5.12.2 Embedding of L^1_ν in $\mathcal{D}'_m$

Lemma 5.327 *(i) Let $f \in L^1_{\nu_0}$ (cf. Remark 5.324). Then $f \in \mathcal{D}'_{m,\nu}$ for all $\nu > \nu_0$.*

(ii) $\mathcal{D}(\mathbb{R}^+\backslash\mathbb{N}) \cap L^1_\nu(\mathbb{R}^+)$ is dense in $\mathcal{D}'_{m,\nu}$ with respect to the norm $\|\|_\nu$.

Proof.

Note that if for some ν_0 we have $f \in L^1_{\nu_0}(\mathbb{R}^+)$, then

$$\int_0^p |f(s)|\mathrm{d}s \le \mathrm{e}^{\nu_0 p}\int_0^p |f(s)|\mathrm{e}^{-\nu_0 s}\mathrm{d}s \le \mathrm{e}^{\nu_0 p}\|f\|_{\nu_0} \tag{5.328}$$

to which, application of $\mathcal{P}^{k-1}$ yields

$$\mathcal{P}^k|f| \le \nu_0^{-k+1}\mathrm{e}^{\nu_0 p}\|f\|_{\nu_0} \tag{5.329}$$

Also, $\mathcal{P}\chi_{[n,\infty)}\mathrm{e}^{\nu_0 p} \le {\nu_0}^{-1}\chi_{[n,\infty)}\mathrm{e}^{\nu_0 p}$ so that

$$\mathcal{P}^m\chi_{[n,\infty)}\mathrm{e}^{\nu_0 p} \le {\nu_0}^{-m}\chi_{[n,\infty)}\mathrm{e}^{\nu_0 p} \tag{5.330}$$

so that, by (5.305) (where now F_n and $\chi_{[n,\infty]}F_n$ are in $L^1_{\mathrm{loc}}(0, n+1)$) we have for $n > 1$,

$$|\Delta_n| \le \|f\|_{\nu_0}\mathrm{e}^{\nu_0 p}{\nu_0}^{1-mn}\chi_{[n,n+1]} \tag{5.331}$$

Let now ν be large enough. We have

$$\sum_{n=2}^{\infty}\nu^{mn}\int_0^\infty |\Delta_n|\mathrm{e}^{-\nu p}\mathrm{d}p \le \nu_0\|f\|_{\nu_0}\sum_{n=2}^{\infty}\int_n^{n+1}\mathrm{e}^{-(\nu-\nu_0)p}\left(\frac{\nu}{\nu_0}\right)^{mn}\mathrm{d}p$$
$$\le \frac{\nu^{2m}e^{-2\nu+2\nu_0}}{\nu_0^{2m-1}(\nu-\nu_0-m\ln(\nu/\nu_0))}\|f\|_{\nu_0} \tag{5.332}$$

For $n = 0$ we simply have $\|\Delta_0\| \le \|f\|$, while for $n = 1$ we write

$$\|\Delta_1\|_\nu \le \|1^{*(m-1)} * |f|\|_\nu \le \nu^{-m+1}\|f\|_\nu \tag{5.333}$$

Combining the estimates above, the proof of (i) is complete. To show (ii), let $f \in \mathcal{D}'_{m,\nu}$ and let k_ϵ be such that $c_m\sum_{i=k_\epsilon}^\infty \nu^{im}\|\Delta_i\|_\nu < \epsilon$. For each $i \le k_\epsilon$ we take a function δ_i in $\mathcal{D}(i, i+1)$ such that $\|\delta_i - \Delta_i\|_\nu < \epsilon 2^{-i}$. Then $\|f - \sum_{i=0}^{k_\epsilon}\delta_i^{(mi)}\|_{m,\nu} < 2\epsilon$. □

The proof of continuity of $f(p) \mapsto pf(p)$: If $f(p) = \sum_{k=0}^\infty \Delta_k^{(mk)}$ then $pf = \sum_{k=0}^\infty(p\Delta_k)^{(mk)} - \sum_{k=0}^\infty mk\mathcal{P}(\Delta_k^{(mk)}) = \sum_{k=0}^\infty(p\Delta_k)^{(mk)} - 1*\sum_{k=0}^\infty(mk\Delta_k)^{(mk)}$. The rest is obvious from continuity of convolution, the embedding shown above and the definition of the norms.