Remark 5.273 *Let ψ be a function satisfying the conditions stated in Proposition 5.259 and assume that $p = 1$ is a branch point of ψ. Then ($-+ \equiv -^1+$),*

$$(\psi * \psi)^{-+} \neq \psi^{-+} * \psi^{-+} \tag{5.274}$$

Proof

Indeed, by (5.269) and (5.265) (cf. note on p. 197;"·" means multiplication),

$$(\psi * \psi)^{-+} = \psi^+ * \psi^+ + 2[\mathcal{H} \cdot (\psi^+ * \psi_1^+)] \circ \tau_{-1} \neq \psi^{-+} * \psi^{-+} =$$
$$[\psi^+ + (\mathcal{H} \cdot \psi_1^+) \circ \tau_{-1}]^{*2} = \psi^+ * \psi^+ + 2[\mathcal{H} \cdot (\psi^+ * \psi_1^+)] \circ \tau_{-1} + [\mathcal{H} \cdot (\psi_1^+ * \psi_1^+)] \circ \tau_{-2} \tag{5.275}$$

since in view of (5.265), in our assumptions, $\psi_1 \not\equiv 0$ and thus $\psi_1 * \psi_1 \not\equiv 0$.

□

There is also the following intuitive reasoning leading to the same conclusion. For a generic system of the form (5.51), $p = 1$ is a branch point of $\mathbf{Y}_0$ and so $\mathbf{Y}_0^- \neq \mathbf{Y}_0^{-+}$. On the other hand, if AC_{-+} commuted with convolution, then $\mathcal{L}(\mathbf{Y}_0^{-+})$ would provide a solution of (5.51). By Lemma 5.240, $\mathcal{L}(\mathbf{Y}_0^-)$ is a different solution (since $\mathbf{Y}_0^- \neq \mathbf{Y}_0^{-+}$). As $\mathbf{Y}_0^-$ and $\mathbf{Y}_0^{-+}$ coincide up to $p = 2$ we have $\mathcal{L}(\mathbf{Y}_0^{-+}) - \mathcal{L}(\mathbf{Y}_0^-) = e^{-2x(1+o(1))}$ as $x \to +\infty$. By Theorem 5.120, however, no two solutions of (5.51) can differ by less than $e^{-x(1+o(1))}$ without actually being equal (also, heuristically, this can be checked using formal perturbation theory), contradiction.

5.11b Derivation of the equations for the transseries for general ODEs

Consider first the scalar equation

$$y' = f_0(x) - y - x^{-1}By + \sum_{k=1}^{\infty} g_k(x)y^k \tag{5.276}$$

For $x \to +\infty$ we take

$$y = \sum_{k=0}^{\infty} y_k \mathrm{e}^{-kx} \tag{5.277}$$

where y_k can be formal series $x^{-s_k} \sum_{n=0}^{\infty} a_{kn} x^{-n}$, with $a_{k,0} \neq 0$, or actual functions with the condition that (5.277) converges uniformly. Let y_0 be the first term in (5.277) and $\delta = y - y_0$. We have