Exercise 2.5 *Assume f is entire, $|f(z)| \le C_1 e^{|az|}$ in $\mathbb{C}$ and $|f(z)| \le C e^{-|z|}$ in a sector of opening more than π. Show that f is identically zero. (A similar statement holds under much weaker assumptions; see Exercise 2.29.)*

2.2 Laplace and inverse Laplace transforms

Let $F \in L^1(\mathbb{R}^+)$ (meaning that $|F|$ is integrable on $[0,\infty)$). Then the Laplace transform

$$(\mathcal{L}F)(x) := \int_0^\infty e^{-px} F(p)dp \tag{2.6}$$

is analytic in $\mathbb{H}$ and continuous in $\overline{\mathbb{H}}$. Note that the substitution allows us to work in space of functions with the property that $F(p)e^{-|\alpha|p}$ is in L^1, correspondingly replacing x by $x - |\alpha|$.

Proposition 2.7 *If $F \in L^1(\mathbb{R}^+)$, then*
(i) $\mathcal{L}F$ is analytic in $\mathbb{H}$ and continuous on the imaginary axis $\partial\mathbb{H}$.
(ii) $\mathcal{L}\{F\}(x) \to 0$ as $x \to \infty$ along any ray $\{x : \arg(x) = \theta\}$ if $|\theta| \le \pi/2$.

Proof. (i) Continuity and analyticity are preserved by integration against a finite measure ($F(p)dp$). Equivalently, these properties follow by dominated convergence[2], as $\epsilon \to 0$, of $\int_0^\infty e^{-isp}(e^{-ip\epsilon} - 1)F(p)dp$ and of $\int_0^\infty e^{-xp}(e^{-p\epsilon} - 1)\epsilon^{-1}F(p)dp$, respectively, the last integral for $\mathrm{Re}\,(x) > 0$.

If $|\theta| < \pi/2$, (ii) follows easily from dominated convergence; for $|\theta| = \pi/2$ it follows from the Riemann-Lebesgue lemma; see Proposition 3.55. □

Remark 2.8 Extending F on $\mathbb{R}^-$ by zero and using the continuity in x proved in Proposition 2.7, we have $\mathcal{L}\{F\}(it) = \int_{-\infty}^\infty e^{-ipt}F(p)\mathrm{d}p = \hat{\mathcal{F}}F(t)$. In this sense, the Laplace transform can be identified with the (analytic continuation of) the Fourier transform, restricted to functions vanishing on a half-line.

First inversion formula

Let $\mathcal{H}$ denote the space of analytic functions in $\mathbb{H}$.

Proposition 2.9 *(i) $\mathcal{L} : L^1(\mathbb{R}^+) \mapsto \mathcal{H}$ and $\|\mathcal{L}F\|_\infty \le \|F\|_1$.*
(ii) $\mathcal{L} : L^1 \mapsto \mathcal{L}(L^1)$ is invertible, and the inverse is given by

$$F(x) = \hat{\mathcal{F}}^{-1}\{\mathcal{L}F(it)\}(x) \tag{2.10}$$

for $x \in \mathbb{R}^+$ where $\hat{\mathcal{F}}$ is the Fourier transform (in distributions if $\mathcal{L}F \notin L^1$).

[2] See e.g. [52]. Essentially, if the functions $|f_n| \in L^1$ are bounded uniformly in n by $g \in L^1$ and they converge pointwise (except possibly on a set of measure zero), then $\lim f_n \in L^1$ and $\lim \int f_n = \int \lim f_n$.