$$\sum_{j=1}^{n-1}\frac{1}{j}-\gamma=\ln n+\int_0^\infty e^{-np}\Big(\frac{1}{p}-\frac{1}{1-e^{-p}}\Big)dp=\frac{\Gamma'(n)}{\Gamma(n)} \tag{4.63}$$

Exercise: The Zeta function. Use the same strategy to show that

$$(n-1)!\zeta(n)=\int_0^\infty p^{n-1}\frac{e^{-p}}{1-e^{-p}}dp=\int_0^1\frac{\ln^{n-1}s}{1-s}ds \tag{4.64}$$

4.3.1 The Euler-Maclaurin summation formula

Assume $f(n)$ does not increase too rapidly with n and we want to find the asymptotic behavior of

$$S(n+1)=\sum_{k=k_0}^{n}f(k) \tag{4.65}$$

for large n. We see that $S(k)$ is the solution of the difference equation

$$S(k+1)-S(k)=f(k) \tag{4.66}$$

To be more precise, assume f has a level zero transseries as $n\to\infty$. Then we write $\tilde{S}$ for the transseries of S which we seek at level zero (see p. 93). Then $\tilde{S}(k+1)-\tilde{S}(k)=\tilde{S}'(k)+\tilde{S}''(k)/2+...+\tilde{S}^{(n)}(k)/k!+...=\tilde{S}'(k)+L\tilde{S}'(k)$ where

$$L=\sum_{j=2}^{\infty}\frac{1}{j!}\frac{d^{j-1}}{dk^{j-1}} \tag{4.67}$$

is contractive on level zero transseries (check) and thus

$$\tilde{S}'(k)=f(k)-L\tilde{S}'(k) \tag{4.68}$$

has a unique solution,

$$\tilde{S}'=\sum_{j=0}^{\infty}(-1)^jL^jf=:\frac{1}{1+L}f \tag{4.69}$$

(check that there are no transseries solutions of higher level). From the first few terms, or using successive approximations, that is writing $S'=g$ and

$$g_l=f-\frac{1}{2}g_l'-\frac{1}{6}g_l''-\cdots \tag{4.70}$$

we get

$$\tilde{S}'(k)=f(k)-\frac{1}{2}f'(k)+\frac{1}{12}f''(k)-\frac{1}{720}f^{(4)}(k)+\cdots=\sum_{j=0}^{\infty}C_jf^{(j)}(k) \tag{4.71}$$