(ii) If $f \in L^1(\mathbb{R})$ *and* $|x|^\eta f(x) \in L^1(\mathbb{R})$ *with* $\eta \in (0,1]$*, then its Fourier transform* $\hat{f} = \int_{-\infty}^{\infty} f(s)e^{-ixs}ds$ *is in* $C^\eta(\mathbb{R})$*.*

(iii) Let $f \in L^1(\mathbb{R})$*. If* $x^n f \in L^1(\mathbb{R})$ *with* $n-1 \in \mathbb{N}$ *then* $\hat{f}$ *is* n *times differentiable, with the* $n-1$*th derivative Lipschitz continuous. If* $e^{|Ax|}f \in L^1(\mathbb{R})$ *then* $\hat{f}$ *extends analytically in a strip of width* $|A|$ *centered on* $\mathbb{R}$*.*

PROOF (i) We have as $x \to \infty$ ($\lfloor \cdot \rfloor$ denotes the integer part)

$$\left|\int_0^1 f(s)e^{ixs}ds\right| = \left|\sum_{j=0}^{\lfloor \frac{x}{2\pi}-1\rfloor}\left(\int_{2j\pi x^{-1}}^{(2j+1)\pi x^{-1}} f(s)e^{ixs}ds + \int_{(2j+1)\pi x^{-1}}^{(2j+2)\pi x^{-1}} f(s)e^{ixs}ds\right)\right| + O(x^{-1})$$
$$= \left|\sum_{j=0}^{\lfloor \frac{x}{2\pi}-1\rfloor}\int_{2j\pi x^{-1}}^{(2j+1)\pi x^{-1}} \big(f(s) - f(s+\pi/x)\big)e^{ixs}ds\right| + O(x^{-1})$$
$$\le \sum_{j=0}^{\lfloor \frac{x}{2\pi}-1\rfloor} a\left(\frac{\pi}{x}\right)^\eta \frac{\pi}{x} \le \frac{1}{2}a\pi^\eta x^{-\eta} + O(x^{-1}) \quad (3.58)$$

(ii) We see that

$$\left|\frac{\hat{f}(s)-\hat{f}(s')}{(s-s')^\eta}\right| = \left|\int_{-\infty}^{\infty}\frac{e^{-ixs}-e^{-ixs'}}{x^\eta(s-s')^\eta}x^\eta f(x)dx\right| \le \int_{-\infty}^{\infty}\left|\frac{e^{-ixs}-e^{-ixs'}}{(xs-xs')^\eta}\right|\left|x^\eta f(x)\right|dx \quad (3.59)$$

is bounded. Indeed, by elementary geometry we see that for $|\phi_1 - \phi_2| < 1$ we have

$$|\exp(i\phi_1) - \exp(i\phi_2)| \le |\phi_1 - \phi_2| \le |\phi_1 - \phi_2|^\eta \quad (3.60)$$

while for $|\phi_1 - \phi_2| \ge 1$ we see that

$$|\exp(i\phi_1) - \exp(i\phi_2)| \le 2 \le 2|\phi_1 - \phi_2|^\eta$$

(iii) Follows in the same way as (ii), using dominated convergence. ☐

Exercise 3.61 *Complete the details of this proof. Show that for any* $\eta \in (0,1]$ *and all* $\phi_{1,2} \in \mathbb{R}$ *we have* $|\exp(i\phi_1) - \exp(i\phi_2)| \le 2|\phi_1 - \phi_2|^\eta$*.*

Note. In Laplace type integrals Watson's lemma implies that it suffices for a function to be continuous to ensure an $O(x^{-1})$ decay of the integral, whereas in Fourier-like integrals, the considerably weaker decay (3.57) is optimal as seen in the exercise below.