Note 4.146 As we see, *control over the analytic properties of $\mathcal{B}\tilde{f}$ near $p = 0$ is essential to Borel summability and, it turns out,* BE *summability.* Certainly, mere inverse Laplace transformability of a function with a given asymptotic series, in however large a sector, does not ensure Borel summability of its series. We know already that for any power series, for instance one that is not Gevrey of finite order, we can find a function f analytic and asymptotic to it in more than a half-plane (in fact, many functions). Then $(\mathcal{L}^{-1}f)(p)$ exists, and is analytic in an open sector in p, origin not necessarily included. Since the series is not Gevrey of finite order, it can't be Borel summable. What goes wrong is the behavior of $\mathcal{L}^{-1}f$ at zero.

4.6 Borel summation as analytic continuation

There is another interpretation showing that Borel summation should commute with all operations. Returning to the example $\sum_{k=0}^{\infty} k!(-x)^{-k-1}$, we can consider the more general sum

$$\sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(\beta k + \beta)}{x^{k+1}} \tag{4.147}$$

which for $\beta = 1$ agrees with (1.48). For $\beta = i$, (4.147) *converges* if $|x| > 1$, and the sum is, using the integral representation for the Gamma function and dominated convergence,

$$x^{\beta-1} \int_0^{\infty} \frac{e^{-xp} p^{\beta-1}}{1 + p^{\beta} x^{\beta-1}} dp \tag{4.148}$$

Analytic continuation of (4.148) back to $\beta = 1$ becomes precisely (1.52).

Exercise 4.149 *Complete the details in the calculations above. Show that continuation from i and from $-i$ gives the same result (1.52).*

Thus Borel summation should commute with all operations with which analytic continuation does. This latter commutation is very general, and comes under the umbrella of the vaguely stated "principle of permanence of relations" which can hardly be formulated rigorously without giving up some legitimate "relations."

Exercise 4.150 (*) *Complete the proof of Theorem 4.136.*