3.9e Spontaneous singularities: The Painlevé's equation P_I

In nonlinear differential equations, the solutions may be singular at points x where the equation is regular. For example, the equation

$$y' = y^2 + 1$$

has a one parameter family of solutions $y(x) = \tan(x + C)$; each solution has infinitely many poles. Since the location of these poles depends on C, thus on the solution itself, these singularities are called *movable* or *spontaneous*.

Let us analyze local singularities of the Painlevé equation P_I,

$$y'' = y^2 + x \tag{3.164}$$

We look at the local behavior of a solution that blows up, and will find solutions that are meromorphic but not analytic. In a neighborhood of a point where y is large, keeping only the largest terms in the equation (*dominant balance*) we get $y'' = y^2$ which can be integrated explicitly in terms of elliptic functions and its solutions have double poles. Alternatively, we may search for a power-like behavior

$$y \sim A(x - x_0)^p$$

where $p < 0$ obtaining, to leading order, the equation $Ap(p-1)(x-x_0)^{p-2} = A^2(x-x_0)^{2p}$ which gives $p = -2$ and $A = 6$ (the solution $A = 0$ is inconsistent with our assumption). Let's look for a power series solution, starting with $6(x - x_0)^{-2}$: $y = 6(x - x_0)^{-2} + c_{-1}(x - x_0)^{-1} + c_0 + \cdots$. We get: $c_{-1} = 0, c_0 = 0, c_1 = 0, c_2 = -x_0/10, c_3 = -1/6$ and c_4 is undetermined, thus free. Choosing a c_4, all others are uniquely determined. To show that there indeed is a convergent such power series solution, we follow the remarks in §3.8b. Substituting $y(x) = 6(x - x_0)^{-2} + \delta(x)$ where for consistency we should have $\delta(x) = o((x - x_0)^{-2})$ and taking $x = x_0 + z$ we get the equation

$$\delta'' = \frac{12}{z^2}\delta + z + x_0 + \delta^2 \tag{3.165}$$

Note now that our assumption $\delta = o(z^{-2})$ makes $\delta^2/(\delta/z^2) = z^2\delta = o(1)$ and thus the nonlinear term in (3.165) is *relatively* small. Thus, *to leading order*, the new equation is linear. This is a general phenomenon: taking out more and more terms out of the local expansion, the correction becomes less and less important, and the equation is better and better approximated by a linear equation. It is then natural to separate out the large terms from the small terms and write a fixed point equation for the solution based on this separation. We write (3.165) in the form

$$\delta'' - \frac{12}{z^2}\delta = z + x_0 + \delta^2 \tag{3.166}$$

and integrate as if the right side was known. This leads to an equivalent integral equation. Since all unknown terms on the right side are chosen to