_{tt} - u_{xx} = 0;\ u(x,0) = f(x), \quad u_t(x,0) = g(x) \tag{1.165}$$

When $f \in C^2(\mathbb{R})$ and $g \in C^1(\mathbb{R})$, the change of variable $\xi = x - t$, $\eta = x + t$ leads to the well-known D'Alembert solution

$$u(x,t) = \tfrac{1}{2}f(x+t) + \tfrac{1}{2}f(x-t) + \tfrac{1}{2}\int_{x-t}^{x+t} g(s)ds \tag{1.166}$$

Without smoothness of $f$ and $g$ (1.166) is interpreted as a weak solution. In the same way we can solve the wave equation with a source,

$$u_{tt} - u_{xx} = S(x,t);\ u(x,0) = f(x), \quad u_t(x,0) = g(x) \tag{1.167}$$

to obtain

$$u(x,t) = \tfrac{1}{2}f(x+t) + \tfrac{1}{2}f(x-t) + \tfrac{1}{2}\int_{x-t}^{x+t} g(s)ds + \frac{1}{2}\int_0^t \int_{x-t+t_1}^{x+t-t_1} S(x_1,t_1)dx_1dt_1 \quad (1.168)$$

The wave equation with potential arises naturally in a number of physical problems, ranging from electrodynamics to the wave evolution in the presence of a black hole. It reads

$$u_{tt} - u_{xx} + V(x)u(x,t) = 0;\; u(x,u) = f(x), \quad u_t(x,0) = g(x) \qquad (1.169)$$

Clearly, at least for general $V$ we cannot expect to solve (1.169) in closed form.

Here we assume that $V \in L^\infty(\mathbb{R})$ and $f, g$ are in $L^1(\mathbb{R})$ and show that (1.169) has a global solution $u(\cdot,t) \in L^1(\mathbb{R})$ and $\|u(\cdot,t)\|_{L^1}$ grows at most exponentially in $t$. That exponential growth is possible for some potentials can be seen in the following way. Looking for solutions in the form $u(x,t) = e^{\lambda t}U(x)$ we obtain

$$-U'' + V(x)U = -\lambda^2 U \qquad (1.170)$$

Eq. (1.170) is the time-independent Schrödinger equation; in that setting it is natural to assume that $V$ decays as $x \to \infty$. An $L^2$ solution of (1.170) for $\lambda \neq 0$ is called a *bound state* of the quantum *Hamiltonian* $-\frac{d^2}{dx^2} + V(x)$, and for many potentials of interest these do exist.

We can use (1.171) to rewrite (1.169) in integral form,

$$u(x,t) = \tfrac{1}{2}f(x+t) + \tfrac{1}{2}f(x-t) + \tfrac{1}{2}\int_{x-t}^{x+t} g(s)ds - \frac{1}{2}\int_0^t \int_{x-t+t_1}^{x+t-t_1} V(x_1)u(x_1,t_1)dx_1dt_1 =: \mathcal{A}[u](x,t) \quad (1.171)$$

**Proposition 1.172** *Assume the initial conditions $f(x) = u(t,0)$ and $g(x) = u_t(x,0)$ are in $L^1(\mathbb{R})$ and $V \in L^\infty(\mathbb{R})$. Then, if $\nu > \sqrt{2}\|V\|_\infty^{\frac{1}{2}}$ we have* $\sup_{t>0} e^{-\nu t}\|u(t,\cdot)\|_{L_1} < \infty$.

**PROOF** We write the Duhamel formula as

$$u = \mathcal{A}u;\; \mathcal{A}u := \frac{f(x-t)+f(x+t)}{2} + \tfrac{1}{2}\int_{-\infty}^{\infty} \chi_t(y-x)g(y)dy + \frac{1}{2}\int_0^t \int_{-\infty}^{\infty} u(y,s)V(y)\chi_{t-s}(y-x))dyds \quad (1.173)$$

where $\chi_a$ is the characteristic function of the interval $[-a,a]$. Consider the Banach space

$$\mathcal{B} = \{u : \|\|u\|_\nu := \sup_{t\in\mathbb{R}^+} e^{-\nu t}\|u(t,\cdot)\|_1 < \infty\};\quad (\nu > \sqrt{2}\|V\|_\infty^{\frac{1}{2}}) \qquad (1.174)$$

Applying Fubini to integrate first in $x$, we see that $\| \int_{-\infty}^{\infty} \chi_t(y-x)g(y)dy\|_1 \le 2t\|g\|_1$ and (since by definition $\|u(\cdot,s)\|_1 \le \|u\|_\nu e^{\nu s}$)

$$\sup_{t>0} e^{-\nu t}\left\|\int_0^t\int_{-\infty}^{\infty} u(y,s)V(y)\chi_{t-s}(y-x)dydt\right\|_1$$
$$\le \|V\|_\infty\|u\|_\nu \sup_{t>0} e^{-\nu t}\int_0^t 2(t-s)e^{\nu s}ds \le 2\|V\|_\infty \nu^{-2}\|u\|_\nu \quad (1.175)$$

Using (1.175) we see that $\mathcal{A}:\mathcal{B}\to\mathcal{B}$ is contractive. Also, assuming $f, g$ and $V$ are smooth, the solution is seen to be smooth too: since $u \in L^1$, Duhamel's formula shows that it is continuous; then, as usual, using continuity we derive differentiability, and inductively, we see that $u$ is smooth.

□

**Exercise 1.176** Complete the details by showing that this result implies global existence of a solution of (1.169).

**Exercise 1.177** (i) Assume $V \in L^2(\mathbb{R})$. Prove a similar result with $\|u\|$ given by $\sup_{t\ge 0} e^{-\nu t}\left(\int_{-\infty}^{\infty}|u(x,t)|^2dx\right)^{1/2}$. Use this result to estimate the largest possible eigenvalue of $V$.

### 1.5c Regular versus singular perturbations

Consider first two elementary problems: finding the roots of the polynomials $P_1(x;\varepsilon) = x^5 - x - \varepsilon$ and $P_2(x;\varepsilon) = \varepsilon x^5 - x - \varepsilon$ for small $\varepsilon$.

We see that $P_1(x;0)$ has five roots, $\rho = 0, \pm 1, \pm i$. We choose one of them, say $\rho = 1$ and look for roots of $P_1(x;\varepsilon)$ in the form $\rho(\varepsilon) = 1 + \sum_{k\geqslant 1} c_k\varepsilon^k$. Substituting in the equation $P_1 = 0$ we get $(4c_1-1)\varepsilon + (4c_2+10c_1^2)\varepsilon^2 + (4c_3 + 20c_1c_2 + 10c_1^3))\varepsilon^3 = 0$, and solving for the coefficients $c_1, \ldots, c_3, \ldots$ we get

$$c_1 = \frac{1}{4},\ c_2 = -\frac{5}{32},\ c_3 = \frac{5}{32}, \ldots \quad (1.178)$$

The series of $\rho(\varepsilon)$ is actually convergent. It would not be very convenient to prove this directly from the recurrence relation, though this is possible. A better way is to substitute $\rho = 1 + \delta$ into the equation, placing the largest term containing $\delta$ on the left side, and showing that the equation for $\delta$ is contractive for small $\delta$, in a space of functions analytic in $\varepsilon$ at $\varepsilon = 0$. We leave the details as an exercise.This is a typical behavior in regularly perturbed problems: the roots of the leading order equation $P_1(x;0)$ give the leading behavior of the actual roots of $P_1(x,\varepsilon)$ as $\varepsilon \to 0$.

By contrast, $P_2(x;0)$ has only one root, $x = 0$. Four solutions of the quintic polynomial $P_2(x,\varepsilon)$ are lost by setting $\varepsilon = 0$ in the equation; this is an example of singular perturtbation since $P_2(x;0)$ does not capture all the behavior of of the five roots of $P_2(x,\varepsilon)$ as $\varepsilon \to 0$. We can find the missing roots by applying a formal dominant balance argument: clearly $\varepsilon x^5$ has to be part of the balance. Balancing $\varepsilon x^5$ with $\varepsilon$ leads to an inconsistency, since $-x$ would turn out to be much larger. We must then have $\varepsilon x^5 \sim x$ or $\varepsilon x^4 \sim 1$. To obtain the higher order corrections, we substitute $x = \varepsilon^{-1/4} y$ and we get

$$y^5 = y + \eta; \quad (\eta = \varepsilon^{5/4}) \tag{1.179}$$

Now the limiting ($\eta \to 0$) equation, $y^5 = y$, has five roots as expected of a quintic polynomial. In fact, the equation (1.179) is $P_1(y;\eta) = 0$ and, if we take $y = 1 + \delta$ we get a convergent expansion $\delta = \frac{1}{4}\eta - \frac{5}{32}\eta^2 + \frac{5}{32}\eta^3 + \ldots$. Substituting to get $\delta$, we see that $\delta(\varepsilon)$ is not analytic; nonetheless it has a convergent expansion in powers of $\varepsilon^{5/4}$. By contrast, we will find that in singular perturbation of differential equations, where a small parameter typically multiples the highest derivative, the asymptotic expansions are generally divergent.

An equation can be regularly perturbed in some regimes and singularly perturbed in some others.

An interesting example is the pendulum of slowly variable length. A model equation is

$$\ddot{q} + \frac{g}{l_0 + \varepsilon t} q = 0 \tag{1.180}$$

where $q$ is the generalized position, $g$ is the gravitational acceleration and $l_0$ is the initial length. A proper treatment of this problem will have to wait until we study adiabatic invariants.

By changing units and $\varepsilon$ we can assume without loss of generality $l_0 = g = 1$. The limiting equation $\ddot{q} + q = 0$ has a two dimensional family of solutions, $y = A \sin t + B \cos t$. Assuming that $y(0) = 0$ and $\dot{y}(0) = 1$ we choose $q_0(t) = \sin t$. We look for solutions $y$ in the form of power series in powers of $\varepsilon$,

$$y(t) = \sin t + \sum_{k=1}^{\infty} \varepsilon^k y_k(t) \tag{1.181}$$

Solving order by order in $\varepsilon$ and using the initial condition $y(0) = 0$ and $\dot{y}(0) = 1$, translating to $y_k(0) = 0$, $y_k'(0) = 0$ for $k \geq 1$, we get

$$\begin{aligned} q(t) = \sin t &+ \left(\frac{1}{4} t \sin t - \frac{1}{4} t^2 \cos t\right) \varepsilon \\ &+ \left[\left(\frac{3}{32} - \frac{3}{32} t^2 - \frac{1}{32} t^4\right) \sin t - \left(\frac{3}{32} t - \frac{1}{16} t^3\right) \cos t\right] \varepsilon^2 + \cdots \end{aligned} \tag{1.182}$$

We see that the validity of the expansion is limited by the condition $t^2 \varepsilon <$ *const.*, where *const.* needs to be relatively small, since otherwise we end up with a series of successively growing terms, and the expansion would be useless: we would have to look at the complete expansion, to all orders in $\varepsilon$, to hope to understand anything about $q$.

In a region where $t < \delta\varepsilon^{-1/2}$ with $\delta$ small enough, we can set up a contractive mapping argument to justify the expansion, which will turn out to be convergent. We leave this as an exercise as well.

Note also that in the time interval $0 < t \ll \sqrt{\varepsilon}$ we have $l = l_0 + O(\sqrt{\varepsilon}$, that is, the length does not change much; this region is not very interesting. A proper treatment of this problem will have to wait until we study adiabatic invariants.

Now, when $t \sim \varepsilon^{-1/2}$ it is natural to take $t\sqrt{\varepsilon} = \tau$ as a new variable, $q(t) = Q(\tau)$ that will not be necessarily small. The equation for $Q$ reads.

$$\varepsilon\ddot{Q} + \frac{Q}{1+\sqrt{\varepsilon}\tau} = 0 \tag{1.183}$$

Now the limit $\varepsilon \to 0$ is singular: in this limit equation (1.183) would become $\frac{Q}{1+\sqrt{\varepsilon}\tau} = 0$; here, as in the case of $P_2(x;\varepsilon)$ we lose most solutions. Furthermore, the surviving solution $Q = 0$ is not very interesting, and it does not satisfy the initial condition. We need to do something else, in this case WKB, which we introduce in §1.5c.1 below.

### 1.5c.1 Singularly perturbed differential equations

Consider first the very simple equation

$$\varepsilon^2 y'' + y = 0; \quad \varepsilon \ll 1 \tag{1.184}$$

which can be of course solved in closed form, which we will do after we explore some qualitative features. The limit $\varepsilon \to 0$ is singular: taking $\varepsilon = 0$ in (1.184) leaves us with $y = 0$. Most solutions of (1.184) are lost in this limit. This is one of the indications that an equation is singularly perturbed. The other one, that we will return to, is non-analytic behavior in $\varepsilon$.

Similarly, the equation

$$\frac{d^2y}{dx^2} - a^2 y = 0 \tag{1.185}$$

is singularly perturbed as $x \to \infty$, since the change of variable $x = 1/z$ brings it to

$$z^4\frac{d^2y}{dz^2} + 2z^3\frac{dy}{dz} - a^2 y(z) = 0 \tag{1.186}$$

and we see that for small $z$ the coefficients of the derivatives on the left side of the equation vanish at $z = 0$, and if we ignored these terms we would be once more left with a scalar equation, $y = 0$.

The eigenvalue problem for the one-dimensional Schrödinger equation

$$-\hbar^2\psi'' + V(x)\psi = E\psi \tag{1.187}$$

is singularly perturbed in the when the Planck constant is taken to the limit $\hbar \to 0$ (its physical value is $\approx 6.626068 \times 10^{-34} m^2 kg/s$). Here $\psi$

is the wave function and it has the physical interpretation that $|\psi(x)|^2$ is probability density function for a particle and the total probability is one: $\|\psi\|_2 = \int_{-\infty}^{\infty} |\psi(x)|^2 dx = 1$. For a typical potential $V$ going to zero as $x \to \infty$, Eq. (1.187) is also singularly perturbed when $x \to \infty$. Indeed, scaling out $\hbar$ now and taking $z = 1/x$ we get

$$-z^4 \frac{d^2\psi}{dz^2} - 2z^3 \frac{d\psi}{dz} + V(1/z)\psi = E\psi \tag{1.188}$$

and for $z = 0$ we are left with the scalar equation $E\psi = 0$. The solution $\psi = 0$ is not physically acceptable, as it violates $\|\psi\|_2 = 1$.

We can analyze (1.188) using dominant balance. It is clear that $V(1/z)\psi$ cannot be part of the dominant balance, since it is necessarily much smaller than $E\psi$. We are left with three possible balances, only one of them consistent. If we assume $z^4 \frac{d^2\psi}{dz^2} \sim -2z^3 \frac{d\psi}{dz}$, *i.e.* $V(1/z)\psi, E\psi << z^4 \frac{d^2\psi}{dz^2}$, we get $\psi \sim \text{const}/z$, but then this violates the assumption $E\psi << z^4 \frac{d^2\psi}{dz^2}$, unless $E = 0$. If instead we balance $2z^3 \frac{d\psi}{dz} = E\psi$ we get $\psi \sim e^{-\frac{1}{4}Ez^{-2}}$, assuming other terms in (1.188) to be smaller, then $z^4 \frac{d^2\psi}{dz^2} \gg E\psi$ contrary to the assumption. We are left with $z^4 \frac{d^2\psi}{dz^2} \sim E\psi$.

To analyze the balance $z^4 \frac{d^2\psi}{dz^2} \sim E\psi$ it is useful to note that the same balance is the only consistent one in (1.186) which can be solved exactly, as it is equivalent to (1.185): the solution is $y(x) = \exp(\pm ax) = \exp(\pm a/z)$. The solutions do not have asymptotic power series for small $z$, but their logs do. An exponential substitution, $y = e^{w(z)}$ is suggested, and this *WKB ansatz* is very helpful in singularly perturbed equations.

We will proceed formally first, and then prove a result for (1.187). So, consider again (1.187) and substitute $\psi(x) = e^{w(x)}$. After dividing by $e^{w(x)}$ we get

$$-\hbar^2(w'' + {w'}^2) = E - V(x) \tag{1.189}$$

or, with $w' = f$, we get the first order nonlinear ODE

$$-\hbar^2(f' + f^2) = E - V(x) \tag{1.190}$$

We analyze (2.264) by dominant balance. We first assume for simplicity that $E > V(x)$ for all $x$; a similar argument works if $E < V(x)$ for all $x$, with $\sqrt{V(x) - E}$ replacing $i\sqrt{E - V(x)}$. The situations in which $E = V(x)$ has nontrivial solutions, called *turning points* are important, and we will study them separately.

**Note 1.191** In a WKB ansatz, we have $w'' \ll {w'}^2$. Indeed, the balance $-\hbar^2 w'' \sim E - V(x)$ would give $w = O_s(\hbar^{-2})$ and then ${w'}^2 = O_s(\hbar^{-4})$ showing that this choice is inconsistent. The balance $w'' \sim -{w'}^2$ does not work either since then $w, w', w'' = O(1)$ whereas, under our assumptions we have $\hbar^{-2}(E - V(x)) = O_s(\hbar^{-2})$. We are left with the balance ${w'}^2 \sim -\hbar^{-2}(E - V(x))$ with $w'' \ll {w'}^2$ (we have already seen that $w'' \not\sim {w'}^2$ ).

According to Note 1.191 we place $w''$ on the right side of the equation, treated as being relatively small. With $f = w'$, (1.190) implies

$$f = \pm\frac{i}{\hbar}\sqrt{E - V(x) + \hbar^2 f'} \tag{1.192}$$

where we choose one sign at a time, say plus for now, and we expand (1.192), by the usual Picard-like asymptotic iterations,

$$f^{[n+1]} = \frac{i}{\hbar}\sqrt{E - V(x) + \hbar^2 f^{[n]\prime}} \tag{1.193}$$

The fact that the highest order derivative is on the right side of the iteration strongly indicates that the expansion thus obtained is divergent.

Expanding in $\hbar$ to three orders we get

$$f^{[n+1]} = \tfrac{i}{\hbar}\sqrt{E - V(x)} + \frac{i\hbar f^{[n]\prime}}{2\sqrt{E-V(x)}} - \frac{i(f^{[n]\prime})^2\hbar^3}{8(E-V(x))^{3/2}} + \cdots \tag{1.194}$$

In this way we get

$$f^{[0]} = \tfrac{i}{\hbar}\sqrt{E - V(x)} \tag{1.195}$$

$$f^{[1]} = \tfrac{i}{\hbar}\sqrt{E - V(x)} + \frac{1}{4}\frac{V'}{E-V} \tag{1.196}$$

$$f^{[2]} = \tfrac{i}{\hbar}\sqrt{E - V(x)} + \frac{1}{4}\frac{V'}{E-V} + \hbar\frac{\frac{5i}{32}V'^2 + \frac{i}{8}V''(E-V))}{(E-V(x))^{5/2}} \tag{1.197}$$

To two orders, this gives

$$w^{[0]} = \tfrac{i}{\hbar}\int_{x_0}^{x}\sqrt{E - V(s)}ds + C \tag{1.198}$$

$$w^{[1]} = \tfrac{i}{\hbar}\int_{x_0}^{x}\sqrt{E - V(s)}ds - \frac{1}{4}\ln(E - V(x)) + C \tag{1.199}$$

or

$$\psi = C_1(E - V(x))^{-1/4}e^{\frac{i}{\hbar}\int_{x_0}^{x}\sqrt{E-V(s)}ds}\,(1 + o(1)) \tag{1.200}$$

### 1.5c.2 Proof of existence of a solution of (1.187) in the form (1.200)

One way of proving the expansion is to return to (1.190) where we substitute for $f(x) = f^{[j]}(x) + \delta(x)$, $j \geqslant 1$ [12]. Here we choose $j = 1$; assuming that the regularity of $V$ allows for calculating higher order terms which involve higher

[12]The minimum $j$ needed depends on the problem; in some settings, $j = 0$ suffices. As a rule, the more terms we pull out, the more contractive the operator becomes, at the expense of getting a more involved algebra.

derivatives of $V$ as seen in (1.195) taking $j = j_1 > 1$ would allow for proving asymptoticity of the expansion with $j_1 + 1$ terms.

$$f(x) = \tfrac{i}{\hbar}\sqrt{E - V(x)} + \frac{1}{4}\frac{V'}{E - V} + \delta(x) \tag{1.201}$$

The equation for $\delta(x)$ is

$$\hbar\delta' + 2i\sqrt{E - V(x)}\delta + \frac{\hbar V'}{2(E - V(x))}\delta = -\hbar g - \hbar\delta^2$$
$$\text{where } \; g(x) := \frac{5}{16}\left(\frac{V'}{E - V(x)}\right)^2 + \frac{V''}{4[E - V(x)]} \tag{1.202}$$

We note that we need to keep the highest derivative term, here $\hbar\delta'$ on the left side of (1.202) even though it is multiplied by small $\hbar$. In fact, in general, in a singularly perturbed problem, the singularly perturbed term cannot be discarded.

We then write the equation in integral form. Let $J = \frac{2i}{\hbar}\int_{x_0}^{x}\sqrt{E - V(s)}ds$ and $\mu(x) = (E - V(x))^{-1/2}$. Using the integrating factor for the left side of 1.202, we get

$$\delta(x) = -\frac{e^{-J(x)}}{\mu(x)}\int_{\infty}^{x}\mu(s)e^{J(s)}g(s)ds - \frac{e^{-J(x)}}{\mu(x)}\int_{\infty}^{x}e^{J(s)}\mu(s)\delta^2(s)ds$$
$$:= \delta_0 + \mathcal{N}\delta \tag{1.203}$$

To prove a rigorous result we need some assumptions.

**Assumption 1.204** For simplicity. we let $V : \mathbb{R} \to \mathbb{C}$, $V \in C^2(\mathbb{R})$, and $V$ is $O(1/x^{1+\varepsilon})$ for large $x$ and it "acts like a symbol" essentially meaning that we can differentiate the asymptotics: $V' = O(1/x^{2+\varepsilon})$ and $V'' = O(1/x^{3+\varepsilon})$. We work on an interval, say $[x_0, \infty)$, where $E - V(x) > a > 0$. We note that under these assumptions we have $g(x) = O(x^{-3-\varepsilon})$ for large $x$.

We introduce the Banach space

$$\mathcal{B} = \{\delta : [x_0, \infty) \to \mathbb{C} \mid \|\delta\| := \sup_{x \geqslant x_0} |x^{1+\varepsilon}\delta(x)| < \infty\} \tag{1.205}$$

We first prove that the term

$$\delta_0 := \frac{e^{-J(x)}}{\mu(x)}\int_{x}^{\infty}\mu(s)e^{J(s)}g(s)ds \tag{1.206}$$

in (1.203) decays as $\hbar \to 0$ or as $x_0 \to +\infty$.

**Lemma 1.207** *Under Assumption 1.204, we have* $\lim_{\hbar\to 0}\|\delta_0\| = 0$. *Furthermore, for any* $\hbar \neq 0$, $\lim_{x_0\to\infty}\|\delta_0\| \to 0$.

**PROOF** Let $t = t(x) = \int_{x_0}^x \sqrt{E - V(t)}dt$. Note that $t : [x_0, \infty] \to [0, \infty]$ is increasing since $t'(x) = \sqrt{E - V(x)} = 1/\mu(x)$ is bounded away from zero. Let

$$f(x) := x^{1+\varepsilon} \int_x^\infty \mu(s) e^{J(s)} g(s) ds = \int_t^\infty e^{2it'/\hbar} g(x(t')) \left[x(t)\right]^{1+\varepsilon} \mu^2(x(t')) dt' \tag{1.208}$$

Since $|x^{1+\varepsilon} e^{J(s)} g(s)\mu(s)| \le s^{1+\varepsilon}\mu(s)|g(s)|$ is in $L^1$, the first equality in (1.208) implies

$$\lim_{x\to\infty} f(x) = 0 \tag{1.209}$$

Using the fact that $\frac{1}{\mu(x)} e^{-J(x)}$ is bounded, (1.205), (1.203) and (1.209) imply $\lim_{x_0\to\infty} \|\delta_0\| = 0$ for any $\hbar \neq 0$. Now, consider the case of $\hbar \to 0$ with $x_0 > 0$ fixed.

We claim that for any $\varepsilon > 0$, there exists $\hbar_0$ such that $|f(x)| \le \varepsilon$ for any $x$ if $|\hbar| \le \hbar_0$.

First, from (1.209), it follows that for large enough $M$ and $x > M$ we have $|f(x)| \le \varepsilon$. Then, the Riemann-Lebesgue lemma implies that for $x \in [x_0, M]$ we have $\lim_{\hbar\to 0} f(x) = 0$. Since $f$ is uniformly continuous on compact sets, convergence is uniform in $x$, *i.e.* there exists $\hbar_0$ so that $|\hbar| \le \hbar_0$ implies $|f(x)| \le \varepsilon$ for any $x$. Therefore, $\lim_{\hbar\to 0} \|\delta_0\| = 0$ since $e^{-J(x)}/\mu(x)$ is bounded. ☐

**Theorem 1.210** *Under Assumption 1.204, if $x_0$ is large enough or $\hbar$ is small enough, then two linearly independent solutions of (1.187) for $x \in (x_0, \infty)$, $\psi = \psi_1$ and $\psi = \psi_2$, satisfy*

$$\psi_1(x) = [E - V(x)]^{-1/4} \exp\left[\frac{i}{\hbar}\int_{x_0}^x [E - V(t)]^{1/2}\, dt\right] \{1 + o(1)\} \tag{1.211}$$

$$\psi_2(x) = [E - V(x)]^{-1/4} \exp\left[-\frac{i}{\hbar}\int_{x_0}^x [E - V(t)]^{1/2}\, dt\right] \{1 + o(1)\} \tag{1.212}$$

**PROOF** We only prove the result for $\psi_1$ since the proof for $\psi_2$ is the same after changing the sign of $i$. Since $\psi_1 = e^W$, $W' = f^{[1]} + \delta$, it is enough to show that (1.203) has a solution in a ball where $\|\delta\|$ is small:

$$B_\varepsilon = \{\delta \in \mathcal{B} | \|\delta\| \leqslant \varepsilon\} \tag{1.213}$$

Using Lemma 1.207, we see that for any $\varepsilon > 0$, if we $x_0$ is large enough or $\hbar$ is small enough, then $\|\delta_0\| \le \frac{1}{2}\varepsilon$. Now, for any $\delta \in B_\varepsilon$, we have

$$|x^{1+\varepsilon}\mathcal{N}[\delta]| \le \|\delta\|^2 x^{1+\varepsilon} \int_x^\infty \frac{\mu(s)}{\mu(x)} s^{-2-2\varepsilon} ds \le C\|\delta\|^2 \le C\varepsilon^2,$$

where $C$ is independent of $\varepsilon$. Choosing $\varepsilon < \frac{1}{2C}$, this implies that $\|\delta_0 + \mathcal{N}[\delta]\| < \varepsilon$ and $\|\mathcal{N}[\delta_1] - \mathcal{N}[\delta_2]\| \leq 2C\varepsilon\|\delta_1 - \delta_2\| < \|\delta_1 - \delta_2\|$. Therefore, the contraction mapping theorem implies that there exists a unique solution in $B_\varepsilon$. ∎

If, as mentioned at the beginning of the section we took $j = j_1 > 1$ instead, then the remainder $g$ in the map (1.224) will be of higher order in $\hbar$. With this change, contractivity is proved in the same way, to obtain an asymptotic expansion with $j_1 + 1$ terms.

**Remark 1.214** *Replacing* $i$ *by* $-i$ *in in (1.200) gives the behavior of a second independent independent solution of (1.187) in* $(x_0, \infty)$.

**Remark 1.215** *(i) No decay assumption on* $V$ *is necessary for Theorem 1.210 to apply for* $x$ *in a fixed (*$\hbar-$*independent interval* $[a, b]$. *(ii) The assumption* $x_0 > 0$ *in Theorem 1.210 is not needed. To allow for* $x_0 < 0$ *the proof is largely the same. Assuming* $V(x) = O(|x|^{-1-\varepsilon})$ *as* $x \to -\infty$, *we would instead use the norm* $\|\delta\| = \sup_{x\in(x_0,\infty)} |1 + |x|)^{1+\varepsilon}|\delta(x)|$.

### 1.5d The case $V(x) - E \geq a > 0$

In this case, if $V \in C^2$, the arguments given in §1.5c.1 that lead to WKB solution of (1.187) may be applied in this case again to give the result

$$\psi = C_1 \left[V(x) - E\right]^{-1/4} \exp\left[\pm\frac{1}{\hbar}\int_{x_0}^{x}\sqrt{V(t) - E}dt\right][1 + o(1)] \quad \text{for } x \in (x_0, \infty) \tag{1.216}$$

either for $x_0 \to +\infty$ or $\hbar \to 0$. The precise result is given below:

**Theorem 1.217** *For* $V \in C^2(x_0, \infty)$, *and* $V(x) - E \geq a > 0$, *as* $x_0 \to +\infty$ *or* $\hbar \to 0^+$, *Two independent solutions of (1.187) are given by* $\psi = \psi_1$ *and* $\psi = \psi_2$, *where*

$$\psi_1(x) = \left[V(x) - E\right]^{-1/4} \exp\left\{\frac{1}{\hbar}\int_{x_0}^{x}\left[V(t) - E\right]^{1/2} dt\right\}\{1 + o(1)\} \tag{1.218}$$

$$\psi_2(x) = \left[V(x) - E\right]^{-1/4} \exp\left\{-\frac{1}{\hbar}\int_{x_0}^{x}\left[V(t) - E\right]^{1/2} dt\right\}\{1 + o(1)\} \tag{1.219}$$

The proof of this theorem is similar to the proof of Theorem 1.210. The only difference is that in the proof for $\psi \sim \psi_1$, in the equation for $\delta$ defined by $W' = \sqrt{V(x) - E} - \frac{V'}{4[V(x)-E]} + \delta$, where $\psi = e^W$, it is necessary to put the integral equation for $\delta$ in the following form

$$\delta = -\frac{e^{-J(x)}}{\mu(x)}\int_{x_0}^{x} g(s)\mu(s)e^{J(s)}ds - \frac{e^{-J(x)}}{\mu(x)}\int_{x_0}^{x} e^{J(s)}\mu(s)\delta^2(s)ds =: \delta_0 + \mathcal{N}\delta \tag{1.220}$$

where $J(x) = \frac{2}{\hbar}\sqrt{V(x)-E}$ and $\mu = (V(x)-E)^{-1/2}$, On the other hand, to prove $\psi \sim \psi_2$, On the other hand, when $W' = -\sqrt{V(x)-E} - \frac{V'}{4[V(x)-E]} + \delta$, where $\psi = e^W$, it is necessary we replace the integration limit $x_0$ in (1.220) by $\infty$, or the right end of whatever $x$ interval one is concerned with, since in this case, $J(x) = -\frac{2}{\hbar}\sqrt{V(x)-E}$– these choices of limits ensure that $e^{J(x)-J(s)} \leq 1$.

**Exercise 1.221** Prove Theorem 1.217. You may want to use the fact that for locally integrable $q$, $\lim_{\hbar\to 0^+}\int_{x_0}^x e^{J(x)-J(s)}q(s)ds \to 0$ when $J = \frac{2}{\hbar}\sqrt{V(x)-E}$.

### 1.5d.1 Turning points

In the previous subsection we assumed that $E - V$ is bounded below. This assumption is in fact necessary, otherwise the asymptotic behavior of the solutions is different. If we examine the procedure used to derive (1.194) from (1.193), we see that the expansion is only valid if $\hbar^2 f^{[n]'} \ll E - V(x)$, that is, to have $f \approx f^{[0]}$ we need $\hbar(E - V(x))^{-1/2} \ll E - V(x)$, that is, $E - V(x) \gg \hbar^{2/3}$. Something else must be done when the latter condition fails.

In our assumption $V$ is smooth. Generically, near a zero of $V(x) - E$, also referred to as a turning point, $V(x) = \alpha(x - x_t) + O(x - x_t)^2$, where $\alpha \neq 0$. Without loss of generality we can take $x_t = 0$ and $\alpha = -1$ through translation and scaling. The region where our WKB does not hold is given by $|x| \lesssim \hbar^{2/3}$. It is natural to change variables to $t = x/\hbar^{2/3}$ in (1.187); we get, after dividing by $\hbar^{2/3}$,

$$-\psi''(t) - t\psi(t) = \hbar^{2/3}t^2\varphi_1(x(t))\psi(t) \tag{1.222}$$

where $\varphi_1(x) = x^{-2}[E - V(x) - x]$. To leading order in small $\hbar$, $\psi$ satisfies $-\psi_0''(t) - t\psi_0(t) = 0$ with the general solution

$$\psi_0(t) = C_1\mathrm{Ai}(-t) + C_2\mathrm{Bi}(-t) \tag{1.223}$$

Since the right hand side of (1.223) is a regular perturbation in $\hbar^{2/3}$ for $t$ in any finite interval, we can obtain higher order corrections in $\hbar$ as usual.

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## 1.6 Borderline region: $x \gg \hbar^{2/3}$

Assume a turning point at $x = 0$, *i.e.* , $E = V(0)$ and that $E - V(x) > 0$ for $x > 0$. Then, for $x > x_0 > 0$, independent of $\hbar$, Theorem 1.210 applies. We now write a mapping for an interval $(a, x_0)$ where $a$ is allowed to depend on $\hbar$:

$$\delta(x) = -\frac{e^{-J(x)}}{\mu(x)}\int_a^x e^{J(s)}\mu(s)g(s)ds - \frac{e^{-J(x)}}{\mu(x)}\int_a^x e^{J(s)}\mu(s)\delta(s)^2 ds := \delta_0 + \mathcal{N}\delta \tag{1.224}$$

The reasoning is similar to that in §1.5c.2. We choose $a$ as small as possible, while still allowing the right side of (1.224) to be contractive. For this to be the case, we need $|g| \lesssim |x|^{-2}$ and we choose $a$ so that $\delta^2 \ll g$; when this is possible, as shown at the end of the argument, the results of Theorem 1.210 extend to the interval $(a, x_0)$. To determine what this condition entails, we use dominant balance in (1.202): $\delta \ll \hbar |g x^{-1/2}| \ll \hbar |x|^{-5/2}$, and thus $\delta^2 \ll g$ implies $\hbar^2 |x|^{-5} \ll \hbar |x|^{-2}$, that is $|x| \gg \hbar^{2/3}$. For contractivity we need, as in §1.5c.2, $|\delta_1 + \delta_1| \ll 1$ which for $\delta_1, \delta_2 = O\left(\hbar x^{-5/2}\right)$ holds if $x \gg \hbar^{2/5}$. This condition is stringent than $|x| \gg \hbar^{2/3}$. We then choose $a = \nu \hbar^{2/3}$ with $\nu$ sufficiently large, and with this, the map is contractive on $(a, x_0)$. We leave the details as an exercise.

### 1.6a Inner region: Rigorous analysis

$$-\psi'' - t\psi = -\hbar^{2/3} t^2 \varphi_1(\hbar^{2/3} t)\psi := f(t) \tag{1.225}$$

which can be transformed into an integral equation in the usual way,

$$\psi(t) = \pi \mathrm{Ai}(-t) \int^t f(s)\mathrm{Bi}(-s)\psi(s)ds - \pi \mathrm{Bi}(-t) \int^t f(s)\mathrm{Ai}(-s)\psi(s)ds + C_1 \mathrm{Ai}(-t) + C_2 \mathrm{Bi}(-t) \tag{1.226}$$

where Ai, Bi are the Airy functions, with the integral representations:

$$\mathrm{Ai}(z) = \frac{1}{2\pi i} \int_{\infty e^{-\pi i/3}}^{\infty e^{\pi i/3}} e^{\frac{1}{3}t^3 - zt} dt \tag{1.227}$$

$$\mathrm{Bi}(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty e^{\pi i/3}} e^{\frac{1}{3}t^3 - zt} dt + \frac{1}{2\pi} \int_{-\infty}^{\infty e^{-\pi i/3}} e^{\frac{1}{3}t^3 - zt} dt \tag{1.228}$$

The integral representations allow us to derive the *global* behavior at $\infty$, that is, the asymptotic expansion in any direction towards infinity, with explicit constants. With $\zeta = \frac{2}{3}|t|^{\frac{3}{2}}$ we have

$$\mathrm{Ai}(-t) \sim \frac{1}{2\sqrt{\pi}} |t|^{-1/4} e^{-\zeta}; \quad \mathrm{Bi}(-t) \sim \frac{1}{\sqrt{\pi}} |t|^{-1/4} e^{\zeta}; \quad t \to -\infty \tag{1.229}$$

[1] and

$$\mathrm{Ai}(-t) \sim \frac{1}{\pi^{1/2} t^{1/4}} \sin(\zeta + \tfrac{\pi}{4}),\ \mathrm{Bi}(-t) \sim \frac{1}{\pi^{1/2} t^{1/4}} \cos(\zeta + \tfrac{\pi}{4})\ (t \to \infty) \tag{1.230}$$

as $t \to -\infty$. We have to choose the limits of integration in (1.226) in order for the right side of (1.226) to be a contractive mapping. The general prescription is that the maximum point of the integration contour should be at the variable point of integration, if the integrand behaves exponentially. We note that we cannot quite choose infinity as an upper limit since the Airy-type behavior was derived in the inner region $|x| \ll \hbar^{2/3}$ and in general is not expected to be the same outside. We will choose as large a $t$-interval $(-M_1, M_2)$, possibly depending on $\hbar$ for which the leading order behavior $\psi \sim C_1 Ai(-t) + C_2 Bi(-t)$ can be shown. We rewrite (1.225) in the integral form

$$\psi(t) = \pi \mathrm{Ai}(-t) \int_0^t f(s)\mathrm{Bi}(-s)\psi(s)ds - \pi \mathrm{Bi}(-t) \int_{-M_1}^t f(s)\mathrm{Ai}(-s)\psi(s)ds \\ + C_1\mathrm{Ai}(-t) + C_2\mathrm{Bi}(-t) = J\psi + \psi_0 \quad (1.231)$$

Next, to control the norm of $J$, for large $M_1$ the estimate

$$|t|^{-1/4} e^{-\frac{2}{3}|t|^{3/2}} \hbar^{2/3} \int_0^{|t|} s^2 s^{-1/4} e^{\frac{2}{3}s^{3/2}} ds \lesssim \hbar^{2/3} M_1, \ (t \to -\infty) \quad (1.232)$$

follows from Watson's Lemma after the change of variable $p = 1 - s^{3/2}/|t|^{3/2}$, and similarly

$$|t|^{-1/4} e^{\frac{2}{3}|t|^{3/2}} \hbar^{3/2} \int_{|t|}^{M_1} s^2 s^{-1/4} e^{-\frac{2}{3}s^{3/2}} ds \lesssim \hbar^{2/3}|t| \lesssim \hbar^{2/3} M_1 \quad (1.233)$$

The right sides of (1.234) and (1.233) are small if $M_1 \ll \hbar^{-2/3}$. For $t \to +\infty$, estimating crudely $|\sin|, |\cos|$ by one, we get

$$t^{-1/4} \hbar^{2/3} \int_0^t |s^2 s^{-1/4}| ds \lesssim \hbar^{2/3} t^{5/2} \lesssim \hbar^{2/3} M_2^{5/2}, \quad (1.234)$$

which is small for $M_2 \ll \hbar^{-4/15}$. We now work in the sup norm on $[-M_1, M_2]$ and obtain, in the usual way, the following result

**Proposition 1.235** *If $\hbar$ is small enough, then $J$ defined in Eq. (1.231) is contractive in $L^\infty(-M_1, M_2)$ when $\hbar^{2/3}M_1$ and $\hbar^{4/15}M_2$ are small enough.*

We leave the details as an exercise. We see that the region of contractivity for $t < 0$ simply requires $|x| \ll 1$. On the other hand, the same is true for $t > 0$, with the price of making the argument quite a bit more involved.

**Note 1.236** The contractivity of the map for $x < 0$ only requires $|x| \ll 1$. However, the norm used, $L^\infty$ does not allow for controlling *the asymptotic behavior of solutions as $t$ becomes large.* In particular, we would like to understand for what range of (large, negative) $t$ does the solution of (1.225) have the behavior described by Airy function asymptotics, (1.229). The behavior (1.229) does not follow from our arguments, and in fact it is not even correct if $|t| \gg \hbar^{-4/15}$ as we will see in §1.6b.

### 1.6b Matching region

Let's analyze the behavior of solutions in the region $1 \ll |t| \ll \hbar^{-4/15}$. We will only analyze $t < 0$, as for $t > 0$ the analysis is similar (in fact, slightly simpler).

We first write $t = -u$ to make the analysis clearer. We get

$$-\psi'' + u\psi = -\hbar^{2/3}u^2\varphi_1(-\hbar^{2/3}u)\psi \tag{1.237}$$

We next bring (1.225) to a form that is best suited for looking at large $t$, a process called *normalization*. In the region where solutions have Airy-like asymptotic behavior, roughly $u^{-1/4}e^{\pm\frac{2}{3}u^{3/2}}$, we change variables so that the leading behavior is of the form $e^s$. A way to do this is simply by rescaling the dependent and independent variables, $\psi(u) = u^{-1/4}g(\frac{2}{3}u^{3/2})$.

With $s = \frac{2}{3}u^{3/2}$, this leads to the equation

$$g'' - g = -\frac{5}{36}s^{-2}g(s) + \frac{18^{1/3}}{2}\hbar\phi_1(s)s^{2/3}g(s) = F(s)g(s) \tag{1.238}$$

where $\phi_1$ is bounded. Choosing $s_0$ large enough, we write (2.45) in the integral form:

$$g = Ae^s + Be^{-s} + \frac{1}{2}\left(e^s\int_M^s F(v)e^{-v}g(v)dv - e^{-s}\int_{s_0}^s F(v)e^v g(v)dv\right) \tag{1.239}$$

where $M$ will be "large but not too large" so that two solutions with asymptotic behavior $e^s$ and $e^{-s}$ respectively exist for $s \in [s_0, M]$.

We now look for a solution with the behavior $g(s) = e^{-s}$ for large $s$. The adapted norm to measure this type of behavior is $\|g\| = \sup_{s>s_0}|g(s)e^s|$. We should take $A = 0$ in (1.239), since the norm of $e^s$ is very large, of order $e^{2M}$. To check for the contractivity of the map in this norm, we use the fact that, by the definition of the norm, $|g(v)| \le \|g\|e^{-v}$. For the first integral we have

$$e^s\left|e^s\int_M^s F(v)e^{-v}g(v)dv\right| \lesssim \|g\|e^{2s}\int_M^s(\hbar^{2/3}v^{2/3} + v^{-2})e^{-2v}dv$$
$$\lesssim \|g\|(\hbar^{2/3}s^{2/3} + s^{-2}) \lesssim \|g\|(\hbar^{2/3}M^{2/3} + s_0^{-2}) \tag{1.240}$$

where we used Watson's lemma. In order for the norm of this part of the operator to be less than one, we need $s_0$ to be large, which we assumed already, and, once more, $|x| \lesssim 1$.

For the second integral, we see that the exponential in the definition of the norm cancels the exponential which was already in the integrand and we get

$$e^s\left|e^{-s}\int_{s_0}^s F(v)e^v g(v)dv\right| \lesssim \|g\|\int_{s_0}^s(\hbar^{2/3}v^{2/3} + v^{-2})ds \lesssim \|g\|\hbar^{2/3}s^{5/3} + s_0^{-1}$$
$$\lesssim x\hbar^{-1}|x|^{5/2} + s_0^{-1} \tag{1.241}$$

which can be made small if $s_0$ is large, as before, and if $|x| \lesssim \hbar^{2/5}$. The mapping is now contractive in a smaller region– the one that we have obtained before in the oscillatory regime.

**Exercise 1.242** *Complete the details of the analysis, and do a similar analysis for the behavior* $e^s$ *(where now the norm would be* $\|g\| = \sup_s |e^{-s}g(s)|$*). Show the existence of solutions of* (1.225) *with the behavior of the Airy functions Ai and Bi, cf.* (1.229) *in the region* $|x| \lesssim \hbar^{2/5}$.

Note now that, when approaching $x = 0$ from the outer region, we have $E - V(x) = ax + o(x^2)$ where, by scaling we chose $a = 1$; then $i\hbar^{-1}\int\sqrt{E-V(x)} = i\hbar^{-1}\frac{2}{3}(x^{3/2} + O(x^{5/2}))$ and

$$(E - V(x))^{-1/4} e^{i\hbar^{-1}\int\sqrt{E-V(x)}} = x^{-1/4} e^{i\hbar^{-1}\frac{2}{3}(x^{3/2}+O(x^{5/2}))} \tag{1.243}$$

and by switching to the variable $t = \hbar^{-2/3}x$ we get the behavior of a linear combination of Ai and Bi in the oscillatory region. Similarly, changing $i$ to $-i$ in the analysis above we get a linearly independent solution, with the behavior given by a different combination of Ai and Bi. This was to be expected since we are, after all, dealing with the same equation in the inner and outer region, up to these changes of variables, and the behaviors should correspond to each other.

Matching means simply finding the concrete values of the constants so that an outer solution equals an inner one.

We note that there is a difference between the oscillatory outer region and the one with growing/decaying exponential behavior. If only the decaying exponential is present in the outer solution, the matching is straightforward: it corresponds simply to the solution with the behavior Ai in the inner region (Bi should not be present since it *grows* exponentially). But if the outer solution has both growing and decaying components, matching becomes more delicate since the small exponential is masked by the larger one to all orders of an asymptotic expansion in $\hbar$ and finding the correspondence between constants cannot be done by classical asymptotic means. One has to go to the complex domain if the potential is analytic or use exponential asymptotic tools.

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## 1.7 Recovering actual solutions from formal ones

Consider the simple ODE

$$y' = y + 1/x \tag{1.244}$$

(1.244) has an irregular singularity at infinity. If we look for formal asymptotic series solutions $\tilde{y} = \sum_{k\geqslant 0} c_k x^{-k}$ we get $c_0 = 0$, $c_k = (-1)^k(k-1)!$, that is

$$\tilde{y} = \sum_{k=0}^{\infty} \frac{(-1)^k k!}{x^{k+1}} \tag{1.245}$$

This series has empty domain of convergence. Nonetheless, we can do the following. Writing

$$k! = \int_0^\infty e^{-t} t^k dt \Rightarrow \frac{(-1)^k k!}{x^{k+1}} = \int_0^\infty e^{-px} p^k dp \tag{1.246}$$

and inserting (1.246) into (1.247), we get

$$\tilde{y} = \sum_{k=0}^{\infty} \int_0^\infty e^{-px} p^k dp \tag{1.247}$$

This following step requires serious justification, but for now we formally interchange summation and integration,

$$\tilde{y} = \int_0^\infty e^{-px} \sum_{k=0}^{\infty} p^k dp = \int_0^\infty \frac{e^{-px}}{1+p} dp = e^x \mathrm{Ei}_1(x) \tag{1.248}$$

If our sole purpose was to solve (1.244) we could bypass the intermediate steps and any need for justification, and simply check that the function we obtained at the end, $e^x\mathrm{Ei}_1(x)$, satisfies the ODE. For the general solution of (1.244), we just add $Ce^x$, the solution of the associated homogeneous equation, to $e^x\mathrm{Ei}_1(x)$.

Of course however, (1.244) is very simple and we could have solved it by variation of constants or other elementary means. The questions are (1) Can we extend this to a much more general procedure, applicable to generic ODEs near irregular singularities? (the answer is yes) and (2) Can we justify the formal steps that led from (1.247) to the function in (1.248)? (the answer is yes again). We leave these issues for later, now we simply note that there is another way to interpret the operations that led to "summing" the divergent series: (1) we took the formal inverse Laplace transform of the series, that is, term-by-term; (indeed $\mathcal{L}^{-1}x^{-k-1} = p^k/k!$, (2) we summed the geometric series $\sum_{k=0}^{\infty}(-p)^k = (1+p)^{-1}$, and, since the radius of convergence of this geometric series is one, we extended $(1+p)^{-1}$ analytically $(1+p)^{-1}$ on $\mathbb{R}^+$, and (3) we took the Laplace transform of $\mathcal{L}$ the result. Since $\mathcal{L}\mathcal{L}^{-1} = I$ the identity, and at a formal level what we did is just that, $\mathcal{L}\mathcal{L}^{-1}$, we expect that if $\tilde{y}$ satisfied an ODE, so will the $\mathcal{L}\mathcal{L}^{-1}\tilde{y}$. This is the route we will take in justifying this procedure.

We also note that the formal series $\tilde{y}$ is divergent since it is obtained by repeatedly differentiating a function which is not entire: the iterative asymptotic process leading to $\tilde{y}$ is $y^{[n+1]} = -1/x + \partial_x y^{[n]}$. The inverse Laplace transform is a Fourier transform in the imaginary direction, and the Fourier transform is the unitary operator that diagonalizes differentiation. After a form of Fourier transform, repeated differentiation becomes repeated multiplication by the "symbol" of the differential operator, denoted by $p$ here. This can only lead to geometric behavior of the terms of the formal series, something we know much more about: this is dealt with by analytic function theory.

Finally, and this is another important point, in this and many problems, applying the inverse Laplace transform has a regularizing effect. Indeed, the formal solution $\sum_{k=0}^{\infty}(-1)^k k! x^{-k-1}$ becomes, after applying $\mathcal{L}^{-1}$, $\sum_{k=0}^{\infty}(-p)^k$ which is convergent. Whatever problem the new series is a solution of, that new problem is expected to have at mot a regular singularity, given this convergence. Indeed, taking $\mathcal{L}^{-1}$ in (1.244) we get, with $\mathcal{L}^{-1}y = Y$,

$$(p+1)Y = 1 \tag{1.249}$$

an ordinary equation with meromorphic solutions.

The same can be dome in the context of PDEs. Let's take the heat equation,

$$h_t = h_{xx}; \quad \text{with } h(0,x) = \frac{1}{1+x^2} \tag{1.250}$$

Since the equation if parabolic, the Cauchy-Kowalesky does not apply. In fat, looking for power series solutions

$$h = \sum_{k=0}^{\infty} H_k(x) t^k \tag{1.251}$$

we obtain the recurrence

$$H_{k+1}(x) = \frac{H_k''(x)}{k+1}; \quad H_0(x) = \frac{1}{1+x^2} = \mathrm{Re}\left(\frac{1}{1+ix}\right) \tag{1.252}$$

where we wrote the initial condition in a way that facilitates taking high order derivatives. We get for $H_k$,

$$H_{k+1} = \frac{H_k''}{k+1} \Rightarrow H_k = \frac{H_0^{(2k)}(x)}{k!} = (-1)^k \frac{(2k)!}{k!} \mathrm{Re}\left((1+ix)^{-2k-1}\right) \tag{1.253}$$

and (1.253) shows that, with the given initial condition, (1.254) is divergent.

Denoting $t = 1/T$ we write

$$\tilde{h} = T \sum_{k=0}^{\infty} H_k(x) T^{-k-1} = T \sum_{k=0}^{\infty} (-1)^k \frac{(2k)!}{k!} \mathrm{Re}\left((1+ix)^{-2k-1}\right) T^{-k-1} \tag{1.254}$$

and we apply to the sum in (1.254) the procedure we used in (1.248), (1.247), (1.246), with $x = T$. We get

$$
\begin{aligned}
T\sum_{k=0}^{\infty}(-1)^k\frac{(2k)!}{k!}\mathrm{Re}\left((1+ix)^{-2k-1}\right)T^{-k-1} \\
= t^{-1}\int_0^{\infty} e^{-\frac{p}{t}}\sum_{k=0}^{\infty}\frac{(-1)^k(2k)!(1+ix)^{-2k-1}p^k}{k!^2}dp \\
= t^{-1}\int_0^{\infty} e^{-\frac{p}{t}}F(p,x)dp : \\
F(p,x)) = -2\mathrm{Re}\left(\frac{4p}{\xi^3+\xi^2\sqrt{\xi^2-4p}-4p\xi}\right)\ \xi = (1+ix) \quad (1.255)
\end{aligned}
$$

# Chapter 2

## Borel summation: an introduction

### 2.1 The Borel transform $\mathcal{B}$

The Borel transform, is defined on formal power series in the reciprocal of a variable, say $x$, with values in the space of formal power series in a dual variable, that we will often denote by $p$. By definition,

$$\mathcal{B}x^{-s} = \frac{p^{s-1}}{\Gamma(s)} \tag{2.1}$$

in $\mathbb{C}$ (or, if $s$ is not an integer, on the universal covering of $\mathbb{C} \setminus \{0\}$ [1]). The Borel transform is similar to a (formal) inverse Laplace transform, except that the latter vanishes in the left half plane: if $\mathrm{Re}\,(s) > -1$ then

$$\mathcal{L}^{-1}x^{-s} = \begin{cases} p^{s-1}/\Gamma(s) & \text{for } \mathrm{Re}\ p > 0 \\ 0 & \text{otherwise} \end{cases} \tag{2.2}$$

For a power series $\tilde{f} = \sum_{k=0}^{\infty} c_k x^{-k-1}$, the Borel transform is applied, by definition, term-by-term:

$$\mathcal{B}\sum_{k=0}^{\infty} \frac{c_k}{x^{k+1}} = \sum_{k=1}^{\infty} \frac{c_k}{k!}p^k \tag{2.3}$$

Because the $k$-th coefficient of $\mathcal{B}\{\tilde{f}\}$ is smaller by a factor $k!$ than the corresponding coefficient of $\tilde{f}$, $\mathcal{B}\{\tilde{f}\}$ may converge even if $\tilde{f}$ does not. Note also that $\mathcal{L}\mathcal{B}$ is *formally* $\mathcal{L}\mathcal{L}^{-1}$, the identity operator. These two facts account for the central role played by $\mathcal{L}\mathcal{B}$ in summability of factorially divergent series.

If $\mathcal{B}\{\tilde{f}\}$ *converges* to a function$f$ which is Laplace transformable, and we apply $\mathcal{L}$ to $f$, we effectively get an identity-like operator from series to functions. More generally, we can allow for noninteger power series. If for instance $0 < \mathrm{Re}\,(s_k) < \mathrm{Re}\,(s_{k+1})$ for all $k \in \mathbb{N}$, then we define

$$\mathcal{B}\sum_{k=1}^{\infty} \frac{c_k}{x^{s_k}} = \sum_{k=1}^{\infty} \frac{c_k}{\Gamma(s_k)}p^{s_k-1} \tag{2.4}$$

[1] This consists of classes of curves in $\mathbb{C} \setminus \{0\}$, where two curves are equivalent if they can be continuously deformed into each other without crossing 0.

## 2.2 Definition of Borel summation and basic properties

We define Borel summation of integer power series, but the definition extends to more general series, see Note 2.11 below.

Borel summation is relative to a direction; see §2.2a. The same formal series $\tilde{f}$ may yield different functions by Borel summation in different directions.

**Borel summation along** $\mathbb{R}^+$ consists of three operations, assuming (2) and (3) are possible:

1. Borel transform, $\tilde{f} \mapsto \mathcal{B}\{\tilde{f}\}$.
2. Convergent summation of the series $\mathcal{B}\{\tilde{f}\}$ and analytic continuation along $\mathbb{R}^+$ (denote the continuation by $F$ and by $\mathcal{D}$ an open set in $\mathbb{C}$ containing $\mathbb{R}^+ \cup \{0\}$ where $F$ is analytic).
3. Laplace transform, $F \mapsto \int_0^\infty F(p)e^{-px}\mathrm{d}p =: \mathcal{LB}\{\tilde{f}\}$, which requires exponential bounds on $F$, defined in some half-plane $\mathrm{Re}\,(x) > x_0$.

**Definition 2.5** *The* domain *of Borel summation along* $\mathbb{R}^+$ *is the subspace* $S_{\mathcal{B}}$ *of series for which the conditions for 2. and 3. above are met. For step 3 we can require that for some constants* $C_F$, $\nu_F$ *we have* $|F(p)| \le C_F e^{\nu_F p}$. *Or we can require that* $\|F\|_\nu < \infty$ *where, for* $\nu > 0$ *we define*

$$\|F\|_\nu := \int_0^\infty e^{-\nu p}|F(p)|dp \tag{2.6}$$

*If* $\tilde{f}$ *is Borel summable, then the Borel sum of* $\tilde{f}$, *denoted by* $\mathcal{LB}\tilde{f}$, *is defined to be* $\mathcal{L}F$.

We note that $L^1_\nu := \{f : \|f\|_\nu < \infty\}$ forms a Banach space, and it is easy to check that

$$L^1_\nu \subset L^1_{\nu'} \text{ if } \nu' > \nu \tag{2.7}$$

and that

$$\|F\|_\nu \to 0 \text{ as } \nu \to \infty \tag{2.8}$$

the latter statement following from dominated convergence.

**Definition 2.9** *If* $\tilde{f} = \sum_{k=1}^n c_k x^{s_k}$ *(a finite sum) then we define* $\mathcal{LB}\tilde{f} = \sum_{k=1}^n c_k x^{s_k}$, *that is,* $\mathcal{LB}$ *applied to finite sums is just the identity.*

**Remark 2.10** *Borel summation is defined on series starting with (finitely many) positive powers of* $x$ *by relating to Definition 2.9:*

$$\mathcal{LB}\sum_{k=1}^{n} c_k x^{-s_k-r} = \sum_{k=1}^{n} c_k x^{-s_k-r} + \mathcal{LB}\sum_{k=n+1}^{\infty} c_k x^{-s_k-r}$$

*where* $0 < \mathrm{Re}\,(s_k - r) < \mathrm{Re}\,(s_{k+1} - r)$ *for all* $k > n$. *In case some of the* $s_j$ *are noninteger, the definition of* $\mathcal{LB}$ *is essentially the same, replacing analyticity at zero with ramified analyticity.*

**Note 2.11** Series of the form

$$\tilde{f} = \sum_{k_i \geqslant 0} c_{k_1 k_2 \dots k_m} x^{-\beta_1 k_1 - \dots - \beta_m k_m - 1} = \sum_{\mathbf{k} \geqslant 0} c_{\mathbf{k}} x^{-\mathbf{k}\boldsymbol{\beta} - \mathbf{1}}$$

with $\mathrm{Re}\,(\beta_j) > 0$ frequently arise as formal solutions of differential systems. Here $\boldsymbol{\beta} = (\beta_1, ..., \beta_m)$ and $c_{\mathbf{k}} = c_{k_1 k_2 \dots k_m}$. We define

$$\mathcal{B} \sum_{\mathbf{k} \geqslant 0} c_{\mathbf{k}} x^{-\mathbf{k}\cdot\boldsymbol{\beta} - \mathbf{1}} = \sum_{\mathbf{k} \geqslant 0} c_{\mathbf{k}} p^{\mathbf{k}\cdot\boldsymbol{\beta}} / \Gamma(\mathbf{k}\cdot\boldsymbol{\beta} + 1) \tag{2.12}$$

*

## 2.2a Borel summation along other directions

Borel summation along other directions in $\mathbb{C}$ is most easily defined by changes of variables. We say that a power series in inverse powers of $x$, $\tilde{f} = \tilde{f}(x)$, is Borel summable as $x \to \infty e^{i\theta}$ or Borel summable along the ray $e^{i\theta}$ if $\tilde{f}(ye^{i\theta})$ is Borel summable for $y$ along $\mathbb{R}^+$. We write $\mathcal{LB}_\theta[\tilde{f}(x)] = \mathcal{LB}[\tilde{f}(ye^{i\theta})]$.

In general, $\mathcal{LB}_\theta$ depends nontrivially on $\theta$. We can take as an illustration the formal series

$$\tilde{f}_1 = \sum_{k=0}^{\infty} \frac{k!}{x^{k+1}} \tag{2.13}$$

that we have examined before. A straightforward calculation shows that the Borel sum of $\tilde{f}_1$ in the direction $\theta$ is

$$\mathcal{LB}_\theta \tilde{f}_1 = \mathcal{LB} \sum_{k=0}^{\infty} \frac{k!}{(ye^{i\theta})^{k+1}} = \int_0^\infty \frac{e^{-yp}dp}{e^{i\theta} - p} = \int_0^{\infty e^{-i\theta}} \frac{e^{-xp}dp}{1 - p} \tag{2.14}$$

This is well defined if $\theta \neq 0 \pmod{2\pi}$.

Taking a $\theta \in (0, 2\pi)$ we note that by the residue theorem (and Jordan's lemma, allowing us to deform the contour of the improper integral in (2.14)) we have

$$[\mathcal{LB}_\theta - \mathcal{LB}_{-\theta}]\, \tilde{f}_1 = 2\pi i e^{-x} \tag{2.15}$$

that is, the Borel sums of $\tilde{f}_1$ on the two sides of $\mathbb{R}^+$ differ by an exponentially small term. This is a manifestation of the Stokes phenomenon. Divergent expansions are generally associated with different behaviors as the singular point (here the point at infinity) is approached from different directions. For instance, we have the following simple result.

**Proposition 2.16** *Assume $f$ is analytic in $\mathcal{D} = \{x : |x| > R$ and $f \sim \tilde{f} = \sum_{k=0}^\infty c_k x^{-k}$ as $x \to \infty e^{i\theta}$ for some $\theta$, where $\tilde{f}$ has empty domain of convergence. Then $x^{-m} f$ is unbounded in $\mathcal{D}$ for any $m$.*

**PROOF** To get a contradiction, assume $x^{-m}f$ is bounded in $\mathcal{D}$. Taking $z = 1/x$ we see that $g(z) = z^{m+1}f(1/z)$ is single-valued and analytic for $|z| < 1/R$ except perhaps at zero. Since $g(z) \to 0$ as $z \to 0$, $z = 0$ is a removable singularity and thus $g(z) = \sum_{k=1}^{\infty} a_k z^k$, where the series converges for $|z| < 1/R$. This implies $f(x) \sim \sum_{k=-m}^{\infty} a_k x^{-k}$ for $|x| > R$. Since the asymptotic expansion of a function is unique, we have $a_k = c_k$ for all $k$ and in particular $\sum_{k=0}^{\infty} c_k x^{-k}$ would converge for $|x| > x_0$ contradicting the assumption on $\tilde{f}$. ∎

**Exercise 2.17** The function

$$g(x) = \int_0^x e^{t^2} dt \tag{2.18}$$

is entire. Use the saddle point method or reduction to Watson's lemma to study the behavior of $g$ as $x \to \infty e^{i\theta}$ for $\theta \in [0, 2\pi)$ and study the change in the asymptotic expansion as $\theta$ changes.

**Note 2.19** A *function* $f$ is sometimes called Borel summable (by slight abuse of language), if it is analytic and suitably decaying in a half-plane (say $\mathbb{H}$), and its inverse Laplace transform $F$ is analytic in a neighborhood of $\mathbb{R}^+ \cup \{0\}$. Such functions are clearly into a one-to-one correspondence with their asymptotic series. Indeed, if the asymptotic series coincide, then their Borel transforms — which are convergent series—coincide, and their analytic continuation is the same in a neighborhood of $\mathbb{R}^+ \cup \{0\}$. The two functions are equal.

## 2.2b Limitations of classical Borel summation; BE summation

The need of extending Borel summation arises because the domain of definition of Borel summation is not wide enough. We see that $\tilde{f}_1$ in (2.13) is not summable along $\mathbb{R}^+$. Yet there is nothing singular in the behavior of $f_1 := \mathcal{L}\mathcal{B}_\theta \tilde{f}_1$, calculated for some $\theta \neq 0$ and then analytically continued to $\theta = 0$.

While in a particular context one can deform the contour of integration of $\mathcal{L}$ in the complex plane to avoid going through singularities, there is of course no *single* ray of integration that would allow for summation of general series $\tilde{f}$ when $\mathcal{B}\tilde{f}$ has singularities in $\mathbb{C}$ (this is the more interesting case).

If the ray of integration has to depend on $\tilde{f}$, then the resulting operator would not have good algebraic properties, in particular it would fail to commute with complex conjugation.

To overcome the limitations of classical Borel summation, Écalle has found general averages of analytic continuations in Borel plane, which do not depend on the origin of the formal series, such that, replacing the Laplace transform along a *singular rays* with averages of Laplace transforms of these continuations, the properties of Borel summation are preserved, and its domain is vastly widened. The fact that such averages exist is nontrivial, though many averages are quite simple and explicit.

Multiplicativity of the summation operator is the main difficulty that is overcome by these special averages. Perhaps surprisingly, convolution (the image of multiplication through $\mathcal{L}^{-1}$), *does not* commute in general with analytic continuation along curves passing between singularities! (See §**??**.)

A simplified form of medianization, the balanced average, which works for generic ODEs (but not in the generality of Écalle's averages) is discussed in §**??**.

Another difficulty is the possibility of super-exponential growth of the Borel transform.

**Example 2.20** If we substitute $x = t^2$ in $\tilde{f}_1$ and take the Borel transform from $t$ to $p$ we get

$$\mathcal{B}\sum_{k=0}^{\infty}\frac{k!}{t^{2k+2}} = \sum_{k=0}^{\infty}\frac{k!p^{2k+1}}{\Gamma(2k+2)} = \sqrt{\pi}e^{p^2/4}\mathrm{erf}(p/2) \tag{2.21}$$

where

$$\mathrm{erf}(p) = 2\pi^{-1/2}\int_0^p e^{-s^2}ds \sim 1 - p^{-1}\pi^{-1/2}e^{-p^2}(1+o(1)) \text{ as } p \to +\infty \tag{2.22}$$

(a way to calculate the last sum in (2.22) is to note that it satisfies the ODE $y' = 1 + \frac{1}{2}py$). Whereas in (2.14) we can deform the contour of integration to give a meaning to the integral when $\theta = 0$, there is nothing obvious we can do to Laplace transform $e^{p^2/4}\mathrm{erf}(p/2)$ which grows like $\sqrt{\pi}e^{p^2/4}$ when $p \to \infty$.

Super-exponential growth, as it turns out, can be generically dealt with by changes of variable, here essentially by undoing the $x \mapsto t$ transformation. There are exceptional cases however when mixtures of different factorial rates of divergence in the same series, preclude this simple fix.

Acceleration and multisummation (the latter considered independently, from a cohomological point of view by Ramis; see also §**??**), universal processes too, were introduced by Écalle to deal with this problem in many contexts. Essentially BE summation is Borel summation, supplemented by averaging and acceleration when needed.

More generally we can allow for expansions containing exponentials by defining $\mathcal{LB}\exp(ax) = \exp(ax)$.

These generalizations of Borel summation which allow for the analysis of quite complicated functions are known as Borel-Écalle summation, or BE summation.

## 2.3 Borel summation as an isomorphism

We start with a discussion about the relation between convergent power series and their sums. Rarely does one make a distinction between a convergent series as a formal object and its sum as an actual function. Some mention of this distinction is made when we need to be careful about the radius of convergence, such as in writing $(1-p)^{-1} = \sum_{k=0}^{\infty} p^k$ if $|p|<1$. The right side of this equality is already interpreted as the sum of the underling formal series. We can understand this if we look at the properties of $\mathsf{S}$, the operator that associates to a formal convergent series its sum. If we restrict series to one sided Taylor series (such as expansions of meromorphic functions at zero), $\mathsf{S}$ is a differential algebra isomorphism, that is, it commutes with multiplication, division, differentiation, etc. For instance $\mathsf{S}(\tilde{f}'\tilde{g}') = \mathsf{S}(\tilde{f})'\mathsf{S}(\tilde{g})'$. If a linear or nonlinear ODE $L(y, y', ..., y^{(n)}, z) = 0$ with analytic coefficients is solved by a formal series whose coefficients grow at most geometrically, then, by the isomorphism, $\mathsf{S}L(\tilde{f}, \tilde{f}', ..., \tilde{f}^(n), z) = 0$ iff $L(\mathsf{S}\tilde{f}, \mathsf{S}\tilde{f}', ..., \mathsf{S}\tilde{f}^{(n)}, z) = 0$, that is, iff $\mathsf{S}\tilde{f}$ is an actual solution. This is true in several variables as well, in solving PDEs for instance. There is essentially no point in speaking about the operator $\mathsf{S}$ since we cannot make distinctions based on algebraic and differential operations between formal convergent series and their sums.

Borel summable series are in a similar relation with their associated functions obtained by Borel summation.

Let $S_{\mathcal{B}}$ be the set of Borel summable series along some direction, say along $\mathbb{R}^+$.

**Proposition 2.23** *(i) $S_{\mathcal{B}}$ is a differential field and so is $\mathcal{LB}\{S_{\mathcal{B}}\}$ .*[2]

*$\mathcal{LB} : S_{\mathcal{B}} \mapsto \mathcal{LB}S_{\mathcal{B}}$ commutes with differential field operations, that is, $\mathcal{LB}$ is a differential algebra isomorphism.*

*(ii) Let $S_c$ be the differential algebra of convergent power series. Then $S_c \subset S_{\mathcal{B}}$ and $\mathcal{LB}_{|S_c} = \mathsf{S}$, the usual summation.*

*(iii) In addition, for $\tilde{f} \in S_B$, $\mathcal{LB}\{\tilde{f}\} \sim \tilde{f}$ as $|x| \to \infty$, $\mathrm{Re}\,(x) > 0$.*

(ii) above implies that $S_{\mathcal{B}}$ and $\mathcal{LB}$ are proper extensions of $S_c$ and $\mathsf{S}$: Borel summable series have the same algebraic properties as convergent series and their sums have properties similar to those of analytic functions.

Some of the properties such as linearity of $\mathcal{LB}$ and commutation with differentiation are straightforward and we leave them as an exercise. For multiplication and division, we need to look more closely at convolutions.

**Definition 2.24 (Inverse Laplace space convolution)** *If $F, G \in L^1_{loc}$, then*

$$(F * G)(p) := \int_0^p F(s)G(p-s)ds \tag{2.25}$$

[2] with respect to formal addition, multiplication, and differentiation of power series.

If $F$ and $G$ are exponentially bounded, then so is $FG$. Indeed, Clearly, if $|F_1| \leqslant C_1 e^{\nu_1 p}$ and $|F_2| \leqslant C_2 e^{\nu_2 p}$, then

$$|F_1 F_2| \leqslant C_1 C_2 p e^{(\nu_1+\nu_2)p} \leqslant C_1 C_2 e^{(\nu_1+\nu_2+1)p}$$

**Lemma 2.26** *Convolution is continuous in* $\|\cdot\|_\nu$, *namely*

$$\|F * G\|_\nu \leq \|F\|_\nu \|G\|_\nu$$

*Furthermore,* $L^1_\nu \subset L^1_{\nu'}$ *if* $\nu' > \nu$, *and* $\mathcal{L}(F*G) = \mathcal{L}(F)\mathcal{L}(G)$. *Furthermore,*

$$\|F\|_\nu \to 0 \quad \text{as } \nu \to \infty \tag{2.27}$$

**PROOF** Note that

$$\int_0^\infty e^{-\nu p} \left| \int_0^p F(s)G(p-s)ds \right| dp \leq \int_0^\infty e^{-\nu s} e^{-\nu(p-s)} \int_0^p |F(s)||G(p-s)|dsdp$$
$$= \int_0^\infty \int_0^\infty e^{-\nu s}|F(s)|e^{-\nu\tau}|G(\tau)|d\tau ds = \|F\|_\nu \|G\|_\nu \tag{2.28}$$

by Fubini. The fact that $\mathcal{L}(F*G) = \mathcal{L}(F)\mathcal{L}(G)$ follows by a calculation similar to (2.28). Eq. (2.27) follows using the dominated convergence theorem. ∎

**Lemma 2.29** *The space of functions which are in* $L^1([0,a)$ *for any* $a > 0$ *and real-analytic on* $[0,\infty)$ *is closed under convolution.*

**PROOF** This simply follows from the rewriting

$$\int_0^p f(s)g(p-s)ds = p\int_0^1 f(pt)g(p(1-t))dt \tag{2.30}$$

∎

The previous lemma implies that $\mathcal{LB}(\tilde{f}\tilde{g}) = \mathcal{LB}(\tilde{f})\mathcal{LB}(\tilde{g})$.
We next analyze division of series in $\mathcal{S}_\mathcal{B}$.

**PROOF of Proposition 2.23** The only nontrivial part is to show that if $\tilde{f}$ is a Borel summable series, then so is $1/\tilde{f}$. We have $f = Cx^m(1+s)$ for some $m$ where $s$ is a small series, that is a series only involving negative powers of $x$. We naturally define $1/\tilde{f} = x^m/(1+s)$ and

$$\frac{1}{1+s} = 1 - s + s^2 - s^3 \cdots . \tag{2.31}$$

First, note that this infinite series is well defined formally. Indeed, assuming for simplicity that $s = \sum_{k=0}^\infty c_k x^{-k-1}$, the coefficient of $x^{-k}$ for any fixed $k$

is collected only from the terms $s^m$ with $j \geqslant k$. This is because the highest power of $x$ in $s^{k+j}$ is $-k-j$. Straightforward algebra shows that

$$(1+s)(1-s+s^2\cdots)=1$$

We want to show that

$$1-s+s^2-s^3+\cdots \tag{2.32}$$

is Borel summable, or that

$$s_1=-s+s^2-s^3+\cdots \tag{2.33}$$

is Borel summable.

Now,

$$s_1=-s+s^2-s^3+\cdots=\sum_{k>1}C_k x^{-k} \tag{2.34}$$

where $C_k$ is the coefficients of $x^{-k}$ in the finite sum $-s+s^2-s^3-...s^k$. Let $\mathcal{B}s=H$. We examine $\mathcal{B}s_1$, or, in fact the function series

$$S=-H+H*H-H^{*3}+\cdots \tag{2.35}$$

where $H^{*n}$ is the self-convolution of $H$ $n$ times. Each term of the series is analytic, by Lemma 2.29. Let $K$ be an arbitrary compact subset of $\mathcal{D}$. If $\max_{p\in K}|H(p)|=m$, then it is easy to see that

$$|H^{*n}|\le m^n 1^{*n}=m^n\frac{p^{n-1}}{(n-1)!} \tag{2.36}$$

Thus the function series in (2.35) is absolutely and uniformly convergent in $K$ and the limit is analytic. Let now $\nu$ be large enough so that $\|H\|_\nu<1$ (see (2.8)). Then the series in (2.35) is norm convergent, thus an element of $L^1_\nu$.

**Exercise 2.37** *Check that* $(1+\mathcal{L}H)(1+\mathcal{L}S)=1$.

It remains to show that the asymptotic expansion of $\mathcal{L}(F*G)$ is indeed the product of the asymptotic series of $\mathcal{L}F$ and $\mathcal{L}G$. This is, up to a change of variable, a consequence of Lemma 1.31.

(ii) Since $\tilde{f}_1=\tilde{f}=\sum_{k=0}^\infty c_k x^{-k-1}$ is convergent, then $|c_k|\le CR^k$ for some $C,R$ and $F(p)=\sum_{k=0}^\infty c_k p^k/k!$ is entire, $|F(p)|\le\sum_{k=0}^\infty CR^k p^k/k!=Ce^{Rp}$ and thus $F$ is Laplace transformable for $|x|>R$. By dominated convergence we have for $|x|>R$,

$$\mathcal{L}\{\sum_{k=0}^\infty c_k p^k/k!\}=\lim_{N\to\infty}\mathcal{L}\{\sum_{k=0}^N c_k p^k/k!\}=\sum_{k=0}^\infty c_k x^{-k-1}=f(x)$$

(iii) This part follows simply from Watson's lemma. ∎

### 2.3.1 Convergent series composed with Borel summable series

**Proposition 2.38** *Assume $A$ is an analytic function in the disk of radius $\rho$ centered at the origin, $a_k = A^{(k)}(0)/k!$, and $\tilde{s} = \sum s_k x^{-k}$ is a small series which is Borel summable along $\mathbb{R}^+$. Then the formal power series obtained by reexpanding*

$$\sum a_k \tilde{s}^k$$

*in powers of $x$ is Borel summable along $\mathbb{R}^+$.*

**PROOF** Let $S = \mathcal{B}s$ and choose $\nu$ to be large enough so that $\|S\|_\nu < \rho$ in $L^1_\nu$. Then

$$\|F\|_\nu := \|A(*S)\|_\nu := \|\sum_{k=0}^{\infty} a_k S^{*k}\|_\nu \le \sum_{k=0}^{\infty} a_k \|S\|_\nu^k \le \sum_{k=0}^{\infty} a_k \rho^k < \infty \quad (2.39)$$

thus $A(*S) \in L^1_\nu$. Similarly, $A(*S)$ is in $L^1_\nu([0,a))$ and in $\mathcal{A}_{K,\nu}([0,a))$ for any $a$. ∎

**Note 2.40** (i) To ensure Borel summability of a series, the rule of thumb is that we first change the independent variable so that the new series has factorial divergence, with power of factorial one. This would ensure that the Borel transform will be convergent, but not entire of an exponential order that would prevent us from taking a final Laplace transform, see Example 2.20.

(ii) *If* Borel summability succeeds, then, by deforming the contour of integration past a singularity of the Borel transform, we collect exponentially small terms, as in (2.15). Those exponentially small terms, since they come from residues or from contours surrounding branch points are of the form $Ce^{-p_0 x}$ where $p_0$ is the position of the singularity.

(iii) When we are dealing with linear equations (e.g., linear ODEs), we note that the sum of two solutions is a solution. Assume that we get one solution by Borel summation in the direction $\theta$, $y = \mathcal{LB}_\theta Y$ and that $\mathcal{LB}_\theta Y \ne \mathcal{LB}_{-\theta} Y$. Then, by linearity, the function

$$y_2 = \mathcal{LB}_\theta Y - \mathcal{LB}_{-\theta} Y \quad (2.41)$$

is also a solution of the equation. Thus, by the discussion in (ii), $y_2 \sim e^{-p_0 x(1+o(1))}$ for some $p_0$. A similar estimate holds in the nonlinear case as we will see though in this case the sum of solutions is not a solution.

(iv) In view of (iii), if we are solving equations using Borel summation tools, we first need to normalize the equation, by changes of dependent and independent variables, so that the behavior of solutions as $x \to \infty$ is of the form $e^{-p_0 x(1+o(1))}$. For instance, for the Airy equation discussed next, solutions behave like $e^{\pm 2/3x^{3/2}}$; for Borel summation, we should switch to the variable $t = x^{3/2}$.

(v) Still when solving equations, instead of finding a formal series first and then summing it is more convenient to take the Borel transform (formal inverse Laplace) of (2.46). Now: if we obtain a solution of the transformed equation in the form of a function which is ramified analytic near the origin and analytic and bounded along $\mathbb{R}^+$ (or along some other direction $\theta$), we will have proved Borel summability (in the direction $-\theta$).

**Note 2.42 (On regularization)** Since the Borel transform maps factorially divergent expansions into convergent ones, it is natural that the equations satisfied by the transformed series have milder singularities, and are, in general, simpler than the original ones. For example, our prototypical equation $y' + y = 1/x$ is transformed by $\mathcal{B}$ to the trivial equation $(1-p)Y = 1$ while the Bessel equation becomes the first order ODE (2.54). Sometimes we end up with explicit representations of the solutions and most often with integral representations which, even if not explicit, allow for a detailed study of the asymptotic behavior of solutions.

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## 2.4 Some examples

In the following we will derive convenient representations for a number of special functions. As discussed in §1.2, special functions are distinguished by the existence of integral representations allowing for detailed global description, in particular for calculating explicitly connection constants, giving the precise relation between the behavior of a given function in various directions towards infinity. From the point of view of a Borel summation approach, the fact that irregular singularities become regular ones and the fact that these special functions solve typically second order differential equations implies that the Borel transform satisfies first or second order equations with simpler singularities, allowing for explicit solutions.

### 2.4.2 The Airy equation

Let us look again at the Airy equation,

$$y'' - xy = 0 \tag{2.43}$$

Here, the behavior of solutions at infinity, that we have already obtained by WKB is

$$y \sim Cx^{-\frac{1}{4}} e^{-\frac{2}{3}x^{3/2}} \tag{2.44}$$

We use the transformation $y(x) = g(\frac{2}{3}x^{\frac{3}{2}})$ to achieve the normalization described in Note 2.40 (iv), and get

$$g'' + \frac{1}{3t}g' - g = 0 \tag{2.45}$$

In view of (2.44) we have

$$t^{-\frac{1}{6}}e^{\pm t}(1+s)$$

where $s$ is a small series.

To eliminate the exponential behavior of one solution, say of the decaying one, we substitute $g = he^{-t}$, and get

$$h'' - \left(2 - \frac{1}{3t}\right)h' - \frac{1}{3t}h = 0 \tag{2.46}$$

To obtain a second solution, we can resort to the substitution $g = he^{t}$, or we can rely on the Stokes phenomenon to obtain it from the one above, as we will do in §2.5. Now $h$ behaves like a small power series, which we would Borel sum. We apply the strategy outlined in Note 2.40 (v), and apply $\mathcal{B}$ to (2.46). We get

$$p(2+p)H' + \frac{5}{3}(1+p)H = 0 \tag{2.47}$$

with the solution

$$H = Cp^{-\frac{5}{6}}(2+p)^{-\frac{5}{6}} \tag{2.48}$$

and thus

$$h(t) = \mathcal{L}\left(Cp^{-\frac{5}{6}}(2+p)^{-\frac{5}{6}}\right) \tag{2.49}$$

and, comparing the asymptotic expansion obtained from (2.49) with that of Airy functions (1.230) to identify the solution we $h$ we get

$$\mathrm{Ai}(x) = \frac{3^{-\frac{1}{6}}}{\pi^{\frac{1}{2}}\Gamma(\frac{1}{6})}\int_0^\infty e^{-\frac{2}{3}x^{\frac{3}{2}}p}p^{-\frac{5}{6}}(2+p)^{-\frac{5}{6}}dp \tag{2.50}$$

from which it is easy to derive the global behavior of Airy functions. We note that $\mathrm{Ai}(x)$ is entire, yet the fact that the integrand in (2.71) has a singularity at $p = -2$ entails a change of asymptotic behavior similar to that discussed after (2.15) (Stokes phenomenon) when the contour of integration crosses $\mathbb{R}^-$. We return to this in §2.5.

### 2.4.3 Bessel functions

If we consider, instead of (2.45), the more general equation

$$g'' + \frac{2\nu+1}{t}g' - g = 0 \tag{2.51}$$

the general solution is

$$g = t^{-\nu}\left(C_1 I_\nu(t) + C_2 K_\nu(t)\right) \tag{2.52}$$

The substitution $g(t) = e^{-t}h(t)$ in (2.51) leads to

$$h'' - \left(2 - \frac{2\nu+1}{t}\right)h' - \frac{2\nu+1}{t}h = 0 \tag{2.53}$$

with inverse Laplace transform

$$p(p+2)H' + (1-2\nu+p(1-2\nu))H = 0 \Rightarrow \; H = Cp^{\nu-\frac{1}{2}}(2+p)^{\nu-\frac{1}{2}} \tag{2.54}$$

Using Watson's lemma, we see that

$$h = \mathcal{L}H \sim C2^{\nu-\frac{1}{2}}\Gamma(\nu+\tfrac{1}{2})t^{-\nu-\frac{1}{2}}; \quad (t\to\infty) \tag{2.55}$$

while

$$K_\nu(t) \sim \sqrt{\frac{\pi}{2t}}e^{-t} \quad (t\to\infty) \tag{2.56}$$

Comparing (2.55) to (2.56), we see that

$$K_\nu(t) = e^{-t}\frac{t^\nu\sqrt{\pi}}{2^\nu\Gamma(\nu+\frac{1}{2})}\int_0^\infty p^{\nu-\frac{1}{2}}(2+p)^{\nu-\frac{1}{2}}e^{-tp}dp;\ \mathrm{Re}\,(\nu) > -1/2 \tag{2.57}$$

For $\mathrm{Re}\,\nu < -1/2$ we can write $K_\nu$ as a loop integral around zero on a path $C$ from $\infty e^{-i\pi}$ to $\infty e^{i\pi}$,

$$K_\nu(t) = \frac{e^{-t}t^\nu\sqrt{\pi}}{2^{\nu+1}i\cos(\pi\nu)\Gamma(\nu+1/2)}\int_C s^{\nu-1/2}(2-s)^{\nu-1/2}e^{st}ds \tag{2.58}$$

The representations (2.57) or (2.58) are very convenient when studying the global properties of $K_\nu$ in $\mathbb{C}$.

## 2.4a Whittaker functions

The equation

$$y'' - \left(\frac{1}{4} - \frac{\kappa}{z} - \frac{\frac{1}{4}-\mu^2}{z^2}\right)y = 0 \tag{2.59}$$

has as solutions the Whittaker functions $M_{\kappa,\mu}(z)$ and $W_{\kappa,\mu}(z)$, [?], eq.13.1.31; for large positive $z$, $M_{\kappa,\mu}(z) \sim e^{z/2(1+o(1))}$ and $W_{\kappa,\mu}(z) \sim e^{-z/2(1+o(1))}$. To analyze $W$, we make the substitution $y(z) = z^{\mu+1/2}e^{-z/2}g(z)$ in (2.59). The power is chosen to eliminate the term in $z^{-2}$ and simplify the Borel transform. For $M$ we would substitute $y(z) = z^{\mu+1/2}e^{z/2}g(z)$. We obtain

$$g'' - \left(1 - \frac{2\mu+1}{z}\right)g' + \frac{\kappa-\mu-\frac{1}{2}}{z}g = 0 \tag{2.60}$$

with Borel transform

$$p(p+1)G' + \left[(1-2\mu)p + \kappa - \mu + \tfrac{1}{2}\right] G = 0 \tag{2.61}$$

with general solution

$$G = Cp^{-\kappa+\mu-1/2}(p+1)^{\mu-1/2+\kappa} \tag{2.62}$$

giving

$$W_{\kappa,\mu}(z) = Cz^{\mu+1/2}e^{-z/2}\int_0^\infty p^{\mu-\kappa-1/2}(p+1)^{\mu-1/2+\kappa}e^{-zp}dp \tag{2.63}$$

which holds for $\mathrm{Re}\,(\mu-\kappa-1/2) > -1$; otherwise $W$ can be defined by using the substitution $y(z) = z^{-\mu+1/2}e^{z/2}g(z)$ or, more generally by analytic continuation. Using Watson's lemma in (2.63) we get

$$W_{\kappa,\mu}(z) \sim C\Gamma(\mu+\tfrac{1}{2}-\kappa)z^{\kappa}e^{-z/2}(1+o(1)); \quad ??\mathrm{Re}\, z \to \infty \tag{2.64}$$

Since $W(z) \sim z^{-\kappa}e^{-z/2}$ ?? as $z \to +\infty$,[1]??, we get $C = 1/\Gamma(\mu-\kappa+\frac{1}{2})$, or

$$W_{\kappa,\mu}(z) = \frac{z^{\mu+1/2}e^{-z/2}}{\Gamma(\mu-\kappa+\frac{1}{2})}\int_0^\infty p^{\mu-\kappa-1/2}(p+1)^{\mu-1/2+\kappa}e^{-zp}dp \tag{2.65}$$

(for $\mathrm{Re}\,(\mu-\kappa-1/2) > -1$).

Parabolic cylinder functions, occurring naturally in turning point asymptotics solve the equation

$$y'' - \left(\tfrac{1}{4}x^2 + a\right)y = 0 \tag{2.66}$$

whose general solution is

$$x^{-1/2}\left(C_1 M_{-\frac{a}{2},\frac{1}{4}}(x^2/2) + C_2 W_{-\frac{a}{2},\frac{1}{4}}(x^2/2)\right) \tag{2.67}$$

from which we can obtain an integral representation for the solutions of (2.66).

### 2.4b Hypergeometric functions

Hypergeomtric functions solve Fuchsian equations, with only regular singular points on the Riemann sphere. Thus all series expansions are convergent. Nonetheless, we can obtain integral representations for them by relating them to other functions, such as Whittaker, in the following way. The more general substitution $y(z) = z^{\beta+3/2}e^{-z/2}g_2$ in (2.59) and inverse Laplace transform leads to the equation

$$p(p+1)G_2'' + [(1-2\beta)p + \tfrac{1}{2} + \kappa - \beta]G_2' + (\beta^2-\mu^2)G_2 = 0 \tag{2.68}$$

with one solution ${}_2F_1(a,b;c;p+1)$ where $a=-\mu-\beta$, $b=\mu-\beta$ and $c=\frac{1}{2}-\beta-\kappa$. These equations are nondegenerate and $\mu,\kappa,\beta$ can be expressed as a function of $a,b,c$. Comparing with the substitution $y(z)=z^{\mu+1/2}e^{-z/2}g(z)$ in §2.4a we see that $g_2=z^{\mu-1-\beta}g$; inverse Laplace transforming this relation we get

$$\Gamma(\beta-\mu+1)G_2(p)=p^{\beta-\mu}*G(p)=$$
$$C\int_0^p (p-s)^{\beta-\mu}s^{-\kappa+\mu-\frac{1}{2}}(s+1)^{\mu-\frac{1}{2}+\kappa}ds$$
$$=Cp^{\beta-\kappa+\frac{1}{2}}\int_0^1 t^{-\kappa+\mu-1/2}(1-t)^{\beta-\mu}(1+pt)^{\mu-1/2+\kappa}dt$$
$$=Cp^{c-a-b}\int_0^1 t^{c-a-1}(1-t)^{-b}(1+pt)^{b-c}dt, \quad (2.69)$$

from which we get an integral representation for ${}_2F_1$.

### 2.4c The Gamma function

The function $f(x)=\log\Gamma(x)$ satisfies the recurrence $f(x+1)-f(x)=\log x$. To get a recurrence for which the formal solution is a power series, with the goal of taking the Borel transform, we first subtract out the terms of $f(x)$ that are large (we can get the first few terms by Euler-Maclaurin): let $g(x)=f(x)-(x\log x-x-\frac{1}{2}\log x)$.
The recurrence satisfied by $g$ is

$$g(x+1)-g(x)=q(x)=1-\Big(\frac{1}{2}+x\Big)\ln\Big(1+\frac{1}{x}\Big)=-\frac{1}{12x^2}+\frac{1}{12x^3}+...$$

First note that $\mathcal{L}^{-1}q=p^{-2}\mathcal{L}^{-1}q''$ with

$$q''=\frac{1}{x}-\frac{1}{x+1}-\frac{1}{2}\Big(\frac{1}{(x+1)^2}+\frac{1}{x^2}\Big)\Rightarrow\mathcal{L}^{-1}g''=1-\frac{p}{2}-\Big(\frac{p}{2}+1\Big)e^{-p}$$

Thus, with $\mathcal{L}^{-1}g(x):=G$ we get

$$(e^{-p}-1)G(p)=\frac{1-\frac{p}{2}-\Big(\frac{p}{2}+1\Big)e^{-p}}{p^2}$$

$$g(x)=\int_0^\infty\frac{1-\frac{p}{2}-\Big(\frac{p}{2}+1\Big)e^{-p}}{p^2(e^{-p}-1)}e^{-xp}dp$$

(It is easy to check that the integrand is analytic at zero; its Taylor series is $\frac{1}{12}-\frac{1}{720}p^2+O(p^3)$.)

The integral is well defined, and it follows that

$$f(x) = C + x(\log x - 1) - \frac{1}{2}\log x + \int_0^\infty \frac{1 - \frac{p}{2} - \left(\frac{p}{2}+1\right)e^{-p}}{p^2(e^{-p}-1)} e^{-xp}dp$$

solves our recurrence. The constant $C = \frac{1}{2}\ln(2\pi)$ is most easily obtained by comparing with Stirling's formula and we thus get the identity

$$\log\Gamma(x) = x(\log x - 1) - \frac{1}{2}\log x + \frac{1}{2}\log(2\pi) + \int_0^\infty \frac{1 - \frac{p}{2} - \left(\frac{p}{2}+1\right)e^{-p}}{p^2(e^{-p}-1)} e^{-nx}dp \tag{2.70}$$

which holds with $x$ replaced by $z \in \mathbb{C}$ as well.

This represents, as it will be clear from the definitions, the Borel summed version of Stirling's formula.

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## 2.5 Stokes phenomena

Near an essential singularity, the behavior of an analytic function depends on the direction of approach of the singularity. For instance, $e^x$ is decreasing in the left half plane and growing in the right half plane, and a transition occurs as $i\mathbb{R}$ is crossed. The Stokes phenomenon describes more subtle phenomena. One of the simplest instances is provided by

$$\mathrm{Ei}(x) = e^x PV \int_0^\infty \frac{e^{-xp}dp}{1-p} = \tfrac{1}{2}e^x \left(\int_0^{\infty+0i} + \int_0^{\infty-0i}\right) \frac{e^{-xp}dp}{1-p}$$

As $x \to$ The Stokes phenomenon is sometimes described as the change in asymptotic behavior of an analytic function as the sector of analysis changes. In this general sense, Typically, This definition is The Stokes phenomenon relates to the fact that the solution that is asymptotic to one fixed formal solution is generally different in different sectors at infinity. We illustrate this in the following analysis of the solutions of the Airy equation.

### 2.5a The Airy equation

We work for now with $t = \frac{2}{3}x^{\frac{3}{2}}$, and write $f(t) = \pi^{\frac{1}{2}}3^{\frac{1}{6}}\Gamma(\frac{1}{6})e^t\mathrm{Ai}(x)$,

$$f(t) = \int_0^\infty e^{-tp}p^{-\frac{5}{6}}(2+p)^{-\frac{5}{6}}dp \tag{2.71}$$

If we analytically continue $t$ anticlockwise, the $p$ contour is rotated homotopically clockwise by the same angle, to keep $tp$ real and positive and have in the process the integral presented in a form suitable for Watson's lemma.

A rotation in the $p$-plane by more than $-\pi$ requires crossing the negative real line where the integrand has a branch point. With $J_-$ the integral along a ray of angle $-\pi+0$ and $J_+$ the same integral along a ray of angle $-\pi-0$, we have

$$J_- = J_+ - (J_+ - J_-) = J_+ + f_2(t);\ \ f_2(t) := -\oint_{\mathcal{C}} e^{-tp} p^{-\frac{5}{6}} (2+p)^{-\frac{5}{6}} dp \quad (2.72)$$

where $\mathcal{C}$ is a curve starting at $\infty e^{-(\pi-0)i}$, goes around $p=-2$ and then to $\infty e^{-(\pi+0)i}$, see Fig. 2.1.

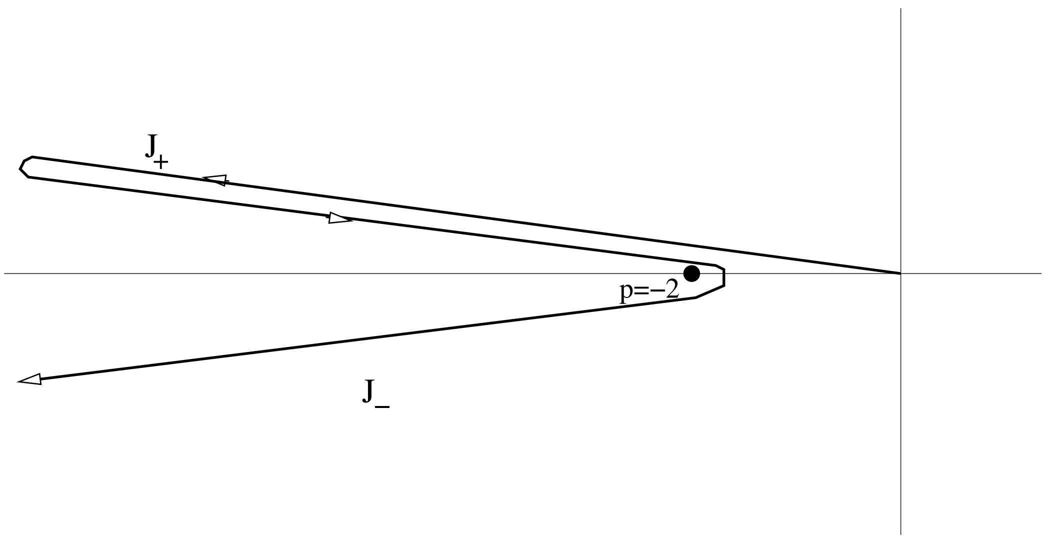


**FIGURE 2.1**: Deformation of contour for (2.71).

Reasoning as in Note 2.40 (iii), the function

$$f_2(t) = -\oint_{\mathcal{C}} e^{-tp} p^{-\frac{5}{6}} (2+p)^{-\frac{5}{6}} dp \quad (2.73)$$

provides, after multiplying it by $e^{-t}$ and changing variable back to $x$, a linearly independent solution of the Airy equation.

We note that the contour in $f_2$ can be deformed all the way to $p=-2$, where we have an integrable singularity.

In the part of the integral $f_2$ which is above $\mathbb{R}^-$ and to the left of $\mathrm{Re}\,(p) = -2$ we have by construction $\arg p = -\pi - 0$, $\arg(2+p) = \pi + 0$, and thus $p^{-\frac{5}{6}}(2+p)^{-\frac{5}{6}} = |p|^{-\frac{5}{6}}|p+2|^{-\frac{5}{6}}$. In the part below $\mathbb{R}^-$ we have $\arg p = -\pi+0$, $\arg(2+p) = -\pi+0$ and thus $p^{-\frac{5}{6}}(2+p)^{-\frac{5}{6}} = e^{\pi i/3}|p|^{-\frac{5}{6}}|p+2|^{-\frac{5}{6}}$. Noting that $1 - e^{5\pi i/3} = e^{\pi i/3}$, and changing variables to $s = e^{i\pi}p$, we get

$$f_2(t) = e^{\pi i/3} \int_2^{\infty} s^{-\frac{5}{6}}(s-2)^{-\frac{5}{6}} e^{ts} ds = e^{\pi i/3} e^{2t} \int_0^{\infty} s^{-\frac{5}{6}}(s+2)^{-\frac{5}{6}} e^{ts} ds \quad (2.74)$$

with $\arg(s) = 0$. In $J_+$, thought of an integral along the upper side of $\mathbb{R}^-$, $p^{-\frac{5}{6}}(2+p)^{-\frac{5}{6}}$ equals $e^{5\pi i/6}|p|^{-\frac{5}{6}}|p+2|^{-\frac{5}{6}}$ from $\operatorname{Re} p = 0$ to $\operatorname{Re} p = -2$ and, as before, $|p|^{-\frac{5}{6}}|p+2|^{-\frac{5}{6}}$ to the left of $\operatorname{Re}(p) = -2$. Changing variables to $s = e^{i\pi}p$, this is the same as the integral below $\mathbb{R}^-$

$$J_+ = -e^{5\pi i/6} \int_0^{\infty - i0} s^{-\frac{5}{6}}(2-s)^{-\frac{5}{6}} e^{ps} ds \quad (2.75)$$

with the natural choice of the argument, that is starting with $\arg s = 0$, $\arg(2-s) = 0$ for small $s > 0$. With this choice, we note that $\arg(2-s) = \pi$ for $s > 2$ in the integrand in (2.75). Thus,

$$f(t) = -e^{5\pi i/6} \int_0^{\infty e^{-0i}} s^{-\frac{5}{6}}(2-s)^{-\frac{5}{6}} e^{ts} ds + e^{\pi i/3} e^{2t} \int_0^{\infty} s^{-\frac{5}{6}}(s+2)^{-\frac{5}{6}} e^{ts} ds \quad (2.76)$$

for $\arg(t) > \pi$. By the change of variables, if $p$ rotates clockwise, so does $s$.

To reach $\arg x = \pi$, *i.e.* $\arg t = \frac{3\pi}{2}$, we rotate further $s$ by $-\pi/2$. Then, for $x \in \mathbb{R}^-$, after the change of variable $s = e^{-i\pi/2}u$, we get

$$\begin{aligned} \pi^{\frac{1}{2}} 3^{\frac{1}{6}} \Gamma(\tfrac{1}{6}) \operatorname{Ai}(-|x|) & \\ &= e^{-\frac{\pi i}{4}} e^{i|t|} \int_0^{\infty} u^{-\frac{5}{6}}(2+iu)^{-\frac{5}{6}} e^{-u|t|} du \\ &\quad + e^{\frac{\pi i}{4}} e^{-i|t|} \int_0^{\infty} u^{-\frac{5}{6}}(2-iu)^{-\frac{5}{6}} e^{-u|t|} du; \quad t = \tfrac{2}{3} x^{\frac{3}{2}} \end{aligned} \quad (2.77)$$

which is real-valued as expected.

For large $|x|$, applying Watson's Lemma to (2.77), one obtains the asymptotic behavior

$$\begin{aligned} \operatorname{Ai}(-|x|) &= \frac{1}{2\sqrt{\pi}} |x|^{-1/4} e^{i\frac{2}{3}|x|^{2/3} - i\frac{\pi}{4}} \left[1 + O\left(|x|^{-3/2}\right)\right] \\ &\quad + \frac{1}{2\sqrt{\pi}} |x|^{-1/4} e^{-i\frac{2}{3}|x|^{2/3} + i\frac{\pi}{4}} \left[1 + O\left(|x|^{-3/2}\right)\right] \end{aligned} \quad (2.78)$$

Equation (2.78) provides the *connection formula* for Ai: It gives, with explicit constants, the behavior at $-\infty$, given the behavior at $+\infty$. This is not possible for more complicated ODEs, even linear ones. The possibility of solving connection problems distinguishes special functions from mere solutions of linear ODEs, and is almost always linked to the existence of underlying integral representations. More generally, for nonlinear integrable systems, solving connection problems can be linked to the existence of Riemann-Hilbert reformulations.

**Note 2.79** When we will study more general equations, we will see that the fact that the Borel transform of the normalized Airy equation has two singularities in the Borel plane results in the existence of two linearly independent solutions of any equation with coefficients analytic at infinity having Ai as a solution. Thus, there can be no "simpler" equation with these analytic properties which as Ai as a solution.

### 2.5b Nonlinear Stokes phenomena

When a differential equation is nonlinear, typically, there are infinitely many singularities in the $p$ plane, as we will see in more detail later.

For now let us take a reverse-engineered example. The function $y(x) = e^{-x}\mathrm{Ei}(x)$ satisfies the model equation $y' + y = 1/x$ that we studied before, and thus $v = 1/(1-y)$ satisfies the equation

$$v' - v + v^2 = \frac{v^2}{x} \tag{2.80}$$

We know that the asymptotic series of $y(x)$ is Borel summable in any direction other that $\mathbb{R}^+$, and thus, by the proof of Proposition 2.23 so is $v$. On the other hand with $v = \mathcal{L}V, y = \mathcal{L}Y$ we have

$$v = 1 + y + y^2 + \cdots \Rightarrow V = 1 + Y + Y * Y + \cdots \tag{2.81}$$

where $Y(p) = 1/(1-p)$. Then

$$Y * Y(p) = \int_0^p \frac{1}{1-s}\frac{1}{1-(p-s)}ds = \frac{2\ln(1-p)}{p-2}$$

which is singular at $p = 1$ *and* $p = 2$. Likewise, $Y^{*3}$ introduces a singularity at $p = 3$ and so on, and without going through a rigorous proof for now, we claim that the last term in (2.81) is singular (on a Riemann surface) exactly at $p \in \mathbb{N}$. What would be the Stokes phenomenon for $v$? (!!!Now I tried at the beginning of the Airy section!!!) If $J_+$ ($J_-$) is the Borel sum of the asymptotic series of $y$ for $\arg x = 0_+$ ($\arg x = 0_-$, resp.), then

$$\frac{1}{1-J_+} = \frac{1}{1-J_- + 2\pi i e^{-x}} = \frac{1}{1-J_-} - \frac{2\pi i e^{-x}}{(1-J_-)^2} + \frac{(2\pi i)^2 e^{-2x}}{(1-J_-)^3} - \cdots \tag{2.82}$$

that is,

$$\frac{1}{1-J_+} = \sum_{k=0}^{\infty} C^k y_k e^{-kx}, \quad C = -2\pi i \tag{2.83}$$

where $y_k$ are Borel summable. We can check that in fact (2.83) is a solution of (2.80) for any $C$. The infinite sum in (2.83) is a *Borel summed transseries* and is the prototypical form of a solution of a nonlinear ODE.

We also note that $v$ has infinitely many singular points (poles) near the imaginary line, where the denominator $1-y$ in the definition of $v$ is zero. In general, this will be seen as being the effect of pile-up of oscillatory exponentials in (2.83).

### 2.5c The Gamma function

Difference equations also produce infinitely many singularities in $p$ plane. Let us look at (2.70). Consider the part

$$g(x)=\int_0^\infty \frac{1-\frac{p}{2}-\left(\frac{p}{2}+1\right)e^{-p}}{p^2(e^{-p}-1)}e^{-xp}dp \tag{2.84}$$

We analytically continue (2.84) to complex $x$ by choosing the integration path to be the ray $\arg p=\theta=-\arg x$ (except when $x\in i\mathbb{R}$ when we would take $\arg p=\theta=-\arg x+i0$ instead) thus ensuring $px$ to be real and positive. When $x$ crosses the negative imaginary axis and moves to the third quadrant, $\arg p$ crosses $\frac{\pi}{2}$ and in the process residues at poles $p=2in\pi$ for $n\in\mathbb{N}$ have to be collected; this gives

$$\begin{aligned} g(x)=\int_0^{\infty e^{i\theta}} G(p)e^{-xp}dp+2\pi i\sum_{j=1}^{\infty}\mathrm{Res}G(p)e^{-xp}|_{p=2j\pi i} \\ =\int_0^{\infty e^{i\theta}} G(p)e^{-xp}dp+\sum_{j=1}^{\infty}\frac{1}{je^{2xj\pi i}} \\ =\int_0^{\infty e^{i\theta}} G(p)e^{-xp}dp-\ln(1-\exp(-2x\pi i)), \end{aligned} \tag{2.85}$$

We note that both the integral and the sum are convergent when $\arg x=-\theta=-\pi/2-\varepsilon$ for $\varepsilon\in\left(0,\frac{\pi}{2}\right)$. Then from (2.70) we get

$$\Gamma(x)=\frac{1}{1-\exp(-2x\pi i)}\sqrt{2\pi}x^{x-\frac{1}{2}}e^{-x}\exp\left(\int_0^{\infty e^{i\theta}} G(p)e^{-xp}dp\right) \tag{2.86}$$

Since $\varepsilon\in\left(0,\frac{\pi}{2}\right]$, it is manifest from (2.86) that $\Gamma(x)$ is analytic for $\arg x\in\left(-\pi,-\frac{\pi}{2}\right)$ while analyticity for $\arg x\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ is manifest in (2.84). When $\arg x=-\frac{\pi}{2}$, we skirt the singularities of the integrand on $\arg p=\frac{\pi}{2}$ by choosing instead integration along $\arg p=\frac{\pi}{2}-\varepsilon$ to conclude analyticity of $\Gamma(x)$ on the negative imaginary axis. Setting $\theta=\pi$ in (2.86), it is clear that $\Gamma(x)$ is real valued and meromorphic on $e^{-i\pi}\mathbb{R}^+$, with simple poles at negative integers. Since $\Gamma(x)$ is real valued on $\mathbb{R}$, from Schwartz reflection principle, $\Gamma(x)$ is analytic in the lower half plane and $\Gamma(x)$ is singular only at simple poles at negative integers. Since Watson's Lemma may be applied to $\int_0^{\infty e^{i\theta}} G(p)e^{-px}$, Stirling's formula holds for large $|x|$ when $\arg\ x\in(-\pi,0]$ since $e^{-2x\pi i}\ll 1$ in this regime. From reflection, the same is true for $\arg\ x\in[0,\pi)$, and the only exception is $\mathbb{R}^-$.

Furthermore, taking $x=-y$ with $y\notin\mathbb{N}$ and then setting $\theta=\pi$ in the expression for $\Gamma(-x)$, we get, by straightforward algebra from (2.86)

$\Gamma(-x)\Gamma(x) = -\dfrac{\pi}{x\sin(\pi x)}$ from which it follows that

$$\Gamma(1-x)\Gamma(x) = \frac{\pi}{\sin(\pi x)} \tag{2.87}$$

We can, especially by analogy with the ODE case, think of the reflection formula (2.87) as the connection formula for the Gamma function.

---

## 2.6 Analysis of convolution equations

### 2.6a Properties of the $p$ plane convolution, Definition 2.24

We will study convolution equations along $\mathbb{R}^+$, in spaces of functions which are locally integrable and exponentially bounded at infinity, or in sectorial domains in $\mathbb{C}$, typically containing a neighborhood of the origin or in compact subsets of such sectorial domains. In all these cases, convolution is well-defined; see also cf. Lemma 2.29 for analyticity properties.

**Lemma 2.88** *In the settings above, functions form a commutative algebra with respect to convolution and addition.*

**PROOF** The needed properties can be checked directly, but it is convenient to use injectivity of the Laplace transform, see Corollary 1.53 and the fact that $\mathcal{L}[f * g] = (\mathcal{L}f)(\mathcal{L}g)$. For function defined in (pre)compact subsets of $\mathbb{C}$ we can extend them with zero to take the Laplace transform.

For instance convolution is associative since

$$\mathcal{L}[f*(g*h)] = \mathcal{L}[f]\mathcal{L}[g*h] = \mathcal{L}[f]\mathcal{L}[g]\mathcal{L}[h] = \mathcal{L}[(f*g)*h] \tag{2.89}$$

and by injectivity, we get

$$f*(g*h) = (f*g)*h \tag{2.90}$$

∎

Similarly, it is commutative and distributive.

### 2.6b Banach convolution algebras

We now define a family of suitable norms so that the Banach spaces induced by these norms are Banach algebras with respect to addition and convolution (defined to be the multiplication in this algebra). The Banach algebra structure requires furthermore that convolution be continuous, that is, $\|F * G\| \le \|F\|\|G\|$.

Some spaces arise naturally and are well suited for the study of convolution equations.

(1) Let $\nu \in \mathbb{R}$ !!!In fact negative is fine too, I guess!!! and define $L^1_\nu := \{f : \mathbb{R}^+ \to \mathbb{C} : f(p)e^{-\nu p} \in L^1(\mathbb{R}^+)\}$; then the norm $\|f\|_\nu$ is defined as $\|f(p)e^{-\nu p}\|_1$ where $\|\cdot\|_1$ denotes the $L^1$ norm.

Lemma 2.26 shows that $L^1_\nu$ is a Banach algebra with respect to convolution.

We see that the norm $\|\cdot\|_\nu$ is the Laplace transform of $|f|$ evaluated at large argument $\nu$, and it is, in this sense, a Borel dual of the sup norm in the original space— since for $\mathrm{Re}\, x > \nu$, $|\mathcal{L}[f](x)| \le \int_0^\infty e^{-\nu p}|f(p)|dp$.

(2) *The space* $L^1_\nu(\mathbb{R}^+e^{i\phi})$. By definition $f \in L^1_\nu(\mathbb{R}^+e^{i\phi})$ if $f_\phi(t) := f(te^{i\phi}) \in L^1_\nu$. Convolution along $\mathbb{R}^+e^{i\phi}$ can be expressed directly as

$$(f*g)(|p|e^{i\phi}) = \int_0^{|p|e^{i\phi}} f(s)g(|p|e^{i\phi}-s)ds = \\ e^{i\phi}\int_0^{|p|} f(te^{i\phi})g(e^{i\phi}(|p|-t))dt = e^{i\phi}(f_\phi * g_\phi)(|p|) \quad (2.91)$$

It is clear that $L^1_\nu(\mathbb{R}^+e^{i\phi})$ is a Banach algebra.

(3) Similarly, we say that $f \in L^1_\nu(S)$ where $S = \{te^{i\phi} : t \in \mathbb{R}^+, \phi \in (a,b)\}$ if $f \in L^1_\nu(\mathbb{R}^+e^{i\phi})$ for all $\phi \in (a,b)$. We define $\|f\|_{\nu,S} = \sup_{\phi\in(a,b)} \|f\|_{L^1_\nu(\mathbb{R}^+e^{i\phi})}$. The space $L^1_\nu(S) = \{f : \|f\|_{\nu,S} < \infty\}$ is also a Banach algebra.

(4) The $L^1_\nu$ spaces can be restricted to an initial interval along a ray, or a compact subset of $S$, restricting the norm to an appropriate set. For instance,

$$L^1_\nu([0,1]) = \left\{f : \int_0^1 e^{-\nu s}|f(s)|ds < \infty\right\} \quad (2.92)$$

These spaces are Banach algebras as well. Obviously, if $A \subset B$, $L^1_\nu(B)$ is naturally embedded (cf. footnote 4 on p. 76) in $L^1_\nu(A)$.

(5) Another important space is $\mathcal{A}_{K;\nu}(\mathcal{E})$, the space of analytic functions in a star-shaped[3] neighborhood $\mathcal{E}$ of the disk $\{p : |p| \le K\}$ in the norm ($\nu \in \mathbb{R}^+$)

$$\|f\| = K \sup_{p\in\mathcal{E}} \left|e^{-\nu|p|}f(p)\right|$$

**Note.** This norm is topologically equivalent with the sup norm (convergent sequences are the same), but better behaved for finding exponential bounds.

**Proposition 2.93** *The space* $\mathcal{A}_{K;\nu}$ *is a Banach algebra with respect to convolution.*

[3]Containing every point $p$ together with the segment linking it to 0.

**PROOF** For analyticity, see Lemma 2.29 To estimate the norm of convolution we write, with $P = |p|$,

$$\left|Ke^{-\nu P}\int_0^p f(s)g(p-s)ds\right| = \left|Ke^{-\nu P}\int_0^P f(te^{i\phi})g((P-t)e^{i\phi})dt\right|$$
$$= \left|K^{-1}\int_0^P Kf(te^{i\phi})e^{-\nu t}Kg((P-t)e^{i\phi})e^{-\nu(P-t)}dt\right|$$
$$\leq K^{-1}\|f\|\|g\|\int_0^K dt = \|f\|\|g\| \quad (2.94)$$

∎

Note that $\mathcal{A}_{K;\nu} \subset L^1_\nu(\mathcal{E})$.
(6) Finally, we note that the space $\mathcal{A}_{K,\nu;0}(\mathcal{E}) = \{f \in \mathcal{A}_{K,\nu}(\mathcal{E}) : f(0) = 0\}$ is a closed subalgebra of $\mathcal{A}_{K,\nu}$.

**Remark 2.95** *If $f$ is a bounded function, then*

$$\|fg\| \leq \|g\| \sup_{p\in\mathbb{R}^+} |f|$$

*in $L^1_\nu$. The same holds if $f$ is holomorphic in $\mathcal{E}$, with* sup *now over $\mathcal{E}$, for the spaces $\mathcal{A}_{K;\nu}$ and $\mathcal{A}_{K,\nu;0}$.*

### 2.6c Spaces of sequences of functions

In Borel summing more general expansions (transseries), it is convenient to look at sequences of vector-valued functions belonging to one or more of the spaces introduced before. For instance, in the scalar case, when the transseries is given by $\tilde{y} = \sum_{j=0}^{\infty} \tilde{y}_j z^j$ with $z = e^{-\lambda x}$, we have

$$y^m = \sum_{j=0}^{\infty} z^j \sum_{k_1+k_2+\cdots+k_m=j} y_{k_1}y_{k_2}\cdots y_{k_m} \tag{2.96}$$

It is convenient to represent $y$ as a vector

$$\mathsf{y} = \{y_j\}_{j\geq 0} \ \ (\mathsf{y}_j := y_j) \tag{2.97}$$

and introduce the product of sequences $\mathsf{fg}$ by

$$(\mathsf{fg})_j = \sum_{j_1+j_2=j} \mathsf{f}_{j_1}\mathsf{g}_{j_2} \tag{2.98}$$

To Borel transform a transseries we look at the sequence of Borel transforms of each $\mathsf{y}_j =: y_j$ above. Thus the Borel dual of $\mathsf{y}$ is the sequence

$$\mathsf{Y} = \{Y_k\}_{k\geq 0}\,; \tag{2.99}$$

where $\mathsf{Y}_j = \mathcal{B}\mathsf{y}_j$. For $\mu > 0$ we define

$$L^1_{\nu,\mu} = \{\mathsf{Y} \in (L^1_\nu)^{\mathbb{N}\cup\{0\}} : \sum_{k\geq 0} \mu^{-k} \|\mathsf{Y}_k\|_\nu < \infty\} \tag{2.100}$$

and introduce the following convolution on $L^1_{\nu,\boldsymbol{\mu}}$

$$(\mathsf{F} \circledast \mathsf{G})_k = \sum_{j=0}^{k} \mathsf{F}_j * \mathsf{G}_{k-j} \tag{2.101}$$

which, as we see, is a double convolution: in $p$ through $*$ and a discrete one in the index.

**Exercise 2.102** *Show that*

$$\|\mathsf{F} \circledast \mathsf{G}\|_{\nu,\mu} \leq \|\mathsf{F}\|_{\nu,\mu} \|\mathsf{G}\|_{\nu,\mu} \tag{2.103}$$

*and* $(L^1_{\nu,\mu}, +, \circledast, \|\,\|_{\nu,\mu})$ *is a Banach algebra. Show that the subspace* $L^1_{\nu,\mu;n} := \{\mathsf{y} \in L^1_{\nu,\mu} : \mathsf{y}_0 = \cdots = \mathsf{y}_{n-1} = 0\}$ *is closed, and thus a Banach algebra too.*

In the vectorial case we have, in general, $m$ exponentials $e^{-\lambda_1 x}, \cdots, e^{-\lambda_m x}$ and the solution is vector valued, with values in say $\mathbb{C}^n$. We then define sequences $\{\mathbf{y}\}_{\mathbf{k}\in(\mathbb{N}\cup\{0\})^m}$ where $\mathsf{y}_{\mathbf{k}} \in \mathbb{C}^n$. When writing the Borel transform of the transseries solution, once more we do so componentwise, for each $\mathbf{k}$ separately. Furthermore, the nonlinear terms in the differential equation are of the form $\mathbf{g}^{\mathbf{l}} = g_1^{l_1} \cdots g_n^{l_n}$ which are *scalar*. See also [44], §2.1.3.

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## 2.7 Focusing spaces and algebras

An important property of the norms (1)–(4) and (6) in §2.6 is that for any $f$ we have $\|f\|_\nu \to 0$ as $\nu \to \infty$. This is used to control nonlinear terms: for large enough $\nu$ they become negligibly small.

A family of norms $\|\|_\nu$ depending on a parameter $\nu \in \mathbb{R}^+$ is **focusing** if for any $f$ with $\|f\|_{\nu_0} < \infty$ for some $\nu_0$ we have

$$\|f\|_\nu \downarrow 0 \text{ as } \nu \uparrow \infty \tag{2.104}$$

($\downarrow$ means monotonically decreasing to the limit, $\uparrow$ means increasing).

Let $\mathcal{V}$ be a linear space and $\{\|\|_\nu\}$ a family of norms satisfying (2.104). For each $\nu$ we define a Banach space $\mathcal{B}_\nu$ as the completion of $\{f \in \mathcal{V} : \|f\|_\nu < \infty\}$. Enlarging $\mathcal{V}$ if needed, we may assume that $\mathcal{B}_\nu = \mathcal{V}$. For $\alpha < \beta$, (2.104) shows

$\mathcal{B}_\alpha$ is naturally embedded in $\mathcal{B}_\beta$.[4] Let $\mathcal{F} \subset \mathcal{V}$ be the projective limit of the $\mathcal{B}_\nu$. That is to say

$$\mathcal{F} := \bigcup_{\nu>0} \mathcal{B}_\nu \tag{2.105}$$

where a sequence is convergent if it converges in *some* $\mathcal{B}_\nu$. We call $\mathcal{F}$ a **focusing space**.

Consider now the case when $(\mathcal{B}_\nu, +, *, \|\|_\nu)$ are commutative Banach algebras. Then $\mathcal{F}$ inherits a structure of a commutative algebra, in which $*$ is continuous. We say that $(\mathcal{F}, *, \|\|_\nu)$ is a **focusing algebra**.

**Examples.** The spaces $\bigcup_{\nu>0} L^1_\nu$ and $\bigcup_{\nu>0} \mathcal{A}_{K;\nu;0}$ and $L^1_{\nu,\mu}$ are focusing algebras. The last space is focusing as $\nu \to \infty$ and $\mu \to \infty$.

**Exercise 2.106** Show that there is no $L^1$ identity element with respect to convolution, that is no $u$ s.t. $u * f = f$ for all $f \in L^1$.

An extension to distributions, very useful in studying singular convolution equations, is the space of staircase distributions $\mathcal{D}'_{m,\nu}$; see [44].

**Remark 2.107** The following simple observation is useful when we want to show that one solution has a number of different properties: analyticity, boundedness, etc. Let $f$ be defined on $S_1 \cup S_2$, and assume that the equation $f(x) = 0$ has a unique solution $x_1$ in $S_1$, a unique solution $x_2$ in $S_2$ and a unique solution $x_3$ in $S_1 \cap S_2$. Then $x_1 = x_2 = x_3 \in S_1 \cap S_2$. Of course, we can equivalently analyze the equation $f$ in $S_1 \cap S_2$ to start with but when we are dealing with more sets when only various combinations of $S_i$ intersect non-emptily, the first approach is more economical.

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## 2.8 Borel summation analysis of nonlinear ODEs

We select a number of relatively simple yet illustrative examples and send for the general theory to [44].

Consider first the ODE

$$y' = x^{-2} - y + y^3 \tag{2.108}$$

This equation was chosen so that it is already in normalized form (more about this later) and it is not solvable by any known methods (in fact, it is nonintegrable in the sense of Painlevé, see §**??**); yet the analysis of this relatively

[4] That is, we can naturally identify $\mathcal{B}_\alpha$ with a subset of $\mathcal{B}_\beta$ which is isomorphic to it.

simple problem illustrates the main aspects of the transseries analysis for a generic system of nonlinear ODEs [44]. The introduction of focusing algebras and loop integral representations (as will be seen later) compactifies earlier presentation [44] and avoids a detour into distribution theory.

We can look for formal power series solutions in the usual way, by inserting a series with unknown coefficients and identifying them, or by iteration. Estimating the coefficients, one would see that the series is divergent, and in this sense, the point at infinity is an irregular singular point. Since we will not base the analysis on the formal series but rather on the *Borel transform of the equation*, and, furthermore, the formal series and its asymptotic properties will emerge as a byproduct, we skip this step now.

As in the linear cases, the logic will be that we apply formally the inverse Laplace transform (Borel transform) to the equation, find a solution of the transformed (Borel plane) equation and then show that the solution of this new equation, when Laplace transformed, results in a solution of the ODE.

Since we have a Banach algebra structure in Borel plane, differential equations become effectively algebraic equations (with convolution $*$ acting as multiplication), which is much easier to deal with. In our case, the formal inverse Laplace of (2.108) is

$$-pY + Y = p + Y^{*3}; \Leftrightarrow Y = \frac{p}{1-p} + \frac{1}{1-p}Y^{*3} := \mathcal{N}(Y) \tag{2.109}$$

where $\mathcal{L}^{-1}y = Y$ and $Y^{*3} = Y * Y * Y$.

**Definition 2.110** *Let $[a,b] \subset (0,2\pi)$, and $S^+ = \{p : \arg(p) \in (a,b)\}$, $S_K^+ = \{p \in S^+ : |p| < K\}$, $\mathbb{D}_\alpha = \{p : |p| < \alpha < 1\}$. Similarly for $[a,b] \subset (-2\pi, 0)$, and $S^- = \{p : \arg(p) \in (a,b)\}$, $S_K^- = \{p \in S^- : |p| < K\}$, $\mathbb{D}_\alpha = \{p : |p| < \alpha < 1\}$.*

**Proposition 2.111** *(i) For large enough $\nu$, (2.109) has a unique solution $Y_0^+$ in the following spaces: $L^1_\nu(S^+), \mathcal{A}_{K;\nu,0}(\mathbb{D}_\alpha), \mathcal{A}_{K;\nu,0}(S_K^+ \cup \mathbb{D}_\alpha)$. (ii) The solution $Y_0^+$ is thus analytic in $S^+ \cup \mathbb{D}_\alpha$ and Laplace transformable along any direction in $S^+$. The Laplace transform is a solution of (2.108).*

*A similar result holds for $Y_0^-$ in $L^1_\nu(S^-), \mathcal{A}_{K;\nu,0}(\mathbb{D}_\alpha), \mathcal{A}_{K;\nu,0}(S_K^- \cup \mathbb{D}_\alpha)$. (in general, $Y_0^- \neq Y_0^+$).*

**Note 2.112** By the uniqueness of the solution in $\mathbb{D}_\alpha$, $Y_0^+ = Y_0^- = Y_0$ in $\mathbb{D}_\alpha$. Thus, $Y_0^+$ $(Y_0^-)$ is simply the analytic continuation of $Y_0$ in the upper (lower, respectively) half-plane. As we will see, in general $Y_0(p)$ has singularities at $p = 1, 2, ...$, and the limiting values of $Y_0^\pm$ on $(0,\infty)$, when they exist, do not coincide.

**PROOF** The proof is only based on the focusing properties of these spaces, and is thus effectively the same for all. We first check, which is straightforward by the results in §2.6c and §2.7, that $\mathcal{N}$ is well defined and with values in these spaces; see also Remark 2.95. (i) Choose $\varepsilon$ small enough. Then for large enough $\nu$ we have

$$\left\|\frac{p}{1-p}\right\|_\nu < \varepsilon/2 \tag{2.113}$$

Let $\mathfrak{B}$ be the ball of radius $\varepsilon$ in the norm $\nu$ and $F$ be a function in $\mathfrak{B}$. Then,

$$\|\mathcal{N}(F)\|_\nu \le \left\|\frac{p}{1-p}\right\|_\nu + \max\left|\frac{1}{p-1}\right| \|F\|_\nu^3 = \varepsilon/2 + c\varepsilon^3 \le \varepsilon \tag{2.114}$$

if $\varepsilon$ is small enough (that is, if $\nu$ is large). Furthermore, for large $\nu$, $\mathcal{N}$ is contractive in $\mathfrak{B}$ for we have, for small $\varepsilon$,

$$\|\mathcal{N}(F_1) - \mathcal{N}(F_2)\|_\nu \lesssim \|F_1^{*3} - F_2^{*3}\|_\nu = \|(F_1 - F_2) * (F_1^{*2} + F_1 * F_2 + F_2^{*2})\|_\nu \\ \lesssim 3\varepsilon^2 \|(F_1 - F_2)\|_\nu \tag{2.115}$$

(ii) We have the following embeddings: $\mathcal{A}_{\nu,0}(S_K^+ \cup \mathbb{D}_\varepsilon) \subset L^1_\nu(S^+)$ (extending the elements of $\mathcal{A}_{\nu,0}$ by zero) and $\mathcal{A}_{\nu,0}(S_K^+ \cup \mathbb{D}_\varepsilon) \subset \mathcal{A}_{\nu,0}(\mathbb{D}_\varepsilon)$ . Thus, by Remark 2.107, in each of these spaces there exists a unique solution $Y_0$ of (2.109), the same for all these spaces.

Thus $Y$ is analytic in $S^+$ and belongs to $L^1_\nu(S^+)$, in particular it is Laplace transformable. The Laplace transform is a solution of (2.108) as it is easy to check.

It also follows that the formal power series solution $\tilde{y}$ of (2.108) is Borel summable in any sector not containing $\mathbb{R}^+$, which is a Stokes ray. We have, indeed, $\mathcal{B}\tilde{y} = Y$ (check!). ☐

### 2.8a Borel summation of the transseries solution

Let $\tilde{y}_0$ the asymptotic series of $\mathcal{L}Y_0$. Looking for a transseries solution,

$$\tilde{y} = \tilde{y}_0 + \sum_{k=1}^{\infty} C^k e^{-kx} \tilde{y}_k \tag{2.116}$$

we insert (2.116) in (2.108) and equate the coefficients of $e^{-kx}$; this results in the system of equations

$$\tilde{y}_0 = -\tilde{y}_0 + x^{-2} + \tilde{y}_0^3 \tag{2.117}$$

$$\tilde{y}_k' + (1 - k - 3\tilde{y}_0^2)\tilde{y}_k = 3\tilde{y}_0 \sum_{j=1}^{k-1} \tilde{y}_j \tilde{y}_{k-j} + \sum_{|j|=k; j_i \ge 1} \tilde{y}_{j_1} \tilde{y}_{j_2} \tilde{y}_{j_3}; \quad k \geqslant 1 \tag{2.118}$$

where, as usual, $|j| = j_1 + j_2 + j_3$ and $\sum_\emptyset = 0$. The equation for $\tilde{y}_1$ is linear and homogeneous:

$$\tilde{y}_1' = 3\tilde{y}_0^2 \tilde{y}_1 \tag{2.119}$$

Thus

$$\tilde{y}_1 = Ce^{\tilde{s}}; \quad \tilde{s} := \int_{\infty}^{x} 3\tilde{y_0}^2(t)dt \tag{2.120}$$

Since $\tilde{s} = O(x^{-3})$, by Proposition 2.23 and Proposition 2.38, $e^{\tilde{s}}$ is Borel summable in $\mathbb{C} \setminus \mathbb{R}^+$. We note that $\tilde{y}_1 = C(1 + o(1))$ and we cannot take the inverse Laplace transform of $\tilde{y}_1$ directly. But the series $x^{-1}\tilde{y}_1$ is Borel summable (say to $\check{\Phi}_1$) see Proposition 2.38. It is convenient[5] to make the substitution $\tilde{y}_k = x^k\tilde{\varphi}_k$ We get

$$\tilde{\varphi}_k' + (1-k-3\tilde{\varphi}_0^2 + kx^{-1})\tilde{\varphi}_k = 3\tilde{\varphi}_0 \sum_{j=1}^{k-1} \tilde{\varphi}_j\tilde{\varphi}_{k-j} + \sum_{j_1+j_2+j_3=k; j_i \ge 1} \tilde{\varphi}_{j_1}\tilde{\varphi}_{j_2}\tilde{\varphi}_{j_3} \tag{2.121}$$

where clearly $\tilde{\varphi}_0 = \tilde{y}_0$, $\tilde{\varphi}_1 = x^{-1}\tilde{y}_1$, with $\tilde{y}_1$ given in (2.120). We choose $\Phi_1 = C\mathcal{B}x^{-1}e^{\tilde{s}}$, define for a choice of sign,

$$\begin{aligned} \boldsymbol{Y}_0 = (Y_0^{\pm}, 0, ..., 0, ...); \quad \boldsymbol{Y}_1 = (0, \Phi_1^{\pm}, ..., 0, ...), 1 = (1, 0, ...); \\ \boldsymbol{\Phi} = (0, 0, \Phi_2, \Phi_3, ...); \quad \hat{k}\boldsymbol{\Phi} = (0, 0, 2\Phi_2, 3\Phi_3, 4\Phi_4, \cdots) \end{aligned} \tag{2.122}$$

and, after Borel transform, we get

$$\begin{aligned} -p\boldsymbol{\Phi} + (1-\hat{k})\boldsymbol{\Phi} = -\hat{k}\ \mathbf{1}{*}\boldsymbol{\Phi} + 3\boldsymbol{Y}_0{*}\boldsymbol{Y}_1^{*2} + \boldsymbol{Y}_1^{*3} \\ + 3(\boldsymbol{Y}_0^{*2} + 2\boldsymbol{Y}_0{*}\boldsymbol{Y}_1 + \boldsymbol{Y}_1^{*2}){*}\boldsymbol{\Phi} + 3(\boldsymbol{Y}_0 + \boldsymbol{Y}_1){*}\boldsymbol{\Phi}^{*2} + \boldsymbol{\Phi}^{*3}; \end{aligned} \tag{2.123}$$

We treat (2.123) as an equation in $L^1_{\mu,\nu;2} \subset L^1_{\mu,\nu}$, the subspace of sequences $\{\Phi_j\}_{j\in\mathbb{N}}$, $\Phi_0 = \Phi_1 = 0$ (and similar subspaces of other focusing algebras).

**Note 2.124** (i) It is important to subtract out $Y_1$, as we have, since its equation allows for a free constant and no contractive mapping argument would work unless the constant $C$ is specified.

(ii) One can show check inductively that

$$\tilde{\varphi}_{2k+1} = O(x^{-2k-1}); \quad \tilde{\varphi}_{2k} = O(x^{-2k-2}); \ \forall \mathbb{N} \ni k \ge 1 \tag{2.125}$$

or

$$\tilde{y}_{2k+1} = O(1); \quad \tilde{y}_{2k} = O(x^{-2}); \ \forall \mathbb{N} \ni k \ge 1 \tag{2.126}$$

where the constants implicit in (2.128) and (2.126) can be calculated in closed form for any given $k$.

**Proposition 2.127** *(i) For $\mu$, $\nu$ large enough, eq. (2.123) is contractive in $L^1_{\nu,\mu;2}(S^+)$, $\mathcal{A}_{\nu,\mu,0;2}(S_K^+ \cup \mathbb{D}_\varepsilon)$ and $\mathcal{A}_{\nu,\mu,0;2}(\mathbb{D}_\varepsilon)$. Thus (2.123) has a unique solution $\boldsymbol{\Phi}^+$ in this space. Similarly, it has a unique solution in these spaces.*

[5] In this problem, $\tilde{y}_1 = 1 + \tilde{v}_1$ would suffice to ensure $O(x^{-2})$ decay of $(\tilde{v}_1, \tilde{y}_2, ...)$.

*Likewise, there is a unique solution* $\mathbf{\Phi}^-$ *in the corresponding spaces in the lower half-plane*[6]*.*

*(ii) Thus there is a* $\nu$ *large enough so that for all* $k$

$$\varphi_k^-(x) = \int_0^{\infty e^{-i\arg(x)}} e^{-xp}\Phi_k^+(p)dp \tag{2.128}$$

*exist for* $|x| > \nu$*. The functions* $\varphi_k^-(x)$ *are analytic in* $x$ *for* $\arg(x) \in (-2\pi - \pi/2, \pi/2)$*. The similarly obtained* $\varphi_k^+(x)$ *by Laplace transforming* $\Phi_k^-$ *along a ray in the fourth quadrant are analytic in* $x$*,* $\arg(x) \in (-\pi/2, 2\pi + \pi/2)$*.*

*(iii) The function series*

$$y^+(x; C_+) = \sum_{k=0}^{\infty} C_+^k e^{-kx} x^k \varphi_k^+(x) \tag{2.129}$$

*and*

$$y^-(x; C_-) = \sum_{k=0}^{\infty} C_-^k e^{-kx} x^k \varphi_k^-(x) \tag{2.130}$$

*converge for sufficiently large* $\operatorname{Re} x$*,* $\arg(x) \in (-\pi/2, \pi/2)$ *and solve (2.108). (See also Proposition 2.132 below.). In fact,* $|\varphi_k^-(x)| \le \mu^k e^{-(\operatorname{Re}(x)-\nu)}$ *if* $\operatorname{Re}(x)$ *is large enough.*

**Note.** The solution cannot be written in the form (2.129) or (2.130) in a sector of opening more than $\pi$ centered on $\mathbb{R}^+$ because the exponentials would become large and convergence is not ensured anymore. Growing exponentials implies, generically, blow-up of the actual solutions; see §**??**. All the $\varphi_k$ however are well behaved.

**Exercise 2.131 (*)** *!!! Agree, let's discuss which!!! Prove Proposition 2.127.*

**Proposition 2.132** *Any solution of (2.108) which is* $o(1)$ *as* $x \to +\infty$ *can be written in the form (2.129) or, equally well, in the form (2.130).*

**PROOF** The method we use here can be extended relatively straightforwardly to general systems, see [44]. Let $y_0 := y^+$ be the solution of (2.108) of the form (2.129) with $C = 0$. Let $y$ be another solution which is $o(1)$ as $x \to +\infty$ and let $\delta = y - y^+$. We have

$$\delta' = -\delta + 3y_0^2\delta + 3y_0\delta^2 + \delta^3 \tag{2.133}$$

which we write in integral form

$$\delta = Ce^{-x} + e^{-x}\int_{x_0}^{x} e^s \left(3y_0(s)^2\delta(s) + 3y_0\delta(s)^2 + \delta(s)^3\right) ds \tag{2.134}$$

[6]Like $Y_0$, the functions $\Phi_k$ are analytic for $|p| < 1$, but generally have branch points at $1, 2, \dots$

For any $C > 0$, eq. (2.134) is contractive on $[x_0, \infty)$ in a ball of radius $\varepsilon$ small enough, in the norm $\sup \|x^2\delta(x)\|$ if $x_0$ is large. This is shown in the usual way. For instance, for some $c$ that is bounded as $x, x_0$ go to infinity we have

$$\left| e^{-x} \int_{x_0}^{x} e^s y_0(s)^2 \delta(s) ds \right| \leqslant c \|x^2 \delta\|_\infty e^{-x} \int_{x_0}^{x} e^s s^{-6} ds \leqslant \frac{2c}{x^6} \|x^2 \delta\|_\infty \quad (2.135)$$

Now, with this unique $\delta = \delta_C$ determined, and the information that $\delta_C \lesssim x^{-2}$ we return to the equation (2.133) which can also be written as

$$\delta' = -\delta + 3y_0^2 \delta + 3y_0 \delta_C \delta + \delta_C^2 \delta \quad (2.136)$$

which is linear and to which we can apply the linear asymptotic theory of ODES [82], which would allow us to deal, in a similar way, with higher order equations. We can deal with this equation straightforwardly however. We obtain the one-dimensional space of solutions by integrating

$$(\ln \delta)' = -1 + 3y_0^2 + 3y_0 \delta_C + \delta_C^2 = -1 + O(x^{-4}) \quad (2.137)$$

It follows that

$$\delta = ce^{-x}(1 + h(x)); \quad h(x) = O(x^{-4}) \quad (2.138)$$

for some $c \in \mathbb{C}$. (where from the construction, $h$ does not depend on $C_1$). On the other hand, the solution $y_{C_1}$ given by (2.129) with $C_+ = C_1$ also satisfies $y_{C_1} - y_+ = C_1 e^{-x}(1 + h(x))$. It follows that $\delta_1 = y - y_{C_1}$ satisfies (2.136) and $\delta_1 = o(e^{-x})$. On the other hand, repeating the argument that led to (2.138), it follows that $\delta_1 = c_1 e^{-x}(1 + h(x))$. This implies $c_1 = 0$ and $\delta_1 \equiv 0$.

$\square$

## 2.8b Analytic structure along $\mathbb{R}^+$

The approach sketched in this section is simple, but of limited scope as it relies substantially on the ODE origin of the convolution equations.

A different, complete proof, that uses the differential equation only minimally is given in [44].

*

By Proposition 2.111, $Y = Y_0^+$ is analytic in any region of the form $\mathbb{D}_\varepsilon \cup S_K^+$. We now sketch a proof that $Y_0$ has analytic continuation along curves that do not pass through the integers.

For this purpose we use (2.129) and (2.130) in order to derive the behavior of $Y$. It is a way of exploiting what Écalle has discovered in more generality, *bridge equations*.

We start with exploring a relatively trivial, nongeneric possibility, namely that $y_0^+ = y_0^- =: y_0$. (This is not the case for our equation, though we will not prove it here; we still analyze this case since it may occur in other equations.)

We take the larger of the values of $\nu^{\pm}$ corresponding to $Y^{\pm}$ in the proof of Proposition 2.111. In our problem, by symmetry w.r.t. complex conjugation, $\nu^{+} = \nu^{-} = \nu$. Since $y_0^+ = y_0^- = y$ we have

$$y = \int_0^{\infty e^{ia'}} Y^+(p)e^{-px}dp = \int_0^{\infty e^{-ia'}} Y^-(p)e^{-px}dp \tag{2.139}$$

where we take $a' \in (a, \pi/2)$ with $a$ defined in Proposition 2.111. Then $y$ is analytic and $O(x^{-2})$ for $|x| > x_0 := \nu m$; $m^{-1} = \min\{\sin(a'/2), \cos(a'/2)\}$ and $\arg(x) \in (-\pi/2 - a'/2, \pi/2 + a'/2)$. Now, by Proposition 1.56 (ii), since $Y = \mathcal{L}^{-1}(y)$, $Y$ is analytic in the sector $S_p = \{p : \arg p \in (-a'/2, a'/2)\}$ and bounded by $e^{\nu m|p|}$. Proposition 2.111 shows that $Y \in \mathcal{A}_{K;\nu,0}(S_K^+ \cup \mathbb{D}_\alpha)$ for any $K$, thus it is analytic in $\{p : |p| < 1\} \cup \{p \arg(p) \in (a, 2\pi - a)$ and bounded by $|p|e^{\nu|p|}$ (since $K = \max\{|p| : p \in S_K^+\}$). Thus $Y$ is entire, and, for large $\nu$, uniformly bounded by $e^{\nu m|p|}$ for all $p$. The estimates on Taylor series coefficient of $Y$ gives rise to factorial decay for large $k$. Applying Laplace transform to the Taylor series of $Y(p)$ term by term, which is justified by dominating convergence theorem, it follows that the series $\tilde{y}_0$ is convergent.

We arrived at the following result:

**Proposition 2.140** *The formal series $\tilde{y}_0$ is convergent iff (nongenerically) $y_+ = y_-$ and factorially divergent otherwise.*

**PROOF** The proof is contained in the analysis preceding the Proposition, except for the factorial divergence, which follows from the fact that $Y_+ \neq Y_-$ implies that the Taylor series of $Y$ at $p = 0$ has finite radius of convergence and applying Watson's lemma to $\int_0^{\infty e^{i\theta}} Y(p)e^{-px}dp$, one obtains a divergent series $\tilde{y}_0$. ∎

### 2.8c The analytic continuations of $Y_0$

By Proposition 2.132, $y_+$ can be represented in the form (2.130). Thus, there exists a constant $S \neq 0$ (called *Stokes multiplier*) such that

$$y^+ = y^- + \sum_{k=1}^{\infty} S^k e^{-kx} x^k \varphi_k^-(x) \tag{2.141}$$

implying

$$y^+ - y^- = Se^{-x}x\varphi_1^-(x) + O(x^2e^{-2x}) \tag{2.142}$$

More generally, for any $C_+$ there is $C_-$ such that we have

$$y^+ + \sum_{k=1}^{\infty} C_+^k e^{-kx} x^k \varphi_k^+(x) = y^- + \sum_{k=1}^{\infty} C_-^k e^{-kx} x^k \varphi_k^-(x) \tag{2.143}$$

from which we obtain

$$y^+ - y^- = (C_- - C_+)e^{-x}x\varphi_1^-(x) + O(x^2e^{-2x}) \tag{2.144}$$

Comparing with (2.142) we get

$$C_- - C_+ = S \tag{2.145}$$

Recalling that $\phi_0^\pm = y_0^\pm$, and denoting $C = C_+$ (2.143) implies the identity

$$\sum_{k=0}^{\infty} C^k e^{-kx} x^k \varphi_k^+(x) = \sum_{k=0}^{\infty} (C+S)^k e^{-kx} x^k \varphi_k^-(x) \tag{2.146}$$

which holds for any $C$! From the convergence of $\mathbf{\Phi}$ in the space $L_{\mu,\nu}$ for large $\mu, \nu$, it follows that the expansions (2.146) are analytic in $C$ if $x$ is chosen large enough. This means we get a series of identities by taking the derivatives of (2.146) at $C = 0$. In particular, we get

$$\varphi_k^+ - \varphi_k^- = \sum_{j=1}^{\infty} \binom{k+j}{j} S^j x^j e^{-jx} \varphi_{k+j}^- \tag{2.147}$$

### 2.8d Local analytic structure of $Y_0, \Phi_k$ along $\mathbb{R}^+$

**Definition 2.148** *We introduce the notation* $\oint_{a;c}^{\infty}$ *to denote an integral along a contour encircling* $a + \mathbb{R}^+$ *in a counter-clockwise manner, with a minimal distance* $0 < c < 1$ *from this set;* $\oint_{a;c,\pm\phi}$ *denote a similar integral, except that it approaches* $\infty e^{i\phi}$ *on the upper side of* $\mathbb{R}^+$ *and* $\infty e^{-i\phi}$ *on the lower-side. Finally, for* $\phi \neq 0$, $\oint_{a;c,\phi}$ *denotes an integral encircling* $a + e^{i\phi}\mathbb{R}^+$ *in the anti-clockwise direction, but not* $a + l$ *for* $l \in \mathbb{Z}^+$*, and keeping minimal distance* $0 < c < 1$ *from each of the sets* $a + e^{i\phi}\mathbb{R}^+$ *and* $a + l$.

We take $\nu' > \nu$, $y = x - \nu'$ with $\mathrm{Re}\,(y) > 0$, $H_0 = e^{-\nu' p} Y_0$. Noting that $y_-(x) = \mathcal{L}Y_+[x]$ and $y_+(x) = \mathcal{L}Y_-[x]$, we may write ((2.141)) as

$$-\oint_{1;c;\pm\alpha}^{\infty} H_0 e^{-yp} dp = \sum_{k=1}^{N} S^k e^{-kx} x^k \varphi_k^-(x) + a_N(y), \tag{2.149}$$

where

$$a_N(y) = \sum_{k=N+1}^{\infty} S^k e^{-k\nu'} e^{-ky} (\nu' + y)^k \phi_k^-(\nu' + y) = O\left(y^{N+1} e^{-(N+1)y}\right) \tag{2.150}$$

Now, for $\alpha \in (a, \pi/2)$ we have

$$S^k x^k e^{-kx}\varphi_k^-(x) = -\frac{S^k}{2\pi i} x^k e^{-kx} \oint_{0;c,\alpha}^{\infty} \Phi_k^+(p) e^{-px} \ln p \; dp$$
$$= -\frac{S^k e^{-kx}}{2\pi i} \oint_{0;c,\alpha}^{\infty} \big[\Phi_k^+(p)\ln p\big]^{(k)} e^{-px} dp$$
$$= -\frac{S^k e^{-ky-k\nu'}}{2\pi i} \oint_{0;c,\alpha}^{\infty} \big[\Phi_k^+(p)\ln p\big]^{(k)} e^{-\nu' p} e^{-py} dp \quad (2.151)$$

Now, since for large $\nu'$, $\big[\Phi_k^+(p)\ln p\big]^{(k)} e^{-\nu' p}$ and $H_0$ are in $L^1$, we can take the Laplace transform ($f \mapsto \int_0^\infty f(y)e^{zy}dy$, where $\mathrm{Re}\, z < 0$) on both sides of (2.149). We note that $a_N(y)$ is analytic for $y > 0$, and it is $O(y^{N+1}e^{-(N+1)y})$. Therefore, $A_N(z) = \int_0^\infty e^{zy} a_N(y)dy$ is analytic for Re $z < N+1$. We can interchange the $(y, p)$ order of variables by Fubini, and get

$$\oint_{1;c;\pm\alpha}^{\infty} \frac{H_0(p)}{p-z} dp = \sum_{k=1}^{N} \frac{S^k e^{-k\nu'}}{2\pi i} \oint_{0;c,\alpha}^{\infty} \frac{\big[\Phi_k^+(p)\ln p\big]^{(k)}}{p-z+k} e^{-\nu' p} dp - A_N(z) \quad (2.152)$$

We start with $\mathrm{Re}\,(z) < N+1$ and $\mathrm{Im}\, z$ large enough. All integrals are manifestly analytic in $z$ in this region.

We start with large negative $\mathrm{Im}\, z$. The right side of (2.152) is analytic for $\mathrm{Re}\,(z) < N+1$, except at the points $z = k$: this is clear since $\Phi_k^+(p)\ln p$ are analytic along the contour except at zero, and the contour of integration can be deformed to accommodate for the variation of $z$, except if $z$ approaches $k$, for some $k = 1, 2, ..., N$.

We start with large negative $\mathrm{Im}\, z$. The right side of (2.152) is analytic for $\mathrm{Re}\,(z) < N+1$, except at the points $z = k$: this is clear since $\Phi_k^+(p)\ln p$ are analytic along the contour except at zero, and the contour of integration can be deformed after collecting residue at $p = z - k$ and that the right side is analytic at $p = z - k$, except when $z = k$ for some $k = 1, 2, ..N$.

To see what happens in a neighborhood of $z = k$, we first define

$$G(z) = \oint_{0;c,\alpha}^{\infty} \frac{\big[\Phi_k^+(p)\ln p\big]^{(k)}}{p-z+k} e^{-\nu' p} dp \quad (2.153)$$

to be the analytic function when $z$ is outside the loop $\oint_{0;c\infty}^{\infty}$. The analytic continuation of $G(z)$ for $z$ inside the loop can be found through contour deformation, and in the process collecting a residue, implying

$$\frac{S^k e^{-k\nu'}}{2\pi i} G(z) = S^k e^{-\nu' z} \big[\Phi_k^+(z-k)\ln(z-k)\big]^{(k)}$$
$$+ \frac{S^k e^{-k\nu'}}{2\pi i} \oint_{0;c,\alpha}^{\infty} \frac{\big[\Phi_k^+(p)\ln p\big]^{(k)}}{p-z+k} e^{-\nu' p} dp \quad (2.154)$$

where $z$ is now inside the contour of integration and the integral term is manifestly analytic for $z$ near $k$.

We can likewise find the analytic continuation of $\oint_{1;c,\pm\alpha}^{\infty} \frac{H_0(p)dp}{p-z}$ for $z$ inside the loop gives rise to $\oint_{1;c,\pm\alpha}^{\infty} \frac{H_0(p)dp}{p-z} + 2\pi i H_0(z)$ for $z$ inside the loop, where the integral term is manifestly analytic near $z = k$. Equating the two sides of (2.152) after analytic continuation inside the loop, we obtain[7] in a (ramified) neighborhood of $z = k$, as $z \uparrow \mathbb{R}^+$,

$$Y_0^-(z) = \frac{S^k}{2\pi i}\Big[\Phi_k^+(z-k)\ln(z-k)\Big]^{(k)} + \tilde{A}_k(z) \tag{2.155}$$

**Remark 2.156** In (2.155), $\tilde{A}_k(z)$ is analytic near $z = k$ and in a neighborhood of the ray $\{z = k + te^{i\alpha} : t \in \mathbb{R}^+\}$, manifestly so, because of the analyticity of all $\Phi_k^+$.

Thus, the only singularities of $Y_0$ on the first Riemann sheet are at $p = k$, and the singular structure is given in (2.158) below. It follows from (2.128) and the fact that $\phi_k(x) = O(x^{-k-2}$ for even $k$ and $\phi_k(x) = O(x^{-k})$ for odd $k$ for large $x$ that $\Phi_k^+(p) = p^{k-\sigma}B_k(p)$ with $\sigma = 1$ if $k$ is odd and $\sigma = -1$ if $k$ is even. Thus,

$$Y_0^-(z) = \frac{S^k}{2\pi i}\Big[(z-k)^{k-\sigma}B_k(z-k)\ln(z-k)\Big]^{(k)} + A_{k;1}(z) \tag{2.157}$$

where $B_k$ and $A_{k;1}$ are analytic near $z = k$, or, finally,

$$Y_0^-(z) = \frac{S^k}{2\pi i}\Big[(z-k)^{1-\sigma}B_{k;2}(z-k)\ln(z-k)\Big]' + A_{k;2}(z) \tag{2.158}$$

$B_{k;2}$ and $A_{k;2}$ are analytic near $z = k$. In particular, $Y_0^+$ is analytic on the Riemann surface of a punctured neighborhood of $p = k$.

Similarly, from (2.147) we get, near $z = k$,

$$\Phi_j^-(z) = \frac{S^k}{2\pi i}\binom{j+k}{j}\Big[\Phi_{j+k}^+(z-k)\ln(z-k)\Big]^{(k)} + A_{k;j}(z) \tag{2.159}$$

with $A_{k;j}(z)$ some functions analytic for $|z-k| < 1$. They are also analytic when $z - k$ is inside the whole contour of integration of $\Phi_k$.

**Note 2.160** We see that the formal series $\tilde{y}_0$ generates, at least in principle, the full transseries and the one-parameter family of small solutions of (2.108).

[7] We quote here results for $Y_0^-(z)$ since we approach $z = \mathbb{R}^+$ from $\operatorname{Im} z < 0$. Obviously similar results can be found for $Y_0^+(z)$ by approaching $\mathbb{R}^+$ from above.

Indeed

$$e^{-kx}x^k\varphi_k^-(x) = x^k e^{-kx}\oint_{0;c,\alpha}^{\infty}\Phi_k^+(p)e^{-px}\ln p\; dp$$
$$= \frac{1}{2\pi i}\oint_{k;c,\alpha}^{\infty}\big[\Phi_k^+(p-k)\ln(p-k)\big]^{(k)}e^{-px}dp = S^{-k}\oint_{k;c,\alpha}^{\infty}Y_0^{-+}(p)e^{-px}dp \tag{2.161}$$

where $-+$ indicates continuation from the fourth quadrant into the first one. Thus, the general small solution is given by

$$y^- + \sum_{k=1}^{\infty}C_-^k e^{-kx}x^k\varphi_k^-(x) = \int_0^{\infty e^{ia}}Y_0^+(p)e^{-px}dp + \sum_{k=1}^{\infty}c^k\oint_{k;c,\alpha}^{\infty}Y_0^{-+}(p)e^{-px}dp \tag{2.162}$$

where $c = C_-/S$.

**Note 2.163** It would not be hard to show for (2.108) that a solution that behaves like $\tilde{y}_0$ as $x \to i\infty$ will have the asymptotic behavior $Se^{-x}(1+o(1))$ close to the negative imaginary line, in a narrow region where $x^{-1} \ll e^{-x} \ll 1$. The value of the Stokes multiplier $S$ is crucial to completely solving the *connection problem, here:* given the behavior at $i\infty$ find the behavior at $-i\infty$. There are several very efficient ways to determine this important parameter numerically, but a closed form expression for it is only expected *in integrable systems* to be discussed in the next section.

**Note 2.164** Thus, the whole information about the small solutions of (2.108) is contained in $\tilde{y}_0$. The singularities of $Y_0$ are determined also by formal analysis of $\tilde{y}_k$, which in turn can be determined from $\tilde{y}_0$ *up to the same constant* $S$.

### 2.8e Determining singularities in Borel plane from asymptotics of Laplace integrals

**Lemma 2.165** *(i) Let $H$ be analytic in the region $\{z : dist(\mathbb{R}^+, z) \in (0,c)\}$ and such that for some $\nu$ we have $\sup_{0<|a|<c}\|H(p+ia)\|_\nu < \infty$. Let*

$$h(x) = \oint_{0;c}^{\infty}e^{-px}H(p)dp \quad (\mathrm{Re}\,(x) > \nu) \tag{2.166}$$

*Assume further that*

$$h(x) = O(e^{-rx}) \text{ as } x \to +\infty \tag{2.167}$$

*where $r < c$. Then $H$ is analytic in $\mathbb{D}_r$.*

*(ii) The same holds in the following other cases:*

*(a)* $\oint_{0;c}^{\infty}$ *is replaced by* $\oint_{0;c;\phi}^{\infty}$ *and* $H$ *is analytic inside the curve, except perhaps for* $\mathbb{R}^+e^{i\phi}$ $\pm\infty$*;*

*(b)* $\oint_{0;c}^{\infty}$ *is replaced by* $\oint_{0;c;\pm\phi}^{\infty}$ *where now the contour surrounds* $\mathbb{R}^+$ *at distance at least* $c$ *and approaches* $\infty$ *at an angle* $\pm\phi$ *and* $H$ *is analytic inside the curve, except perhaps for* $\mathbb{R}^+$*.*

**PROOF** Note first that $h_1(x) := h(x+\nu+\varepsilon)$ is analytic in a neighborhood of $(-\varepsilon, \infty)$. This and (2.345) show that $\check{h}_1(q) = \int_0^\infty h_1(x)e^{-qx}dx$ exists, and the integrand in the definition of $\check{h}_1(q)$ satisfies the hypotheses of Fubini's theorem and

$$\check{h}_1(q) = \oint_0^\infty \frac{e^{-\nu p-\varepsilon p}H(p)}{p+q}dp =: \oint_0^\infty \frac{H_1(p)}{p+q}dp \tag{2.168}$$

**Note 2.169** Eq. (2.345) implies that the Laplace transform $\check{H} := \int_0^\infty h(x)e^{-qx}dx$ exists and is analytic in the half plane $\mathrm{Re}\, q > -r$.

We start with large $\mathrm{Re}\, q$ and approach the origin. To enter the disk of radius $r$, $q$ crosses the contour of integration. We bend the contour inward allowing $q$ to approach the origin at a distance $0 < c' < c$ and then pass the contour through $q$, collecting the residue $2\pi i H(q)$, and then return to the original contour. Thus, for $|q| < r$ we have

$$\check{h}_1(q) = 2\pi i H_1(q) + \oint_{0;c}^\infty \frac{H_1(p)}{p+q}dp \tag{2.170}$$

where now $q$ is in $\mathbb{D}_r$. By Note 2.342 $\check{h}_1(q)$ is analytic in $\mathbb{D}_r$ and so is the integral on the right side of (2.343), manifestly so due to the fact that the contour is outside $\mathbb{D}_r$. But then $H_1(q)$ and therefore $H(q)$ is analytic in $\mathbb{D}_r$.

(ii) The proof is very similar to that of (i). ∎

**Exercise 2.171** *Adapt the proof above to the weaker condition*

$$h(k) = O(e^{-rk}) \text{ as } \mathbb{N} \ni k \to +\infty \tag{2.172}$$

*Hint: consider instead the properties of the generating function* $\sum_{k=k_0}^\infty h_1(k)z^k$*.*

---

## 2.9 Spontaneous singularities and the Painlevé property

In nonlinear differential equations, the solutions may be singular at points $x$ where the equation is regular. For example, the equation

$$y' = y^2 + 1 \tag{2.173}$$

has a one parameter family of solutions $y(x) = \tan(x + C)$; each solution has infinitely many poles. Since the location of these poles depends on $C$, thus on the solution itself, these singularities are called *movable* or *spontaneous*. Whether these spontaneous singularities are poles or essential singularities, particularly branch points, is crucial for the integrability of the equation.

Let us analyze local singularities of the Painlevé equation $\mathrm{P_I}$, ** changed $x$ to $z$**

$$y'' = y^2 + z \tag{2.174}$$

We look at the local behavior of a solution that blows up, and will find solutions that are meromorphic but not analytic. In a neighborhood of a point where $y$ is large, keeping only the largest terms in the equation (*dominant balance*) we get $y'' = y^2$ which can be integrated explicitly in terms of elliptic functions and its solutions have double poles. Alternatively, we may search for a power-like behavior

$$y \sim A(z - z_0)^p$$

where $p < 0$ obtaining, to leading order, the equation $Ap(p-1)(z-z_0)^{p-2} = A^2(z-z_0)^{2p}$ which gives $p = -2$ and $A = 6$ (the solution $A = 0$ is inconsistent with our assumption). Let's look for a power series solution, starting with $6(z-z_0)^{-2} : y = 6(z-z_0)^{-2} + c_{-1}(z-z_0)^{-1} + c_0 + \cdots$. We get: $c_{-1} = 0, c_0 = 0, c_1 = 0, c_2 = -z_0/10, c_3 = -1/6$ and $c_4$ is undetermined, thus free.

### 2.9a The Painlevé property

To address the question whether nonlinear equations can define new functions, Fuchs had the idea that a crucial criterion now known as the *Painlevé property* (PP), is the absence of movable (meaning their position is solution-dependent) essential singularities, primarily branch-points, see [32]. First order equations were classified with respect to the PP by Fuchs, Briot and Bouquet, and Painlevé by 1888, and it was concluded that they give rise to no new functions. Painlevé took this analysis to second order, looking for all equations of the form $u'' = F(t, u, u')$, with $F$ rational in $u'$, algebraic in $u$, and analytic in $t$, having the PP [39, 40]. His analysis, revised and completed by Gambier and Fuchs, found some fifty types (types since some have free parameters) with this property and succeeded to solve all but six of them in terms of previously known functions. The remaining six types are now known as the Painlevé equations, and their solutions, the Painlevé *transcendents*, play a fundamental role in many areas of pure and applied mathematics.

### 2.9b Analysis of a modified $\mathrm{P_I}$ equation

The Painlevé property is very demanding. It is of course beyond the scope of this course to analyze a general second order equation. We can however experiment with a general analytic function $A(z) = \sum_{k=0}^{\infty} a_k z^k$ instead of $z$:

$$y'' = y^2 + A(z) \tag{2.175}$$

We can see that $A(0)$ can be eliminated by a shift of the independent variable, and $A'(0)$, if nonzero, can be normalized to 1. Looking for singular solutions, the dominant balance is the same as in the beginning of §2.9 and thus the local expansion starts with the same term, $6(z-z_0)^{-2}$. Substituting $y = \sum_{k\geq -2} c_k(z-z_0)^k$ in (2.175) and identifying the coefficients, the $c_k$ with $k<4$ can be determined order by order. The coefficient of $(z-z_0)^2$ however does not involve $c_k$ at all; it is

$$-\sum_{k=2}^{\infty} \frac{k(k-1)}{2} a_k z_0^k \tag{2.176}$$

Since (2.176) must vanish, the possibilities, up to linear changes of variables are $A(z)=0$, $A(z)=1$ and $A(z)=z$. The first two give the equation of elliptic functions and the third is $\mathrm{P_I}$ itself .

#### 2.9b.1 The Painlevé test, further discussion

S. Kovalewsky searched for cases of the spinning top having the PP. She found a previously unknown integrable case and solved it in terms of hyperelliptic functions. Her work [36], [37], [38] was so outstanding that not only did she receive the 1886 Bordin Prize of the Paris Academy of Sciences, but the associated financial award was almost doubled.

The method pioneered by Kovalevskaya to identify integrable equations using the Painlevé property is now known as the *Painlevé test*, which she combined with Liouville's results on integrability of Hamiltonian systems. As mentioned , the Painlevé equations, as well as others with the PP were subsequently rederived from linear problems. Why this is so often the case is not completely understood.

However, at an informal level, we note that the Painlevé property guarantees some form of integrability of the equation, in the following sense. Consider for simplicity a second order equation $y''=F(x,y,y')$. Denote by $Y(x;x_0;C_1,C_2)$ the solution with initial conditions $y(x_0)=C_1, y'(x_0)=C_2$. Let $x_1$ be any point which is not a fixed singularity (singularity of the equation). We have, locally for any $x$, $y(x_1)=Y(x_1;x;y(x),y'(x))$ and $y'(x_1)=\partial_1 Y(x_1;x;y(x),y'(x))$. Of course, $y(x_1)=C_1$ and $y'(x_1)=C_2$ are constants, and we have thus found two locally conserved quantities. This local property is essentially what the flowbox theorem provides. The word local is crucial. A system is integrable if *global* conserved quantities exist. Note first that $C_1$ and $C_2$ are obtained by solving the same ODE! What prevents $C_1$ and $C_2$ to extend to global conserved quantities? Once more, it is the possibility of spontaneous singularities. In principle, $C_1$ and $C_2$ might only exist in a bounded domain, bordered by a natural boundary of solutions.

Since the solutions of $\mathrm{P_I}$ however are meromorphic, it is always possible, in principle, to extend $C_1$ and $C_2$ by solving the equation along a path avoiding the poles, and the possible fixed singularities. For the fixed singularities of the equation, we select one common path for all solutions. With $x_1$ kept fixed, of course $Y(x_1;x;y(x),y'(x))$ and $\partial_1 Y(x_1;x;y(x),y'(x))$ are constant along trajectories, and we have obtained (by no means in closed form, but this does not matter) two constants of motion in $\mathbb{C}$, analytic except for lower dimensional manifolds.

On the contrary, movable branch-points have the potential to prevent the existence of well-behaved constants of motions for the following reason. Suppose $y_0$ satisfies a meromorphic (second order, for concreteness) ODE and $K(x;y,y')$ is a constant of motion. If $x_0$ is a branch point for $y_0$, then $y_0$ can be continued past $x_0$ by avoiding the singular point, or by going around $x_0$ any number of times before moving away. This leads to different branches $(y_0)_n$ of $y_0$, all of them, by simple analytic continuation arguments, solutions of the same ODE. By the definition of $K(x;y,y')$ however, we should have $K(x;(y_0)_n,(y_0)'_n) = K(x;y_0,y'_0)$ for all $n$, so $K$ assumes the same value on this infinite set of solutions. We can proceed in the same way around other branch points $x_1, x_2, ...$ possibly returning to $x_0$ from time to time. Generically, we expect to generate a family of $(y_0)_{n_1,...,n_j}, (y_0)'_{n_1,...,n_j}$ which is dense in the phase space. This is an expectation, to be proven in specific cases. To see whether an equation falls in this generic class M. Kruskal introduced a test of nonintegrability, the *poly-Painlevé test* which measures indeed whether branching is "dense." Properly interpreted and justified the Painlevé property measures whether an equation is integrable or not. See, e.g., [18]. In case an the solutions of an equation exhibit dense branching, it is also to be expected that the behavior in the complex domain along paths that encircle many branch points, because of this density, will have a chaotic character.

**Exercise 2.177** ***Show that the solution of $y' = y^5 - 1$ has no single-valued conserved quantity in $K(x,y)$ in $\mathbb{C}^2$: solve the differential equation implicitly and show that by winding around the five logarithmic singularities of $x(y)$ in suitable ways, $K(x,y_j(x))$ takes the same value on a family of $y_j(x)$ which is dense in $\mathbb{C}$.***

Choosing a $c_4$, all others are uniquely determined. At this stage, we have two free parameters in the solution: $x_0$ and $c_4$. We can expect no further undetermined coefficient, since the equation is second order.

### 2.9b.2 The list of Painlevé equations

The six classes of Painlevé transcendents, identified by Painlevé (P), Gambier (G) and R. Fuchs (F) are

$$\frac{d^2y}{dt^2} = 6y^2 + t \ (I;P) \tag{2.178}$$

$$\frac{d^2y}{dt^2} = 2y^3 + ty + \alpha \ (II;P) \tag{2.179}$$

$$ty\frac{d^2y}{dt^2} = t\left(\frac{dy}{dt}\right)^2 - y\frac{dy}{dt} + \delta t + \beta y + \alpha y^3 + \gamma t y^4 \ (III;P) \tag{2.180}$$

$$y\frac{d^2y}{dt^2} = \tfrac{1}{2}\left(\frac{dy}{dt}\right)^2 + \beta + 2(t^2-\alpha)y^2 + 4ty^3 + \tfrac{3}{2}y^4 \ (IV;G) \tag{2.181}$$

$$\begin{aligned}\frac{d^2y}{dt^2} &= \left(\frac{1}{2y} + \frac{1}{y-1}\right)\left(\frac{dy}{dt}\right)^2 - \frac{1}{t}\frac{dy}{dt} \\ &\quad + \frac{(y-1)^2}{t}\left(\alpha y + \frac{\beta}{y}\right) + \gamma\frac{y}{t} + \delta\frac{y(y+1)}{y-1} \ (V;G)\end{aligned} \tag{2.182}$$

$$\begin{aligned}\frac{d^2y}{dt^2} &= \tfrac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t}\right)\left(\frac{dy}{dt}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t}\right)\frac{dy}{dt} \\ &\quad + \frac{y(y-1)(y-t)}{t^2(t-1)^2}\left(\alpha + \beta\frac{t}{y^2} + \gamma\frac{t-1}{(y-1)^2} + \delta\frac{t(t-1)}{(y-t)^2}\right) \ (VI;F)\end{aligned} \tag{2.183}$$

In these equations, $\alpha, \beta, \gamma, \delta$ are arbitrary parameters in $\mathbb{C}$.

Beginning in the 1980's, almost a century after their discovery, these problems were solved, using their striking relation to linear problems[8], by various methods including the powerful techniques of isomonodromic deformation and reduction to Riemann-Hilbert problems [28], [29], [34].

### 2.9b.3 Linearization of the Painlevé equations

The Painlevé equations are related to linear problems in a number of ways, in some broad sense equivalent: isomonodromic deformations, Lax Pairs, Riemann-Hilbert problems and others, and in view of the connection to solvable linear problems are considered themselves to be solvable. The following is one of the simplest to explain such links [27]. Consider the system of equa-

[8]Some linear problems conducive to Painlevé equations were known already at the beginning of last century. In 1905 Fuchs found a linear isomonodromic problem leading to $P_{VI}$.

tions:

$$\frac{\partial \mathbf{\Psi}}{\partial \lambda} = \mathbf{A}(t,\lambda)\mathbf{\Psi} \tag{2.184}$$

$$\frac{\partial \mathbf{\Psi}}{\partial t} = \mathbf{B}(t,\lambda)\mathbf{\Psi} \tag{2.185}$$

where in which **A** and **B** are matrices and $(t,\lambda)$ are independent variables. Then we have a compatibility equation

$$\frac{\partial^2 \mathbf{\Psi}}{\partial t \partial \lambda} = \frac{\partial^2 \mathbf{\Psi}}{\partial \lambda \partial t} \tag{2.186}$$

or

$$\frac{\partial \mathbf{A}}{\partial t} - \frac{\partial \mathbf{B}}{\partial \lambda} + \mathbf{AB} - \mathbf{BA} = 0 \tag{2.187}$$

The equation $P_I$ is the compatibility condition for

$$\mathbf{A}(t,\lambda) = (4\lambda^4 + 2y^2 + t)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - i(4\lambda^2 y + 2y^2 + t)\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$
$$- \left(2\lambda y' + \frac{1}{2\lambda}\right)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$\mathbf{B}(t,\lambda) = \left(\lambda + \frac{y}{\lambda}\right)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \frac{iy}{\lambda}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{2.188}$$

while for $P_{II}$, we have

$$\mathbf{A}(t,\lambda) = -i(4\lambda^2 + 2y^2 + t)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - 2y'\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \left(4\lambda y - \frac{\alpha}{\lambda}\right)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$\mathbf{B}(t,\lambda) = \begin{pmatrix} -i\lambda & y \\ y & i\lambda \end{pmatrix} \tag{2.189}$$

### 2.9c Rigorous analysis of the meromorphic expansion for $P_I$

Substituting $y(x) = 6(x - x_0)^{-2} + \delta(x)$, with $\delta(x) = o((x - x_0)^{-2})$ and taking $x = x_0 + z$ we obtain

$$\delta'' = \frac{12}{z^2}\delta + z + x_0 + \delta^2 \tag{2.190}$$

To find the dominant balance we note that our assumption $\delta = o(z^{-2})$ makes $\delta^2/(\delta/z^2) = z^2\delta = o(1)$ and thus the nonlinear term in (2.190) is *relatively* small. Thus, *to leading order*, the new equation is linear. This is a general phenomenon: taking out more and more terms out of the local expansion, the correction becomes less and less important, and the equation is better and better approximated by a linear equation. We then rewrite (2.190) as

$$\delta'' - \frac{12}{z^2}\delta = z + x_0 + \delta^2 \tag{2.191}$$

which we convert into an integral equation. The indicial equation for the Euler equation corresponding to the left side of (2.191) is $r^2 - r - 12 = 0$ with solutions $4, -3$. We get

$$\delta = \frac{D}{z^3} - \frac{1}{10}x_0 z^2 - \frac{1}{6}z^3 + Cz^4 - \frac{1}{7z^3}\int_0^z s^4\delta^2(s)ds + \frac{z^4}{7}\int_0^z s^{-3}\delta^2(s)ds$$
$$= -\frac{1}{10}x_0 z^2 - \frac{1}{6}z^3 + Cz^4 + J(\delta) \quad (2.192)$$

the assumption that $\delta = o(z^{-2})$ forces $D = 0$. The occurrence of a "disallowed" freedom, $D/z^3 \gg \delta$ in this case is related to the phenomenon of negative resonances, quite common in Painlevé analysis; see [31] for a discussion. Now, $C$ is arbitrary. To find $\delta$ formally, we would simply iterate (2.192) as usual: we first take $\delta = 0$ on the right side and obtain $\delta_0 = -\frac{1}{10}x_0 z^2 - \frac{1}{6}z^3 + Cz^4$. Then we take $\delta^2 = \delta_0^2$ and compute $\delta_1$ from (2.192) and so on. This yields:

$$\delta = -\frac{1}{10}x_0 z^2 - \frac{1}{6}z^3 + Cz^4 + \frac{x_0^2}{1800}z^6 + \frac{x_0}{900}z^7 + ... \quad (2.193)$$

To prove convergence of the expansion, we scale out the leading power of $z$ in $\delta$, $z^2$ and write $\delta = z^2 u$. The equation for $u$ is

$$u = -\frac{x_0}{10} - \frac{z}{6} + Cz^2 - \frac{z^{-5}}{7}\int_0^z s^8 u^2(s)ds + \frac{z^2}{7}\int_0^z s u^2(s)ds$$
$$= -\frac{x_0}{10} - \frac{z}{6} + Cz^2 + J(u) \quad (2.194)$$

It is straightforward to check that, given $C_1$ large enough (compared to $x_0/10$ etc.) there is an $\varepsilon$ such that this is a contractive equation for $u$ in the ball $\|u\|_\infty < C_1$ in the space of analytic functions in the disk $|z| < \varepsilon$. We conclude that $\delta$ is analytic and that $y$ is meromorphic near $x = x_0$.
**Note**. The Painlevé property discussed in requires that $y$ is globally meromorphic, and we did *not* prove this. That indeed $y$ is globally meromorphic is in fact true, but the proof is delicate (see e.g. [1]). Generic equations fail even the local Painlevé property. For instance, for the simpler, autonomous, equation

$$f'' + f' - f^2 = 0 \quad (2.195)$$

the same analysis yields a local behavior starting with a double pole, $f \sim -6z^{-2}$. Further terms in a local *power series* expansion are:

$$f = \frac{6}{z^2} - \frac{6}{5z} - \frac{1}{50} - \frac{z}{250} - \frac{7z^2}{5000} - \frac{79}{75000}z^3 + \text{undefined} \quad (2.196)$$

that is, no coefficient of $z^{-4}$ works. More terms have to be pulled out for a contractive mapping approach to work. We take

$$f = \frac{6}{z^2} - \frac{6}{5z} - \frac{1}{50} - \frac{z}{250} - \frac{7z^2}{5000} - \frac{79z^3}{75000} + \delta(z)$$

proceeding as above and integrating by parts the $\delta'$ term we get, after some work,

$$\delta = \frac{18z^4 \ln z}{21875} + Cz^4 + z^5 P(z) + \mathcal{N}(z, \delta(z), \delta(z)^2) \tag{2.197}$$

where $P(z)$ is a polynomial of third degree and $\mathcal{N}$ is a contractive operator in the space of functions which are $O(z^4 \ln z)$. If we pull out fewer terms from $f$ a contraction mapping argument may not work (at least not in a naive way). We see that a log term is generated in the process.

**Note 2.198** Eq. (2.195) does not have the Painlevé property. The log terms generate infinitely many solutions by analytic continuation around one singular point, and suggests the equation is not integrable.

### 2.9d The Painlevé property for PDEs

We briefly discuss the Weiss-Tabor-Carnevale (WTC) method [83] which adapts Painlevé analysis to PDEs. Remarkably, when applied to Burgers' equation and to KdV, their method leads naturally to the solution of these equations, through the Cole-Hopf transformation and Lax pairs respectively. We base the presentation on their aforementioned paper.

Meromorphic functions of several variables are locally ratios $P/Q$ of analytic functions, and singularities occur when the denominator vanishes,

$$Q(z_1, ..., z_n) = 0 \tag{2.199}$$

which is a manifold of complex dimension $N - 1$ (in particular, singularities are not isolated anymore). The WTC test requires that in a neighborhood the manifold described by (2.199) the solution $u$ of the given PDE be single-valued as well, that is, for some analytic functions $u_j(z_1, ..., z_n)$ we have

$$u = Q^{-m} \sum_{j=0}^{\infty} u_j Q^j; \quad \text{for some } m \in \mathbb{N} \tag{2.200}$$

Direct substitution of (2.200) into the PDE determines the compatible values of $\alpha$ and defines recursively the $u_j$, $j \geqslant 0$. In the resulting expression, we assume that $Q^{j+1} \ll Q^j$ and therefore treat it like an asymptotic series in powers of $Q$.

The first example is Burgers' equation

$$u_t + uu_x = \sigma u_{xx} \tag{2.201}$$

As in the case of the Painlevé test, the analysis has to be done carefully, but there is no need for rigor, as we are simply dealing with a practical criterion. Substituting (2.200) into (2.201) and using the assumption of analyticity suggests $m = 1$, since otherwise the most negative power of $Q$ would be unbalanced. With $m = 1$, after the aforementioned substitution, the most negative power of $Q$ is $-3$; the equation for its coefficient to vanish is

$$u_0 = -2\sigma Q_x \tag{2.202}$$

We now take $u = u_0 Q^{-1} + u_1$ where $u_0$ is given by (2.202) and require that the coefficient of $Q^{-2}$. This gives

$$Q_t + u_1 Q_x = \sigma Q_{xx} \tag{2.203}$$

For the coefficient of $Q^{-1}$ we get

$$u_1 Q_{xx} + u_{1x} Q_x + Q_{xt} - \sigma Q_{xxx} = 0 \tag{2.204}$$

We note that this equation does not contain $u_2$; this is similar to the equation for $c_4$ in the case of Painlevé. The equation is either satisfied, or the expansion fails. Thus, the third order term is resonant and if (2.204) holds, then $u_2$ is free. On the other hand, we see that the left side of (2.204) is just the $x$ derivative of (2.203), and thus (2.204) indeed holds:

$$\partial_x (u_1 Q_x + Q_t - \sigma Q_{xx}) = 0 \tag{2.205}$$

The general recurrence for $u_j$ is of the form

$$(j+1)(j-2)\sigma \phi_x^2 u_j = F\left(\{u_k\}_{k<j}; Q_t, \{\partial_x^k Q\}\right) \tag{2.206}$$

where we see two resonances, the negative one corresponding to the freedom in choosing $Q$ and at $j = 2$ we get the identity (2.205). If we modify Burgers' equation by adding, say, $au(t,x)$ to its left side, we get instead of (2.205)

$$\partial_x (u_1 Q_x + Q_t - \sigma Q_{xx} + aQ) = 0 \tag{2.207}$$

and this, for $a \neq 0$, combined with (2.204) implies $Q_x = 0$, and (2.202) would give $u_0 = 0$, and then (2.203) gives $Q_t = 0$ and since $Q$ has a zero, we would have $Q \equiv 0$, a contradiction. For this modified Burgers' equation, a meromorphic expansion (2.200) and the formal Painlevé property fail.

Let's return to $a = 0$; as mentioned, the equation (2.205) for $u_2$ is automatically satisfied, $u_2$ is free, and (2.206) implies that all $u_j$ for $j \geqslant 3$ are uniquely determined. There is a meromorphic local expansion in a neighborhood of the singular manifold, and we could check that under suitable analyticity assumptions on $Q$ it actually converges.

If we set $u_1 = u_2 = 0$ we can check that the $u_j = 0$ for $j \geq 3$ is consistent. With this choice, (2.203) implies

$$Q_t = \sigma Q_{xx}; \quad u(t,x) = -2\sigma Q_x / Q \tag{2.208}$$

which is the Cole-Hopf transform, see [17], [33], [84], mapping (2.201) to the heat equation and providing the closed form solution of Burgers' equation!

Choosing instead $u_1 = Q$ we get

$$Q_t + Q Q_x = \sigma Q_{xx} \text{ and } u = -2\sigma Q_x / Q + Q \text{ imply } u_t + u u_x = \sigma u_{xx} \tag{2.209}$$

which is the Bäcklund transformation for Burgers', discovered by Fokas [30].

### 2.9d.1 KdV

The KdV equation reads

$$u_t + uu_x + \sigma u_{xxx} = 0 \tag{2.210}$$

the consistent power of $Q$ is $-2$. Inserting $u = Q^{-2}\sum_{k=0}^{\infty} Q^k u_k$, the most negative power of $Q$ is $Q^{-5}$; setting the coefficients of $Q^{-5}$ and $Q^{-4}$ to zero, we get

$$u_0 = -12\sigma Q_x^2; \quad u_1 = 12\sigma Q_{xx} \tag{2.211}$$

Setting the coefficient of $Q^{-3}$ to zero gives

$$Q_x Q_t + Q_x^2 u_2 + 4\sigma Q_x Q_{xxx} - 3\sigma Q_{xx}^2 = 0 \tag{2.212}$$

while the similar equation for the coefficient of $Q^{-2}$ can be rewritten as

$$Q_{tx} + u_2 Q_{xx} - u_3 Q_x^2 + \sigma Q_{xxxx} = 0 \tag{2.213}$$

while the equation for the coefficient of $Q^{-1}$ is equivalent to

$$\partial_x(Q_{tx} + u_2 Q_{xx} - u_3 Q_x^2 + \sigma Q_{xxxx}) = 0 \tag{2.214}$$

The equation above corresponds to a resonance, as $u_4$ does not participate; we see that for KdV it is automatically satisfied. The resonances for the $u_j$ are at $j = -1, 4, 6$, and the resonant equation at $j = 6$ is longer and we omit it.

Looking again for truncated series, in this case

$$u_j = 0 \text{ for all } j > 2 \tag{2.215}$$

it can be checked that $u$ given by

$$u = -12\sigma Q_x^2/Q^2 + 12\sigma Q_{xx}/Q + u_2 = 12\sigma(\ln Q)_{xx} + u_2 \tag{2.216}$$

satisfies (2.210) if $u_2$ satisfies KdV:

$$u_{2t} + u_2 u_{2x} + \sigma u_{2xxx} \tag{2.217}$$

which is a Bäcklund transformation for (2.210).

With $u_3 = 0$, (2.213) becomes

$$Q_{tx} + u_2 Q_{xx} + \sigma Q_{xxxx} = 0 \tag{2.218}$$

Solving (2.212) for $Q_t$ and differentiating with respect to $t$ we get

$$Q_{xt} = 2VV_t = -2VV_x u_2 - V^2 u_{2x} - 8\sigma VV_{xxx} \tag{2.219}$$

where we substituted $Q_x = V^2$. Taking the ansatz

$$6\sigma V_{xx} + u_2 V = \lambda V \tag{2.220}$$

in (2.219) KdV is linearized, in a Lax pair form, to

$$\begin{aligned} &6\sigma V_{xx} + u_2 V = \lambda V \\ &2V_t + u_2 V_x + \lambda V_x + 2\sigma V_{xxx} \end{aligned} \tag{2.221}$$

and now, if $u_2$ and $V$ satisfy (2.221), then $u$ defined in (2.216) satisfies (2.210).

## 2.10 Gevrey classes, least term truncation and Borel summation

Let $\tilde{f} = \sum_{k=0}^{\infty} c_k x^{-k}$ be a formal power series, with power-one of factorial divergence, and let $f$ be a function asymptotic to it. The definition (1.3) provides estimates of the value of $f(x)$ for large $x$, within $o(x^{-N})$, $N \in \mathbb{N}$, which are, as we have seen, insufficient to determine a unique $f$ associated to $\tilde{f}$. Simply widening the sector in which (1.3) is required cannot change this situation since, for instance, $\exp(-x^{1/m})$ is beyond all orders of $\tilde{f}$ in a sector of angle almost $m\pi$.

*If*, however, by truncating the power series at some suitable $N(x)$ instead of a fixed $N$, we can sometimes achieve exponentially good approximations in a sector of width more than $\pi$, then uniqueness is ensured, as this exercise shows:

**Exercise 2.222** Assume $f$ is analytic for $|z| > z_0$ in a sector $S$ of opening more than $\pi$ and that $|f(z)| \le Ce^{-a|z|}$ ($a > 0$) in $S$. Show that $f$ is identically zero. Does the conclusion hold if $e^{-a|z|}$ is replaced by $e^{-a\sqrt{|z|}}$?

(This can be shown using Phragmèn-Lindelöf's principle. Without it, take a suitable inverse Laplace transform $F$ of $f$, show that $F$ is analytic near zero and $F^{(n)}(0) = 0$ and use Proposition 1.56).

This leads us to the notion of Gevrey asymptotics.

**Gevrey asymptotics.**

$$\tilde{f}(x) = \sum_{k=0}^{\infty} c_k x^{-k}, \quad x \to \infty$$

is by definition Gevrey of order $1/m$, or Gevrey-$(1/m)$ if

$$|c_k| \le C_1 C_2^k (k!)^m$$

for some $C_1, C_2$ [7]. There is an immediate generalization to noninteger power series.

**Remark 2.223** The Gevrey order of the series $\sum_k (k!)^r x^{-k}$, where $r > 0$, is the same as that of $\sum_k (rk)! x^{-k}$. Indeed, we have, by Stirling's formula,

$$const^{-k} \le (rk)!/(k!)^r \le const^k$$

Taking $x = y^m$ and $\tilde{g}(y) = \tilde{f}(x)$, then $\tilde{g}$ is Gevrey-1 and we will focus on this case. Also, the corresponding classification for series in $z$, $z \to 0$ is obtained by taking $z = 1/x$.

*

**Definition 2.224** *Let $\tilde f$ be Gevrey-one. A function $f$ is* Gevrey-one asymptotic *to $\tilde f$ as $x \to \infty$ in a sector $S$ if for some $C_1, C_2, C_5$, all $x \in S$ with $|x| > C_5$ and all $N$ we have*

$$|f(x) - \tilde f^{[N]}| \le C_1 C_2^{N+1} |x|^{-N-1} (N+1)! \tag{2.225}$$

i.e., if the error $f - \tilde f^{[N]}$ is, up to powers of a constant, of the same size as the first omitted term in $\tilde f$.

Note the *uniformity requirement* in $N$ and $x$; this plays a crucial role.

**Remark 2.226 (Exponential accuracy)** *If $\tilde f$ is Gevrey-one and the function $f$ is Gevrey-one asymptotic to $\tilde f$, then $f$ can be approximated by $\tilde f$ with exponential precision in the following way. Let $N = \lfloor |x/C_2| \rfloor$ ($\lfloor \cdot \rfloor$ is the integer part); then for any $C > C_2$ we have*

$$f(x) - \tilde f^{[N]}(x) = O(x^{1/2} e^{-|x|/C}), \qquad (|x| \text{ large}) \tag{2.227}$$

Indeed, letting $|x| = NC_2 + \varepsilon$ with $\varepsilon \in [0,1)$ and applying Stirling's formula we have

$$N!(N+1)C_2^N |NC_2 + \varepsilon|^{-N-1} = O(x^{1/2} e^{-|x|/C_2}) \ \square$$

**Note 2.228** *Optimal truncation*, or least term truncation, see e.g., [26], is in a sense a refined version of Gevrey asymptotics. It requires *optimal constants* in addition to an improved form of Rel. (2.225). In this way the imprecision of approximation of $f$ by $\tilde f$ turns out to be smaller than the largest of the exponentially small corrections allowed by the problem where the series originated. Thus the cases in which uniqueness is ensured are more numerous. Often, optimal truncation means stopping near the least term of the series, and this is why this procedure is also known as *summation to the least term.*

## 2.10a Connection between Gevrey asymptotics and Borel summation

The following theorem goes back to Watson [59].

**Theorem 2.229** *Let $\tilde f = \sum_{k=2}^{\infty} c_k x^{-k}$ be a Gevrey-one series and assume the function $f$ is analytic for large $x$ in $S_{\pi+} = \{x : |\arg(x)| < \pi/2 + \delta\}$ for some $\delta > 0$ and Gevrey-one asymptotic to $\tilde f$ in $S_{\pi+}$ as in (2.225). Then*

*(i) $f$ is unique.*

*(ii) $\mathcal{B}(\tilde f)$ is analytic (at $p = 0$ and) in the sector $S_\delta = \{p : \arg(p) \in (-\delta, \delta)\}$, and Laplace transformable in any closed subsector.*

*(iii) $\tilde f$ is Borel summable in any direction $e^{i\theta}\mathbb{R}^+$ with $|\theta| < \delta$ and $f = \mathcal{L}\mathcal{B}_\theta \tilde f$.*

*(iv) Conversely, if $\tilde f$ is Borel summable along any ray in the sector $S_\delta$ given by $|\arg(x)| < \delta$, and if $\mathcal{B}\tilde f$ is uniformly bounded by $e^{\nu|p|}$ in any closed subsector of $S_\delta$, then $f$ is Gevrey-1 with respect to its asymptotic series $\tilde f$ in the sector $|\arg(x)| \le \pi/2 + \delta$.*

**Note.** In particular, when the assumptions of the theorem are met, Borel summability follows using only *asymptotic estimates*.

The Nevanlinna-Sokal theorem [77] weakens the conditions sufficient for Borel summability, requiring essentially estimates in a half-plane only. It was originally formulated for expansions at zero, essentially as follows:

**Theorem 2.230 (Nevanlinna-Sokal)** Let $f$ be analytic in $C_R = \{z : \mathrm{Re}\,(1/z) > R^{-1}\}$ and satisfy the estimates

$$f(z) = \sum_{k=0}^{N-1} a_k z^k + R_N(z) \tag{2.231}$$

with

$$|R_N(z)| \le A\sigma^N N!|z|^N \tag{2.232}$$

uniformly in $N$ and in $z \in C_R$. Then $B(t) = \sum_{n=0}^{\infty} a_n t^n/n!$ converges for $|t| < 1/\sigma$ and has analytic continuation to the strip-like region $S_\sigma = \{t : \mathrm{dist}(t, \mathbb{R}^+) < 1/\sigma\}$, satisfying the bound

$$|B(t)| \le K \exp(|t|/R) \tag{2.233}$$

uniformly in every $S_{\sigma'}$ with $\sigma' > \sigma$. Furthermore, $f$ can be represented by the absolutely convergent integral

$$f(z) = z^{-1} \int_0^\infty e^{-t/z} B(t) dt \tag{2.234}$$

for any $z \in C_R$. Conversely, if $B(t)$ is a function analytic in $S_{\sigma''}$ ($\sigma'' < \sigma$) and there satisfying (2.233), then the function $f$ defined by (2.234) is analytic in $C_R$, and satisfies (2.231) and (2.232) [with $a_n = B^{(n)}(t)|_{t=0}$] uniformly in every $C_{R'}$ with $R' < R$.

**Note 2.235** Let us point out first a possible pitfall in proving Theorem 2.229. Inverse Laplace transformability of $f$ and analyticity away from zero in some sector follow immediately from the assumptions. What does not follow immediately is analyticity of $\mathcal{L}^{-1}f$ at zero. On the other hand, $\mathcal{B}\tilde{f}$ clearly converges to an analytic function near $p = 0$. But there is no guarantee that $\mathcal{B}\tilde{f}$ has anything to do with $\mathcal{L}^{-1}f$! This is where Gevrey estimates enter.

**PROOF of Theorem 2.229**

(i) Uniqueness clearly follows once we prove (ii) and (iii).

(ii) and (iii) By a simple change of variables we arrange $C_1 = C_2 = 1$. The series $\tilde{F}_1 = \mathcal{B}\tilde{f}$ is convergent for $|p| < 1$ and defines an analytic function, $F_1$. By Proposition 1.56, the function $F = \mathcal{L}^{-1}f$ is analytic for $|p| > 0, |\arg(p)| < \delta$, and $F(p)$ is analytic and uniformly bounded by $e^{\nu|p|}$ if $\nu > C_5$

and $|\arg(p)| < \delta_1 < \delta$. We now show that $F$ is analytic for $|p| < 1$. (A different proof is seen in §2.10a.1.) Taking $p$ real, $p \in [0,1)$ we obtain in view of (2.225) that

$$|F(p) - \tilde{F}^{[N-1]}(p)| \le \int_{-i\infty+N}^{i\infty+N} d|s| \left| f(s) - \tilde{f}^{[N-1]}(s) \right| e^{\mathrm{Re}\,(ps)}$$
$$\le N! e^{pN} \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{|x+iN|^N} = N! e^{pN} \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{(x^2+N^2)^{N/2}}$$
$$= \frac{N! e^{pN}}{N^{N-1}} \int_{-\infty}^{\infty} \frac{\mathrm{d}\xi}{(\xi^2+1)^{N/2}} \le C N^{3/2} e^{(p-1)N} \to 0 \text{ as } N \to \infty \quad (2.236)$$

for $0 \le p < 1$. Thus $\tilde{F}^{[N-1]}(p)$ converges. Furthermore, the limit, which by definition is $F_1$, is seen in (2.236) to equal $F$, the inverse Laplace transform of $f$ on $[0,1)$. Since $F$ and $F_1$ are analytic in a neighborhood of $(0,1)$, $F = F_1$ wherever *either* of them is analytic[9]. The domain of analyticity of $F$ is thus, by (ii), $\{p : |p| < 1\} \cup \{p : |p| > 0, |\arg(p)| < \delta\}$.

(iv) Let $|\phi| < \delta$. We have, by integration by parts,

$$f(x) - \tilde{f}^{[N-1]}(x) = x^{-N} \mathcal{L} \frac{d^N}{dp^N} F \quad (2.237)$$

On the other hand, $F$ is analytic in $S_a$, some $a = a(\phi)-$neighborhood of the sector $\{p : |\arg(p)| < |\phi|\}$. Estimating Cauchy's formula on a radius-$a(\phi)$ circle around the point $p$ with $|\arg(p)| < |\phi|$ we get, for some $\nu$,

$$|F^{(N)}(p)| \le N! a(\phi)^{-N} \|F(p) e^{-\nu \mathrm{Re}\ p}\|_{\infty, S_a} e^{\nu \mathrm{Re}\ p}$$

Thus, by (2.237), with $\theta$, $|\theta| \le |\phi|$, chosen so that $\gamma = \cos(\theta - \arg(x))$ is maximal we have

$$\left| f(x) - \tilde{f}^{[N]} \right| = \left| x^{-N} \int_0^{\infty \exp(-i\theta)} F^{(N)}(p) e^{-px} dp \right|$$
$$\le const N! a^{-N} |x|^{-N} \|F e^{-\nu|p|}\|_{\infty;S_a} \int_0^{\infty} e^{-p|x|\gamma + \nu|p| + \nu a} dp$$
$$= const. N! a^{-N} \gamma^{-1} |x|^{-N-1} \|F e^{-\nu\gamma|p|}\|_{\infty;S_a} \quad (2.238)$$

for large enough $x$.

∎

[9] Here and elsewhere we identify a function with its analytic continuation.

### 2.10a.1 Sketch of the proof of Theorem 2.230

We can assume that $f(0) = f'(0) = 0$ since subtracting out a finite number of terms of the asymptotic expansion does not change the problem. Then, we take to $x = 1/z$ (essentially, to bring the problem to our standard setting). Let

$$F = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} f(1/x)e^{px}dx = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} g(x)e^{px}dx$$

We now want to show analyticity in $S_\sigma$ of $F$. That, combined with the proof of Theorem 2.229 completes the argument.

We have

$$f(1/x) = \sum_{j=2}^{N-1}\frac{a_j}{x^j} + R_N(x)$$

and thus,

$$F(p) = \sum_{j=2}^{N-1}\frac{a_j p^{j-1}}{(j-1)!} + \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} R_N(1/x)e^{px}dx$$

and thus

$$|F^{(N-2)}(p)| = \left|a_{N-1} + \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{N-2}R_N(1/x)e^{px}dx\right| \le A_2\sigma^N N!;\quad p\in\mathbb{R}^+$$

and thus $|F^{(n)}(p)/n!| \le A_3 n^2\sigma^n$, and the Taylor series of $F$ at any point $p_0 \in \mathbb{R}^+$ converges, by Taylor's theorem, to $F$, and the radius of convergence is $1/\sigma$. The bounds at infinity follow in the usual way: let $c = R^{-1}$. Since $f$ is analytic for $\operatorname{Re} x > c$ and is uniformly bounded for $\operatorname{Re} x \ge c$, we have

$$\left|\int_{c-i\infty}^{c+i\infty} f(1/x)e^{px}dx\right| \le K_1 e^{cp}\int_{-\infty}^{\infty}\frac{dx}{x^2+1} \le K_2 e^{cp} \tag{2.239}$$

for $p \in \mathbb{R}^+$. In the strip, the estimate follows by combining (2.239) with the local Taylor formula.

**Note 2.240** As we see, *control over the analytic properties of* $\mathcal{B}\tilde{f}$ *near* $p = 0$ *is essential to Borel summability and, it turns out,* BE *summability.* Certainly, mere inverse Laplace transformability of a function with a given asymptotic series, in however large a sector, does not ensure Borel summability of its series. We know already that for any power series, for instance one that is not Gevrey of finite order, we can find a function $f$ analytic and asymptotic to it in more than a half-plane (in fact, many functions). Then $(\mathcal{L}^{-1}f)(p)$ exists, and is analytic in an open sector in $p$, origin not necessarily included. Since the series is not Gevrey of finite order, it can't be Borel summable. What goes wrong is the behavior of $\mathcal{L}^{-1}f$ at zero.

## 2.11 Multiple scales; Adiabatic invariants

We now look at slightly perturbed equations with periodic solutions. This is a represents a class of asymptotic problems in its own right, with many applications from celestial mechanics to the study of oscillators with changing parameters. Interestingly, there are still open problems in this area, see [3]. One of the simplest problems in this category is the slowly changing length $L(t)$.We sketch the derivation of the equations without getting into details (see e.g. [4]). One writes the position in polar coordinates,

$$x(t) = L(t)\sin\theta(t); \;\; y(t) = -L(t)\cos\theta(t) \tag{2.241}$$

writes the kinetic energy $T = \frac{m}{2}(\dot{x}^2 + \dot{y}^2), U = mgy(t)$, then the Lagrangian $\mathcal{L} = T - U$. The motion is described by the Euler-Lagrange equation

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} - \frac{\partial \mathcal{L}}{\partial \theta} = 0$$

which in appropriate units to eliminate $m$ and $g$ reads

$$\ddot{\theta} + \frac{2\dot{L}}{L}\dot{\theta} + \frac{1}{L}\sin\theta = 0 \tag{2.242}$$

and in the approximations $\sin\theta \approx \theta$ we get

$$\ddot{\theta} + \frac{2\dot{L}}{L}\dot{\theta} + \frac{1}{L}\theta = 0 \tag{2.243}$$

The function $L$ is slowly changing; we will take $L(t) = \varphi(\varepsilon t)$. Let's take for simplicity $L(t) = 1 + \varepsilon t$. The equation becomes

$$\ddot{\theta} + \frac{2\varepsilon}{1+\varepsilon t}\dot{\theta} + \frac{1}{1+\varepsilon t}\theta = 0 \tag{2.244}$$

### 2.11a The problem as a regularly perturbed equation; secular terms

Superficially, this appears to be a regularly perturbed problem. So let us see first what regular perturbation theory gives. We substitute

$$\theta = \theta_0 + \varepsilon\theta_1 + \varepsilon^2\theta_2 + \cdots \tag{2.245}$$

in (2.244) and get

$$\theta_0'' + \theta_0 = 0 \tag{2.246}$$

with the general solution $Ae^{it} + Be^{-it}$. By linearity, it suffices to analyze the sequence of equations for $\theta_j$ when $\theta_0 = e^{\pm it}$ and by conjugation symmetry, we can reduce the analysis to the case $\theta_0 = e^{it}$ (this choice is simpler than a trigonometric function, say $\theta_0 = \cos t$ since $d/dt$ generates $\sin t$ as well). Then, $\theta_1$ satisfies

$$\theta_1'' + \theta_1 = (t - 2i)e^{it} \tag{2.247}$$

with the general solution

$$\theta_1 = Ae^{it} + Be^{-it} - (\tfrac{1}{4}it + \tfrac{3}{4})te^{it} = -(\tfrac{1}{4}it + \tfrac{3}{4})te^{it} \tag{2.248}$$

where without loss of generality, we took $A = B = 0$ (other combinations would simply be absorbed in a more general, $\varepsilon-$ dependent, initial condition). Next, we get

$$\theta_2'' + \theta_2 = -(\tfrac{i}{4}t^3 + \tfrac{9}{4}t^2 + \tfrac{9}{2}t + \tfrac{3}{2})e^{it} \tag{2.249}$$

Again choosing the free constants to be zero, we get

$$\theta_2 = -(\tfrac{1}{32}t^3 - \tfrac{5i}{16}t^2 - \tfrac{21}{32}t + \tfrac{3i}{32})te^{it} \tag{2.250}$$

By induction we easily see that $\theta_k$ grows with $t$ like $t^{2k+2}$. For the expansion (2.245) to stay asymptotic we need $t^2\varepsilon \ll 1$ that is, $t \ll \varepsilon^{-1/2}$. But this time is too short for anything interesting to happen, since $L = 1 + \varepsilon t$ is at most of order $1 + O(\varepsilon^{1/2})$, barely away from the initial value 1, and then the expansion becomes invalid. The terms in the expansion that are not periodic in $t$ and lead to grow are called "secular" terms (from Latin –temporal as opposed to eternal). There are various ways to eliminate them (compensate for them would be more accurate). In any case, the solution seems to grow with $t$, but the accuracy does not allow to determine if this is a problem with the expansion or with the solution.

### 2.11b The Poincaré-Lindstedt method

We consider a choice of "local" time variable adapted to the changing frequency: instead of $t$ we use

$$\tau = t + \sum_{l=1}^{k}\sum_{k=1}^{N} a_{kl}\varepsilon^l t^k \tag{2.251}$$

and write again $\theta = \theta_0(\tau) + \varepsilon\theta_1(\tau) + \varepsilon^2\theta_2(\tau) + \cdots$. We try to determine the $a_{kl}$ together with the $\theta_j$ so that the equation is formally satisfied, and so that $\theta_j$ contain no secular terms. The leading equation is the same:

$$\theta_0'' + \theta_0 = 0 \Rightarrow \theta_0 = e^{i\tau} \tag{2.252}$$

The equation for $\theta_1$ is now

$$\theta_1'' + \theta_1 = -\left[-6a_{31}\tau^2 + (6ia_{31} - 4a_{21} - 1)\tau + 2i + 2ia_{21} - 2a_{11}\right]e^{i\tau} \tag{2.253}$$

The right side of (2.253) has to vanish, since $e^{i\tau}$ is *resonant* (it is a solution of the homogeneous equation) and its presence would create secular terms; this determines $a_{31}, a_{21}$ and $a_{11}$. The procedure works order by order, while becoming more and more cumbersome of course. To $o(\varepsilon^2)$ we get

$$\tau = \tau_+(t) = t - (\tfrac{1}{4}t^2 + \tfrac{3i}{4}t)\varepsilon + (\tfrac{1}{8}t^3 - \tfrac{3i}{8}t^2 - \tfrac{3}{32}t)\varepsilon^2 + (-\tfrac{5}{64}t^4 + \tfrac{9}{128}t^2 + \tfrac{3i}{64}t)\varepsilon^3 \tag{2.254}$$

We can proceed similarly with a second solution of (2.244) starting with $e^{-it}$ and get

$$\tau_-(t) = \left(1 - \frac{3i}{4}\varepsilon - \frac{3}{32}\varepsilon^2\right) t - \left(\frac{1}{4}\varepsilon - \frac{3i}{8}\varepsilon^2\right) t^2 + \frac{1}{8}\varepsilon^2 t^3 + o(\varepsilon^2) \tag{2.255}$$

and finally obtain a general (approximate) solution in the form

$$\theta = C_+ e^{i\tau_+(t)} + C_- e^{i\tau_-(t)} \tag{2.256}$$

If carried to all orders, this covers a time interval $t\varepsilon \ll 1$, where the length still changes very little, but show that there is no growth of the solution on this larger time scale.

### 2.11c Multi-scale analysis

We describe this briefly here (see [8] for more details), but will not elaborate since for many equations coming for applications (such as Hamiltonian systems) there are better approaches. Multi-scale analysis apparently also goes back to Poincaré and Lindstedt and is meant to make the series calculations more systematic. The problem at hand presents *two scales*: the fast one, $t \sim 1$ related to the period of cos and a scale of order $t\varepsilon = \tau \sim 1$ where it appears that the *period starts to change*. We then write, with $\tau(t) = t\varepsilon$,

$$\theta(t) \sim \sum_{k=0}^{\infty} \Theta_k(t, \tau(t))\varepsilon^k \tag{2.257}$$

treat $t$ and $\tau$ as if they were independent variables and get a system of PDEs:

$$\partial_t^2 \Theta_0 + (1+\tau)^{-1}\Theta_0 = 0 \tag{2.258}$$

The equation for $\Theta_1$ is

$$\partial_t^2 \Theta_1 + (1+\tau)^{-1}\Theta_1 = -2\partial_{t\tau}^2\Theta_0 - 2(1+\tau)^{-1}\partial_t\Theta_0 \tag{2.259}$$

with general solution

$$\Theta_0 = F_+(\tau)e^{it(1+\tau)^{-1/2}} + F_-(\tau)e^{-it(1+\tau)^{-1/2}} \tag{2.260}$$

To obtain further terms, one substitutes (say) $\Theta_0 = F_+(\tau)e^{it(1+\tau)^{-1/2}}$ in the equation for $\Theta_1$ and determines $F(\tau)$ to eliminate secular terms. This is not always possible, and the reason is that the method of multiple scales does not allow for substantial changes in oscillation frequency; they have to be addressed by changes of dependent variable if they do, see [8]. Our equation is one that would need to be modified, since it leads to the inconsistent equation

$$\frac{dF}{d\tau} + \left( \frac{1}{2(1+\tau)} - \frac{it}{2(1+\tau)^{3/2}} \right) F = 0 \tag{2.261}$$

which is inconsistent because of the presence of $t$ while $F$ is a function of $\tau$ only. Once more, we will be content with (2.260), which we will compare with results gotten by more systematic methods, and send to [8] on how to amend the approach.

### 2.11d How do the parameters of the motion change when $L$ is doubled?

Of course none of the methods above gives us any information on this; we have some series expansions that are not known to converge, and the question is no simpler than trying to predict the behavior of an analytic function beyond the disk of analyticity, when only estimates on the Taylor coefficients are provided. A representation with wider range of validity is needed.

### 2.11e WKB

What we saw in (2.256) an (2.255) is that in fact if we use a power series inside an exponential instead of just a ordinary power series, the domain of validity increases. This suggests that there is an underlying WKB setting. Since the problem is linear, WKB is applicable. The variable that needs to be small in (2.256) and (2.255) for the expansions to be valid is the “slow” variable $s = t\varepsilon$. We change thus to this variable, $s = t\varepsilon$, $\theta(t) = v(s)$, and we get

$$\varepsilon^2 v'' + 2\varepsilon^2 \frac{v'}{1+s} + \frac{v}{1+s} \tag{2.262}$$

which is singularly perturbed. Performing first a Liouville transformation $v = hg$ and choosing $h = (1+s)^{-1}$ so that the first derivative term vanishes we get

$$g'' + \varepsilon^{-2} \frac{g}{1+s} = 0 \tag{2.263}$$

where, as usual, we substitute $g = e^w$, $w' = f$, and obtain

$$f = \pm i \sqrt{\varepsilon^{-2}(1+s)^{-1} + f'} \tag{2.264}$$

wherefrom, by iteration, we get

$$w = \pm 2i(1+s)^{1/2}\varepsilon^{-1} + \frac{1}{4}\ln(1+s) \pm \frac{3i\varepsilon}{16\sqrt{1+s}} + \frac{3\varepsilon^2}{64(1+s)} \tag{2.265}$$

and thus, expressing the results in terms of the original $\theta$, we get two asymptotic solutions,

$$\theta_{\pm} = (1+s)^{-3/4} \exp\left(\pm\frac{2i}{\varepsilon}\sqrt{1+s}\right)\left(1+\frac{3i\varepsilon}{16\sqrt{1+s}}+\frac{15\varepsilon^2}{512(1+s)}+\cdots\right) \tag{2.266}$$

At this stage we note that taking

$$\theta = A(\varepsilon)\theta_+; \text{ where } A(\varepsilon) = \exp\left(-\frac{2i}{\varepsilon}-\frac{3i\varepsilon}{16}-\frac{3\varepsilon^2}{64}\right) \tag{2.267}$$

the Taylor series of $\theta$ at $\varepsilon = 0$ is

$$\theta = e^{it}\left[1-\left(\frac{i}{4}t+\frac{3}{4}\right)t\varepsilon+\left(-\frac{1}{32}t^3+\frac{5i}{16}t^2+\frac{21}{32}t-\frac{3i}{32}\right)t\varepsilon^2+\cdots\right] \tag{2.268}$$

which is what we get by combining (2.248) and (2.250). Likewise, if we take

$$\varphi = \ln(\theta_+) - \frac{2i}{\varepsilon} - \frac{3i\varepsilon}{16} - \frac{3\varepsilon^2}{64} \tag{2.269}$$

and expand $\varphi$ in series we get

$$\varphi = it - \left(\frac{it}{4}+\frac{3}{4}\right)t\varepsilon + \left(\frac{it^2}{8}+\frac{3t}{8}-\frac{3i}{32}\right)t\varepsilon^2+\cdots \tag{2.270}$$

which is the expansion (2.254). We see that the regular perturbation expansion containing secular terms and the Poincaré-Lindstedt series simply correspond to various re-expansions of the WKB solutions. The range of validity of the Poincaré-Lindstedt series, if calculated to all orders is $t\varepsilon \ll 1$ since we are expanding $\ln\theta_+$ for small $\tau$. The regular perturbation expansion has an even smaller range of validity, as we are also expanding out the exponential. The term $e^{const.t\varepsilon^2}$ cannot be expanded asymptotically in $\varepsilon$ when $t\varepsilon^2 \not\ll 1$. Thus, the very narrow domain of validity of the regular perturbation expansion is explained by the fact that exponential behavior cannot be approximated by power series, in a region where the exponent is large. The fact that the expansion breaks down is only a sign of the mismatch in behavior type, and not an actual change in the solutions.

**Note 2.271** The form (2.266) is valid for all $|t\varepsilon$ provided no singularities or turning points are crossed. Here, this simply means $t \in \mathbb{R}^+$.

### 2.11f The adiabatic invariant

For a pendulum of fixed length, (2.243) takes the form

$$\ddot{\theta} + L^{-1}\theta =: \ddot{\theta} + \omega^2\theta = 0 \tag{2.272}$$

Multiplying by $L^2\dot\theta$ and integrating we get

$$E = \tfrac{1}{2}L^2\dot\theta^2 + L\tfrac{1}{2}\theta^2 = const. \approx \tfrac{1}{2}v^2 + L(1-\cos\theta) \tag{2.273}$$

where $E$ is the total energy. This is a conserved quantity, or an invariant of the motion. What happens when $L = L(t)$? Are there conserved quantities? Clearly, since the general solution is

$$\theta = C_+\theta_+ + C_-\theta_- \tag{2.274}$$

any combination of $C_+$ and $C_-$ is conserved. We then solve for $C_+$ and $C_-$ as a function of $\theta, \dot\theta$. Up to numerical constants of no relevance, and to leading order in $\varepsilon$, we have –recall that $\dot\theta$ denotes $\frac{d\theta}{dt}$

$$C_+ \sim (1+s)^{3/4}(\theta + i\dot\theta\sqrt{1+s})\exp\left(-\frac{2i}{\varepsilon}\sqrt{1+s}\right) \tag{2.275}$$

$$C_- \sim (1+s)^{3/4}(\theta - i\dot\theta\sqrt{1+s})\exp\left(\frac{2i}{\varepsilon}\sqrt{1+s}\right) \tag{2.276}$$

Certainly $C_+$ and $C_-$ are constant to the order presented, see Note 2.271. Both of them oscillate rapidly on the slow, $s$, scale. We notice however that $C_+C_-$ does not oscillate, and in fact any constant of motion that does not change rapidly on the $s$ scale is a function of $C_+C_-$:

$$C_+C_- \sim L(t)^{3/2}\theta^2 + L(t)^{5/2}\dot\theta^2 = L(t)^{1/2}E(t) = \frac{E(t)}{\omega(t)} = E_0(1+o(1)) \tag{2.277}$$

see (2.273). The quantity $E(t)/\omega(t)$ ($\omega(t)$ being the instantaneous frequency) is an *adiabatic invariant*: it is constant *to leading order* along the solutions of (2.243). How does the pendulum behave when $L \to \lambda L$, $\lambda > 1$? Since $E = L\theta_{max}^2/2$, (2.282) implies that the amplitude $\theta_{max}$ decreases by a factor of $\lambda^{3/4}$. To find the position at a time $t$, one needs calculate a few orders in the asymptotic expansion of $C_+$ and $C_-$ until enough accuracy is obtained to determine $\theta(t)$. This gives the solution of the main *connection problem*, relating two positions after a very long time.

Having obtained the absolute position at time $t$, for a small number of periods centered at $t$ the behavior of the pendulum is then well approximated by one of length $L(t)$, energy $E_0L(t)^{-1/2}$ and initial location calculated above. We note that a numerical approach would require integration over a very long time, with a high number of digits to avoid accumulation of errors, a demanding task.

## 2.11g Solution for more general $L$

We now write the length as $L(\varepsilon t)$; with the change of variable $s = t/\varepsilon$, $\theta(t) = y(s)$,(2.243) and $'$ denoting $d/ds$ we get

$$y'' + \frac{2L'}{L}y' + \frac{1}{\varepsilon^2 L}y' = 0 \tag{2.278}$$

which we solve for small $\varepsilon$ by the usual WKB substitution $y(s) = \exp(w(s)/\varepsilon)$; this leads to

$$w' = \pm \frac{i}{\varepsilon\sqrt{L}} \sqrt{1 + \varepsilon^2 L \left( w'' + \frac{2L'w'}{L} \right)} \tag{2.279}$$

where we iterate in the usual way,

$$w'^{[n+1]} = \pm \frac{i}{\varepsilon\sqrt{L}} \sqrt{1 + \varepsilon^2 L \left( w''^{[n]} + \frac{2L'w'^{[n]}}{L} \right)} \tag{2.280}$$

and this yields

$$y_{\pm} = L^{-\frac{3}{4}} \exp\left( \pm \frac{i}{\varepsilon} \int \frac{1}{\sqrt{L(u)}} du \right) \left[ 1 \mp \frac{3i\varepsilon}{8} \left( \frac{L'}{L^{\frac{1}{2}}} + \frac{3}{4} \int \frac{L'(u)^2}{L(u)^{3/2}} \right) + \cdots \right] \tag{2.281}$$

Taking $y = C_+y_+ + C_-y_-$, and solving for $C_{\pm}$ in terms of $y(s) = \theta(t), \varepsilon \frac{d}{ds} y = \dot{\theta}$, we get

$$K(\theta, \dot{\theta}, t) = C_-C_+ = \frac{1}{4}\frac{E(t)}{\omega(t)} + \frac{3}{20}\frac{dL^{\frac{5}{2}}}{ds} \varepsilon\, \theta\dot{\theta} + o(\varepsilon);$$
$$\omega(t)^2 = L(\varepsilon t),\ \ E(t) = L(\varepsilon t)^2 \dot{\theta}^2 + L(\varepsilon t)\theta(t)^2 \tag{2.282}$$

Using (2.278) we get that the variation of $K$ is of order $\varepsilon^2$:

$$\frac{d}{dt} K(\theta, \dot{\theta}, t) = \tfrac{3}{2} L^2 \tfrac{d}{ds} \left( L^{-\frac{1}{2}} \tfrac{dL}{ds} \right) \theta\dot{\theta} \tag{2.283}$$

### 2.11h Working with action-angle variables

The WKB method allows for a rigorous, precise and uniform asymptotic analysis. A serious limitation of the method is that is does not easily extend to nonlinear problems. We discuss, at an informal level a method that generalizes to nonlinear systems as well. we first look at the pendulum, and then apply the method to the equation $\mathrm{P_I}$.

In the pendulum problem, the angle is periodic, a natural *angle variable*. It is convenient to pass to the angle as an independent variable to eliminate the oscillation. To work with slowly changing we choose one of them to be a constant of motion for the pendulum of fixed length. In that case

$$\theta\ddot{\theta} + \frac{\theta\dot{\theta}}{L} = 0 \Rightarrow \dot{\theta}^2 + L^{-1}\theta^2 = Q(\theta, \dot{\theta}) = const. \tag{2.284}$$

This is, up to a multiplicative constant, the energy $2E = L^2\dot{\theta}^2 + L\theta^2$; $Q$ or $E$ can be chosen equivalently as a slow variable, and the calculations are quite similar. We prefer to treat the question as a mathematical one, and from this perspective $Q$ is more natural. To eliminate to leading order the oscillatory part of the evolution, we proceed as follows. We perform a hodograph-like transformation, taking $\theta$ to be the independent variable and $t$ and $Q$ to be the dependent ones. Then we analyze the change in $t$ and $Q$ after a complete $\theta$ cycle.

It is simpler to describe the procedure when $L$ is analytic. We then evolve $\theta$ on a positively oriented loop in $\mathbb{C} \setminus J$ where cut $J = (-a, a)$ contains the interval $[-E/L, E/L]$. Without analyticity assumptions, the procedure would be to evolve $\theta$ from $-\sqrt{E/L}$ to $\sqrt{E/L}$ and back to $-\sqrt{E/L}$ changing the sign of the square root every time it becomes zero, to preserve smoothness of the quantities involved.

We obtain

$$\dot{Q} = 2\dot{\theta}\ddot{\theta} + 2L^{-1}\theta\dot{\theta} - L^{-2}\dot{L}\theta^2 \tag{2.285}$$

$$\frac{dt}{d\theta} = \frac{1}{u}; \quad u := \dot{Q} = L^{-\frac{1}{2}}\sqrt{\beta^2 - \theta^2}; \ \beta^2 = QL \tag{2.286}$$

With $\delta = L^{-\frac{3}{2}}\dot{L}$ and substituting $\ddot{\theta}$ from (2.243), we get

$$\frac{dQ}{d\theta} = -\delta\left(4\sqrt{\beta^2 - \theta^2} + (\beta^2 - \theta^2)^{-\frac{1}{2}}\right) \tag{2.287}$$

$$\frac{dt}{d\theta} = L^{\frac{1}{2}}(\beta^2 - \theta^2)^{-\frac{1}{2}} \tag{2.288}$$

It is convenient to use a second slow function of $t$, for instance $L$:

$$\frac{dQ}{d\theta} = -\delta\left(4\sqrt{\beta^2 - \theta^2} + (\beta^2 - \theta^2)^{-\frac{1}{2}}\right)$$
$$\frac{dL}{d\theta} = L^2\delta(\beta^2 - \theta^2)^{-\frac{1}{2}} \tag{2.289}$$

We write (2.289) in integral form, with $\theta_i$ an initial value of $\theta$,

$$Q(\theta) = Q(\theta_i) - \int_{\theta_i}^{\theta} \delta(s)\left(4\sqrt{\beta^2 - s^2} + (\beta^2 - s^2)^{-\frac{1}{2}}\right) ds \tag{2.290}$$

$$L(\theta) = L(\theta_i) + \int_{\theta_i}^{\theta} L^2(s)\delta(s)(\beta^2 - s^2)^{-\frac{1}{2}} ds \tag{2.291}$$

If $\theta$ evolves for say, a loop or less, $\delta$, $L$ and $\beta$ are approximately constant, equal to their value at $\theta_i$ which we denote by $\delta_i, L_i, \beta_i$ respectively. This can be shown in a straightforward way by noticing that the right side of (2.284) is contractive mapping in the sup norm. We omit the straightforward details. To leading order, the integrals can then be calculated explicitly, and the result is

$$Q(\theta) - Q(\theta_i) = -\delta_i\left(\frac{3s}{2}\sqrt{\beta_i^2 - s^2} - \frac{5}{2}\beta_i^2 \arcsin(s/\beta_i)\right)\Big|_{\theta_i}^{\theta}(1 + o(1))$$
$$L(\theta) - L_i = L_i^2\delta_i \arcsin(s/\beta_i)|_{\theta_i}^{\theta}(1 + o(1)) \tag{2.292}$$

To eliminate to leading order the $\theta$ dependence, we can calculate the Poincaré map, that is the change of $Q$ and $L$ after one full loop. We denote by $Q_j$ and $L_j$ the value of these quantities after $j$ loops. In the complement of the cut, $\sqrt{\beta^2 - s^2}$ is single-valued, and thus only the arcsin changes, by $2\pi$. We obtain the recurrence

$$Q_{n+1} - Q_n = -5\pi\delta_n Q_n L_n(1 + o(1));\ L_{n+1} - L_n = 2\pi L_n^2 \delta_n \qquad (2.293)$$

The right side of (2.293) is small and the recurrence is to leading order approximated by a differential equation

$$\frac{dQ}{dL} = -\frac{5}{2}\frac{Q}{L} \Rightarrow QL^{\frac{5}{2}} = EL^{\frac{1}{2}} = \frac{E}{\omega} = const. \qquad (2.294)$$

thus recovering to leading order (2.282). To find the long time behavior of the pendulum with changing length, one calculates the adiabatic invariant with sufficiently many orders as a function of $L$, after $n$ loops; $L$ is known as a function of $t$, and after $n$ complete loops the position $\theta$ is known (through $Q$) and the initial velocity is zero. The missing part of the evolution, the one from loop $n$ to loop $n+1$ is obtained from the integral system (2.290)

The analysis can be carried out without the linearization $\sin\theta \approx \theta$, the calculations now involving elliptic functions.

### 2.11i The Painlevé equations $\mathrm{P_I}$

We analyze now a nonlinear problem– $\mathrm{P_I}$, (2.174)– in the region where solutions have poles [22]. For the analysis of forst order equations, see [23]

We use a different normalization of $\mathrm{P_I}$ so that instead of (2.173), the equation is in the form

$$y'' = 6y^2 + z \qquad (2.295)$$

For an equation allowing for formal, factorially divergent, power series solutions, the normalized form is the one in which the series are Gevrey-1, see Note 2.40 on p.61. This normalization often works best in studying the general solution as well, see Note 2.334 below.

Looking for a power behavior for large $z$, we substitute $y = Az^b$ in (2.295), and this gives $A = \pm\sqrt{-1/6}$ and $b = \frac{1}{2}$. This balance is consistent and leads to formal power series solutions $y \sim \pm\sqrt{\frac{-z}{6}}$ for large $z$.

We will study the family of solutions with $y \sim +\sqrt{\frac{-z}{6}}$ as the opposite sign can be treated similarly. Their transseries can be obtained by determining first the asymptotic power series $\tilde{y}_0$ with leading order $+\sqrt{\frac{-z}{6}}$. Then by linear perturbation theory around it one finds the form of the small exponential, and notices that the exponential is determined up to one multiplicative parameter.

We get the transseries solution

$$\tilde{y} = \sqrt{\frac{-z}{6}} \sum_{k=0}^{\infty} \xi^k \tilde{y}_k \tag{2.296}$$

where

$$\xi = \xi(z) = Cx^{-1/2}\mathrm{e}^{-x}; \quad \text{with} \quad x = x(z) = \frac{(-24z)^{5/4}}{30} \tag{2.297}$$

and $\tilde{y}_k$ are power series, in particular

$$\tilde{y}_0 = 1 - \frac{1}{8\sqrt{6}(-z)^{5/2}} - \frac{7^2}{2^8 \cdot 3}\frac{1}{z^5} - ... - \frac{\tilde{y}_{0;k}}{(-z)^{5k/2}} - ...$$

To normalize the equation, cf. again Note 2.40, the new independent variable $x$ is chosen to be $x$ so that the exponent is linear. It is also convenient to pull out $\sqrt{\frac{-z}{6}}$ from the dependent variable. We take

$$x = \frac{(-24z)^{5/4}}{30}; \; y(z) = \sqrt{\frac{-z}{6}}\, Y(x)$$

and $\mathrm{P_I}$ becomes

$$Y''(x) - \frac{1}{2}Y^2(x) + \frac{1}{2} = -\frac{1}{x}Y'(x) + \frac{4}{25}\frac{1}{x^2}Y(x) \tag{2.298}$$

which, in fact, coincides with Boutroux's form (cf. [61]). To apply the results in [46] and [44], (2.298) needs to be further normalized and to this end we subtract the $O(1)$ and $O(x^{-1})$ terms of the asymptotic behavior of $Y(x)$ for large $x$. It is convenient to subtract also the $O(x^{-2})$ term (since the resulting equation becomes simpler). Then the substitution

$$Y(x) = 1 - \frac{4}{25x^2} + h(x)$$

transforms (2.298) to

$$h'' + \frac{1}{x}h' - h - \frac{1}{2}h^2 - \frac{392}{625x^4} = 0 \tag{2.299}$$

Finally, the results in [44] and in [46] apply to first order systems of the form

$$\mathbf{y}' + \left(\hat{\Lambda} + \frac{1}{x}\hat{B}\right)\mathbf{y} = \mathbf{g}(x^{-1}, \mathbf{y}); \; \hat{\Lambda} = \mathrm{diag}(\lambda_i), \; \hat{B} = \mathrm{diag}(\beta_i) \tag{2.300}$$

(eq. (1.1) [46]) in where $g = O(x^{-2}, y^2)$, rather than to $n-$th order equations. We then write

$$\begin{pmatrix} h \\ h' \end{pmatrix}' = \begin{pmatrix} 0 \\ \frac{392}{625}x^{-4} \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} h \\ h' \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & -\frac{1}{x} \end{pmatrix}\begin{pmatrix} h \\ h' \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{2}h^2 \end{pmatrix} \tag{2.301}$$

Simple algebra shows that the transformation

$$\begin{pmatrix} h \\ h' \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1-\frac{1}{4x} & 1+\frac{1}{4x} \\ -1-\frac{1}{4x} & 1-\frac{1}{4x} \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \tag{2.302}$$

brings (2.301) to the normal form (2.300).

**Note 2.303 (Results from [46] and [45])** (i) By Theorem (ii) in [46] if $\mathbf{y}$ is a solution of the system (2.300) with $\mathbf{y} = o(x^{-3})$[10] for $x \to \infty$, $x \in e^{-i\phi}\mathbb{R}$ (for some $\phi$) then $\mathbf{y}$ has a *unique Borel summed transseries*: for some $C$

$$\mathbf{y}(x; C) = \sum_{k=0}^{\infty} C^k e^{-kx}(\mathcal{L}_\phi \mathbf{Y}_k)(x) \quad \text{for } \ x \in e^{-i\phi}\mathbb{R},\ |x| \text{ large} \tag{2.304}$$

where $\mathbf{Y}_0 = p^3\mathbf{A}_0(p)$ $\mathbf{Y}_k(p) = p^{k/2-1}\mathbf{A}_k(p)$, with $\mathbf{A}_k(p)$ *independent of* $C$ and analytic in $\mathbb{C} \setminus \{\pm1, \pm2, \ldots\}$. With $\mathbf{F}_k(p) = \int_0^p \mathbf{Y}_k(s)ds$, all $\mathbf{F}_k(|p|e^{i\phi})$ are left and right continuous in $\phi$ at $\phi = 0$ and $\phi = \pi$. There exist $\nu$ and $M$ independent of $k$ such that $\sup_{p\in\mathbb{C}\setminus\mathbb{R}} |\mathbf{F}_k(p)e^{-|p|\nu}| \le M^k$.

(ii) The constant $C$ in (2.304) depends on the direction $\phi$: $C = C(\arg x)$ is piecewise constant; it can only change at the Stokes rays.

(iii) We have

$$\mathcal{L}_\phi \mathbf{Y}_k := \mathbf{y}_k \sim \mathbf{c}_k x^{-\frac{k}{2}};\ k \geqslant 1; \mathbf{y}_0 = O(x^{-4}) \ \text{ for } x \to \infty, \quad x \in e^{-i\phi}\mathbb{R} \tag{2.305}$$

where, if $C \neq 0$, $c_1$ is chosen to be 1 by convention, thus fixing $C$.

(iv) For any $\delta > 0$ there is $b > 0$ so that for all $k \geqslant 0$ and $\phi$ in $(-\pi, 0)\cup(0, \pi)$ we have $\int_0^\infty |\mathbf{Y}_k(pe^{-i\phi})|e^{-bp}dp < \delta^k$ (Proposition 20 in [46]).

**Note 2.306** (i) Algebraically, the equation is simpler in variables $(h, h')$ than in $\mathbf{y}$, and it is more convenient to work directly with the second order equation (2.173); the results in [46], [44] and [45] translate easily through the linear substitution (2.302) into results about $h$ and $H := \mathcal{L}^{-1}h$. In particular, (2.308) below holds for solutions $h = o(x^{-3})$, where $H_k$ satisfy all the analyticity properties and, up to constants, bounds satisfied by $\mathbf{Y}_k$.

The following result follows from [45]:

**Lemma 2.307** *(i) Assume $h$ solves* (2.299) *and satisfies* $h(x) = o(x^{-3})$ *as* $x \to \infty$ *with* $\arg x \in (-\frac{\pi}{2}, \frac{\pi}{2})$. *Then* $h(x) \sim C_+x^{-1/2}e^{-x}$ *as* $x \to +i\infty$ *for some* $C_+$.

*Furthermore, for such $h$ there is $C_-$ such that $h \sim C_-x^{-1/2}e^{-x}$ as $x \to -i\infty$, and there exists a unique sequence $\{c_k\}_k$ such that $h \sim \frac{392}{625x^4} + \sum_{k=5}^{\infty} c_k x^{-k}$ as $x \to +\infty$ where the asymptotic expansion is differentiable.*

[10]This is the case for (2.299), where $h = O(x^{-4})$

*A solution $h$ as above has the Borel summed transseries representations:*

$$h(x) = \sum_{k=0}^{\infty} C_{\pm}^{k} e^{-kx} (\mathcal{L}_{\phi} H_k)(x) \quad \textit{for} \ \pm\phi \in (0, \pi/2] \tag{2.308}$$

*where $H_k(p) = p^{\frac{k}{2}-1} A_k(p)$ and $A_k$ are analytic in $\mathbb{C}_r := \mathbb{C} \setminus \{\pm 1, \pm 2, \ldots\}$. The functions $H_k$ satisfy bounds of the type in Notes 2.306 and 2.303(iv).*

For $\phi \in (0, \pi)$, the functions $(\mathcal{L}_{\phi} H_k)(x)$ are analytic in a sector $(-\pi/2, 3\pi/2)$. However, the exponentials $e^{-kx}$ blow up in the left half plane. The transseries expansion (2.308) cannot hold in the left half plane, and close to the direction where its asymptoticity fails we need to match it to a different expansion. As usual, we first identify the critical quantity, call it $\xi$, responsible for loss of asymptoticity. In this case, it is $\xi = Ce^{-x} x^{-1/2}$, where we took into account the dominant behavior of $\mathbf{Y}$, (2.305) or equivalently the behavior of $\mathcal{L}_{\phi} H_k$.

We then re-expand $\mathbf{Y}$ as a function series in powers of $1/x$ and coefficients depending on $\xi$. This is similar to a two scale expansion, except that there is no "external" small parameter $\varepsilon$, its role being played by $x^{-1}$.

Given $h$ as in (2.304), there is a unique constant $C_+$ with the following properties. The leading behavior of $h$ for large $|x|$ with $\arg x$ close to $\pi/2$ is

$$h \sim H_0(\xi) + \frac{H_1(\xi)}{x} + \frac{H_2(\xi)}{x^2} + \cdots \quad (x \to i\infty \text{ with } |\xi - 12| > \varepsilon, \ |\xi| < M) \tag{2.309}$$

Formally, these functions are simply obtained as in a two scale expansion, thinking of $x$ and $\xi$ as being practically independent variables. Substituting (2.309) in (2.299) the equation for $H_0$ is the leading order one, the coefficient of $1/x^0$,

$$H_0'' - H_0 - \frac{1}{2} H_0^2 = 0; \quad H_0(\xi) \sim \xi \text{ as } \xi \to 0 \tag{2.310}$$

The initial condition is obtained from the condition that (2.309) matches (2.304) when $e^{-x} \to 0$. Similarly, the coefficients of $1/x, \ldots$ give

$$H_1(\xi) = \frac{-\frac{1}{60}\xi^4 + 3\xi^3 + 210\xi^2 + 216\xi}{(12 - \xi)^3}, \ldots, H_n(\xi) = \frac{P_n(\xi)}{\xi^n (\xi - 12)^{n+2}} \tag{2.311}$$

with $P_n$ polynomials of degree $3n + 2$. There is an equivalent of the non-secularity condition: each $H_n$ contains a free constant which is determined from the equation of $H_{n+1}$ by requiting that $H_{n+1} = O(1)$ for large $x$ (for a general constants, one would get $H_{n+1} = O(x)$, $H_{n+1} = O(x^2)$ etc. undermining the asymptoticity of the series). In [45], the validity of (2.309) is proved in a general setting.

The first array of poles beyond $i\mathbb{R}^+$ is located at points $x = p_n$ near the solutions $\tilde{p}_n$ of the equation $\xi(x) = 12$ where $H_0$ has a pole:

$$p_n = \tilde{p}_n + o(1) = 2n\pi i - \frac{1}{2} \ln(2n\pi i) + \ln C_+ - \ln 12 + o(1), \quad (n \to \infty) \tag{2.312}$$

Rotating $x$ further into the second quadrant, $h$ develops successive arrays of poles separated by distances $O(\ln x)$ of each other as long as $\arg(x) = \pi/2 + o(1)$ [45].

**Note 2.313** The array of poles developed near the other edge of the sector of analyticity, for $\arg(x) = -\pi/2 + o(1)$, is obtained by the conjugation symmetry.

### 2.11i.1 Region of validity of the expansion (2.309)

This expansion is valid in the transseries region and in a domain containing one, essentially vertically aligned, array of poles. There are in fact infinitely many arrays of poles in the fourth quadrant, and the $H_n$ expansion above fails to be asymptotic after the first array. One can however proceed in a similar manner, finding a new $\tilde{\xi}$ to obtain a valid expansion near the second array of poles. However, very much as in a two-scale expansion, this re-expansion method does not work in a sufficiently wide area. In fact, angularly, it only covers a sector of rough width $x^{-1}\ln x$ after which no further matching with expansions of the form (2.309) is possible. Beyond this narrow region we need to do something else.

Since $WKB$ is not suitable for nonlinear equations, we use a method similar to an adiabatic invariant representation. Note that for large $x$ (2.299) is close to the autonomous Hamiltonian system

$$h'' - h - h^2/2 = 0 \tag{2.314}$$

with Hamiltonian

$$s = {h'}^2 - h^2 - h^3/3 \tag{2.315}$$

The solutions of (2.314) are elliptic functions, doubly periodic in $\mathbb{C}$. With $w = u'$ we first rewrite equation (2.299) as a system

$$\frac{du}{dx} = w \tag{2.316}$$

$$\frac{dw}{dx} = u + \frac{u^2}{2} - \frac{w}{x} + \frac{392}{625}\frac{1}{x^4} \tag{2.317}$$

We expect the solutions to be asymptotically periodic, and $s$ to be a slow varying quantity; this is certainly the case in the region where (2.309) holds. It is then natural to take $h =: u$ as an independent angle-like variable and treat $s$ and $x$ as dependent variables. and then, with

$$R(u,s) = \sqrt{u^3/3 + u^2 + s} \tag{2.318}$$

we transform (2.316), (2.317) into a system for $s(u)$ and $x(u)$:

$$\frac{ds}{du} = -\frac{2w}{x} + \frac{784}{625}\frac{1}{x^4} = -\frac{2R(u,s)}{x} + \frac{784}{625}\frac{1}{x^4} \tag{2.319}$$

$$\frac{dx}{du} = \frac{1}{w} = \frac{1}{R(u,s)} \tag{2.320}$$

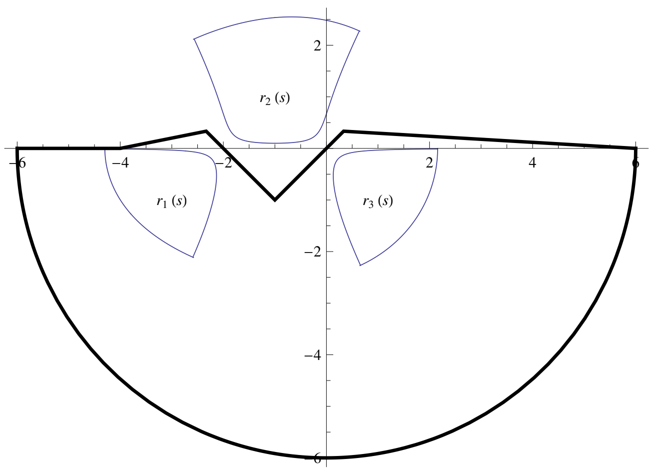


**FIGURE 2.2**: Regions of the roots of $u^3/3 + u^2 + s$ and the contour $\mathcal{C}$. The $r_i$s are the regions where the three roots of $R$ change as $x$ traverses $\Sigma$, and $s = s(x)$ changes accordingly.

**Note 2.321** Given an initial condition $u_I, s(u_I), x(u_I)$ such that the right side of (2.319), (2.320) is analytic, the system $\{(2.319), (2.320)\}$ admits a locally analytic solution $x(u), s(u)$. The function $x(u)$ is analytically invertible by the inverse function theorem since $1/R \neq 0$. Using (2.319) this determines an analytic $s(x)$. From $s$ and $u$, we define an analytic branch of $w = u'$. The systems $\{(2.316), (2.317)\}$ and $\{(2.319), (2.320)\}$ are then equivalent in any domain in which $u, u', s(u), x(u)$ are analytic.

To obtain a nontrivial evolution, we evolve $u$ on a loop around exactly two of the three roots of $R$, the only choice to have an infinitely-sheeted Riemann surface. Indeed the evolution around one square-root branch point would give rise to two sheets; an evolution around all three corresponds to a loop around infinity, which is also a square-root branch point of $R$.

It turns out that there are closed curves $\mathcal{C}$, see Fig. 2.2, similar to the classical *cycles* [35], such that $R(u, s(u))$ does not vanish on $\mathcal{C}$ and $x(u)$ traverses $\Sigma$ from edge to edge as $u$ travels along $\mathcal{C}$ a number $N_m$ times. More precisely, starting with $u_0 \in \mathcal{C}$ and writing $u_n$ instead of $u_0$ to denote that $u$ has traveled $n$ times along $\mathcal{C}$, $s_n = s(u_n)$ and $x_n = x(u_n)$, the following hold: (i) $x_0 = x(u_0)$ is close to the first array of poles near $i\mathbb{R}$, $\arg(x_0) = -\pi/2(1+o(1))$, and $s(u_0)$ is given by (2.315) (recall that $u = h$) (ii) for some $N = N_m(x_0)$, $x_N$ is close to the last array of poles, $\arg(x_N) = -\pi(1 + o(1))$. The size of $|x_n|$ is of the order $|x_0|$ for all $n \leq N$. Two roots of $R(u, s_n), n = 0, 1.., N_m$ are in the interior of $\mathcal{C}$ and a third one is in its exterior. Written in integral form, (2.319) and (2.320) become

$$s(u) = s_n - 2\int_{u_n}^{u} \left( \frac{R(v,s(v))}{x(v)} - \frac{392}{625}\frac{1}{x(v)^4} \right) dv \tag{2.322}$$

$$x(u) = x_n + \int_{u_n}^{u} \frac{1}{R(v,s(v))} dv \tag{2.323}$$

where the integrals are along $\mathcal{C}$.

### 2.11i.2 The Poincaré map

As in the case of the pendulum, an important ingredient is the Poincaré map for (2.322), (2.323): we look at $(s_{n+1}, x_{n+1})$ as a function of $(x_n, s_n)$. With the adiabatic invariants analogy in mind, the Poincaré map is used to eliminate the fast evolution. The asymptotic expansions of $s(u)$ and $x(u)$ when $u$ is between $u_n$ and $u_{n+1}$ are straightforward local expansions of (2.322) and (2.323). We denote

$$J(s) = \oint_{\mathcal{C}} R(v,s)\, dv; \quad L(s) = \oint_{\mathcal{C}} \frac{dv}{R(v,s)} \tag{2.324}$$

It is easily checked that

$$J'' + \frac{1}{4}\rho(s) J = 0; \quad \text{where} \quad \rho(s) = \frac{5}{3s\,(3\,s+4)} \tag{2.325}$$

and, since $J' = L/2$ we get

$$L'' - \frac{\rho'(s)}{\rho(s)} L' + \frac{1}{4}\rho(s) L = 0 \tag{2.326}$$

The points $s = 0$ and $s = -4/3$ are regular singular points of (2.325) (and of (2.326)) and correspond to the values of $s$ for which the polynomial $u^3/3 + u^2 + s$ has repeated roots. Simple asymptotic analysis of (2.322) and (2.323) shows that the Poincaré map satisfies

$$s_{n+1} = s_n - \frac{2J_n}{x_n}\,(1 + o(1)) \qquad \text{with } J_n = J(s_n) \tag{2.327}$$

$$x_{n+1} = x_n + L_n\,(1 + o(1)) \qquad \text{with } L_n = L(s_n) \tag{2.328}$$

Here, and in the following *heuristic* outline, $o(1)$ stands for terms which are small for large $x_n$ and large $n$. The rigorous justification of these estimates is done in [22].

### 2.11i.3 Solving (2.327) and (2.328); asymptotically conserved quantities

We see from (2.327) that $s_{n+1} - s_n \ll s_n$ and $x_{n+1} - x_n \ll x_n$. Here too it is natural to take a "continuum limit" and approximate $s_{n+1} - s_n$ by $ds/dn$

and $x_{n+1} - x_n$ by $dx/dn$. We get

$$\frac{ds}{dx} = \frac{ds/dn}{dx/dn} = \frac{-2J(s)}{xL(s)}(1 + o(1)) = -\frac{J(s)}{xJ'(s)}(1 + o(1)) \tag{2.329}$$

which implies, by separation of variables and integration,

$$\mathcal{Q}(x, s) := xJ(s) = x_0 J(s_0)\ (1 + o(1)) \tag{2.330}$$

That is, $\mathcal{Q}$ is asymptotically a constant of motion.

In the case of the pendulum, $L(t)$ is given and one constant of motion suffices: together with the value of $L$, it provides two independent conditions for a second order equation; for $\mathrm{P_I}$, to fully solve the equation we need to control another quantity; a second (nonautonomous) one is obtained using (2.327) and (2.330) as follows. We write

$$\frac{1}{J^2(s)}\frac{ds}{dn} = -\frac{2}{x_0 J(s_0)}(1 + o(1)) \tag{2.331}$$

Let $\hat{J}$ be an independent solution of (2.325) with $\hat{J}(0) = 0$. Since the first order derivative in (2.325) is missing, the Wronskian $W$ of any two solutions is a constant. Thus $(\hat{J}/J)' = W/J^2$ and $1/J^2$ is a perfect derivative. Denote

$$\mathcal{K}(s) := \kappa_0 \int_0^s \frac{ds}{J(s)^2} = \frac{\hat{J}(s)}{J(s)} \tag{2.332}$$

where $\kappa_0$ is the Wronskian of $J$ and $\hat{J}$. Integrating both sides of (2.331) from 0 to $n$ we get

$$\mathcal{K}(s) - \mathcal{K}(s_0) = -\frac{2n}{\kappa_0\, x_0 J(s_0)}(1 + o(1)) \Rightarrow \mathcal{K}(s) + \frac{2n}{\kappa_0\, x_0 J(s_0)} = \mathcal{K}(s_0) + o(1) \tag{2.333}$$

for $n = O(x_0)$. $\mathcal{K}$ is in fact a Schwarzian triangle function.

**Note 2.334** As we saw in the analysis leading to (2.309), the singularities of solutions having asymptotic power series behavior in the right half plane are almost periodic, with the same period as the exponential terms in the transseries. While these solutions form a lower dimensional manifold, spontaneous formation of singularities is a "local" process, and it is expected that singularities are produced with roughly the same spacial spacing for all solutions. For this reason, the normalization based on simplifying the exponentials in the transseries is a reasonable choice even in a transseries free region.

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## 2.12 Appendix

In this book we work in $\mathbb{R}^n$ (or $\mathbb{C}$) and we will state the results in this simpler setting. See [74] for general measure spaces. The integrals we use are Lebesgue integrals. A function is in $L^1(S)$ where $S$ is a measurable set if $\int_S |f(x)|dx < \infty$. The Lebesgue measure $\lambda$ is simply the measure defined first on boxes $B$ by $\lambda(B) = volume(B)$, and then extended to measurable sets by additivity and "continuity" (regularity). A function is measurable if its inverse image of any measurable set is measurable.

### 2.12a The dominated convergence theorem

**Theorem 2.335 (dominated convergence)** *Assume $\{f_n\}_{n\in\mathbb{N}}$ is a family of real-valued functions and that $f_n(x) \to f(x)$ for almost all $x$ in $S$* [11]*. Assume further that for all $n$ $|f_n| \le g$ a.e $[\lambda]$* [11] *, where $g$ is in $L^1(S)$. Then $f \in L^1(S)$ and*

$$\lim_{n\to\infty} \int_S f_n(s)ds \to \int_S f(s)ds \tag{2.336}$$

The Theorem also applies for complex valued functions, when real and imaginary parts have the requisite properties. Furthermore, it is easy to see that a similar statement holds for more general parametric convergence, that is, if $n$ is replaced by a parameter $y$ in a, say, metric space, under similar assumptions: $|f(y,x)| \le g(x)$ for all $(x,y)$ where $g$ is integrable, and $f(y,x) \to f(x)$ as $y \to y_0$ a.e.$[\lambda]$.

**Note 2.337** if $K$ is a compact set in $\mathbb{R}$, then $F \in L^1_\nu(K)$, see (1.45), **iff** $F \in L^1(K)$. Indeed, in this case there exist two positive constants $c \le c_2$ such that $c < e^{-\nu p} < c_2$; the rest is straightforward. Nonetheless, if $F \in L^1([a,b])$, it is still useful to work in $L^1_\nu([a,b])$ $0 \le a < b \in \mathbb{R}$, since $\|F\|_{L^1_\nu([a,b])} \to 0$ as $\nu \to \infty$. Indeed, if $\nu > 0$ we have $|F(p)|e^{-\nu p} \le |F(p)|$ and $|F(p)|e^{-\nu p} \to 0$ on $[a,b]$. Thus Theorem 2.12a applies and $\int_a^b F(p)e^{-xp}dp \to 0$.

**Lemma 2.338** *(i) Let $H$ be analytic in the region $\{z : dist(\mathbb{R}^+, z) \in (0,c)\}$ and such that for some $\nu$ we have $\sup_{0<|a|<c} \|H(p+ia)\|_\nu < \infty$. Let*

$$h(x) = \oint_{0;c}^{\infty} e^{-px} H(p)dp \quad (\mathrm{Re}\,(x) > \nu) \tag{2.339}$$

*where we use the notation $\oint_{a;c}^{\infty}$ for an integral along a contour encircling $\mathbb{R}^+$ counteclockwise, at a distance $c$ of it.*

*Assume further that*

$$h(x) = O(e^{-rx}) \text{ as } x \to +\infty \tag{2.340}$$

*where $r < c$. Then $H$ is analytic in $\mathbb{D}_r$.*

[11] That is, except possibly for a set of measure zero; a set has zero measure if it contained in a union of boxes of arbitrarily small total measure. The notation a.e. $[\lambda]$ simply means for all $x$ except for a zero measure set.

*(ii) The same holds in the following other cases:*

*(a)* $\oint_{0;c}^{\infty}$ *is replaced by* $\oint_{0;c;\phi}^{\infty}$ *where now the contour surrounds* $\mathbb{R}^+$ *at distance at least c and approaches* $\infty$ *at an angle* $\phi$ *and* $H$ *is analytic inside the curve, except perhaps for* $\mathbb{R}^+e^{i\phi}$ $\pm\infty$*;*

*(b)* $\oint_{0;c}^{\infty}$ *is replaced by* $\oint_{0;c;\pm\phi}^{\infty}$ *where now the contour surrounds* $\mathbb{R}^+$ *at distance at least c and approaches* $\infty$ *at an angle* $\pm\phi$ *and* $H$ *is analytic inside the curve, except perhaps for* $\mathbb{R}^+$*.*

**PROOF** Note first that $h_1(x) := h(x+\nu+\varepsilon)$ is analytic in a neighborhood of $(-\varepsilon, \infty)$. This and (2.345) show that $\check{h}_1(q) = \int_0^\infty h_1(x)e^{-qx}dx$ exists, and the integrand in the definition of $\check{h}_1(q)$ satisfies the hypotheses of Fubini's theorem and

$$\check{h}_1(q) = \oint_0^\infty \frac{e^{-\nu p-\varepsilon p}H(p)}{p+q}dp =: \oint_0^\infty \frac{H_1(p)}{p+q}dp \tag{2.341}$$

**Note 2.342** Eq. (2.345) implies that the Laplace transform $\check{H} := \int_0^\infty h(x)e^{-qx}dx$ exists and is analytic in the half plane $\operatorname{Re} q > -r$.

We start with large $\operatorname{Re} q$ and approach the origin. To enter the disk of radius $r$, $q$ crosses the contour of integration. We bend the contour inward allowing $q$ to approach the origin at a distance $0 < c' < c$ and then pass the contour through $q$, collecting the residue $2\pi i H(q)$, and then return to the original contour. Thus, for $|q| < r$ we have

$$\check{h}_1(q) = 2\pi i H_1(q) + \oint_{0;c}^\infty \frac{H_1(p)}{p+q}dp \tag{2.343}$$

where now $q$ is in $\mathbb{D}_r$. By Note 2.342 $\check{h}_1(q)$ is analytic in $\mathbb{D}_r$ and so is the integral on the right side of (2.343), manifestly so due to the fact that the contour is outside $\mathbb{D}_r$. But then $H_1(q)$ and therefore $H(q)$ is analytic in $\mathbb{D}_r$.

(ii) The proof is very similar to that of (i). ∎

**Exercise 2.344** *Adapt the proof above to the weaker condition*

$$h(k) = O(e^{-rk}) \text{ as } \mathbb{N} \ni k \to +\infty \tag{2.345}$$

*Hint: consider instead the properties of the generating function* $\sum_{k=k_0}^\infty h_1(k)z^k$.

## 2.13 Banach spaces and the contractive mapping principle

In rigorously proving asymptotic results about *solutions* of various problems, where a closed form solution does not exist or is awkward, the contractive mapping principle is a handy tool. Once an asymptotic expansion solution has been found, if we use a truncated expansion as a quasi-solution, the remainder should be small. As a result, the complete problem becomes one to which the truncation is an exact solution modulo small errors (usually involving the unknown function). Therefore, most often, asymptoticity to a formal solution can be shown rigorously by rewriting this latter equation as a small perturbation of the identity operator (in a suitable norm) acting on a truncation of the formal solution. Some general guidelines on how to construct this operator are discussed in §2.13b. It is desirable to go through the rigorous proof, whenever possible — this should be straightforward when the asymptotic solution has been correctly found—, one reason being that this quickly signals errors such as omitting important terms, or exiting the region of asymptoticity.

In §2.13.1 we discuss, for completeness, a few basic facts about Banach spaces. There is of course a vast literature on the subject; see e.g. [69].

### 2.13.1 A brief review of Banach spaces

Familiar examples of Banach spaces are the $n$-dimensional Euclidian vector spaces $\mathbb{R}^n$. A norm exists in a Banach space, which has the essential properties of a length: scaling, positivity except for the zero vector which has length zero and the triangle inequality (the sum of the lengths of the sides of a triangle is no less than the length of the third one). Once we have a norm, we can define limits, by reducing the notion to that in $\mathbb{R}$: $x_n \to x$ iff $\|x - x_n\| \to 0$. A normed vector space $\mathcal{B}$ is a Banach space if it is complete, that is every sequence with the property $\|x_n - x_m\| \to 0$ uniformly in $n, m$ (a Cauchy sequence) has a limit in $\mathcal{B}$. Note that $\mathbb{R}^n$ can be thought of as the space of functions defined on the set of integers $\{1, 2, ..., n\}$. If we take a space of functions on a domain containing infinitely many points, then the Banach space is usually infinite-dimensional. An example is $L^\infty[0, 1]$, the space of bounded functions on $[0, 1]$ with the norm $\|f\| = \sup_{[0,1]} |f|$. A function $L$ between two Banach spaces which is linear, $L(x + y) = Lx + Ly$, is bounded (or continuous) if $\|L\| := \sup_{\|x\|=1} \|Lx\| < \infty$. Assume $\mathcal{B}$ is a Banach space and that $S$ is a closed subset of $\mathcal{B}$. In the *induced topology* (i.e., in the same norm), $S$ is a complete normed space.

### 2.13.2 Fixed point theorem

Assume $\mathcal{M} : S \mapsto \mathcal{B}$ is a (linear or nonlinear) operator with the property that for any $x, y \in S$ we have

$$\|\mathcal{M}(y) - \mathcal{M}(x)\| \leq \lambda \|y - x\| \tag{2.346}$$

with $\lambda < 1$. Such operators are called **contractive**. Note that if $\mathcal{M}$ is linear, this just means that the norm of $\mathcal{M}$ is less than one.

**Theorem 2.347** *Assume $\mathcal{M} : S \mapsto S$, where $S$ is a closed subset of $\mathcal{B}$ is a contractive mapping. Then the equation*

$$x = \mathcal{M}(x) \tag{2.348}$$

*has a unique solution in $S$.*

**PROOF** Consider the sequence $\{x_j\}_j \in \mathbb{N}$ defined recursively by

$$\begin{aligned} x_0 &= x_0 \in S \\ x_1 &= \mathcal{M}(x_0) \\ &\cdots \\ x_{j+1} &= \mathcal{M}(x_j) \\ &\cdots \end{aligned} \tag{2.349}$$

We see that

$$\|x_{j+2} - x_{j+1}\| = \|\mathcal{M}(x_{j+1}) - \mathcal{M}(x_j)\| \le \lambda \|x_{j+1} - x_j\| \le \cdots \le \lambda^j \|x_1 - x_0\| \tag{2.350}$$

Thus,

$$\|x_{j+p+2} - x_{j+2}\| \le \left(\lambda^{j+p} + \cdots \lambda^j\right) \|x_1 - x_0\| \le \frac{\lambda^j}{1-\lambda} \|x_1 - x_0\| \tag{2.351}$$

and $x_j$ is a Cauchy sequence, and it thus converges, say to $x$. Since by (2.346) $\mathcal{M}$ is continuous, passing the equation for $x_{j+1}$ in (2.349) to the limit $j \to \infty$ we get

$$x = \mathcal{M}(x) \tag{2.352}$$

that is existence of a solution of (2.348). For uniqueness, note that if $x$ and $x'$ are two solutions of (2.348), by subtracting their equations we get

$$\|x - x'\| = \|\mathcal{M}(x) - \mathcal{M}(x')\| \le \lambda \|x - x'\| \tag{2.353}$$

implying $\|x - x'\| = 0$, since $\lambda < 1$. ∎

**Note 2.354** *Note that contractivity and therefore existence of a solution of a fixed point problem depends on the norm. An adapted norm needs to be chosen for this approach to give results.*

**Definition 2.355** *The norm $\|.\|$ of a linear operator $L : \mathcal{A} \to \mathcal{B}$ is simply defined as*

$$\|L\| = \sup_{\|x\|=1} \|Lx\|$$

**Exercise 2.356** *Show that if $L$ is a linear operator from the Banach space $\mathcal{B}$ into itself and $\|L\| < 1$ then $I - L$ is invertible, that is $x - Lx = y$ has always a unique solution $x \in \mathcal{B}$. "Conversely," assuming that $I - L$ is not invertible, then in whatever norm $\|\cdot\|_*$ we choose to make the same $\mathcal{B}$ a Banach space, we must have $\|L\|_* \ge 1$ (why?).*

### 2.13a Fixed points and vector valued analytic functions

A theory of analytic functions with values in a Banach space can be constructed by almost exactly following the usual construction of analytic functions. For the construction to work, we need the usual vector space operations and a topology in which these operations are continuous. A typical setting is that of a Banach algebra[12]. A detailed presentation is found in [51] and [61], but the basic facts are simple enough for the reader to redo the necessary proofs.

### 2.13b Choice of the contractive map

An equation can be rewritten in a number of equivalent ways. In solving an asymptotic problem, as a general guideline we mention:

- The operator $\mathcal{N}$ appearing in the final form of the equation, which we want to be contractive, should not contain derivatives of highest order, divided differences with small denominators, or other operations poorly behaved with respect to asymptotics, and it should only depend on the sought-for solution in a formally small way. The latter condition should be, in a first stage, checked for consistency: the discarded terms, calculated using the first order approximation, should indeed turn out to be small.

- To obtain an equation where the discarded part is manifestly small it often helps to write the sought-for solution as the sum of the first few terms of the approximation, plus an exact remainder, say $\delta$. The equation for $\delta$ is usually more contractive. It also becomes, up to smaller corrections, linear.

- The norms should reflect as well as possible the expected growth/decay tendency of the solution itself and the spaces chosen should be spaces where this solution lives.

- All freedom in the solution has been accounted for, that is, we should make sure the final equation cannot have more than one solution.

**Note 2.357** At the stage where the problem has been brought to a contractive mapping setting, it usually can be recast without conceptual problems, but perhaps complicating the algebra, to a form where the implicit function theorem applies (and vice versa). The contraction mapping principle is often more natural, especially when the topology, suggested by the problem itself, is not one of the common ones. But an implicit function reformulation might bring in more global information.

[12] A Banach algebra is a Banach space of functions endowed with multiplication which is distributive, associative and continuous in the Banach norm.

## 2.14 Examples

### 2.14a Linear differential equations in Banach spaces

Consider the equation

$$Y'(t) = L(t)Y(t); \quad Y(0) = Y_0 \tag{2.358}$$

in a Banach space $X$, where $L(t) : X \to X$ is linear, norm continuous in $t$ and uniformly bounded,

$$\sup_{t\in[0,\infty)} \|L(t)\| < L \tag{2.359}$$

Then the problem (2.358) has a global solution on $[0,\infty)$, and $\|Y(t)\| \leq \|Y_0\|e^{(L+\varepsilon)t}$.

**PROOF** By comparison with the case when $X = \mathbb{R}$, the natural growth is indeed $Ce^{Lt}$, so we rewrite (2.358) as an integral equation, in a space where the norm reflects this possible growth. Consider the space of continuous functions $Y : [0,\infty) \mapsto X$ in the norm

$$\|Y\|_{\infty,L} = \sup_{t\in[0,\infty)} e^{-Lt/\lambda}\|Y(t)\| \tag{2.360}$$

with $\lambda < 1$ and the auxiliary equation

$$Y(t) = Y_0 + \int_0^t L(s)Y(s)ds =: \mathcal{A}[Y](t) \tag{2.361}$$

which is well defined on $X$ and is contractive there since

$$\begin{aligned} e^{-Lt/\lambda}\left|\int_0^t L(s)Y(s)ds\right| &\leq Le^{-Lt/\lambda}\int_0^t e^{Ls/\lambda}\|Y\|_{\infty,L}ds \\ &= \lambda(1-e^{-Lt/\lambda})\|Y\|_{\infty,L} \leq \lambda\|Y\|_{\infty,L}, \end{aligned} \tag{2.362}$$

and therefore in a ball of radius $(1+\gamma)\|Y_0\|$, for large enough $\gamma$ (in fact, we need $(1+\gamma)(1-\lambda) > 1$),

$$\|\mathcal{A}[Y]\|_{\infty,L} \leq \|Y_0\| + \lambda(1+\gamma)\|Y_0\| < (1+\gamma)\|Y_0\|$$

while

$$\|\mathcal{A}[Y_1] - \mathcal{A}[Y_2]\|_{\infty,L} \leq \lambda\|Y_1 - Y_1\|_{\infty,L},$$

implying $\mathcal{A}$ to be contraction map; and so a unique solution exists for the initial value problem (2.358) with given exponential bounds for growth as given. We note that in linear problems, we do not need to restrict the analysis to a ball. ◻

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