For this purpose we use (5.33) and (5.34) in order to derive the behavior of Y. It is a way of exploiting what Écalle has discovered in more generality, *bridge equations*.

We start with exploring a relatively trivial, nongeneric possibility, namely that $y_0^+ = y_0^- =: y_0$. (This is not the case for our equation, though we will not prove it here; we still analyze this case since it may occur in other equations.) Since in this case

$$y_0^{\pm} = \int_0^{\infty e^{\pm i\epsilon}} Y(p)e^{-px}dp = y_0 \tag{5.44}$$

we have $y \sim \tilde{y}_0$ in a sector of arbitrarily large opening. By inverse Laplace transform arguments, Y is analytic in an arbitrarily large sector in $\mathbb{C} \setminus \{0\}$. On the other hand, we already know that Y is analytic at the origin, and it is thus entire, of exponential order at most one. Then, $\tilde{y}_0$ converges.

Exercise 5.45 *Complete the details in the argument above.*

We now consider the generic case $y^+ \neq y^-$. Then there exists $S \neq 0$ so that

$$y^+ = \int_0^{\infty e^{-i\epsilon}} e^{-px}Y^-(p)dp = y^- + \sum_{k=1}^{\infty} S^k e^{-kx}x^k\varphi_k^-(x) \tag{5.46}$$

Thus

$$\int_{\infty e^{-i\epsilon}}^{\infty e^{+i\epsilon}} e^{-px}Y(p)dp = \int_1^{\infty} e^{-px}(Y^+(p) - Y^-(p))dp = \sum_{k=1}^{\infty} S^k e^{-kx}x^k\varphi_k^-(x) \tag{5.47}$$

In particular, we have (since $\Phi^{\pm} = \Phi$ for $|p| < 1$)

$$\frac{1}{x}\int_1^{\infty} e^{-px}(Y^+(p) - Y^-(p))dp = Se^{-x}\int_0^{\infty e^{-i\epsilon}} e^{-px}\Phi_1(p)dp + O(x^2e^{-2x})$$
$$= S\int_1^{\infty e^{-i\epsilon}} e^{-px}\Phi_1(p-1)dp + O(x^2e^{-2x}) \tag{5.48}$$

Then, by Proposition 2.22, $\int_0^p Y^+ = \int_0^p Y^- + S\Phi_1(p-1)$ on $(1,2)$. (It can be checked that $\int Y$ has lateral limits on $(1,2)$, by looking at the convolution equation in a focusing space of functions continuous up to the boundary.)

Since Φ_1 is continuous, this means (for $p \neq 1$) $\int_0^p Y^+ = S\Phi_1(p-1) + \int_0^p Y^-$ or $Y^+ = Y^- + SY_1(p-1)$, or yet, $Y^+(1+s) = Y^-(1+s) + SY_1(s)$ everywhere in the right half s plane where $Y^-(1+s) + SY_1(s)$ is analytic, in particular in the fourth quadrant. Thus the analytic continuation of Y from the upper plane along a curve passing between 1 and 2 exists in the lower half-plane; it equals the continuation of two functions along a straight line not crossing any singularities. The proof proceeds by induction, reducing the number of crossings at the expense of using more of the functions $Y_2, Y_3,$ *etc.*

This analysis can be adapted to general *differential* equations, and it allows for finding the resurgence structure (singularities in p) by constructing and solving Riemann-Hilbert problems, in the spirit above.