$$AC_\gamma((\psi * \psi)) = AC_\gamma(\psi) *_\gamma AC_\gamma(\psi) \text{ if } \arg(\gamma(t))\text{=const} \neq 0 \tag{5.266}$$

(Since ψ is analytic along such a line). The notation $*_\gamma$ means (2.21) with $p = \gamma(t)$.

Note though that, suggestive as it might be, (5.266) is *incorrect* if the condition stated there is not satisfied and γ is a path that crosses the real line (see §5.11a)! From (5.266) and (5.264), we get

$$(\psi * \psi)^- = \psi^- * \psi^- = \psi^+ * \psi^+ + \sum_{k=1}^{\infty} \left(\mathcal{H} \sum_{m=0}^{k} \psi_m^+ * \psi_{k-m}^+ \right) \circ \tau_{-k} =$$
$$(\psi * \psi)^+ + \sum_{k=1}^{\infty} \left(\mathcal{H} \sum_{m=0}^{k} \left(\psi_m * \psi_{k-m} \right)^+ \right) \circ \tau_{-k} \tag{5.267}$$

Note. *Check that, if f, g are 0 on $\mathbb{R}^-$ and $k \geq m$, then* $\int_0^p f(s-m)g\{p-[s-(k-m)]\}ds = \int_0^{p-m} f(t)g(p-t-k)dt = \int_0^{p-k} f(t)g(p-k-t)dt = (\mathcal{H}\, f * g)(p-k)$.
Now the analyticity of $\psi * \psi$ in $\mathcal{R}_1$ follows: on the interval $p \in (j, j+1)$ we have from (5.265)

$$\left(\psi * \psi\right)^{-^j}(p) = \left(\psi * \psi\right)^-(p) = \left(\psi^{*2}\right)^+(p) + \sum_{k=1}^{j} \sum_{m=0}^{k} \left(\psi_m * \psi_{k-m}\right)^+(p-k) \tag{5.268}$$

Again, formula (5.268) is useful for analytically continuing $\left(\psi * \psi\right)^{-^j}$ along a path as the one depicted in Fig. 5.1. By dominated convergence, $(\psi * \psi)^{\pm} \in \mathcal{Q}^{\pm}_{(0,\infty)}$, (5.206). By (5.265), ψ_m are analytic in $\mathcal{R}_1^+ := \mathcal{R}_1 \cap \{p : \mathrm{Im}\,(p) > 0\}$ and thus by (5.266) the right side . of (5.268) can be continued analytically in $\mathcal{R}_1^+$. The same is then true for $(\psi*\psi)^-$. The function $(\psi*\psi)$ can be extended analytically along paths that cross the real line from below. Likewise, $(\psi*\psi)^+$ can be continued analytically in the lower half-plane so that $(\psi*\psi)$ is analytic in $\mathcal{R}_1$.

Combining (5.268), (5.266) and (5.263) we get a similar formula for the analytic continuation of the convolution product of two functions, f, g satisfying the assumptions of Proposition 5.259

$$(f * g)^{-^j+} = f^+ * g^+ + \sum_{k=1}^{j} \left(\mathcal{H} \sum_{m=0}^{k} f_m^+ * g_{k-m}^+ \right) \circ \tau_{-k} \tag{5.269}$$

Note that (5.269) corresponds to (5.264) and in those notations we have:

$$\left(f * g\right)_k = \sum_{m=0}^{k} f_m * g_{k-m} \tag{5.270}$$