

CONNECTIONS SEMINAR MANIFESTO

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This is a manifesto providing the basis for the new Connections Seminar in the Mathematics Department.

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1 Executive Statement

The new Connections Seminar will initially meet on Friday afternoons. The time and place of the first meeting will be announced shortly. Most speakers will be internal, but we plan to invite selected outside speakers to the seminar at a later time.

The Connections Seminar will begin with two lectures by Ovidiu Costin on Analyzable Functions. Costin is currently writing a book on this topic.

The purpose of the Connections Seminar is to explore and develop the foundations of mathematics **broadly conceived**, with a special focus on fundamental connections between different areas of mathematics.

At the present time, systematic foundations of mathematics are mainly carried out through mathematical logic as developed by Frege, Dedekind, Cauchy, Cantor, Zermelo, Frankel, Gödel, Turing, and many others.

The Connections Seminar is not intended to be a Logic seminar. Instead, we will take a very broad view of "foundations", which is much more focused on the actual activity of mathematicians.

The intention is that in the Connections Seminar, we will try to communicate at the most fundamental level possible, with our colleagues who are experts in different areas of mathematics, taking into account their different perspectives, aims, knowledge, and value systems.

The expectation is that there will emerge:

- i. Many new connections between different areas of mathematics from such an open atmosphere devoted to fundamentals.
- ii. Many improved ways of expositing and organizing mathematics that greatly increase the accessibility of developments to nonexperts.
- iii. Substantial synergy between i and ii.

2 Original purposes, interdisciplinary connections

Most areas of mathematics originated on the basis of a well defined purpose or set of purposes. Normally, these original purposes were well appreciated by the general mathematical community.

As areas evolve, and as the founders of those areas exit from the scene, the original purposes often take on lesser importance.

Generally speaking, original purposes can only be partially realized at the early stages of development of an area of mathematics. It generally takes a long time for the relevant deep methods and machinery to be developed.

The area then takes on a life of its own, where specialized issues internal to the area dominate. The original purposes often become no longer relevant to the ongoing development.

It is at this mature stage of development that one may be able to make great progress by returning to the original purposes. For then, one is heavily armed with a great deal of machinery and experience that was not available before.

When going back to such basics, one naturally comes up with interdisciplinary connections of various sorts, both within and outside mathematics.

A principal idea of the Connections Seminar is to have a group of mathematicians who collectively are experts or at least knowledgeable about a wide range of areas of mathematics, systematically return to the original purposes of these areas.

This would be done partly through presentations that meet certain agreed upon standards of simplicity and adherence to the most fundamental of issues.

These presentations would be made by the seminar members and also outside speakers who become familiar with what we are trying to do, are sympathetic to our philosophy, and are expected to provide inspiration.

We have no doubt that if we have suitably basic and fundamentally oriented talks about original purposes of a given area, then this will easily suggest relevant

ideas from the perspective of other areas of mathematics.

Such an environment should result in the identification of many new seminal connections between various areas of mathematics.

Of course, in order for a new seminal connection to really take hold, there have to be some critical fundamental results or at least some promising critical conjectures, in order to "add meat to the bones".

Since this kind of systematic approach across mathematics has apparently not been pursued for many years, there is likely a considerable opportunity for major progress under this approach. So we expect that many required fundamental results and critical conjectures can emerge from this collaborative effort.

This return to original purposes and search for fundamental interdisciplinary connections is strongly connected with what we have in mind by the phrase "foundational organization" - one of the topics which we hope to analyze in the Connections Seminar.

We believe in the idea of a powerful organizational scheme for mathematics, that would put the myriad advances into some clear general picture that is fully intelligible to the general mathematical community - and largely to other scholarly communities as well.

We know of only two serious attempts to provide such a powerful organization scheme for mathematics as a whole.

1. The 2000 AMS Subject Classification Scheme <http://www.ams.org/msc/>
2. Mathematics Form and Function, by Saunders Mac Lane, 1986, Springer Verlag, 476 pages.

Clearly, 1 is a rather ad-hoc taxonomic amalgam, but still provides obvious service to the mathematical community.

Certainly 2 has some excellent qualities, and there is no question that Mac Lane was trying to address this acute need. This is highly admirable. However, this book is nowhere near systematic enough to really do what we want. It is too descriptive, and seems to be missing key powerful organizational schemes that are yet to be articulated.

Our mathematics textbooks don't even provide the appropriate foundational organizations of limited areas of mathematics. The best of them aim at reasonably efficient presentations of a hodge podge of topics that are known to represent "essential tools of the trade" for the relevant area. To be sure, the best of these textbooks provide essential services to the profession, but do they discuss how to formulate productive research programs, or how to weigh the relative interest and importance of various mathematical developments?

We are optimistic about the possibility of a suitably powerful organizational scheme for mathematics. Clearly the development of a powerful foundational organization of mathematics as a whole has to be a seriously collaborative effort.

The most fundamental principle of foundational organization is that a concept can be introduced only if it is to serve a clearly stated fundamental purpose. This includes a frequently encountered indirect fundamental purpose: the concept is needed to state a theorem, or to prove a theorem, where the theorem serves a clearly stated fundamental purpose.

3 Three sample meta-programs

Meta-programs (examples below) cut across potentially all areas of mathematics. The three particular examples may or may not be a principal focus of the Connections Seminar.

Meta-programs are particularly valuable for the Connections Seminar, in that they present an opportunity for us to focus on one single aspect of mathematics that is readily accessible to all participants, but which touches every area of mathematics in a different way.

One example of a meta-program could be: Give computable forms of mathematical theorems.

In some cases, it makes sense to ask for computability without modifying the theorem. For example, it is not known if the Mordell Conjecture, proved by Faltings, is computable in the sense of there being a computable bound.

In other cases, one seeks the development of, say, an area of analysis, in computational terms. This would require very careful attention to the various relevant notions of computability in a continuous setting.

There are also important negative results: there is no way to compute a bound on solutions to a general Diophantine equation over the integers. This conclusion has not been established over the rationals.

Next, we mention a possible meta-program that is not reasonably launched, except in very narrow contexts:

What is a classification? There may be several distinct notions that are relevant, and which go beyond mere taxonomy. Describe the various interesting notions of classification.

Every area of mathematics has its own fundamental classifications, where it is at least intuitively clear what is being accomplished. Yet it is not clear just how to formulate notions of classification, generally.

One benefit from having some good relevant notions of classification is to enable us to state and prove theorems like this: there is no way to classify the abc's

of an xyz. Or, we can classify all finite Abelian groups, but there is no similar way to classify finite green groups.

How would the Connections Seminar investigate meta-programs like "computability in mathematics" and "classifications in mathematics"?

We would have members and outside speakers survey computability and non-computability in their areas of expertise. The speakers and audience would be particularly sensitive to issues of meaning that cut across all areas.

As a possible third example, we mention the meta-program of tameness.

It is well understood that some mathematical objects are well behaved, or tame, exhibiting no pathology, or a very limited form of pathology; whereas others are not well behaved, or wild, exhibiting lots of pathology.

Here is a rough guide to what we are talking about.

An arbitrary function on the reals can exhibit extreme pathology; an arbitrary Borel measurable function on the reals can exhibit less extreme, but strong pathology; an arbitrary continuous function on the reals can exhibit significant, but not strong, pathology; an arbitrary C^∞ function on the reals can exhibit some pathology; an arbitrary real analytic function on the reals can exhibit limited pathology; and an arbitrary polynomial on the reals can exhibit no pathology - just intricacies.

The opening talks by Ovidiu Costin on Analyzable Functions concern a notion of function that admits pathology only in quite limited ways.

4 Some other kinds of issues

Here is a provisional list of topics that may be useful for the Connections Seminar.

a. Why is calculus so useful? In what sense is calculus useful? What is calculus?

One view of calculus is that calculus is not fairly represented by basic theorems in real analysis such as attainment of maxima and intermediate value. These come directly from the completeness of the real line, which is a logical matter rather than a calculus matter. Rather, it begins in earnest with Taylor series. Also, in the setting of complex functions, twice differentiability follows from differentiability in a way that seems much deeper than the way that elementary theorems of real analysis (intermediate value theorem, attainment of maxima, etc.) follow from the completeness of the real line. It would appear to be a fundamental part of calculus. But what does this mean and how can it be demonstrated?

There are many notions of "useful" for mathematics, and these should be systematically sorted out and analyzed, as a serious enterprise.

b. What is geometry? What is the relation between geometry and analysis? Can we give geometric proofs of various fundamental theorems in analysis? What does that mean?

The normal way in which geometry is treated is in terms of collections of points. Yet points, from the geometrical point of view, appear to be fictions. Already Aristotle rejected points in favor of what we now call intervals - with the point of view that the very idea of an endpoint of an interval makes no sense.

There is also the basic idea that in geometry, the transformations count - not the points. And for transformations, what counts is the group structure. Yet we still cling to the safety of using pointsets as the basis for geometry. Should we or shouldn't we? What is gained or lost by clinging to points? What is gained or lost by avoiding points?

c. To what extent can mathematics be given a strictly finitary treatment that is reasonably simple, informative, and fruitful?

To be sure, a considerable amount of mathematics is normally presented in a finitary way. However, this limits us to finite combinatorics and elementary number theory. Yet even when dealing with mathematics saturated with uses of the real numbers, one gets the feeling that the underlying essence is of a finite combinatorial nature. The huge collection of points on the real number line is a kind of fluff.

We also know that if we are going to compute something, we have to reduce things down to the finite, in some sense. So can we rework everything by starting from the finite at the outset?

This issue arises in an essential way in computer science. For instance, various basic operations (including the usual field operations) are explicitly defined on finite spaces in the IEEE standard for computer arithmetic - for example, in 32 bit arithmetic. Of course, we no longer have a field or even a ring. This must be done because the computer must start and end with finite amounts of information only.

On the other hand, there are various results from logic that indicate at least some sort of limitation to this - in the form of results that say that in certain contexts, infinite objects must be used.

c. Advanced topics.

The Connections Seminar certainly should be involved with more advanced topics of various kinds, but not in a way that would better handled by the many specialized seminars already present in the Department.

Advanced topics should be chosen so that their motivation and essence can be communicated in readily accessible terms to a wide audience, and be related to fundamental issues and purposes. They are particularly suitable if there is an

expectation that something productive can come from a suitable presentation to really diverse listeners that represents many different perspectives.

In the Connections Seminar we will continually try to relate the fundamental discussions with contemporary mathematical developments.

5 Longer term goals

The expectation is that over, say, a five year period,

- i. More new seminal connections within mathematics during this period, originate at OSU, through the Connections Seminar, than in any other center for mathematics, worldwide. Furthermore, these get launched through critical results and conjectures. Seminar notes are made available on the web.
- ii. A meta-program of classification gets firmly launched, well informed by actual classifications across many areas of mathematics.
- iii. A major outline of a foundational organization of mathematics is produced. This includes an articulation of its basic principles.

Normally, even in the most visible mathematics departments in the world, members typically do not interact, intellectually, with other members in different areas of mathematics. Of course, there are going to be some collaborations, but relatively few compared to the potential.

We believe that through the activities of the Connections Seminar, the Department can realize its potential for vastly increased levels of interaction both within the Department and the wider University community.

This requires the willingness to open up without fear of looking ridiculous. Mathematicians have to recognize that there are ideas that come easily and naturally to them, and there are ideas that do not. The mix will generally be different for different mathematicians.

Most mathematicians are relatively uninformed about most mathematics. Those who are not should not be too arrogant. They might not be as ignorant as the average. But they should remember that:

- a. They might be less cognizant than they think, especially in the judgment of relevant specialists.
- b. Imagination and creativity, combined with a sense of what kind of thing is known and how to get more information when needed, is incomparably more powerful than mere non ignorance and non foolishness.

With dedication, perseverance, patience, and good fortune, the Mathematics Department could become a noted fertile and important mathematics center worldwide.