Invariant theory and the two-local cohomology of symmetric groups

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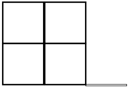
A diagrammatic approach to invariant theory

We may use Young diagrams to represent symmetric polynomials - that is elements of $R[x_1, \cdots, x_n]^{\mathcal{S}_n}$ - through symmetrized monomials.

$$P \leftrightarrow \mathrm{Sym}(x^P),$$

where $\mathrm{Sym}(m)$ denotes the minimal symmetrization of a monomial m, namely $\sum_{[\sigma] \in S_n/H} \sigma \cdot m$ where H fixes m.

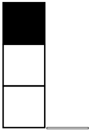


$$\leftrightarrow {x_1}^2 {x_2}^2 + {x_1}^2 {x_3}^2 + {x_2}^2 {x_3}^2.$$

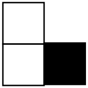
A diagrammatic approach to invariant theory

Multiplication is given by the sum of "stackings" over all possible column matchings (up to automorphism).

This seems ineffective - one doesn't immediately see the fact that these are polynomial rings! But doing this with colored blocks is a fruitful approach to invariants of multiple sets of variables.



$\leftrightarrow {x_1}^2 y_1 + {x_2}^2 y_2.$



$\leftrightarrow {x_1}^2 y_2 + {x_2}^2 y_1.$

A diagrammatic approach to invariant theory

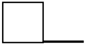
These diagrams immediately define additive bases for these rings of invariants, with multiplication still represented by stacking.

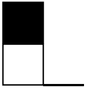
On the other hand, presenting these rings in positive characteristic by generators and relations over has been notoriously difficult.

In particular, Feshbach gave explicit generators and inductively defined relations for these rings over $\mathbb{F}_2$, and to our knowledge such descriptions are still open for these rings over $\mathbb{F}_p$ with $p > 2$.

A diagrammatic approach to invariant theory

A set of generators for $\mathbb{F}_2[x_1, x_2, y_1, y_2]^{\mathcal{S}_2}$ is

$a =$ , $b =$ , $c =$ , $d =$ 

$e =$ , $f =$ ,

With relations:

$$ab = e + f$$

and

$$ef = a^2 d + b^2 c.$$

Hopf ring structure

The two-dimensionality of these diagrams is useful for representing a Hopf ring structure we have noticed on the direct sum (coproduct) of these rings of invariants.

Definition

A Hopf ring is a ring object in the category of coalgebras. Explicitly, a Hopf ring is vector space V with two multiplications, one comultiplication, and an antipode $(\odot, \cdot, \Delta, S)$ such that the first multiplication forms a Hopf algebra with the comultiplication and antipode, the second multiplication forms a bialgebra with the comultiplication, and these structures satisfy the distributivity relation

$$\alpha \cdot (\beta \odot \gamma) = \sum_{\Delta\alpha = \sum a' \otimes a''} (a' \cdot \beta) \odot (a'' \cdot \gamma).$$

Hopf ring structure

In topology, Hopf rings first arose as the homology of infinite loop spaces which represent ring spectra.

Direct sums of rings of symmetric invariants also form Hopf rings.

The $\cdot$ multiplication is the standard one, represented by stackings in our diagrams (and defined to be zero if the number of variables does not agree).

Hopf ring structure

The $\odot$ multiplication is a "shuffle product," represented by concatenating diagrams, with a coefficient of $\binom{i+j}{i}$ if there are $i+j$ repeated columns in the resulting diagram coming from i columns in the first diagram and j in the second.

The comultiplication is defined by partitioning the columns of a diagram to make two new diagrams.

Hopf ring structure

Definition

An $\mathbb{N}$-component Hopf ring is one which as a ring under $\cdot$ is a coproduct of rings A_i, $i \geq 0$.

By distributivity, the second product $\odot$ is thus graded with respect to this decomposition.

Hopf ring structure

Definition

A divided powers Hopf ring generated by a finite set $a_1, \cdots a_k$ is the Hopf ring generated under the two products by variables $a_{i,n}$ with $1 \leq i \leq k$ and $n \geq 1$ with coproducts determined by

$$\Delta a_{i,n} = \sum_{k+\ell=n} a_{i,k} \otimes a_{i,\ell},$$

and $\odot$-products

$$a_{i,n} \odot a_{i,m} = \binom{n+m}{n} a_{i,n+m}.$$

Hopf ring structure

Theorem (GSS)

The direct sum of rings of invariants in k collections of variables (for example, $\bigoplus_n A[x_1, \cdots, x_n, y_1, \cdots, y_n]^{\mathcal{S}_n}$ for $k = 2$) is a free divided powers $\mathbb{N}$-component Hopf ring, with k generators on the first component, given by the variables themselves.

Hopf ring structure

There are similar Hopf ring structures on representation rings of symmetric groups, defined by tensor product, induction product, and restriction coproduct. The induction/restriction Hopf algebra structure was studied by Zelevinsky. (Similar games can be played for general linear groups over finite fields.)

Theorem (Strickland-Turner)

For any ring-theory E^*, the cohomology of symmetric groups $\bigoplus_n E^*(B\mathcal{S}_n)$ forms a (derived) Hopf ring where

- *The $\cdot$ product is the standard product (with zero products between distinct summands).*
- *The coproduct Δ is induced by the standard covering $p : B\mathcal{S}_n \times B\mathcal{S}_m \to B\mathcal{S}_{n+m}$.*
- *The product $\odot$ is the transfer associated to p.*

Mod-two cohomology of symmetric groups

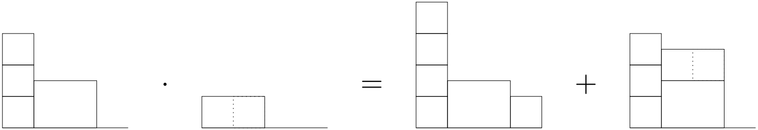
Theorem (GSS)

$\bigoplus_n H^*(B\mathcal{S}_n;\mathbb{F}_2)$ is a free divided powers $\mathbb{N}$-component Hopf ring over $\mathbb{F}_2$ on classes $\gamma_k \in H^{2^k-1}(B\mathcal{S}_{2^k};\mathbb{F}_2)$.

We also determine the action of the Steenrod algebra (and recover classical results such as H-atomicity of QS^0).

Mod-two cohomology of symmetric groups

This Hopf ring has a diagrammatic presentation - which we call that of "skyline diagrams" - similar to that for rings of symmetric invariants in how both products and the coproduct are represented but with boxes of different sizes for the γ_k.

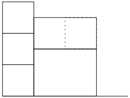


$$(\gamma_{1,1}^3 \odot \gamma_{2,1} \odot 1_2) \cdot (\gamma_{1,2} \odot 1_4) = \gamma_{1,1}^4 \odot \gamma_{2,1} \odot 1_2 + \gamma_{1,1}^3 \odot \gamma_{1,2}\gamma_{2,1} \odot 1_2$$

Mod-two cohomology of symmetric groups

The duality between this basis and the Dyer-Lashof basis for homology is not understood.

These classes have geometric meaning as characteristic classes for finite-sheeted covering maps, akin to classical characteristic classes defined by Shubert cycles.

Consider $\alpha =$  which defines a characteristic class in degree eight for an eight-sheeted covering map as follows for manifolds.

Mod-two cohomology of symmetric groups

- Embed the covering $p : \widetilde{M} \to M$ in some $M \times \mathbb{R}^d \to M$.
- For $m \in M$ consider the unordered configuration $p^{-1}(m) = (x_1, \cdots, x_8)/\sim$ with $x_i \in \mathbb{R}^d$ and where $\sim$ stands for relabeling.
- Consider the locus $\chi_\alpha(p) \subset M$ of m such that in $p^{-1}(m)$ there are two points which share their first three coordinates, and four points which share their first coordinate and can be partitioned into two groups of two which share their second coordinate.
- Then the characteristic class of p associated to α is Poincaré dual to $\chi_\alpha(p)$ (under standard transversality assumptions)

Two-local cohomology of symmetric groups

Theorem (GSS, in preparation)

$\bigoplus_n H^*(B\mathcal{S}_n; \mathbb{Z}_{(2)})$ contains a free divided powers $\mathbb{N}$-component Hopf ring over $\mathbb{Z}_{(2)}$ on classes whose mod-two reductions are ${\gamma_k}^2$.

This subring is thus represented by skyline diagrams for which a building block is two of some mod-two building block stacked vertically.

The divided powers structure governs higher torsion in this sub-Hopf ring.

If a diagram consists of a single column is repeated 2^k times, it represents a 2^{k+1}-torsion class. More generally, if it is repeated n times then it represents $2^{|n|_2+1}$-torsion.

For a general diagram, one takes the minimal order of such (maximal collections of) repeated blocks within the diagram.

Symmetric invariants of the cohomology of $\mathbb{R}P^\infty$

To understand the full two-local story, we turn back to invariant theory.

The similarity between the presentations of the mod-two cohomology of symmetric groups and that of symmetric invariants is not a coincidence, but reflects the standard connection between group cohomology and invariant theory.

Relevant to the two-local cohomology of symmetric groups is $H^*(B\mathcal{S}_2 \times \cdots \times B\mathcal{S}_2; \mathbb{Z})^{\mathcal{S}_n}$.

Symmetric invariants of the cohomology of $\mathbb{R}P^\infty$

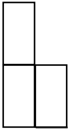
Consider the cochains of $\mathbb{R}P^\infty \times \mathbb{R}P^\infty$.

$$
\begin{tikzcd}
{} & {} & {} & & \\
y^2 \arrow[u, "0"] \arrow[r, "0"] & xy^2 \arrow[u, "0"] \arrow[r, "2"] & x^2y^2 \arrow[u, "0"] \arrow[r, "0"] & x^3y^2 \arrow[r, "2"] & x^4y^2 \\
y \arrow[u, "2"] \arrow[r, "0"] & xy \arrow[u, "-2"] \arrow[r, "2"] & x^2y \arrow[u, "2"] \arrow[r, "0"] & x^3y \arrow[u, "-2"] \arrow[r, "2"] & x^4y \arrow[u, "2"] \\
1 \arrow[u, "0"] \arrow[r, "0"] & x \arrow[u, "0"] \arrow[r, "2"] & x^2 \arrow[u, "0"] \arrow[r, "0"] & x^3 \arrow[u, "0"] \arrow[r, "2"] & x^4 \arrow[u, "0"]
\end{tikzcd}
$$

Symmetric invariants of the cohomology of $\mathbb{R}P^\infty$

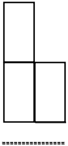
There are two types of $\mathcal{S}_2$-invariant cocyles:

- $x^{2k}y^{2\ell} + x^{2\ell}y^{2k}$ (or $x^{2k}y^{2k}$),
  which we represent by diagrams such as

 $\leftrightarrow x^4y^2 + x^2y^4.$

Symmetric invariants of the cohomology of $\mathbb{R}P^\infty$

- $x^{2k+1}y^{2\ell} - x^{2k}y^{2\ell+1} + x^{2\ell}y^{2k+1} - x^{2\ell+1}y^{2k}$ (or $x^{2k+1}y^{2k} + x^{2k}y^{2k+1}$).
  These are all multiples of cocycles of the first type by $x^2y - xy^2$. We denote this by an additional "Bockstein band" at the bottom of the diagram.

 $\leftrightarrow x^6y^3 - x^5y^4 + x^3y^6 - x^4y^5.$

(Our convention of area representing degree does not hold with Bockstein bands.)

Symmetric invariants of the cohomology of $\mathbb{R}P^\infty$

Theorem (GSS)

Even-height Young diagrams with (non-overlapping) Bockstein bands constitute a basis for $H^*(B\mathcal{S}_2 \times \cdots \times B\mathcal{S}_2; \mathbb{Z})^{\mathcal{S}_n}$.

Multiplication is given by column matching as before along with "band overlap rules."

Conjecture (GSS)

There is an additive basis for the two-local cohomology of symmetric groups given by even skyline diagrams with Bockstein bands (which must contain at least one column of width one or two).

Next steps

- Odd primes, alternating groups (in progress).
- Analyze AHSS for K-theory and Morava K-theory (with Sadofsky?). Differentials compatible with Hopf ring structure.
- Dream: Computation and geometric understanding (in terms of framed cobordism) of Hurewicz homomorphism for QS^0.
- Dream (after rational work with Ben Walter): computable and geometrically meaningful unstable homotopy models/invariants based on bar construction on small cochain models of $B\mathcal{S}_n$.