If $\hat{\Lambda}$ is diagonalizable, then it can be easily diagonalized in (5.289) by the substitution $\mathbf{y} = \hat{C}\mathbf{y}^{[1]}$, where $\hat{C}^{-1}\hat{\Lambda}\hat{C}$ is diagonal.

So we can assume that $\hat{\Lambda}$ is already diagonal. Now, a transformation of the form $\mathbf{y} = (I + x^{-1}\hat{V})\mathbf{y}^{[1]}$ brings (5.289), up to terms of order $\mathbf{y}/x^2$, to an equation of the type

$$\mathbf{y}' = \mathbf{f}_0(x) - \hat{\Lambda}\mathbf{y} + \frac{1}{x}\left(\hat{A} + \hat{V}\hat{\Lambda} - \hat{\Lambda}\hat{V}\right)\mathbf{y} + \mathbf{g}(x, \mathbf{y}) \tag{5.290}$$

Now we regard the map

$$\hat{\hat{\Lambda}}\hat{V} :=\mapsto \hat{V}\hat{\Lambda} - \hat{\Lambda}\hat{V}$$

as a linear map on the space of matrices $\hat{V}$, or, which is the same, on $\mathbb{C}^{2n}$. The equation

$$\hat{\hat{\Lambda}}\hat{V} = \hat{X} \tag{5.291}$$

has a solution *iff* $\hat{X}$ is not in the kernel of $\hat{\hat{\Lambda}}$, which by definition, consists in all matrices such that $\hat{\hat{\Lambda}}\hat{Y} = 0$, or, in other words, all matrices which commute with $\hat{\Lambda}$. Since the eigenvalues of $\hat{\Lambda}$ are distinct, it is easy to check that $\hat{\hat{\Lambda}}\hat{Y} = 0$ implies $\hat{Y}$ is diagonal. So, we can change the *off-diagonal* elements of $\hat{A}$ at will, in particular we can choose them to be zero. By further transformations $\mathbf{y} = (I + x^{-j}\hat{V})\mathbf{y}^{[1]}$, $j = 2...m$, we can diagonalize the coefficients of $x^{-2}\mathbf{y}, ..., x^{-m}\mathbf{y}$.

So, we can assume all coefficients of $x^{-j}\mathbf{y}$ up to any fixed m are diagonal. To show that we can actually assume the coefficients of $x^{-j}\mathbf{y}$, $j = 2...m$, to be zero it is then enough to show that this is possible for a scalar equation

$$y' = f_0(x) - \Lambda y + \frac{1}{x}Ay + (A_2x^{-2} + \cdots + A_mx^{-m})y + \mathbf{g}(x, y) \tag{5.292}$$

As usual, by subtracting terms, we can assume $f_0(x) = O(x^{-M})$ for any choice of M, so for the purpose of this argument, we can see that we can safely assume f_0 is absent.

$$y' = -\Lambda y + \frac{1}{x}Ay + (A_2x^{-2} + \cdots + A_mx^{-m})y + \mathbf{g}(x, y) \tag{5.293}$$

Now, by substituting $y = (1 + c_1/x + c_2/x^2 + \cdots + c_m/x^m)y^{[1]}$ for suitable c_i, the new coefficients $A_j^{[1]}$ vanish (check!).

*5.12 Appendix: The C^*–algebra of staircase distributions, $\mathcal{D}'_{m,\nu}$

Let $\mathcal{D}$ be the space of test functions (compactly supported C^∞ functions on $(0, \infty)$) and $\mathcal{D}(0, x)$ be the test functions on $(0, x)$.