4.8 Borel transform of the solutions of an example ODE, (4.54)

For generic differential systems there exist general results on the Borel summability of formal transseries solutions; see §5. The purpose now is to illustrate a strategy of proof that is convenient and which applies to a reasonably large class of settings.

It would be technically awkward to prove, based on the formal series alone, that the Borel transform extends analytically along the real line and that it has the required exponential bounds towards infinity.

A better approach is to control the Borel transform of $\tilde{y}$ via the equation it satisfies. This equation is the formal inverse Laplace transform of (4.54), namely, setting $Y = \mathcal{B}\tilde{y}$

$$-pY + Y = p + Y * Y * Y := p + Y^{*3} \tag{4.155}$$

We then show that the equation (4.155) has a (unique) solution which is analytic in a neighborhood of the origin and in any sector not containing $\mathbb{R}^+$ in which this solution has exponential bounds. Thus Y is Laplace transformable, except along $\mathbb{R}^+$, and immediate verification shows that $y = \mathcal{L}Y$ satisfies (4.54). Furthermore, since the Maclaurin series $S(Y)$ formally satisfies (4.155), then the formal Laplace (inverse Borel) transform $\mathcal{B}^{-1}S(Y)$ is a *formal* solution of (4.54), and thus equals $\tilde{y}$ since this solution, as we proved in many similar settings, is unique. But since $S(Y) = \mathcal{B}\tilde{y}$ it follows that $\tilde{y}$ is Borel summable, except along $\mathbb{R}^+$, and the Borel sum solves (4.54). The analysis of (4.54) in Borel plane is given in detail in §5.3.

Along $\mathbb{R}^+$ there are singularities, located at $p = 1+\mathbb{N}$. Medianization allows BE summation along $\mathbb{R}^+$ too.

The transformed equations are expected to have analytic solutions—which are therefore more regular than the original ones.

Further analysis of the convolution equations reveals the detailed analytic structure of $\mathcal{B}\tilde{y}$, including the position and type of singularities, needed in understanding the Stokes phenomena in the actual solutions.

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*4.9 Appendix: Rigorous construction of transseries

This section can be largely omitted at a first reading except when rigor, further information, or precise definitions and statements are needed.