Proposition 6.19 *Take $\lambda_1 = 1$, $n = 1$ in (5.51) and let $\xi = x^{\alpha}e^{-x}$. Assume that the corresponding F_0 in (6.13) is not entire (this is generic[1]). Let $\mathbb{D}_r$ be the maximal disk where F_0 is analytic[2]. Assume $\xi_0 \in \partial\mathbb{D}_r$ is a singular point of F_0 such that F_0 admits analytic continuation in a ramified neighborhood of ξ_0. Then y is singular at infinitely many points, asymptotically given by*

$$x_n = 2n\pi i + \alpha_1 \ln(2n\pi i) + \ln C - \ln \xi_0 + o(1) \ (n \to \infty) \tag{6.20}$$

(recall that $C = C_+$).

Remark 6.21 We note that asymptotically y is a function of $Ce^{-x}x^{\alpha}$. This means that there are infinitely many singular points, nearly periodic, since the points x so that $Ce^{-x}x^{\alpha} = \xi_0$ are nearly periodic.

We need the following result which is in some sense a converse of Morera's theorem.

Lemma 6.22 *Assume that $f(\xi)$ is analytic on the universal covering of $\mathbb{D}_r\backslash\{0\}$. Assume further that for any circle around zero $\mathcal{C} \subset \mathbb{D}_r\backslash\{0\}$ and any $g(\xi)$ analytic in $\mathbb{D}_r$ we have $\oint_{\mathcal{C}} f(\xi)g(\xi)d\xi = 0$. Then f is in fact analytic in $\mathbb{D}_r$.*

PROOF Let $a \in \mathbb{D}_r\backslash\{0\}$. It follows that $\int_a^{\xi} f(s)ds$ is single-valued in $\mathbb{D}_r\backslash\{0\}$. Thus f is single-valued and, by Morera's theorem, analytic in $\mathbb{D}_r\backslash\{0\}$. Since by assumption $\oint_{\mathcal{C}} f(\xi)\xi^n d\xi = 0$ for all $n \geq 0$, there are no negative powers of ξ in the Laurent series of $f(\xi)$ about zero: f extends as an analytic function at zero. ∎

PROOF of Proposition 6.19

By Lemma 6.22 there is a circle $\mathcal{C}$ around ξ_s and a function $g(\xi)$ analytic in $\mathbb{D}_r(\xi - \xi_s)$ so that $\oint_{\mathcal{C}} F_0(\xi)g(\xi)d\xi = 1$. In a neighborhood of $x_n \in X$ the function $f(x) = \mathrm{e}^{-x}x^{\alpha_1}$ is conformal and for large x_n

$$-\oint_{f^{-1}(\mathcal{C})} y(x)g(f(x))f'(x)dx$$
$$= \oint_{\mathcal{C}} (F_0(\xi) + O(x_n^{-1}))g(\xi)d\xi = 1 + O(x_n^{-1}) \neq 0 \tag{6.23}$$

It follows that for large enough x_n $y(x)$ is not analytic inside $\mathcal{C}$ either. Since the radius of $\mathcal{C}$ can be taken $o(1)$ the result follows.

∎

[1]After suitable changes of variables; see comments after Theorem 6.57.

[2]By Theorem 6.57 F_0 is always analytic at zero.