Integrability as well as (5.262) follow from (5.265), (5.268) and Remark 5.140. □

By (5.118) and (5.265),

$$\psi^{ba} = \psi^{+} + \sum_{k=1}^{\infty} \frac{1}{2^k} (\psi_k^{+} \mathcal{H}) \circ \tau_{-k}$$

so that

$$(\psi^{ba} * \psi^{ba})(p) = \Big(\psi^{+}(p) + \sum_{k=1}^{\infty} \frac{1}{2^k} (\mathcal{H}(p-k)\psi^{+}(p-k)\Big)^{*2} =$$
$$(\psi^{+} * \psi^{+})(p) + \sum_{k=1}^{\infty} \frac{1}{2^k} \sum_{m=0}^{k} (\mathcal{H}\psi_m^{+})(p-k) * (\mathcal{H}\psi_{k-m}^{+})(p-k+m) = (\psi^{*2})^{ba} \tag{5.271}$$

where we used (5.270) (see also the note on p. 197).

To finish the proof of Theorem 5.120, note that on any finite interval the sum in (5.118) has only a finite number of terms and by (5.271) balanced averaging commutes with any finite sum of the type

$$\sum_{k_1,\dots,k_n} c_{k_1,\dots,k_n} f_{k_1} * \dots * f_{k_n} \tag{5.272}$$

and then, by continuity, with any sum of the form (5.272), with a finite or infinite number of terms, provided it converges in L^1_{loc}. Averaging thus commutes with all the operations involved in the equations (5.222). By uniqueness therefore, if $\mathbf{Y}_0 = \mathbf{Y}^{ba}$ then $\mathbf{Y}_k = \mathbf{Y}_k^{ba}$ for all k. Preservation of reality is immediate since (5.89), (5.92) are real if (5.51) is real, therefore $\mathbf{Y}_0^{ba}$ is real-valued on $\mathbb{R}^+\backslash\mathbb{N}$ (since it is real-valued on $[0,1)\cup(1,2)$) and so are, inductively, all $\mathbf{Y}_k$.

5.11 Appendix

5.11a $AC(f*g)$ versus $AC(f)*AC(g)$

Typically, the analytic continuation along curve in $\mathcal{W}_1$ which is not homotopic to a straight line does not commute with convolution.