Let $N$ be given and choose $z_N$; $|z_N| < 1$ such that $N(z_N) \geq N$. For $|z| < |z_N|$ we have $N(z) \geq N(z_N) \geq N$ and thus

$$\left| f(z) - \sum_{n=0}^{N} a_n z^n \right| = \left| \sum_{n=N+1}^{N(z)} a_n z^n \right| \leq \sum_{j=N+1}^{N(z)} |z^{j/2}| 2^{-j} \leq |z|^{N/2+1/2}$$

Using Lemma 1.13, the proof follows. ∎

The function $f$ is certainly not unique. Given a power series there are many functions asymptotic to it. Indeed there are many functions asymptotic to the (identically) zero power series at zero, in any sectorial punctured neighborhood of zero in the complex plane, and even on the Riemann surface of the log on $\mathbb{C} \setminus \{0\}$, e.g., $e^{-x^{-1/n}}$ has this property in a sector of width $2n\pi$.

**Lemma 3.41 (Borel-Ritt)** *Given a formal power series* $\tilde{f} = \sum_{k=0}^{\infty} \frac{c_k}{x^{k+1}}$ *there exists an entire function* $f(x)$*, of exponential order one (see proof below), which is asymptotic to* $\tilde{f}$ *in* $\mathbb{H}$*, i.e., if* $\phi \in (-\pi/2, \pi/2)$ *then*

$$f(x) \sim \tilde{f} \text{ as } x = \rho e^{i\phi}, \quad \rho \to +\infty$$

**PROOF** Let $\tilde{F} = \sum_{k=0}^{\infty} \frac{c_k}{k!} p^k$, let $F(p)$ be a function asymptotic to $\tilde{F}$ as in Proposition 3.40. Then clearly the function

$$f(x) = \int_0^{\epsilon} e^{-xp} F(p) dp$$

($\epsilon > 0$ small) is entire, bounded by $Const. e^{|x|}$, i.e., the exponential order is one, and, by Watson's lemma it has the desired properties.

∎

**Exercises.**
(1) How can this method be modified to give a function analytic in a sector of opening $2\pi n$ for an arbitrary fixed $n$ which is asymptotic to $\tilde{f}$?

(2) Assume $F$ is bounded on $[0, 1]$ and has an asymptotic expansion $F(t) \sim \sum_{k=0}^{\infty} c_k t^k$ as $t \to 0^+$. Let $f(x) = \int_0^1 e^{-xp} F(p) dp$. (a) Find necessary conditions and sufficient conditions on $F$ such that $\tilde{f}$, the asymptotic power series of $f$ for large positive $x$, is a convergent series for $|x| > R > 0$. (b) Assume that $\tilde{f}$ *converges to* $f$. Show that $f$ is zero. (c) Show that in case (a) if $F$ is analytic in a neighborhood of $[0, 1]$ then $f = \tilde{f} + e^{-x} \tilde{f}_1$ where $\tilde{f}_1$ is convergent for $|x| > R > 0$.

(3) The width of the sector in Lemma 3.41 cannot be extended to more than a half-plane: Show that if $f$ is entire, of exponential order one, and bounded in a