Note 4.141 Let us point out first a possible pitfall in proving Theorem 4.135. Inverse Laplace transformability of f and analyticity away from zero in some sector follow immediately from the assumptions. What does not follow immediately is analyticity of $\mathcal{L}^{-1}f$ at zero. On the other hand, $\mathcal{B}\tilde{f}$ clearly converges to an analytic function near $p=0$. But there is no guarantee that $\mathcal{B}\tilde{f}$ has anything to do with $\mathcal{L}^{-1}f$! This is where Gevrey estimates enter.

PROOF of Theorem 4.135

(i) Uniqueness clearly follows once we prove (ii).

(ii) and (iii) By a simple change of variables we arrange $C_1=C_2=1$. The series $\tilde{F}_1=\mathcal{B}\tilde{f}$ is convergent for $|p|<1$ and defines an analytic function, F_1. By Proposition 2.12, the function $F=\mathcal{L}^{-1}f$ is analytic for $|p|>0, |\arg(p)|<\delta$, and $F(p)$ is analytic and uniformly bounded if $|\arg(p)|<\delta_1<\delta$. We now show that F is analytic for $|p|<1$. (A different proof is seen in §4.5a.1.) Taking p real, $p\in[0,1)$ we obtain in view of (4.131) that

$$\begin{aligned}|F(p)-\tilde{F}^{[N-1]}(p)| &\le \int_{-i\infty+N}^{i\infty+N} d|s| \left|f(s)-\tilde{f}^{[N-1]}(s)\right| e^{\mathrm{Re}\,(ps)}\\ &\le N!e^{pN}\int_{-\infty}^{\infty}\frac{\mathrm{d}x}{|x+iN|^N}=N!e^{pN}\int_{-\infty}^{\infty}\frac{\mathrm{d}x}{(x^2+N^2)^{N/2}}\\ \le \frac{N!e^{pN}}{N^{N-1}}&\int_{-\infty}^{\infty}\frac{\mathrm{d}\xi}{(\xi^2+1)^{N/2}}\le CN^{3/2}e^{(p-1)N}\to 0 \text{ as } N\to\infty \qquad (4.142)\end{aligned}$$

for $0\le p<1$. Thus $\tilde{F}^{[N-1]}(p)$ converges. Furthermore, the limit, which by definition is F_1, is seen in (4.142) to equal F, the inverse Laplace transform of f on $[0,1)$. Since F and F_1 are analytic in a neighborhood of $(0,1)$, $F=F_1$ wherever *either* of them is analytic[9]. The domain of analyticity of F is thus, by (ii), $\{p:|p|<1\}\cup\{p:|p|>0,|\arg(p)|<\delta\}$.

(iv) Let $|\phi|<\delta$. We have, by integration by parts,

$$f(x)-\tilde{f}^{[N-1]}(x)=x^{-N}\mathcal{L}\frac{d^N}{dp^N}F \qquad (4.143)$$

On the other hand, F is analytic in S_a, some $a=a(\phi)-$neighborhood of the sector $\{p:|\arg(p)|<|\phi|\}$. Estimating Cauchy's formula on a radius-$a(\phi)$ circle around the point p with $|\arg(p)|<|\phi|$ we get, for some ν,

$$|F^{(N)}(p)|\le N!a(\phi)^{-N}\|F(p)e^{-\nu\mathrm{Re}\;p}\|_{\infty,S_a}e^{\nu\mathrm{Re}\;p}$$

Thus, by (4.143), with θ, $|\theta|\le|\phi|$, chosen so that $\gamma=\cos(\theta-\arg(x))$ is maximal we have

[9] Here and elsewhere we identify a function with its analytic continuation.