where $\binom{\mathbf{l}}{j} = \prod_{i=1}^{n}\binom{l_i}{j_i}$, $(\mathbf{v})_m$ means the component m of $\mathbf{v}$, and $\sum_{(\mathbf{i}_{mp}:\mathbf{k})}$ stands for the sum over all vectors $\mathbf{i}_{mp} \in \mathbb{N}^n$, with $p \le j_m, m \le n$, so that $\mathbf{i}_{mp} \succ 0$ and $\sum_{m=1}^{n}\sum_{p=1}^{j_m}\mathbf{i}_{mp} = \mathbf{k}$. We use the convention $\prod_{\emptyset} = 1, \sum_{\emptyset} = 0$. With $m_i = 1 - \lfloor \mathrm{Re}\,\beta_i \rfloor$ we obtain for $\mathbf{y_k}$

$$\mathbf{y}_{\mathbf{k}}' + \left(\hat{\Lambda} - \mathbf{k}\cdot\boldsymbol{\lambda}\hat{I} + x^{-1}(\hat{B} + \mathbf{k}\cdot\mathbf{m})\right)\mathbf{y_k} + \sum_{|\mathbf{j}|=1}\mathbf{d_j}(x)(\mathbf{y_k})^{\mathbf{j}} = \mathbf{t_k}(\mathbf{y}) \tag{5.283}$$

There are clearly finitely many terms in $\mathbf{t_k}(\mathbf{y})$. To find a (not too unrealistic) upper bound for this number of terms, we compare with $\sum_{(\mathbf{i}_{mp})'}$ which stands for the same as $\sum_{(\mathbf{i}_{mp})}$ except with $\mathbf{i} \ge 0$ instead of $\mathbf{i} \succ 0$. Noting that $\binom{k+s-1}{s-1} = \sum_{a_1+\ldots+a_s=k} 1$ is the number of ways k can be written as a sum of s integers, we have

$$\sum_{(\mathbf{i}_{mp})} 1 \le \sum_{(\mathbf{i}_{mp})'} 1 = \prod_{l=1}^{n_1}\sum_{(\mathbf{i}_{mp})_l} 1 = \prod_{l=1}^{n_1}\binom{k_l + |\mathbf{j}| - 1}{|\mathbf{j}| - 1} \le \binom{|\mathbf{k}| + |\mathbf{j}| - 1}{|\mathbf{j}| - 1}^{n_1} \tag{5.284}$$

Remark 5.285 *Equation (5.282) can be written in the form (5.91).*

Proof. The fact that only predecessors of $\mathbf{k}$ are involved in $\mathbf{t}(\mathbf{y}_0,\cdot)$ and the homogeneity property of $\mathbf{t}(\mathbf{y}_0,\cdot)$ follow immediately by combining the conditions $\sum \mathbf{i}_{mp} = \mathbf{k}$ and $\mathbf{i}_{mp} \succ 0$. □

The formal inverse Laplace transform of (5.283) is then

$$\left(-p + \hat{\Lambda} - \mathbf{k}\cdot\boldsymbol{\lambda}\right)\mathbf{Y_k} + \left(\hat{B} + \mathbf{k}\cdot\mathbf{m}\right)\mathcal{P}\mathbf{Y_k} + \sum_{|\mathbf{j}|=1}\mathbf{D_j} * (\mathbf{Y_k})^{\mathbf{j}} \;=\; \mathbf{T_k}(\mathbf{Y}) \tag{5.286}$$

with

$$\mathbf{T_k}(\mathbf{Y}) = \mathbf{T}\left(\mathbf{Y}_0, \{\mathbf{Y}_{\mathbf{k}'}\}_{0\prec\mathbf{k}'\prec\mathbf{k}}\right) = \sum_{\mathbf{j}\le k;\ |\mathbf{j}|>1}\Big(\mathbf{g}_{0,\mathbf{j}}\cdot + \mathbf{D_j} *\Big)\sum_{(\mathbf{i}_{mp};\mathbf{k})}\;{}^{*}\!\!\prod_{m=1}^{n_1}\;{}^{*}\!\!\prod_{p=1}^{j_m}\left(\mathbf{Y}_{\mathbf{i}_{mp}}\right)_m \tag{5.287}$$

(as before, "$\cdot$" means usual multiplication) and

$$\mathbf{D_j} = \sum_{\mathbf{l}\ge\mathbf{j}}\binom{\mathbf{l}}{\mathbf{j}}\mathbf{G_l} * \mathbf{Y}_0^{*(\mathbf{l}-\mathbf{j})} + \sum_{\mathbf{l}\ge\mathbf{j};|\mathbf{l}|\ge 2}\binom{\mathbf{l}}{\mathbf{j}}\mathbf{g}_{0,\mathbf{l}}\mathbf{Y}_0^{*(\mathbf{l}-\mathbf{j})} \tag{5.288}$$

5.11c Appendix: Formal diagonalization

Consider again the equation

$$\mathbf{y}' = \mathbf{f}_0(x) - \hat{\Lambda}\mathbf{y} + \frac{1}{x}\hat{A}\mathbf{y} + \mathbf{g}(x,\mathbf{y}) \tag{5.289}$$