Lemma 3.37 *Let* $F \in L^1(\mathbb{R}^+)$, $x = \rho e^{i\phi}$, $\rho > 0$, $\phi \in (-\pi/2, \pi/2)$ *and assume*

$$F(p) \sim p^\beta \quad \textit{as } p \to 0^+$$

with $\mathrm{Re}\,(\beta) > -1$. *Then*

$$\int_0^\infty F(p)e^{-px}dp \sim \Gamma(\beta+1)x^{-\beta-1} \quad (\rho \to \infty)$$

PROOF If $U(p) = p^{-\beta}F(p)$ we have $\lim_{p\to 0} U(p) = 1$. Let χ_A be the characteristic function of the set A and $\phi = \arg(x)$. We choose C and a positive so that $|F(p)| \le C|p^\beta|$ on $[0,a]$. Since

$$\left|\int_a^\infty F(p)e^{-px}\mathrm{d}p\right| \le e^{-a\mathrm{Re}\,x}\|F\|_1 \tag{3.38}$$

we have after the change of variable $s = p|x|$,

$$x^{\beta+1}\int_0^\infty F(p)e^{-px}\mathrm{d}p = e^{i\phi(\beta+1)}\int_0^\infty s^\beta U(s/|x|)\chi_{[0,a]}(s/|x|)e^{-se^{i\phi}}\mathrm{d}s \\ + O(|x|^{\beta+1}e^{-xa}) \to \Gamma(\beta+1) \quad (|x| \to \infty) \tag{3.39}$$

by dominated convergence in the last integral. ∎

3.4a The Borel-Ritt lemma

Any asymptotic series at infinity is the asymptotic series in a half-plane of some (vastly many in fact) entire functions. First a weaker result.

Proposition 3.40 *Let* $\tilde{f}(z) = \sum_{k=0}^\infty a_k z^k$ *be a power series. There exists a function* f *so that* $f(z) \sim \tilde{f}(z)$ *as* $z \to 0$.

PROOF The following elementary line of proof is reminiscent of optimal truncation of series. By Remark 1.32 we can assume, without loss of generality, that the series has zero radius of convergence and $a_0 = 1$. Let $z_0 > 0$ be small enough and for every z, $|z| < z_0$, define $N(z) = \max\{N : \forall\, n \le N,\ |a_n z^{n/2}| \le 2^{-n}$. We have $N(z) < \infty$, otherwise, by Abel's theorem, the series would have nonzero radius of convergence. Noting that for any n we have $n \ln|z| \to -\infty$ as $|z| \to 0$ it follows that $N(z)$ is nondecreasing as $|z|$ decreases and that $N(z) \to \infty$ as $z \to 0$. Consider

$$f(z) = \sum_{n=0}^{N(z)} a_n z^n$$