We note that to get the coefficient of $f^{(n)}$ correctly, using iteration, we need to keep correspondingly many terms on the right side of (4.70) and iterate $n+1$ times.

In this case, we can find the coefficients explicitly. Indeed, examining the way the C_js are obtained, it is clear that they do not depend on f. Then it suffices to look at some particular f for which the sum can be calculated explicitly; for instance $f(k) = e^{k/n}$ summed from 0 to n. By one of the definitions of the Bernoulli numbers we have

$$\frac{z}{1-e^{-z}} = \sum_{j=0}^{\infty} (-1)^j \frac{B_j}{j!} z^j \tag{4.72}$$

Exercise 4.73 *Using these identities, determine the coefficients C_j in (4.71).*

Using Exercise 4.73 we get

$$S(k) \sim \int_{k_0}^{k} f(s)ds + \frac{1}{2}f(n) + C + \sum_{j=1}^{\infty} \frac{B_{2j}}{2j!} f^{(2j-1)}(k) \tag{4.74}$$

Rel. (4.74) is called the Euler-Maclaurin sum formula.

Exercise 4.75 (*) *Complete the details of the calculation involving the identification of coefficients in the Euler-Maclaurin sum formula.*

Exercise 4.76 *Find for which values of $a > 0$ the series*

$$\sum_{k=1}^{\infty} \frac{e^{i\sqrt{k}}}{k^a}$$

is convergent.

Exercise 4.77 (*) *Prove the Euler-Maclaurin sum formula in the case f is C^∞ by first looking at the integral $\int_n^{n+1} f(s)ds$ and expanding f in Taylor at $s = n$. Then correct f to get a better approximation, etc.*

That (4.74) gives the correct asymptotic behavior in fairly wide generality is proved, for example, in [20].

We will prove here, under stronger assumptions, a stronger result which implies (4.74). The conditions are often met in applications, after changes of variables, as our examples showed.

Lemma 4.78 *Assume f has a Borel summable expansion at 0^+ (in applications f is often analytic at 0) and $f(z) = O(z^2)$. Then $f(\frac{1}{n}) = \int_0^\infty F(p)e^{-np}dp$, $F(p) = O(p)$ for small p and*

$$\sum_{k=n_0}^{n-1} f(1/k) = \int_0^\infty e^{-np} \frac{F(p)}{e^{-p}-1} dp - \int_0^\infty e^{-n_0 p} \frac{F(p)}{e^{-p}-1} dp \tag{4.79}$$