(ii) We have the following embeddings: $L^1_\nu(S) \subset L^1_\nu(S_K), \mathcal{A}_{\nu,0}(S_K \cup B) \subset L^1_\nu(S_K)$. Thus, by Remark 5.18, there exists a unique solution Y of (5.19) which belongs to all these spaces.

Thus Y is analytic in S and belongs to $L^1_\nu(S)$, in particular it is Laplace transformable. The Laplace transform is a solution of (4.54) as it is easy to check.

It also follows that the formal power series solution $\tilde{y}$ of (4.54) is Borel summable in any sector not containing $\mathbb{R}^+$, which is a Stokes ray. We have, indeed, $\mathcal{B}\tilde{y} = Y$ (check!). □

5.3b Borel summation of the transseries solution

With $\tilde{y}_0$ the asymptotic series of $\mathcal{L}Y_0$ (note that$\tilde{y}_0 = \tilde{y}$ in (4.56)), we get

$$\tilde{y} = \tilde{y}_0 + \sum_{k=1}^{\infty} C^k e^{-kx} \tilde{y}_k \tag{5.25}$$

in (4.54) and equate the coefficients of e^{-kx} we get the system of equations

$$\tilde{y}_k' + (1 - k - 3\tilde{y}_0^2)\tilde{y}_k = 3\tilde{y}_0 \sum_{j=1}^{k-1} \tilde{y}_j \tilde{y}_{k-j} + \sum_{j_1+j_2+j_3=k;\, j_i \ge 1} \tilde{y}_{j_1}\tilde{y}_{j_2}\tilde{y}_{j_3} \tag{5.26}$$

The equation for $\tilde{y}_1$ is linear and homogeneous:

$$\tilde{y}_1' = 3\tilde{y}_0^2 \tilde{y}_1 \tag{5.27}$$

Thus

$$\tilde{y}_1 = Ce^{\tilde{s}}; \quad \tilde{s} := \int_\infty^x 3\tilde{y}_0{}^2(t)dt \tag{5.28}$$

Since $\tilde{s} = O(x^{-3})$ is the product of Borel summable series (in $\mathbb{C}\setminus\mathbb{R}^+$), then, by Proposition 4.109, $e^{\tilde{s}}$ is Borel summable in $\mathbb{C}\setminus\mathbb{R}^+$. We note that $\tilde{y}_1 = 1 + o(1)$ (with $C = 1$) and we cannot take the inverse Laplace transform of $\tilde{y}_1$ directly. But the series $x^{-1}\tilde{y}_1$ is Borel summable (say to $\check{\Phi}_1$) see Proposition 4.109. It is convenient to make the substitution $\tilde{y}_k = x^k \tilde{\varphi}_k$. We get

$$\tilde{\varphi}_k' + (1 - k - 3\tilde{\varphi}_0^2 + kx^{-1})\tilde{\varphi}_k = 3\tilde{\varphi}_0 \sum_{j=1}^{k-1} \tilde{\varphi}_j \tilde{\varphi}_{k-j} + \sum_{j_1+j_2+j_3=k;\, j_i \ge 1} \tilde{\varphi}_{j_1}\tilde{\varphi}_{j_2}\tilde{\varphi}_{j_3} \tag{5.29}$$

where clearly $\tilde{\varphi}_0 = \tilde{y}_0$, $\tilde{\varphi}_1 = x^{-1}\tilde{y}_1$, with $\tilde{y}_1$ given in (5.28). After Borel transform, we have

$$-p\mathbf{\Phi} + (1 - \hat{k})\mathbf{\Phi} = -\hat{k} * \mathbf{\Phi} + 3Y_0^{*2} * \mathbf{\Phi} + 3Y_0 * \mathbf{\Phi} \mathbf{*} \mathbf{\Phi} + \mathbf{\Phi} \mathbf{*} \mathbf{\Phi} \mathbf{*} \mathbf{\Phi}; (k \ge 2) \tag{5.30}$$

where $\mathbf{\Phi} = \{\Phi_j\}_{j\in\mathbb{N}}$, $(\hat{k}\mathbf{\Phi})_k = k\Phi_k$ and $(F * \mathbf{G})_k := F * G_k$ (cf. (5.13)).