Journal of Algebra 591 (2022) 289–341

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Equivariant perverse sheaves and quasi-hereditary algebras

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A R T I C L E I N F O

Article history:
Received 7 February 2021
Available online 3 November 2021
Communicated by Prakash Belkale

This paper is dedicated to the memory of Edward Thomas Cline, Jr.

MSC:
primary 14L30
secondary 14F20

Keywords:
Linear algebraic group actions
Equivariant perverse sheaves
Quasi-hereditary algebras

A B S T R A C T

Let X denote a quasi-projective variety over a field on which a connected linear algebraic group G acts with finitely many orbits. Then, the G-orbits define a stratification of X. We establish several key properties of the category of equivariant perverse sheaves on X, which have locally constant cohomology sheaves on each of the orbits. Under the above assumptions, we show that this category comes close to being a highest weight category in the sense of Cline, Parshall and Scott and defines a quasi-hereditary algebra. We observe that the above hypotheses are satisfied by *all toric varieties* and by *all spherical varieties* associated to connected reductive groups over any algebraically closed field.
Next we show that the odd dimensional intersection cohomology sheaves vanish on all spherical varieties defined over algebraically closed fields of positive characteristics, extending similar results for spherical varieties defined over the field of complex numbers by Michel Brion and the author in prior work. Assuming that the linear algebraic group G and the action of G on X are defined over a finite field $\mathbb{F}_q$, and where the odd dimensional intersection cohomology sheaves on the orbit closures vanish, we also establish several basic properties of the mixed category of mixed equivariant perverse sheaves so that the associated terms in the weight filtration are fi-

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https://doi.org/10.1016/j.jalgebra.2021.10.027

nite sums of the shifted equivariant intersection cohomology complexes on the orbit closures.

Contents

1. Introduction

It was a few years ago, while exploring the literature for something else, that we came upon the work of Cline, Parshall and Scott on highest weight categories and quasi-hereditary algebras. (See for example, [13], [32].) Soon it became clear that many nice properties for the category of perverse sheaves on a space needed to construct quasi-hereditary algebras are shared by the category of equivariant perverse sheaves on a scheme provided with the action of a linear algebraic group, under some fairly mild conditions on the group action. In fact, in the literature, (for example, [32, section 5]), many of the nice properties for the category of perverse sheaves are shown to hold only for stratified schemes, where the strata are acyclic.[1] When a linear algebraic group acts on a scheme, the orbits form a particularly nice stratification; however, they are seldom acyclic. The goal of the present paper is to show that, nevertheless, the category of equivariant perverse sheaves on a scheme provided with the action of a linear algebraic group has many of these nice properties, when there are only finitely many orbits.

In particular, we show that it is possible to adapt the machinery in [32] this way, to provide an abundant supply of quasi-hereditary algebras, under fairly mild conditions on the group action. For example, we show that this construction applied to all toric varieties or spherical varieties associated to connected reductive groups defined over any algebraically closed field, or all complex spherical varieties associated to complex connected reductive groups, produces quasi-hereditary algebras. See Theorem 1.2 for

[1] There is a separate construction of highest weight categories due to van der Kallen (see [41]) using Schubert varieties and the cohomology of line bundles on them, but the relation between the construction of [32] and [41] is not clear.

more details. This theorem, together with Theorem 1.1, forms the *first main result* of the paper.

The remainder of the paper discusses various related results on equivariant perverse sheaves. The point is that the category of perverse sheaves on a stratified scheme seems unlikely to be the category of modules over a Koszul algebra, unless the strata are acyclic. However, as observed earlier, the orbit stratification on a scheme where there are only finitely many orbits for the action of a linear algebraic group rarely satisfies this condition, except in very special cases like that of Schubert varieties. Therefore, the *goal of the remainder of the paper* is to derive other interesting properties of the category of equivariant perverse sheaves under weaker hypotheses, like the vanishing of odd dimensional intersection cohomology sheaves. We also prove the above vanishing for all spherical varieties over algebraically closed fields of positive characteristics, extending earlier results due to Brion and the author for complex spherical varieties: see [9].

Notations and Conventions. Throughout the paper, k will denote a fixed field, which will always be a perfect field of arbitrary characteristic $p \geq 0$, and of finite ℓ-cohomological dimension for any prime $\ell \neq p$. We will also assume that the following condition holds:

$$\mathrm{H}^{\mathrm{i}}_{\mathrm{et}}(\mathrm{Spec}\, k, \mathbb{Z}/\ell^{\mathrm{n}}) \text{ is finite for each i and each } \ell \neq p. \tag{1.0.1}$$

Clearly these hypotheses are satisfied by any algebraically closed field or any finite field.

We will then restrict to quasi-projective, separated and reduced schemes of finite type over k: note that we are not requiring the schemes to be irreducible or connected. (The assumption they are of finite type over k automatically implies there are only finitely many connected and irreducible components.) *Such schemes may often be referred to as quasi-projective schemes, schemes, and/or (quasi-projective) varieties defined over k.* k will be called the *base field*. The only base fields of characteristic 0 that we allow will be the ones that admit imbeddings into the field of complex numbers. Therefore, without loss of generality, we may assume the base fields k that are of characteristic 0 and algebraically closed, identify with the field of complex numbers $\mathbb{C}$: the main observation here is that by the smooth base change theorem in étale cohomology (see [28, Chapter VI,Corollary 4.3]) the étale cohomology of a scheme X over k and the etale cohomology of the induced scheme $\mathrm{X}_{\mathbb{C}}$ over $\mathbb{C}$ are isomorphic with respect to torsion sheaves.

All the results of the first four sections hold over any algebraically closed base field, and all the results of section 5 hold over any algebraically closed field of positive characteristic. Section 6 discusses the weight filtration, and therefore it becomes necessary to assume there that the base field is a finite field $\mathbb{F}_q$: this also makes it necessary to carry out some of the discussion in sections 2 and 3 also over finite fields. As a result, we will assume in general that the base field in sections 2 and 3 is either algebraically closed (of arbitrary characteristic) or a finite field, with any necessary further restrictions spelled out clearly for each key result discussed there. We will assume it is algebraically closed throughout sections 4 and 5.

Let X denote a quasi-projective scheme provided with an action by a *linear algebraic group* G, all defined over k. Then the derived category of sheaves on X needs to be replaced by the equivariant derived category $\mathrm{D^b_{G,c}(X)}$ of complexes of sheaves with bounded, equivariant and constructible cohomology sheaves.

The key to defining the equivariant derived category is a suitable form of the *Borel construction* $\mathrm{EG \times_G X}$, associated to the action of G on the scheme X. There are two alternate constructions here:

(i) that produces the simplicial scheme $\mathrm{EG \times_G X}$ discussed in section 2.2, where one obtains BG (the *classifying space* of G) on taking $\mathrm{X = Spec\,k}$, and
(ii) that produces $\mathrm{EG \times_G X}$ as an ind-scheme (that is, a direct system of schemes) $\mathrm{\{EG^{gm,m} \times_G X | m \geq 1\}}$, with $\mathrm{BG^{gm,m} = EG^{gm,m} \times_G (Spec\,k)}$ a *finite degree approximation* to the *classifying space* of G, and $\mathrm{EG^{gm,m} \to BG^{gm,m}}$ its universal principal G-bundle. These are defined in Definition 7.1.

In view of its functoriality, we choose to work exclusively with the simplicial model discussed in section 2.2. A discussion of the construction $\mathrm{\{EG^{gm,m} \times_G X | m \geq 1\}}$ and a comparison between the two constructions is left to the Appendix.

The equivariant derived category $\mathrm{D^b_{G,c}(X)}$ is then defined as a certain full subcategory of the derived category of the Borel construction $\mathrm{EG \times_G X}$, or of $\mathrm{EG^{gm,m} \times_G X}$ for m sufficiently large: see section 2.2 and the Appendix for more details. The equivariant derived category $\mathrm{D^b_{G,c}(X)}$ incorporates the group action, whereas the ordinary derived category $\mathrm{D^b_c(X)}$ does not.

Different variants of equivariant derived categories. When $k = \mathbb{C}$ or an algebraically closed field of characteristic 0, the derived category $\mathrm{D^b_{G,c}(X, \mathbb{Q})}$ ($\mathrm{D^b_{G,c}(X, \mathbb{C})}$) is made up of complexes of sheaves of $\mathbb{Q}$-vector spaces ($\mathbb{C}$ vector spaces, respectively). One may observe that the derived category $\mathrm{D^b_c(X, \mathbb{Q})}$ ($\mathrm{D^b_c(X, \mathbb{C})}$) is $\mathbb{Q}$-linear ($\mathbb{C}$-linear, respectively) in this case, meaning the Hom-sets are all $\mathbb{Q}$-vector spaces ($\mathbb{C}$-vector spaces) and the composition of morphisms is $\mathbb{Q}$-linear ($\mathbb{C}$-linear, respectively). In positive characteristics, the derived category $\mathrm{D^b_c(X, \mathbb{Q}_\ell)}$ ($\mathrm{D^b_c(X, \bar{\mathbb{Q}}_\ell)}$, where $\bar{\mathbb{Q}}_\ell$ denotes a fixed algebraic closure of $\mathbb{Q}_\ell$) with ℓ prime to $char(k)$ is defined in Definition 2.1. In this case, the derived category $\mathrm{D^b_{G,c}(X, \mathbb{Q}_\ell)}$ is $\mathbb{Q}_\ell$-linear, meaning the Hom-sets are $\mathbb{Q}_\ell$-vector spaces and the composition of morphisms is $\mathbb{Q}_\ell$-linear. Similarly the derived category $\mathrm{D^b_{G,c}(X, \bar{\mathbb{Q}}_\ell)}$ is $\bar{\mathbb{Q}}_\ell$-linear. We will use the generic notation $\mathrm{D^b_{G,c}(X)}$ to denote any one of the four derived categories: $\mathrm{D^b_{G,c}(X, \mathbb{Q})}$, $\mathrm{D^b_{G,c}(X, \mathbb{C})}$, $\mathrm{D^b_{G,c}(X, \mathbb{Q}_\ell)}$ or $\mathrm{D^b_{G,c}(X, \bar{\mathbb{Q}}_\ell)}$, when the statements we consider hold for all of them. In fact, all the results of sections 2 and 3 hold for any one of the above derived categories.

It is essential to work with the derived category $\mathrm{D^b_{G,c}(X, \mathbb{C})}$ when $k = \mathbb{C}$ and with $\mathrm{D^b_{G,c}(X, \bar{\mathbb{Q}}_\ell)}$ in positive characteristics for the construction of quasi-hereditary algebras. The derived category $\mathrm{D^b_{G,c}(X, \bar{\mathbb{Q}}_\ell)}$ also plays an important role in section 6.

The equivariant derived category comes equipped with a canonical t-structure (defined using the middle perversity function) so that its heart identifies with the abelian category of equivariant perverse sheaves. The simple objects of this category are the equivariant intersection cohomology complexes defined on G-stable subvarieties and studied extensively by the author, in collaboration with Michel Brion, in several earlier papers: see for example, [23], [9], [10]. This construction is also recalled in section 2.4.

In the present paper, we will *restrict* to the case where a linear algebraic group G acts on a scheme X *with finitely many orbits.* □

Here is a sample of the main results in this paper. Let $\mathcal{S}$ denote the stratification by G-orbits on X and let $\mathcal{S}_{\rm G}$ denote the induced stratification of the Borel construction $\rm EG \times_G X$. As discussed in section 2.5, $\rm D^b_{G,c}(X, \mathcal{S}_G)$ denotes the full subcategory of $\rm D^b_{G,c}(X)$ consisting of complexes K whose cohomology sheaves are bounded, G-equivariant and locally constant (that is, lisse in the sense of [16, I.-Pureté], in the étale setting) on the strata: see sections 2.3, 2.5 for more details.

Theorem 1.1. *Let* G *denote a connected linear algebraic group acting on scheme* X *with finitely many orbits. Assume that the base field is an algebraically closed field of arbitrary characteristic or it is a finite field k. Then there is a canonical t-structure on the equivariant derived category* $\rm D^b_{G,c}(X, \mathcal{S}_G)$ *whose heart is the category* $\mathcal{P}_{\rm G}(\rm X, \mathcal{S}_G)$ *of equivariant perverse sheaves defined in* (3.1)*. Moreover, the following hold:*

(i) $\mathcal{P}_{\rm G}(\rm X, \mathcal{S}_G)$ *is an Artinian and Noetherian category.*
(ii) *The simple objects in* $\mathcal{P}_{\rm G}(\rm X, \mathcal{S}_G)$ *are the equivariant intersection cohomology complexes,* $\rm IC^G(\mathcal{L})$*, which denotes the equivariant intersection cohomology complex obtained by starting with the irreducible* G*-equivariant local system* $\mathcal{L}$ *on some stratum* S *as in* (2.4.1)*.*
(iii) *Assume next that the base field is algebraically closed. Then the category* $\mathcal{P}_{\rm G}(\rm X, \mathcal{S}_G)$ *has enough projectives and every object has a projective cover.*

The proof of this theorem is spread out and distributed over several Propositions proven in section 3 and follows roughly along the same lines as in the non-equivariant case (that is, when the group G is trivial).

Next we consider the category of pairs

$$\mathbf{S}_L = \{(\mathcal{O}, \mathcal{L}_{\mathcal{O}}) | \mathcal{O} \,\varepsilon\, \mathcal{S}, \mathcal{L}_{\mathcal{O}} \text{ a G} - \text{equivariant irreducible local system on } \mathcal{O}\}.$$

By replacing the above category by its skeleton category, we may assume that for each fixed orbit $\mathcal{O}$, the distinct objects $(\mathcal{O}, \mathcal{L}_{\mathcal{O}})$ belong to distinct isomorphism classes of G-equivariant irreducible local systems on the orbit $\mathcal{O}$ and that $\mathbf{S}_L$ is a finite set. We show in Proposition 3.10, that one may define a partial order on the set $\mathbf{S}_L$.

Theorem 1.2. *Let* G *denote a connected linear algebraic group acting on a scheme* X *with finitely many orbits and all defined over an algebraically closed field k of arbitrary* *characteristic. Let $\mathcal{S}$ denote the stratification of* X *by the* G*-orbits. Then the following hold:*

(a) *The category $\mathcal{P}_{\mathrm{G}}(\mathrm{X}, \mathcal{S}_{\mathrm{G}})$ contains a full subcategory $\mathcal{C}$, which is closed under extensions, and with the Grothendieck group isomorphic to that of $\mathcal{P}_{\mathrm{G}}(\mathrm{X}, \mathcal{S}_{\mathrm{G}})$.*
(b) *There exists an Artinian highest weight category $\bar{\mathcal{C}}$ with the weight poset the partially ordered finite set $\mathbf{S}_L$ so that the category $\mathcal{C}$ is equivalent to the full subcategory filtered by objects $j^p_{\mathcal{O},G!}(\mathcal{L}_{\mathcal{O}})$ (which is the standard* G*-equivariant perverse sheaf defined in Definition 3.4), as $\mathcal{O}$ varies among the* G*-orbits on* X*, and $\mathcal{L}_{\mathcal{O}}$ varies among the irreducible* G*-equivariant local systems on $\mathcal{O}$.*

Definition 1.3. Throughout the paper we will adopt the following convention regarding the stabilizers. Let G denote a linear algebraic group scheme acting on a scheme X. Then for each point $x \in X$, the *stabilizer at* x, denoted G_x, will denote the *reduced subscheme* of the schematic stabilizer at x.

Remark 1.4. A point that may be worth mentioning is that, since the G-stable strata on a given scheme are all G-orbits, at least for complex varieties, one obtains the isomorphism: $\pi_1(\mathrm{EG} \times_{\mathrm{G}} S, x) \cong \pi_1(\mathrm{BG}_x) \cong \mathrm{G}_x/\mathrm{G}_x^0$ (where G_x is the stabilizer at the point x on the orbit S). The corresponding ℓ-completed fundamental group classifies ℓ-adic G-equivariant local systems on S in positive characteristics, with $\ell \neq char(k)$, k being the base field. Since $\mathrm{G}_x/\mathrm{G}_x^0$ is finite, it follows that the category of G-equivariant ℓ-adic local systems on S is a semi-simple category and every object is projective. (A corresponding result holds for complex varieties with $\mathbb{Q}_\ell$ ($\bar{\mathbb{Q}}_\ell$) replaced by $\mathbb{Q}$ ($\mathbb{C}$, respectively).)

Examples 1.5. Any toric variety or any spherical variety defined over an algebraically closed field of arbitrary characteristic provides examples where the last two theorems apply. However, one can also restrict to various subcategories, for example, the class of simply connected complex spherical varieties considered in [10] to obtain classes of varieties where the last two theorems apply: these examples are discussed in detail in Proposition 3.14.

Construction of Quasi-Hereditary Algebras from Equivariant Perverse Sheaves. Under the assumptions of Theorem 1.2, we may now construct *the associated Quasi-Hereditary Algebra* as follows. First, we may recall that one of the basic hypothesis in [32, (5.9) Theorem], is to start with a triangulated category that is $\mathbb{F}$-linear over an algebraically closed field $\mathbb{F}$. Therefore, when the base field is the field of complex numbers (or of characteristic 0), we will start with the equivariant derived category $\mathrm{D}^{\mathrm{b}}_{\mathrm{G,c}}(\mathrm{X}, \mathbb{C})$. When the base field k is of positive characteristic p, we will start with $\mathrm{D}^{\mathrm{b}}_{\mathrm{G,c}}(\mathrm{X}, \bar{\mathbb{Q}}_\ell)$, with ℓ a prime different from p. We will denote the resulting equivariant derived categories by

$$\overline{\mathrm{D}}^{b}_{\mathrm{G},c}(\mathrm{X}). \tag{1.0.2}$$

Next we consider the category of pairs $\mathbf{S}_L$ again: we will view this as a finite partially ordered set. We will denote the standard objects $j^p_{\mathcal{O},G!}(\mathcal{L}_{\mathcal{O}})$ by $V(\mathcal{O})$: clearly these are indexed by the elements of the above set $\mathbf{S}_L$, with $\mathcal{O}$ a G-orbit and $\mathcal{L}_{\mathcal{O}}$ an irreducible G-equivariant local system on $\mathcal{O}$. Now each $V(\mathcal{O})$ has a projective cover by Theorem 1.1(iii), which we will denote by $\mathrm{P}(\mathcal{O})$.

Then we let $\mathrm{T} = \oplus_{\mathcal{O}}\mathrm{P}(\mathcal{O})$ and $\mathrm{A} = Hom_{\overline{\mathrm{D}}^b_{\mathrm{G},c}(\mathrm{X})}(\mathrm{T},\mathrm{T})$. Propositions 3.10 and 4.1 verify that the hypotheses of [32, (5.9) Theorem] are satisfied, so that this is a quasi-hereditary algebra. The highest weight category $\bar{\mathcal{C}}$ in the above theorem is given by the category of all finitely generated modules over A. We also point out that, despite the above constructions, it is far from clear that the category of equivariant perverse sheaves with respect to a fixed G-stable stratification is a highest weight category, that is, unless the group G is trivial. See Remark 4.2.

The next result we establish is the vanishing of odd dimensional intersection cohomology for all spherical varieties defined over algebraically closed fields of positive characteristics, extending such results in [23, (19) Theorem, (20) Corollary] for all Schubert varieties and in [9, Theorem 4] for all complex spherical varieties.

Theorem 1.6. *Assume the base field k is algebraically closed and of characteristic $p > 0$. Let* G *denote a connected reductive group,* X *a* G*-spherical variety defined and of finite type over* Spec k *and* $\mathcal{L}$ *a* G*-equivariant ℓ-adic local system on the open dense* G*-orbit, with $\ell \neq p$.*

Then $\mathcal{H}^i(\mathrm{IC}^{\mathrm{G}}(\mathrm{X};\mathcal{L})) = 0$ for all odd i.
In case X *is also projective,* $\mathrm{IH}^i(\mathrm{X};\mathcal{L}) = 0$ *for all odd i as well. (Here* $\mathrm{IC}^{\mathrm{G}}(\mathrm{X};\mathcal{L})$ *denotes the equivariant intersection cohomology complex with the middle perversity whose restriction to the open* G*-orbit is* $\mathcal{L}$ *and* $\mathrm{IH}^*(\mathrm{X};\mathcal{L})$ *denotes the corresponding intersection cohomology groups.)*

Remark 1.7. In the setting of [2, 4.0, p. 102], one begins with $\mathcal{L}[d_{\mathcal{O}}]$, where $d_{\mathcal{O}}$ is the dimension of the open orbit $\mathcal{O}$, (that is, instead of $\mathcal{L}$), and applies a perverse extension to define the equivariant intersection cohomology complex $\mathrm{IC}^{\mathrm{G}}(\mathrm{X};\mathcal{L}[\mathrm{d}_{\mathcal{O}}])$. Then the vanishing condition above becomes $\mathcal{H}^i(\mathrm{IC}^{\mathrm{G}}(\mathrm{X};\mathcal{L}[\mathrm{d}_{\mathcal{O}}])) = 0$ for $i + d_{\mathcal{O}}$ *odd.*

In the remainder of the paper we consider the category of mixed equivariant perverse sheaves defined on schemes of finite type over finite fields, and provided with an action by a linear algebraic group scheme G. For the remainder of the paper, the schemes that are defined and of finite type over a finite field will be usually indicated by a subscript o. It is well-known that unless the strata are all acyclic, one cannot expect to show that the category of mixed perverse sheaves on a scheme X_{o} with respect to a given stratification, whose associated terms in the weight filtrations are semi-simple, is Koszul. As observed above, when a linear algebraic group G_{o} acts on a scheme X_{o} (all defined over the finite field $\mathbb{F}_q$), the orbit stratification will seldom have all the strata acyclic. Nevertheless, the following theorem shows that one can recover certain nice properties for this category under the following assumptions.

Assume one of the following situations: (i) either G_o is a linear algebraic group acting on the given scheme X_o, the strata are the G_o-orbits and the geometric stabilizers at points on X_o are all connected or the extension of a geometrically connected subgroup by a finite abelian subgroup, *or* (ii) the group is trivial, the induced strata on X are all simply connected, but not necessarily affine.

Let $IC^{G_o}(\mathcal{L}_{S_o}[d_{S_o}])$ denote the G_o-equivariant intersection cohomology complex on the closure of the stratum S_o of dimension d_{S_o} obtained by starting with the G_o-equivariant local system $\mathcal{L}_{S_o}$ on S_o. Then we will assume the following condition holds for any stratum S_o and any G_o-equivariant ℓ-adic local system $\mathcal{L}_o$ on S_o (with $\ell \neq char(\mathbb{F}_q)$):

$$\mathcal{H}^i(IC^{G_o}(\mathcal{L}_o[d_{S_o}])) = 0 \text{ if } i + d_{S_o} \text{ is odd and } IC^{G_o}(\mathcal{L}_{S_o}[d_{S_o}]) \text{ is pure of weight } d_{S_o}. \tag{1.0.3}$$

The last purity condition above means one in the sense of [2, section 5]. The above conditions, and parts of them, have been verified for large classes of spherical varieties in [23, (19) theorem, (20) Corollary] and [9, Theorem 4]: for example, all of them hold for all toric varieties as well as Schubert varieties (associated to reductive groups) in any characteristic and the vanishing of cohomology sheaves as in (1.0.3) has been shown to hold for all spherical simply connected complex varieties. Moreover, Theorem 1.6 extends such vanishing results to spherical varieties over algebraically closed fields of positive characteristics.

We will let $\tilde{\mathcal{P}}^{G_o}_{mixed}$ denote the full subcategory of the above category of G_o-equivariant perverse sheaves consisting of those P_o so that for each $j \,\varepsilon\, \mathbb{Z}$, $grd^j_W(P_o)$, (that is, the associated graded term in the weight filtration with weight j) is a finite sum of equivariant intersection cohomology complexes $IC^{G_o}(\mathcal{L}_{S_o}[d_S])((d_{S_o} - j)/2)$ if $(d_{S_o} - j)$ is even and trivial otherwise.

Theorem 1.8. *In the situation above, let $\mathcal{S}_o$ denote the given strata on X_o. Let P_o denote an object in $\mathcal{P}_{G_o}(X_o, \mathcal{S}_{o,G_o})$, so that the associated object P in $\mathcal{P}_G(X, \mathcal{S}_G)$ is an indecomposable projective.*

(i) Then P_o can be lifted to $\tilde{\mathcal{P}}^{G_o}_{mixed}$, that is, there exists an object $\tilde{P}_o \,\varepsilon\, \tilde{\mathcal{P}}^{G_o}_{mixed}$ so that its underlying equivariant perverse sheaf is P_o.

(ii) Moreover, $\tilde{P}_o$ is a projective object in $\tilde{\mathcal{P}}^{G_o}_{mixed}$.

Here is a quick outline of the paper. In positive characteristics, we will always work on the étale site and consider cohomology with respect to ℓ-adic sheaves (with ℓ different from the characteristic), while we will work on the transcendental site for complex algebraic varieties with respect to sheaves of $\mathbb{Q}$ or $\mathbb{C}$ vector spaces.

The next section is devoted to a quick review of Equivariant Derived Categories associated to actions of linear algebraic groups on algebraic varieties. We conclude that section with a discussion of t-structures, the heart of which is the category of Equivariant

Perverse Sheaves. As pointed out already, there are at least two distinct competing models for equivariant derived categories: one using a simplicial construction, corresponds to the derived category of the associated quotient stack, whereas the geometric model is defined by using finite degree approximations to the classifying spaces: see for example, [5, Part I, section 2], [30, 4.2] and [39, section 1]. Each has its own advantages, so we have included an appendix that recalls basic comparison results between the two models of equivariant derived categories from [26, section 5].

The next four sections discuss the main results of the paper. Section 3 discusses basic properties of the category of Equivariant Perverse Sheaves concluding with a proof of Theorem 1.1. Section 4 discusses a proof of Theorem 1.2 and the construction of Quasi-Hereditary Algebras from the category of equivariant perverse sheaves on a scheme on which a connected linear algebraic group acts with finitely many orbits. In section 5, we prove Theorem 1.6, and thereby establish the vanishing of odd dimensional intersection cohomology sheaves on all spherical varieties defined over the algebraic closed fields of positive characteristic. Section 6 is devoted to weight filtrations on the Equivariant Derived Category associated to actions of linear algebraic groups over the algebraic closure of finite fields concluding with a proof of Theorem 1.8. An appendix discusses background material on equivariant derived categories.

Acknowledgments. The author wishes to thank Michel Brion for several discussions and correspondence related to this paper. In fact, this paper originated in a joint project with Michel Brion on equivariant derived categories associated to spherical varieties and which appears in [26]. Moreover, the discussion in section 5 owes a great deal to discussions with Brion, who explained the local structure of spherical varieties in positive characteristics to the author. He also thanks Leonard Scott for patiently and quickly answering many of his questions on the work of Cline, Parshall and Scott on highest weight categories and also for suggesting improvements to an earlier version of this paper. Finally, the author also thanks the referee for undertaking a careful reading of the paper and making several valuable suggestions that surely have improved the paper.

2. Basic framework: review of equivariant derived categories

Though for the most part, we will only need to consider schemes of finite type defined over algebraically closed fields, the need to consider weight filtrations makes it necessary to adopt a slightly more general framework to begin with. Therefore, we will always let k denote a field which will be either algebraically closed or a finite field $\mathbb{F}_q$. Over a field of characteristic 0, it is easy to obtain derived categories of complexes of sheaves which are $\mathbb{C}$-linear, that is, where the *Hom*-sets are $\mathbb{C}$-vector spaces and where the pairing induced by composition of morphisms is $\mathbb{C}$-bilinear: we simply take the derived categories of sheaves of $\mathbb{C}$-vector spaces. To do this in positive characteristics, is more involved: it needs the machinery of *adic*-sheaves as in [16, I.-Pureté], [17], [21] and [22] (see also [24, section 3] which closely follows [16, I.-Pureté]). See also [34] and [40] for related results. We proceed to discuss this next following the approach in [16, I.-Pureté].

2.1. The adic formalism

The following discussion is necessary only to consider the case where the base field k is of positive characteristic. Recall our standing assumption that it is perfect and of finite étale cohomological dimension for any prime $\ell \neq char(k) = p$, and that it satisfies the basic assumption in (1.0.1). Let ℓ denote a fixed prime different from $char(k) = p$, let E denote a finite extension of $\mathbb{Q}_\ell$, let R denote the integral closure of $\mathbb{Z}_\ell$ in E and m the maximal ideal in R. In view of our assumptions, for each scheme U of finite type over k, the cohomology groups

$$\mathrm{H}^{\mathrm{i}}_{\mathrm{et}}(\mathrm{U}, \mathrm{R}/\mathrm{m}^{\nu}) \tag{2.1.1}$$

are finite for all i and $\nu > 0$, and there exists a large enough integer N, dependent on U so that $\mathrm{H}^{\mathrm{i}}_{\mathrm{et}}(\mathrm{U}, \mathrm{R}/\mathrm{m}^{\nu}) = 0$ for all $i > N$.

2.1.2. Sheaves

Next let $\mathrm{X}_\bullet$ denote a simplicial scheme over k. Recall this means, we are given schemes $\mathrm{X_n}$, $n \geq 0$, of finite type over k, for each $n \geq 0$ and face maps $d_i : \mathrm{X_n} \rightarrow \mathrm{X_{n-1}}$, $i = 0, \cdots, n$ and degeneracies $s_i : \mathrm{X_{n-1}} \rightarrow \mathrm{X_n}$, $i = 0, \cdots, n-1$, satisfying certain relations as in [4, p. 230]. Given a scheme Y, $\mathrm{Y_{et}}$ will denote the small étale topology on Y: see [28, Chapter II, section 1]. Then the étale topology on a simplicial scheme $\mathrm{X}_\bullet$ consists of objects $\mathrm{U_n} \in \mathrm{X_{n,et}}$, for some n. Given U in $\mathrm{X_{n,et}}$ and V in $\mathrm{X_{m,et}}$, a morphism $u_\alpha : \mathrm{U} \rightarrow \mathrm{V}$ will denote a map lying over some structure map $\alpha : \mathrm{X_n} \rightarrow \mathrm{X_m}$ of the simplicial scheme. The coverings of such a U will be the étale coverings in $\mathrm{X_{n,et}}$. This Grothendieck topology will be denoted $\mathrm{Et}(\mathrm{X}_\bullet)$. A *sheaf* F on $\mathrm{Et}(\mathrm{X}_\bullet)$ is a contravariant functor from $\mathrm{Et}(\mathrm{X}_\bullet)$ to an abelian subcategory of the category of abelian groups satisfying the sheaf axiom. This data then means a sheaf F consists of a collection of sheaves $\{F_n|n\}$, with F_n a sheaf on $\mathrm{X_{n,et}}$, together with structure maps $\phi_\alpha : \alpha^*(F_m) \rightarrow F_n$, for each structure map $\alpha : \mathrm{X_n} \rightarrow \mathrm{X_m}$, satisfying certain compatibility conditions. Such a sheaf $F = \{F_n|n\}$ is *constructible* if each of the sheaves F_n is constructible. Moreover, we say such a sheaf is *cartesian* (or *has descent*) if the maps ϕ_α are isomorphisms for all structure maps α of the simplicial scheme $\mathrm{X}_\bullet$.

Let ℓ be a prime number different from the residue characteristics. For each $\nu > 0$, we let $C^b_c(\mathrm{Et}(\mathrm{X}_\bullet), \mathbb{Z}/\ell^\nu)$ denote the category of bounded complexes of sheaves of $\mathbb{Z}/\ell^\nu$-modules on $\mathrm{Et}(\mathrm{X}_\bullet)$ with *constructible cohomology sheaves.* $C^b_{ctf}(\mathrm{Et}(\mathrm{X}_\bullet), \mathbb{Z}/\ell^\nu)$ will denote the full sub-category $C^b_c(\mathrm{Et}(\mathrm{X}_\bullet), \mathbb{Z}/\ell^\nu)$ of complexes that are *of finite tor dimension.* If E is a finite extension of $\mathbb{Q}_\ell$ and R is the integral closure of $\mathbb{Z}_\ell$ in E, we obtain the categories $C^b_c(\mathrm{Et}(\mathrm{X}_\bullet), \mathrm{R}/\mathrm{m}^\nu)$ and $C^b_{ctf}(\mathrm{Et}(\mathrm{X}_\bullet), \mathrm{R}/\mathrm{m}^\nu)$ in a similar manner. A map $f : K \longrightarrow L$ of complexes in the above categories is a quasi-isomorphism if it induces an isomorphism of the cohomology sheaves. We obtain the derived categories $\mathrm{D^b_c}(\mathrm{Et}(\mathrm{X}_\bullet), \mathbb{Z}/\ell^\nu)$, $\mathrm{D^b_c}(\mathrm{Et}(\mathrm{X}_\bullet), \mathrm{R}/\mathrm{m}^\nu)$, $\mathrm{D^b_{ctf}}(\mathrm{Et}(\mathrm{X}_\bullet), \mathbb{Z}/\ell^\nu)$, $(\mathrm{D^b_{ctf}}(\mathrm{Et}(\mathrm{X}_\bullet), \mathrm{R}/\mathrm{m}^\nu)$ by inverting the quasi-isomorphisms.

Assume the above situation. Observe that the full abelian sub-category of sheaves with *descent* on $\mathrm{Et}(\mathrm{X}_\bullet)$ is closed under extensions in the category of all sheaves on $\mathrm{Et}(\mathrm{X}_\bullet)$. Therefore (see [20, p. 47]), we may let $\mathrm{D^{des}_c}(\mathrm{Et}(\mathrm{X}_\bullet), \mathrm{R}/\mathrm{m}^\nu)$ denote the full subcategory of $\mathrm{D_c}(\mathrm{Et}(\mathrm{X}_\bullet), \mathrm{R}/\mathrm{m}^\nu)$ consisting of complexes K so that each of the cohomology sheaves $H^i(K)$ is a sheaf with *descent*. The category $\mathrm{D^{b,des}_{ctf}}(\mathrm{Et}(\mathrm{X}_\bullet), \mathrm{R}/\mathrm{m}^\nu)$ will be defined to be the *full subcategory* of $\mathrm{D^b_{ctf}}(\mathrm{Et}(\mathrm{X}_\bullet), \mathrm{R}/\mathrm{m}^\nu)$ satisfying a similar condition.

One defines

$$\begin{aligned} &\mathrm{D^b_c}(\mathrm{Et}(\mathrm{X}_\bullet), \mathbb{Z}_\ell) = 2 - \lim_{\infty \leftarrow \nu} \mathrm{D^b_{ctf}}(\mathrm{Et}(\mathrm{X}_\bullet), \mathbb{Z}/\ell^\nu) \\ &(\mathrm{D^b_c}(\mathrm{Et}(\mathrm{X}_\bullet), \mathrm{R}) = 2 - \lim_{\infty \leftarrow \nu} \mathrm{D^b_{ctf}}(\mathrm{Et}(\mathrm{X}_\bullet), \mathrm{R}/\mathrm{m}^\nu)) \end{aligned} \tag{2.1.3}$$

(See [16, p. 148].) (Recall this means the objects of $\mathrm{D^b_c}(\mathrm{Et}(\mathrm{X}_\bullet), \mathrm{R})$ are inverse systems $\{{}_\nu K\}$, with ${}_\nu K \epsilon\ \mathrm{D^b_{ctf}}(\mathrm{Et}(\mathrm{X}_\bullet), \mathrm{R}/\mathrm{m}^\nu)$ so that $\mathrm{R}/\mathrm{m}^\nu \overset{\mathrm{L}}{\underset{\mathrm{R}/\mathrm{m}^{\nu+1}}{\otimes}} ({}_{\nu+1}\mathrm{K}) \simeq {}_\nu\mathrm{K}$. Given two such inverse systems $\{{}_\nu K\}$, $\{{}_\nu L\}$,

$$Hom(\{{}_\nu K\}, \{{}_\nu L\}) = \lim_{\overleftarrow{\nu}} Hom({}_\nu K, {}_\nu L)). \tag{2.1.4}$$

Definition 2.1. (i) $\mathrm{D^b_c}(\mathrm{Et}(\mathrm{X}_\bullet), \mathbb{Q}_\ell)$ $(\mathrm{D^b_c}(\mathrm{Et}(\mathrm{X}_\bullet), \mathrm{E}))$ is the quotient of $\mathrm{D^b_c}(\mathrm{Et}(\mathrm{X}_\bullet), \mathbb{Z}_\ell)$ $(\mathrm{D^b_c}(\mathrm{Et}(\mathrm{X}_\bullet), \mathrm{R})$, respectively) by the *thick* (that is, closed under extensions, as well as sub- and quotient objects) subcategory of torsion sheaves.

(ii) Finally we define the derived category

$$\mathrm{D^b_c}(\mathrm{Et}(\mathrm{X}_\bullet), \bar{\mathbb{Q}}_\ell) = 2 - \lim_{\overrightarrow{\mathrm{E}}} \mathrm{D^b_c}(\mathrm{Et}(\mathrm{X}_\bullet), \mathrm{E}), \tag{2.1.5}$$

where the colimit is over all finite extensions E of $\mathbb{Q}_\ell$. (Recall this means the objects of $\mathrm{D^b_c}(\mathrm{Et}(\mathrm{X}_\bullet), \bar{\mathbb{Q}}_\ell)$ are direct systems $\{K_\mathrm{E}|\mathrm{E}\}$, where $K_\mathrm{E}\ \epsilon\ \mathrm{D^b_c}(\mathrm{Et}(\mathrm{X}_\bullet), \mathrm{E}))$ and that given two such systems $K = \{K_\mathrm{E}|\mathrm{E}\}$ and $L = \{L_\mathrm{E}|\mathrm{E}\}$,

$$Hom(K, L) = \lim_{\overrightarrow{E}} Hom(K_\mathrm{E}, L_\mathrm{E}). \tag{2.1.6}$$

Assuming the above situation one defines $\mathrm{D^{b,des}_{ctf}}(\mathrm{Et}(\mathrm{X}_\bullet), \mathrm{R})$ $(\mathrm{D^{b,des}_c}(\mathrm{Et}(\mathrm{X}_\bullet), \mathrm{E})$, $\mathrm{D^{b,des}_c}(\mathrm{Et}(\mathrm{X}_\bullet), \bar{\mathbb{Q}}_\ell))$ to be the *full-subcategory* of $\mathrm{D^b_{ctf}}(\mathrm{Et}(\mathrm{X}_\bullet), \mathrm{R})$ $(\mathrm{D^b_c}(\mathrm{Et}(\mathrm{X}_\bullet), \mathrm{E})$, $\mathrm{D^b_c}(\mathrm{Et}(\mathrm{X}_\bullet), \bar{\mathbb{Q}}_\ell)$ respectively) consisting of complexes *whose cohomology sheaves have descent.*

Let $\nu > 0$ be a fixed integer and let R be the integral closure of $\mathbb{Z}_\ell$ in a finite extension E of $\mathbb{Q}_\ell$. We now observe the existence of spectral sequences

$$E_1^{p,q}(\nu) = Ext^q({}_\nu K_p,\, {}_\nu L_p) \Rightarrow Ext^{p+q}({}_\nu K,\, {}_\nu L), \tag{2.1.7}$$

where Ext^n is the n-th right derived functor of Hom in $C_c^b(\mathrm{X_{p,et}}, \mathrm{R/m^\nu})$.

Observe that (2.1.7) is a *right-half-plane* spectral sequence. Therefore, if ${}_\nu K$ and ${}_\nu L$ are *bounded* complexes with constructible cohomology sheaves (with ℓ-torsion, ℓ as always different from the residue characteristics), it follows that each $Ext^n({}_\nu K,\ {}_\nu L)$ is *finite* (with ℓ-torsion), since the base field satisfies the finiteness conditions as in (1.0.1). Therefore taking the inverse limit of the spectral sequences in (2.1.7) over $\nu > 0$ provides strongly-convergent spectral sequences

$$E_1^{p,q} = \lim_{\substack{\longleftarrow \\ \nu}} Ext^q({}_\nu K_p,\, {}_\nu L_p) \Rightarrow \lim_{\substack{\longleftarrow \\ \nu}} Ext^{p+q}({}_\nu K,\, {}_\nu L) \tag{2.1.8}$$

The finiteness of the Ext-groups in (2.1.7) shows (in view of [2, Proposition 2.2.15]) that the categories $\mathrm{D_c^b(Et(X_\bullet)}, \mathbb{Z}_\ell)$ and $\mathrm{D_c^b(Et(X_\bullet), R)}$ (and $\mathrm{D_c^{b,des}(Et(X_\bullet)}, \mathbb{Z}_\ell)$ and $\mathrm{D_c^{b,des}(Et(X_\bullet), R)}$ are *triangulated categories*, where the distinguished triangles are inverse systems of distinguished triangles in $\mathrm{D_{ctf}^b(Et(X_\bullet)}, \mathbb{Z}/\ell^\nu)$ and $\mathrm{D_{ctf}^b(Et(X_\bullet), R/m^\nu)})$ respectively. We then obtain the following result (whose proof is skipped, as it can be deduced readily from the above observations):

Proposition 2.2. *The derived category* $\mathrm{D_c^{b,des}(Et(X_\bullet)}, \bar{\mathbb{Q}}_\ell)$ *is* $\bar{\mathbb{Q}}_\ell$*-linear.*

2.2. Equivariant derived categories

We choose to work with the simplicial model discussed in detail in [15, sections 5 and 6], and also in [24, section 6] or [25]. One reason for choosing this model is that the simplicial model is clearly functorial in the group action. In addition, we are able to freely invoke the discussion of the étale fundamental group of simplicial schemes discussed in [18, Proposition 5.6] and [24, (A.3.0) through (A.3.3)]. A detailed comparison of equivariant derived categories defined this way, with the equivariant derived categories constructed using approximations of EG and BG by schemes (as in [39, section 1] or [30, 4.2]: see also [5, Part I, section 2]) appears in [26, section 5]: this is recalled in the appendix. We proceed to briefly recall the derived categories defined using the simplicial model for EG and BG.

Given a linear algebraic group G acting on a scheme X, $\mathrm{EG \times_G X}$ will now denote the simplicial scheme defined by letting $(\mathrm{EG}\underset{\mathrm{G}}{\times}\mathrm{X})_{\mathrm{n}} = \mathrm{G^{\times n} \times X}$ with the face maps induced by the group action $\mu : \mathrm{G \times X \to X}$, the group multiplication $\mathrm{G \times G \to G}$ and the obvious projection $\mathrm{G \times X \to X}$. The i-th degeneracy is induced by sticking in the identity element of the group G in the i-th place. Given a Grothendieck topology, Top, on schemes over k, one defines an induced Grothendieck topology $\mathrm{Top(EG \times_G X)}$ whose objects are $\mathrm{U_n \to (EG \times_G X)_n}$ in $\mathrm{Top((EG \times_G X)_n)}$ for some $n \geq 0$. The maps between two such objects, and coverings for this topology are defined as in the étale case: see 2.1.2. When one chooses the étale topology, this site will be denoted $\mathrm{Et(EG \times_G X)}$.

2.2.1. Sheaves F on the site $\mathrm{Top}(\mathrm{EG}\underset{\mathrm{G}}{\times}\mathrm{X})$ may be defined just as in the étale case: see 2.1.2. This means, $F = \{F_m | m \geq 0\}$ with F_m a sheaf on $\mathrm{Top}((\mathrm{EG}\underset{\mathrm{G}}{\times}\mathrm{X})_{\mathrm{m}})$, provided with structure maps $\alpha^*(F_m) \to F_n$ for each structure map $\alpha : (\mathrm{EG}\underset{\mathrm{G}}{\times}\mathrm{X})_{\mathrm{n}} \to (\mathrm{EG}\underset{\mathrm{G}}{\times}\mathrm{X})_{\mathrm{m}}$ satisfying certain obvious compatibility conditions. We say that a sheaf F is *equivariant* if it has *descent* (or is *cartesian*), that is, if the above maps $\alpha^*(F_m) \to F_n$ are all isomorphisms. $\mathrm{D}_c^b(\mathrm{EG} \times_{\mathrm{G}} \mathrm{X})$ will denote the derived category of complexes of sheaves of $\mathbb{Q}$ or $\mathbb{C}$ vector spaces on the simplicial space $\mathrm{EG} \times_{\mathrm{G}} \mathrm{X}$ with bounded constructible cohomology sheaves when everything is defined over the complex numbers, and will denote one of the derived categories: $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(\mathrm{Et}(\mathrm{EG} \times_{\mathrm{G}} \mathrm{X}), \mathbb{Q}_\ell)$ or $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(\mathrm{Et}(\mathrm{EG} \times_{\mathrm{G}} \mathrm{X}), \bar{\mathbb{Q}}_\ell)$, in general. In this framework, $\mathrm{D}^b_{\mathrm{G,c}}(\mathrm{X})$ will denote the full subcategory of $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(\mathrm{EG}\underset{\mathrm{G}}{\times}\mathrm{X})$ consisting of complexes of sheaves so that the cohomology sheaves are *equivariant*. Moreover, for each finite or infinite interval $\mathrm{I} = [\mathrm{a}, \mathrm{b}]$ of the integers, $\mathrm{D}^{\mathrm{I}}_{\mathrm{G},c}(\mathrm{X})$ will denote the full subcategory of $\mathrm{D}_{\mathrm{G,c}}(\mathrm{X})$ consisting of complexes K for which $\mathcal{H}^i(K) = 0$ for all $i \notin \mathrm{I}$. We will define a local system on $\mathrm{EG} \times_{\mathrm{G}} \mathrm{X}$ to be a sheaf $F = \{F_n | n\}$ so that (i) F_0 is locally constant and constructible (lisse in the ℓ-adic case: see section 2.3 below) on X and (ii) it is equivariant.

Terminology 2.3. We will adopt following terminology throughout the rest of the paper. If G is a linear algebraic group acting on a scheme X, $\mathrm{EG} \times_{\mathrm{G}} \mathrm{X}$ will always denote the simplicial scheme defined above. In particular BG will denote the corresponding simplicial scheme when $\mathrm{X} = \mathrm{Spec}\,\mathrm{k}$. In positive characteristics, we will only consider the derived categories $\mathrm{D}^{\mathrm{b}}_{\mathrm{G,c}}(\mathrm{X}, \mathbb{Q}_\ell)$ or $\mathrm{D}^{\mathrm{b}}_{\mathrm{G,c}}(\mathrm{X}, \bar{\mathbb{Q}}_\ell)$, whereas over the field of complex numbers, we will consider $\mathrm{D}^{\mathrm{b}}_{\mathrm{G,c}}(\mathrm{X}, \mathbb{Q})$ or $\mathrm{D}^{\mathrm{b}}_{\mathrm{G,c}}(\mathrm{X}, \mathbb{C})$.

2.3. Local systems and G*-equivariant local systems*

A local system will denote a locally constant constructible sheaf of $\mathbb{Q}$ or $\mathbb{C}$-vector spaces of finite dimension on schemes over $\mathbb{C}$, while it will denote a constructible *lisse* $\mathbb{Q}_\ell$-sheaf or $\bar{\mathbb{Q}}_\ell$-sheaf (in the sense of [16, I.-Pureté]) in positive characteristics. (Recall that an ℓ-a-dic sheaf is *lisse*, if each sheaf in the corresponding inverse system is locally constant and constructible.) In characteristic 0, the G-*equivariant local systems* on a G-scheme X correspond to representations of the fundamental group $\pi_1(\mathrm{EG} \times_{\mathrm{G}} \mathrm{X}, x)$ in $\mathbb{C}$ or $\mathbb{Q}$-vector spaces where x is a fixed point of X. In positive characteristics, the G-*equivariant ℓ-adic local systems* on a G-scheme X correspond to ℓ-adic representations of the étale fundamental group $\pi_1(\mathrm{EG} \times_{\mathrm{G}} \mathrm{X}, x)$. Similarly local systems on a variety X correspond to representations (ℓ-adic representations) of the fundamental group $\pi_1(\mathrm{X}, \quad x)$ which is the usual fundamental group in characteristic 0 (the étale fundamental group in positive characteristics, respectively). (See [33].)

When the scheme X itself is an orbit for the G-action on a larger scheme, $\mathrm{X} \cong \mathrm{G}/\mathrm{G}_x$, where x denotes a fixed point on X and G_x denotes the stabilizer at x. In this case, $\pi_1(\mathrm{EG} \times_{\mathrm{G}} \mathrm{X}, \quad x) \cong \pi_1(\mathrm{BG}_x)$. When G_x is connected, $\pi_1(\mathrm{BG}_x)$ will be a quotient of

$Gal_k(\bar{k})$ which is trivial if $k = \bar{k}$ is algebraically closed and is $\hat{\mathbb{Z}}$ if $k = \mathbb{F}_q$. In either case, when G_x is connected, the G-equivariant irreducible local systems on the orbit G/G_x are constant and 1-dimensional, induced from representations of $Gal_k(\bar{k})$. (Here one may make use of the techniques in the proof of [26, Theorem 1.6] to show that one obtains the usual long-exact sequence involving the completed étale fundamental groups of EG_x, BG_x and G_x which will provide the above conclusions. The completion will be at a prime $\ell \neq char(k)$.)

We will often make the following assumption on G-equivariant local systems $\mathcal{L}$ on a scheme X provided with a G-action.

$$\pi_1(EG\underset{G}{\times}X, x) \text{ acts on the stalk } \mathcal{L}_x \text{ through a finite quotient group } F. \tag{2.3.1}$$

Proposition 2.4. *(i) Under the above assumption* (2.3.1)*, every* G*-equivariant local system is semi-simple.*

(ii) The assumption (2.3.1) *holds if the base field is algebraically closed and* X *has a transitive action by* G*, so that* X *identifies with an orbit for the* G*-action.*

Proof. Under the assumption (2.3.1), the local system $\mathcal{L}$ defines a representation of F on the $\mathbb{Q}_\ell$-vector space (or $\bar{\mathbb{Q}}_\ell$-vector space) associated to $\mathcal{L}_x$. Since F is finite, this splits up into the sum of irreducible representations of F on $\mathbb{Q}_\ell$-vector spaces (or $\bar{\mathbb{Q}}_\ell$-vector spaces). One may show by standard arguments that each of the summands corresponds to an irreducible ℓ-adic representation of F and therefore to a G-equivariant irreducible local system on $EG\underset{G}{\times}X$. This proves (i) in positive characteristics and a similar argument proves it over the field of complex numbers.

Next we will consider (ii). Let x denote a fixed point of X and let G_x denote the corresponding stabilizer at x. Then $EG \times_G X \cong BG_x$. Since G_x^0 is connected, BG_x^0 is simply connected. One may then readily see that $\pi_1(EG\underset{G}{\times}X, x) \cong G_x/G_x^0$ in characteristic 0 ($\pi_1(EG\underset{G}{\times}X, x)\hat{}_\ell \cong (G_x/G_x^0)\hat{}_\ell$, in positive characteristic p, respectively, with $\hat{}_\ell$ denoting the completion at $\ell \neq p$.). Since G_x/G_x^0 is a finite group, we see that the assumption (2.3.1) holds. □

2.4. Equivariant intersection cohomology: see [23, (11)]

If $\mathcal{L}$ is a local system on an open G-orbit, $IH^*(X; \mathcal{L})$ will denote the corresponding intersection cohomology with the middle perversity. In case $\mathcal{L}$ is also G-equivariant, we will denote by $H^*_G(X; \mathcal{L})$ ($IH^*_G(X; \mathcal{L})$) the corresponding equivariant cohomology (the equivariant intersection cohomology with the middle perversity, respectively: see [23, (11)] or [9, (1.4.1)]). We quickly recall this construction for the convenience of the reader. Let $U_0 \overset{j_0}{\to} U_1 \overset{j_1}{\to} \cdots \overset{j_n}{\to} U_n = X$ denote a filtration of the given scheme by G-stable open subschemes. Then one may apply the Borel construction to each term of the above sequence of schemes to obtain the diagram:

$$\mathrm{EG}\times_{\mathrm{G}}\mathrm{U}_0 \xrightarrow{j_0^{\mathrm{G}}} \mathrm{EG}\times_{\mathrm{G}}\mathrm{U}_1 \xrightarrow{j_1^{\mathrm{G}}} \cdots \xrightarrow{j_n^{\mathrm{G}}} \mathrm{EG}\times_{\mathrm{G}}\mathrm{X} \tag{2.4.1}$$

all provided with a structure map to the classifying space BG. One starts with a G-equivariant local system $\mathcal{L}$ on $\mathrm{EG}\times_{\mathrm{G}}\mathrm{U}_0$ and applies a *perverse extension* to obtain the corresponding equivariant intersection cohomology complex $\mathrm{IC}^{\mathrm{G}}(\mathcal{L})$. The hypercohomology of $\mathrm{EG}\times_{\mathrm{G}}\mathrm{X}$ with respect to $\mathrm{IC}^{\mathrm{G}}(\mathcal{L})$ will be denoted $\mathrm{IH}^*_{\mathrm{G}}(\mathrm{X};\mathcal{L})$.

Both $\mathrm{H}^*_{\mathrm{G}}(\mathrm{X})$ and $\mathrm{IH}^*_{\mathrm{G}}(\mathrm{X};\mathcal{L})$ are modules over $\mathrm{H}^*(\mathrm{BG})$, the cohomology ring of BG. If $\mathcal{L}$ is a local system, $\mathcal{L}^\vee$ will denote its dual with respect to $\mathbb{Q}$ ($\mathbb{Q}_\ell$).

2.5. The role of stratifications

Next we will consider the role of stratifications. A G-stratification of the scheme X is a decomposition of X into finitely many locally closed smooth and G-stable subschemes called *strata* so that the closure of a stratum is the union of lower dimensional strata. Let the stratification be denoted $\mathcal{S}=\{S_\alpha|\alpha\}$. Since the Borel construction is functorial, such a stratification of X defines a stratification $\{\mathrm{EG}\underset{\mathrm{G}}{\times}S_\alpha|\alpha\}$ of the Borel construction $\mathrm{EG}\underset{G}{\times}X$. This stratification of $\mathrm{EG}\times_{\mathrm{G}}\mathrm{X}$ will be denoted $\mathcal{S}_{\mathrm{G}}$.

Given a G-stratification, and an interval I as above, we let $\mathrm{D}^{\mathrm{I}}(\mathrm{EG}\underset{\mathrm{G}}{\times}\mathrm{X},\mathcal{S}_{\mathrm{G}})$ denote the full subcategory of $\mathrm{D}^{\mathrm{I}}(\mathrm{EG}\underset{\mathrm{G}}{\times}\mathrm{X})$ consisting of

$$\{K\,\varepsilon\,D^{\mathrm{I}}(\mathrm{EG}\underset{\mathrm{G}}{\times}\mathrm{X},\mathcal{S}_{\mathrm{G}})|\mathcal{H}^{\mathrm{i}}(\mathrm{K}) \text{ are local systems on each of the strata } \mathcal{S}_\alpha \text{ and for all i}\}. \tag{2.5.1}$$

A perversity function p defined on a stratified scheme Y will be defined as a non-decreasing function on codimension of the strata, so that the value on the open stratum will be 0. We will view p as defined on the strata themselves. (We will only consider the middle perversity, which is defined by $m(S)$ = the codimension of S in Y.) Recall that the standard t-structure on a derived category of complexes with bounded cohomology is one whose heart consists of complexes that have non-trivial cohomology only in degree 0. Then one may start with standard t-structures defined on each of the strata S, shifted by the perversity $p(S)$, and obtain a *non-standard t-structure* on the bounded derived category, $\mathrm{D}^b(\mathrm{Y})$ by *gluing* as in [2, 1.4 Recollement].

Next we discuss how the t-structures on the equivariant derived category $\mathrm{D}^b_{\mathrm{G}}(\mathrm{X},\mathcal{S}_{\mathrm{G}})$ behave as one varies the stratifications.

Given a G-stratification $\mathcal{S}=\{S_\alpha|\alpha\}$ of X, let $i_{S_\alpha}=id\underset{\mathrm{G}}{\times}i:\mathrm{EG}\underset{\mathrm{G}}{\times}S_\alpha\to\mathrm{EG}\underset{\mathrm{G}}{\times}\mathrm{X}$ denote the induced closed immersion.

Proposition 2.5. *Let $\mathcal{S}$ denote a fixed* G*-stratification of the* G*-scheme* X *and let* $\mathcal{S}_{\mathrm{G}}$ *denote the induced stratification on* $\mathrm{EG}\underset{\mathrm{G}}{\times}\mathrm{X}$ *for the action of* G *on* X*. Let* $\mathcal{T}=\{\mathcal{T}_\beta\}$ *denote a* G*-stratification which is a refinement of the* G*-stratification* $\mathcal{S}$*. Then, any complex in*

$\mathrm{D}^b_{\mathrm{G}}(\mathrm{X})$ *whose cohomology sheaves are local systems on each stratum of* $\mathcal{S}_{\mathrm{G}}$ *clearly belongs to* $\mathrm{D}^b(\mathrm{EG}\underset{\mathrm{G}}{\times}\mathrm{X}, \mathcal{T}_{\mathrm{G}})$*. This induces the inclusion functor*

$$\mathrm{D}^{\mathrm{b}}_{\mathrm{G}}(\mathrm{X}, \mathcal{S}_{\mathrm{G}}) \to \mathrm{D}^{\mathrm{b}}_{\mathrm{G}}(\mathrm{X}, \mathcal{T}_{\mathrm{G}}),$$

and this functor preserves the t-structures on either side obtained by gluing.

Proof. This follows from [2, Proposition 2.1.14]. □

Proposition 2.6. *Assume in addition to the above hypotheses that* G *acts with finitely many orbits on the scheme* X*. Let* $\mathcal{S}$ *denote the stratification of* X *by the* G*-orbits. Then* $\mathrm{D}^b_{\mathrm{G}}(\mathrm{X}, \mathcal{S}_{\mathrm{G}}) = \mathrm{D}^{\mathrm{b}}_{\mathrm{G}}(\mathrm{X})$*.*

Proof. This is clear since any G equivariant sheaf is a local system on each orbit. □

3. Equivariant perverse sheaves: proof of Theorem 1.1

We will work implicitly with the middle perversity throughout the rest of the paper. We will assume that one is given a G-stable stratification $\mathcal{S}$ of X and let $\mathcal{S}_{\mathrm{G}}$ denote the induced stratification of $\mathrm{EG} \times_{\mathrm{G}} \mathrm{X}$. In this case, the t-structure obtained by gluing on $\mathrm{D}^{\mathrm{b}}_{\mathrm{G}}(\mathrm{X}, \mathcal{S}_{\mathrm{G}}) = \mathrm{D}^{\mathrm{b}}(\mathrm{EG}\underset{\mathrm{G}}{\times}\mathrm{X}, \mathcal{S}_{\mathrm{G}})$ is such that

$$\mathrm{D}^{\mathrm{b}, \leq 0}_{\mathrm{G}}(\mathrm{X}, \mathcal{S}_{\mathrm{G}}) = \{K \,\varepsilon\, \mathrm{D}^{\mathrm{b}}(\mathrm{X}, \mathcal{S}_{\mathrm{G}}) | \mathcal{H}^{\mathrm{i}}(\mathrm{i}^{(\mathrm{m})})^*_{\mathrm{S}}(\mathrm{K})) = 0, \mathrm{i} > \mathrm{codim}(\mathrm{S})\} \text{ and} \tag{3.0.1}$$

$$\mathrm{D}^{\mathrm{b}, \geq 0}_{\mathrm{G}}(\mathrm{X}, \mathcal{S}_{\mathrm{G}}) = \{K \,\varepsilon\, \mathrm{D}^{\mathrm{b}}(\mathrm{X}, \mathcal{S}_{\mathrm{G}}) | \mathcal{H}^{\mathrm{i}}(\mathrm{Ri}^{(\mathrm{m})})^!_{\mathrm{S}}(\mathrm{K})) = 0, \mathrm{i} < \mathrm{codim}(\mathrm{S})\}.$$

The former (the latter) is often called the *aisle* (the *co-aisle*, respectively) of the t-structure.

Definition 3.1. (Perverse sheaves on the Borel construction and Equivariant Perverse sheaves)

(i) In the case of a G-stratification $\mathcal{S}$, we let

$$\mathcal{P}_{\mathrm{G}}(\mathrm{X}, \mathcal{S}_{\mathrm{G}}) = \mathrm{D}^{\mathrm{b}, \leq 0}_{\mathrm{G}}(\mathrm{X}, \mathcal{S}_{\mathrm{G}}) \cap \mathrm{D}^{\mathrm{b}, \geq 0}_{\mathrm{G}}(\mathrm{X}, \mathcal{S}_{\mathrm{G}})$$

and call this the category of perverse sheaves on $\mathrm{EG}\underset{\mathrm{G}}{\times}\mathrm{X}$ with respect to the stratification $\mathcal{S}_{\mathrm{G}}$.

(ii) Assuming that we are given a compatible family of G-stable stratifications $\mathfrak{S} = \{\mathcal{S}_{\mathrm{G},i} | i \,\varepsilon\, I\}$, one may take the 2-colimit over i to define $\mathcal{P}_{\mathrm{G}}(\mathfrak{S})$. (Here I is assumed to be a small filtered category, and key use is made of Proposition 2.5.)

Remark 3.2. For much of our work in this paper, we need to fix a stratification and consider complexes whose cohomology sheaves are local systems on each stratum: hence the above definitions. In view of Proposition 2.5, one may take the 2-colimit over all stratifications to obtain a category of perverse sheaves that is intrinsically defined, and does not depend on the stratification used.

Proposition 3.3. *(See [2, Chapters 1 and 2], [24, section 4].) (i) For a fixed* G*-stable stratification* $\mathcal{S}_{\mathrm{G}}$ *of* X*, the category* $\mathcal{P}_{\mathrm{G}}(\mathrm{X}, \mathcal{S}_{\mathrm{G}})$ *is an Artinian and Noetherian abelian category.*

(ii) For a small filtered category I*, and a family of* G*-stable stratifications* $\mathfrak{S} = \{\mathcal{S}_{\mathrm{G},i}|i\}$*, the category* $\mathcal{P}_{\mathrm{G}}(\mathfrak{S})$ *is also an Artinian and Noetherian abelian category.*

(iii) For a fixed G*-stable stratification* $\mathcal{S}$ *of* X*,*

$$\mathcal{P}_{\mathrm{G}}(\mathrm{X}, \mathcal{S}_{\mathrm{G}}) = \{\mathrm{P}\,\varepsilon\,\mathrm{D}^{\mathrm{b}}_{\mathrm{G}}(\mathrm{X})|\pi^*(\mathrm{P}) \cong \mathrm{p}_2^*(\mathrm{L}), \mathrm{L}\,\varepsilon\,\mathcal{P}(\mathrm{X}, \mathcal{S})\}$$

where $\mathcal{P}(\mathrm{X}, \mathcal{S}_{\mathrm{G}})$ *denotes the category of perverse sheaves on* X *with respect to the* G*-stable stratification* $\mathcal{S}_{\mathrm{G}}$ *and* $\pi : \mathrm{EG} \times \mathrm{X} \to \mathrm{EG}\underset{\mathrm{G}}{\times}\mathrm{X}$ *(*$\mathrm{p}_2 : \mathrm{EG} \times \mathrm{X} \to \mathrm{X}$*) is the quotient for the group-action (projection to the second factor, respectively).*

(iv) The simple objects in $\mathcal{P}_{\mathrm{G}}(\mathrm{X}, \mathcal{S}_{\mathrm{G}})$ *are the equivariant intersection cohomology complexes,* $\mathrm{IC}^{\mathrm{G}}(\mathcal{L})$*, which denotes the equivariant intersection cohomology complex obtained by starting with the irreducible* G*-equivariant local system* $\mathcal{L}$ *on some stratum* S *in* $\mathcal{S}$*.*

Proof. The statements in (i), (ii) and (iv) may be deduced readily from the usual results on the category of perverse sheaves on a scheme provided with a stratification: see [2], making use of the following ideas. One invokes the comparison of equivariant derived categories in Theorem 7.2, which shows that one may make use of $\mathrm{EG}^{gm,m}$ and $\mathrm{BG}^{gm,m}$, with $m >> 0$, which are approximations to EG and BG to define the equivariant derived categories. Then $\mathrm{EG} \times_{\mathrm{G}} \mathrm{X}$ gets replaced by the scheme $\mathrm{EG}^{gm,m} \times_{\mathrm{G}} \mathrm{X}$, for $m >> 0$. Now the equivariant derived category $\mathrm{D}^{\mathrm{b}}_{\mathrm{G,c}}(\mathrm{X})$ corresponds to the derived category considered in (7.1.2). Then a G-equivariant perverse sheaf corresponds to a perverse sheaf P on the scheme $\mathrm{EG}^{gm,m}\underset{\mathrm{G}}{\times}\mathrm{X}$, so that its pull-back to $\mathrm{EG}^{gm,m}\times\mathrm{X}$ is isomorphic to the pull-back of a perverse sheaf Q on the scheme X by the projection $p_{2,m} : \mathrm{EG}^{gm,m}\times\mathrm{X} \to \mathrm{X}$. As a result, all the results in [2, Chapters 1 and 2] carry over to the equivariant derived category and equivariant perverse sheaves readily, thereby proving statements (i), (ii) and (iv).

The definition of the equivariant derived category $\mathrm{D}^b_{\mathrm{G}}(\mathrm{X}, \mathcal{S}_{\mathrm{G}})$ as in subsection 2.5 and (7.1.2) proves (iii). It is also possible to readily rework the above properties in the simplicial framework as is done in [24, sections 3 and 4]. □

Definition 3.4. (The standard and co-standard objects) (i) Let Y denote a stratified scheme and let $j_S : S \to \mathrm{Y}$ denote the locally closed immersion of a locally closed stratum into Y. Let $d_S = dim_k(S)$. If $\mathcal{L}_S$ is a local system on S, we let

$j^p_{S!}(\mathcal{L}_S[d_S]) = \tau^p_{\geq 0} j_{S!}(\mathcal{L}_S[d_S]) = {}^p\mathcal{H}^0(j_{S!}(\mathcal{L}_S[d_S])) \,\varepsilon\, \mathcal{P}(\mathrm{X}, \mathcal{S})$ be called the *standard perverse sheaf* associated to $\mathcal{L}_S[d_S]$. We also let $j^p_{S*}(\mathcal{L}_S[d_S]) = \tau^p_{\leq 0} Rj_{S*}(\mathcal{L}_S[d_S]) = {}^p\mathcal{H}^0(Rj_{S*}(\mathcal{L}_S[d_S]) \,\varepsilon\, \mathcal{P}(\mathrm{X}, \mathcal{S})$ be called the *co-standard perverse sheaf* associated to $\mathcal{L}_S[d_S]$. Here the truncation functor $\tau^p_{\leq 0}$ ($\tau^p_{\geq 0}$) denotes the perverse truncation functor.

(ii) Let X denote a scheme with the action of a linear algebraic group G, let $\mathcal{S}$ denote a G-stable stratification of X. If $j_S : S \to \mathrm{X}$ denotes the G-equivariant locally closed immersion of a locally closed stratum into X, we will let $j_{S,G} : \mathrm{EG}\underset{\mathrm{G}}{\times} S \to \mathrm{EG}\underset{\mathrm{G}}{\times}\mathrm{X}$ denote the induced map. If $\mathcal{L}$ is a G-equivariant local system on S, we let $\Delta(\mathcal{L}) = j^p_{S,G!}(\mathcal{L}_S[d_S]) = \tau^p_{\geq 0} j_{S,G!}(\mathcal{L}_S[d_S]) = {}^p\mathcal{H}^0(j_{S,G!}(\mathcal{L}_S[d_S])) \,\varepsilon\, \mathcal{P}_{\mathrm{G}}(\mathrm{X}, \mathcal{S}_{\mathrm{G}})$ be called the *standard equivariant perverse sheaf* associated to $\mathcal{L}_S[d_S]$. We also let $\nabla(\mathcal{L}) = j^p_{S,G*}(\mathcal{L}_S[d_S]) = \tau^p_{\leq 0} Rj_{S,G*}(\mathcal{L}_S[d_S]) = {}^p\mathcal{H}^0(Rj_{S*}(\mathcal{L}_S[d_S]) \,\varepsilon\, \mathcal{P}_{\mathrm{G}}(\mathrm{X}, \mathcal{S}_{\mathrm{G}})$ be called the *co-standard equivariant perverse sheaf* associated to $\mathcal{L}_S[d_S]$. Here the truncation functor $\tau^p_{\leq 0}$ ($\tau^p_{\geq 0}$) denotes the perverse truncation functor.

Lemma 3.5. *Suppose $f : \mathrm{X} \to \mathrm{Y}$ is a* G*-equivariant map between schemes with actions by the linear algebraic group* G*. If f is affine, so are the induced maps $f_{\mathrm{G},m} : (\mathrm{EG}\underset{\mathrm{G}}{\times}\mathrm{X})_m \to (\mathrm{EG}\underset{\mathrm{G}}{\times}\mathrm{Y})_m$ for all m.*

Proof. Recall $(\mathrm{EG}\underset{\mathrm{G}}{\times}\mathrm{X})_m = \mathrm{G}^{\mathrm{m}} \times \mathrm{X}$ and $(\mathrm{EG}\underset{\mathrm{G}}{\times}\mathrm{Y})_m = \mathrm{G}^m \times \mathrm{Y}$. Therefore the assertion in the Lemma is clear. □

Proposition 3.6. *Let* X *denote a scheme provided with an action by a linear algebraic group* G*. Let $\mathcal{S} = \{S\}$ denote a decomposition of* X *into locally closed smooth subschemes (called strata), where each stratum S is* G*-stable. Assume that $j_S : S \to X$ and $j_{\mathrm{T}} : T \to X$ are the immersions of two strata. Let $j_{S,\mathrm{G}} : \mathrm{EG}\underset{\mathrm{G}}{\times} S \to \mathrm{EG}\underset{\mathrm{G}}{\times}\mathrm{X}$ and $j_{T,\mathrm{G}} : \mathrm{EG}\underset{\mathrm{G}}{\times} T \to \mathrm{EG}\underset{\mathrm{G}}{\times}\mathrm{X}$ denote the corresponding induced immersions and let $Hom^n_{\mathrm{D^b_G}(\mathrm{X},\mathcal{S}_{\mathrm{G}})}$ denote the Hom in the corresponding equivariant derived category. Let $\mathcal{L}$ denote a* G*-equivariant local system on any of the strata. Then the following hold.*

(i) $\mathrm{Hom}^{\mathrm{n}}_{\mathrm{D^b_G}(\mathrm{X},\mathcal{S}_{\mathrm{G}})}(\mathrm{j^p_{S,G!}}(\mathcal{L}_{\mathrm{S}}[\mathrm{d_S}]), \mathrm{j^p_{T,G*}}(\mathcal{L}_{\mathrm{T}}[\mathrm{d_T}])) = 0, \mathrm{n} < 0$.

(i') More generally, $\mathrm{Hom}^{\mathrm{n}}_{\mathrm{D^b_G}(\mathrm{X},\mathcal{S}_{\mathrm{G}})}(\mathrm{P}', \mathrm{P}) = 0$*, for all $n < 0$ if* P', P *are two* G*-equivariant perverse sheaves, or if* $\mathrm{P}' \,\varepsilon\, \mathrm{D^{b,\leq 0}_G}(\mathrm{X}, \mathcal{S}_{\mathrm{G}})$ *and* $\mathrm{P} \,\varepsilon\, \mathrm{D^{b,\geq 0}_G}(\mathrm{X}, \mathcal{S}^{\mathrm{G}})$.

(ii) $\mathrm{Hom}_{\mathrm{D^b_G}(\mathrm{X},\mathcal{S}_{\mathrm{G}})}(\mathrm{j^p_{S,G!}}(\mathcal{L}_{\mathrm{S}}[\mathrm{d_S}]), \mathrm{j^p_{T,G*}}(\mathcal{L}_{\mathrm{T}}[\mathrm{d_T}])) = \mathrm{Hom}_{\mathrm{D^b_G}(\mathrm{X},\mathcal{S}_{\mathrm{G}})}(\mathrm{j_{S,G!}}(\mathcal{L}_{\mathrm{S}}[\mathrm{d_S}]), \mathrm{Rj_{T,G*}}(\mathcal{L}_{\mathrm{T}}[\mathrm{d_T}]))$.

(iii) The canonical map

$$\mathrm{Hom}^1_{\mathrm{D^b_G}(\mathrm{X},\mathcal{S}_{\mathrm{G}})}(\mathrm{j^p_{S,G!}}(\mathcal{L}_{\mathrm{S}}[\mathrm{d_S}]), \mathrm{j^p_{T,G*}}(\mathcal{L}_{\mathrm{T}}[\mathrm{d_T}])) \to$$
$$\mathrm{Hom}^1_{\mathrm{D^b_G}(\mathrm{X},\mathcal{S}_{\mathrm{G}})}(\mathrm{j_{S,G!}}(\mathcal{L}_{\mathrm{S}}[\mathrm{d_S}]), \mathrm{Rj_{T,G*}}(\mathcal{L}_{\mathrm{T}}[\mathrm{d_T}]))$$

is injective.

(iii') The canonical map

$$\mathrm{Hom}^{\mathrm{i}}_{\mathrm{D^b_G(X,\mathcal{S}_G)}}(\mathrm{j^p_{S,G!}}(\mathcal{L}_\mathrm{S}[\mathrm{d_S}]),\mathrm{P}) \to \mathrm{Hom}^{\mathrm{i}}_{\mathrm{D^b_G(X,\mathcal{S}_G)}}(\mathrm{j_{S,G!}}(\mathcal{L}_\mathrm{S}[\mathrm{d_S}]),\mathrm{P})$$

is bijective for $i = 0$ and injective for $i = 1$ for any G*-equivariant perverse sheaf* P*.*

(iv)

$$\begin{aligned}&\mathrm{Hom}^{\mathrm{n}}_{\mathrm{D^b_G(X,\mathcal{S}_G)}}(\mathrm{j_{S,G!}}(\mathcal{L}_\mathrm{S}[\mathrm{d_S}]),\mathrm{Rj_{T,G*}}(\mathcal{L}_\mathrm{T}[\mathrm{d_T}]))\\ &\quad\cong \mathrm{Hom}^{\mathrm{n}}_{\mathrm{D^b_G(X,\mathcal{S}_G)}}(\mathcal{L}_\mathrm{S}[\mathrm{d_S}],\mathrm{Rj^!_{S,G}}\mathrm{Rj_{T,G*}}(\mathcal{L}_\mathrm{T}[\mathrm{d_T}]))\\ &\quad\cong \mathrm{Hom}^{\mathrm{n}}_{\mathrm{D^b_G(X,\mathcal{S}_G)}}(\mathrm{j^*_{T,G}}(\mathrm{j_{S,G!}}\mathcal{L}_\mathrm{S}[\mathrm{d_S}]),\mathcal{L}_\mathrm{T}[\mathrm{d_T}])\end{aligned}$$

This is trivial unless $S = T$.

(v) $\mathrm{Hom}^{\mathrm{n}}_{\mathrm{D^b_G(X,\mathcal{S}_G)}}(\mathrm{j^p_{S,G!}}(\mathcal{L}_\mathrm{S}[\mathrm{d_S}]),\mathrm{j^p_{T,G*}}(\mathcal{L}_\mathrm{T}[\mathrm{d_T}])) = 0,$ *for* $\mathrm{S}\neq\mathrm{T},\mathrm{n}\leq 1.$

(vi) $\mathrm{Hom}^{\mathrm{n}}_{\mathrm{D^b_G(X,\mathcal{S}_G)}}(\mathrm{j^p_{S,G!}}(\mathcal{L}_\mathrm{S}[\mathrm{d_S}]),\mathrm{j^p_{S,G*}}(\mathcal{L}_\mathrm{S}[\mathrm{d_S}])) = \mathrm{Hom}^{\mathrm{n}}_{\mathrm{D^b_G(S)}}(\mathcal{L}_\mathrm{S}[\mathrm{d_S}],\mathcal{L}_\mathrm{S}[\mathrm{d_S}]),\mathrm{n}\leq 0.$

(vii) The obvious map

$$\mathrm{Hom}^{1}_{\mathrm{D^b_G(X,\mathcal{S}_G)}}(\mathrm{j^p_{S,G!}}(\mathcal{L}_\mathrm{S}[\mathrm{d_S}]),\mathrm{j^p_{S,G*}}(\mathcal{L}'_\mathrm{S}[\mathrm{d_S}])) \to \mathrm{Hom}^{1}_{\mathrm{D^b_G(S)}}(\mathcal{L}_\mathrm{S}[\mathrm{d_S}],\mathcal{L}'_\mathrm{S}[\mathrm{d_S}])$$

is also an injection.

(viii) Suppose $j_S : S \to X$ and $j_\mathrm{T} : T \to X$ are both affine maps. Then

$$\mathrm{Hom}^{\mathrm{n}}_{\mathrm{D^b_G(X,\mathcal{S}_G)}}(\mathrm{j^p_{S,G!}}(\mathcal{L}_\mathrm{S}[\mathrm{d_S}]),\mathrm{j^p_{T,G*}}(\mathcal{L}_\mathrm{T}[\mathrm{d_T}])) \cong 0 \text{ for all n if } \mathrm{S}\neq\mathrm{T}.$$

Proof. (i) is a basic property of perverse sheaves: see [2, Corollaire 2.1.21]. In fact, this holds for any perverse sheaves replacing the standard and co-standard objects as in (i). This observation also proves (i'). (A more detailed proof may be obtained by considering the spectral sequence:

$$E_1^{u,v} = \mathrm{Hom}^{v}_{\mathrm{D^b(EG\underset{G}{\times}X}_u,(\mathcal{S}_G)_u)}(\mathrm{P}'_u,\mathrm{P}_u) \Rightarrow \mathrm{Hom}^{u+v}_{\mathrm{D^b_G(X,\mathcal{S}_G)}}(\mathrm{P}',\mathrm{P}).$$

Here P'_u (P_u) denotes the restriction of P' (P) to $(\mathrm{EG}\underset{\mathrm{G}}{\times}\mathrm{X})_u$. The $E_1^{u,v}$-terms are trivial for all $v < 0$, by [2, Corollaire 2.1.21] since P'_u and P_u are perverse sheaves on $(\mathrm{EG}\underset{\mathrm{G}}{\times}\mathrm{X})_u$. Since $v \geq 0$, it follows that for $u + v < 0$, one needs $u < 0$. But clearly $u \geq 0$.)

(ii) and (iii) follow by diagram chase making use of (i) and the last statement in (i'). (One needs to consider the distinguished triangles

$$\begin{aligned}\tau^p_{\leq -1} j_{S,G!}(\mathcal{L}_S[d_S]) \to j_{S,G!}(\mathcal{L}_S[d_S]) \to j^p_{S,G!}(\mathcal{L}_S[d_S]) = \tau^p_{\geq 0} j_{S,G!}(\mathcal{L}_S[d_S])\\ \to \tau^p_{\leq -1} j_{S,G!}(\mathcal{L}_S[d_S])[1] \text{ as well as}\\ j^p_{S,G*}(\mathcal{L}_S[d_S]) = \tau^p_{\leq 0} Rj_{S,G*}(\mathcal{L}_S[d_S]) \to Rj_{S,G*}(\mathcal{L}_S[d_S]) \to \tau^p_{\geq 1}(Rj_{S,G*}(\mathcal{L}_S[d_S])\\ \to j^p_{S,G*}(\mathcal{L}_S[d_S])[1]\end{aligned}$$

and then make use of (i) as well as the last statement in (i').)

(iii') follows from (i') by a similar diagram-chase. The isomorphisms in (iv) follow readily by the adjunctions between the functors there. Here we have to consider two cases: in both cases $S \neq T$. In case $S \subseteq \bar{T}$ or if $S \cap \bar{T} = \phi$, the composite functor $Rj^!_{S,G}Rj_{T,G*}$ is identically zero, since $S \cap T = \phi$. In case $T \subseteq \bar{S}$ or $T \cap \bar{S} = \phi$, the composite functor $j^*_{T,\mathrm{G}}j_{S,G!}$ is identically zero since again $S \cap T = \phi$. This proves (iv). Now (v) follows readily by combining (ii), (iii) and (iv).

Observe that both the right-hand-side and left-hand-side of (vi) are trivial for $n < 0$. For $n = 0$, the isomorphism in (vi) follows from (ii) and the adjunction between the functors there. (vii) follows from (iii) and the adjunction between the functors there. (viii) follows from (iv) and Lemma 3.5, making use of [2, Theorem 4.1.1 and Corollaire 4.1.2]. □

We will add the following (rather well-known) result here, which will be used several times later on.

Lemma 3.7. *Let* $\mathrm{D} = \mathrm{D}^{\mathrm{b}}_{\mathrm{G}}(\mathrm{X}, \mathcal{S}_{\mathrm{G}})$ *and let* $C = \mathcal{P}_{\mathrm{G}}(\mathrm{X}, \mathcal{S}_{\mathrm{G}})$ *as before. Then given two objects* $K, L \in C$*, the natural map* $Ext^i_C(K, L) \to Ext^i_D(K, L)$ *is an isomorphism for* $i = 0, 1$ *and is an injection for* $i = 2$.

Proof. For $i = 0$, this is clear since C being the heart of the triangulated category D, is a full subcategory of the latter. For $i = 1$, this is discussed in [2, Remarque 3.1.17(ii)]. For $i = 2$, this may be then readily deduced from the case $i = 1$ as in [3, Lemma 3.2.4]. □

3.1. Existence of enough projectives

We begin with the following general result. Let $\mathbb{F}$ denote a field and let $\mathbf{A}$ denote an $\mathbb{F}$-linear abelian category, that is, $\mathbf{A}$ is an abelian category, where each Hom between two objects is an $\mathbb{F}$-vector space, and the composition $Hom_{\mathbf{A}}(a, b) \times Hom_{\mathbf{A}}(b, c) \to Hom_{\mathbf{A}}(a, c)$ is $\mathbb{F}$-linear.

Proposition 3.8. *Assume that the following additional hypotheses are satisfied.*

- (i) *Each object in* $\mathbf{A}$ *has finite length.*
- (ii) *There are only finitely many isomorphism classes of simple objects in* $\mathbf{A}$.
- (iii) *The endomorphisms of simple objects in* $\mathbf{A}$ *are reduced to scalars. Let* $\{L(s)|s\varepsilon S\}$ *represent the simple isomorphism classes in* $\mathbf{A}$. *Assume a partial order* $\leq$ *is given on* S *and* S *is equipped with the order topology, that is, closed subsets* $\mathrm{T} \subseteq \mathrm{S}$ *are characterized by* $s\varepsilon T, s' \leq s \Rightarrow s'\varepsilon T$. *For any closed* $\mathrm{T} \subseteq \mathrm{S}$*, let* $\mathbf{A}_T$ *denote the full subcategory of* $\mathbf{A}$ *of objects supported on* T*, that is, all of whose simple subquotients have parameter in* T.

  Assume we are given for all $s\varepsilon S$*, objects* $\Delta(s)$*,* $\nabla(s)$ *and morphisms* $\Delta(s) \to L(s)$*,* $L(s) \to \nabla(s)$ *in* $\mathbf{A}$ *such that*

(iv) *whenever* $\mathrm{T} \subseteq \mathrm{S}$ *is closed and* $s\varepsilon T$ *is maximal,* $\Delta(s) \to L(s)$ *is a projective cover and* $L(s) \to \nabla(s)$ *is an injective hull of* $L(s)$ *in* $\mathbf{A}_T$*. (In particular, both* $\Delta(s)$ *and* $\nabla(s)$ *are indecomposable.)*

(v) $ker(\Delta(s) \to L(s))$ *and* $coker(L(s) \to \nabla(s))$ *lie in* $\mathbf{A}_{<s}$*, for every* $s\varepsilon S$*.*

We call $\Delta(s)$ $(\nabla(s))$ *the standard (co-standard, respectively) objects.*

Then the abelian category $\mathbf{A}$ *has enough projectives,*

Proof. The proof that the above hypotheses imply the above conclusions is rather well-known: see, for example, [3, 3.2, also the remarks following the statement of Theorem 3.2.1]. □

In the remainder of this section we will assume that a not necessarily connected linear algebraic group G acts on the scheme X with finitely many orbits. Let $\mathcal{S}$ denote the corresponding G-stable stratification of X defined by the orbits. Next we consider the category of pairs

$$\mathbf{S}_L = \{(\mathcal{O}, \mathcal{L}_{\mathcal{O}}) | \mathcal{O}\, \varepsilon\, \mathcal{S}, \mathcal{L}_{\mathcal{O}} \text{ a G} - \text{equivariant irreducible local system on } \mathcal{O}\}. \quad (3.1.1)$$

By replacing the above category by its skeleton category, we may assume that for each fixed orbit $\mathcal{O}$, the distinct objects $(\mathcal{O}, \mathcal{L}_{\mathcal{O}})$ belong to distinct isomorphism classes of G-equivariant irreducible local systems on the orbit $\mathcal{O}$ and that $\mathbf{S}_L$ is a set. We proceed to define an order relation on $\mathbf{S}_L$.

Definition 3.9. We say $(\mathcal{O}', \mathcal{L}'_{\mathcal{O}'}) \leq (\mathcal{O}, \mathcal{L}_{\mathcal{O}})$ if

$$(i)\, \mathcal{O}' \subseteq \bar{\mathcal{O}} \text{ and} \quad (3.1.2)$$
$$(ii) \text{ there is a map } \mathcal{L}'_{\mathcal{O}'}[dim(\mathcal{O}')] \to Rj^{!}_{\mathcal{O}'} j^{p}_{\mathcal{O}!}(\mathcal{L}_{\mathcal{O}}[dim(\mathcal{O})]) \text{ in } \mathrm{D_G}(\mathrm{X}, \mathcal{S}_{\mathrm{G}})$$

that induces a monomorphism on cohomology sheaves in degree $-dim(\mathcal{O}')$. (Here $j' : \mathcal{O}' \to \mathrm{X}$ and $j : \mathcal{O} \to \mathrm{X}$ are the obvious immersions.)

Proposition 3.10. *The above relation is a partial-order on* $\mathbf{S}_L$*. The set* $\mathbf{S}_L$ *is finite.*

Proof. It is clear that the above relation is *reflexive*. Next we proceed to show it is *transitive*. Observe first that if $\mathcal{O}' \subseteq \bar{\mathcal{O}}$ and $\mathcal{O} \subseteq \bar{\mathcal{O}}''$, then $\mathcal{O}' \subseteq \bar{\mathcal{O}}''$.

Next observe that, by adjunction, the existence of the map in (3.1.2)(ii) is equivalent to the existence of a map $j_{\mathcal{O}'!}(\mathcal{L}'_{\mathcal{O}'}[dim(\mathcal{O}')]) \to j^{p}_{!}(\mathcal{L}_{\mathcal{O}}[dim(\mathcal{O})])$ in $\mathrm{D_G}(\mathrm{X}, \mathcal{S}_{\mathrm{G}})$. However, since $j^{p}_{\mathcal{O}!} = \tau^{p}_{\geq 0} j_{\mathcal{O}!}$, and $j^{p}_{\mathcal{O}'!} = \tau^{p}_{\geq 0} j_{\mathcal{O}'!}$, where $\tau^{p}_{\geq 0}$ denotes the perverse truncation, it follows from [2, Proposition 1.3.3] that the last map in fact induces a map $j^{p}_{\mathcal{O}'!}(\mathcal{L}'_{\mathcal{O}'}[dim(\mathcal{O}')]) \to j^{p}_{\mathcal{O}!}(\mathcal{L}_{\mathcal{O}}[dim(\mathcal{O})])$ in $\mathrm{D_G}(\mathrm{X}, \mathcal{S}_{\mathrm{G}})$. Therefore, giving the two maps in $\mathrm{D_G}(\mathrm{X}, \mathcal{S}_{\mathrm{G}})$:

$$\mathcal{L}'_{\mathcal{O}'}[dim(\mathcal{O}')] \to Rj^!_{\mathcal{O}'}j^p_{\mathcal{O}!}(\mathcal{L}_{\mathcal{O}}[dim(\mathcal{O})]) \text{ and } \mathcal{L}_{\mathcal{O}}[dim(\mathcal{O})] \to Rj^!_{\mathcal{O}}j^p_{\mathcal{O}''!}(\mathcal{L}''_{\mathcal{O}''}[dim(\mathcal{O}'')])$$

correspond to giving maps

$$j^p_{\mathcal{O}'!}(\mathcal{L}'_{\mathcal{O}'}[dim(\mathcal{O}')]) \to j^p_{\mathcal{O}!}(\mathcal{L}_{\mathcal{O}}[dim(\mathcal{O})]) \text{ and } j^p_{\mathcal{O}!}(\mathcal{L}_{\mathcal{O}}[dim(\mathcal{O})]) \to j^p_{\mathcal{O}''!}(\mathcal{L}''_{\mathcal{O}''}[dim(\mathcal{O}'')])$$

in $\mathrm{D_G}(\mathrm{X}, \mathcal{S}_\mathrm{G})$. Clearly the last two maps may composed to obtain the map

$$j^p_{\mathcal{O}'!}(\mathcal{L}'_{\mathcal{O}'}[dim(\mathcal{O}')]) \to j^p_{\mathcal{O}''!}(\mathcal{L}''_{\mathcal{O}''}[dim(\mathcal{O}'')]) \text{ in } \mathrm{D_G}(\mathrm{X}, \mathcal{S}_\mathrm{G}).$$

Since $j^p_{\mathcal{O}'!} = \tau^p_{\geq 0} j_{\mathcal{O}'!}$ and $j^p_{\mathcal{O}''!} = \tau^p_{\geq 0} j_{\mathcal{O}''!}$, this is equivalent to giving a map $j_{\mathcal{O}'!}(\mathcal{L}'_{\mathcal{O}'}[dim(\mathcal{O}')]) \to j^p_{\mathcal{O}''!}(\mathcal{L}''_{\mathcal{O}''}[dim(\mathcal{O}'')])$ and therefore to giving a map $\mathcal{L}'_{\mathcal{O}'}[dim(\mathcal{O}')] \to Rj^!_{\mathcal{O}'}j^p_{\mathcal{O}''!}(\mathcal{L}''_{\mathcal{O}''}[dim(\mathcal{O}'')])$.

Moreover, one may see that the last map be obtained by applying $Rj^!_{\mathcal{O}'}j^p_{\mathcal{O}!}$ to the map $\mathcal{L}_{\mathcal{O}}[\mathrm{dim}(\mathcal{O})] \to Rj^!_{\mathcal{O}}j^p_{\mathcal{O}''!}(\mathcal{L}''_{\mathcal{O}''}[dim(\mathcal{O}'')])$ and precomposing with the map $\mathcal{L}'_{\mathcal{O}'}[\mathrm{dim}(\mathcal{O}')] \to Rj^!_{\mathcal{O}'}j^p_{\mathcal{O}!}(\mathcal{L}_{\mathcal{O}}[\mathrm{dim}(\mathcal{O})])$, since the adjunction $j_{\mathcal{O}!}Rj^!_{\mathcal{O}}K \to K$ factors through $j^p_{\mathcal{O}!}Rj^!_{\mathcal{O}}K \to K$, if $K \in \mathrm{D}^{\mathrm{p}, \geq 0}_{\mathrm{G}}(\mathrm{X}, \mathcal{S}_\mathrm{G})$. These complete the proof of the transitivity.

Clearly if $\mathcal{O}' \subseteq \bar{\mathcal{O}}$ and $\mathcal{O} \subseteq \bar{\mathcal{O}}'$, then $\mathcal{O}' = \mathcal{O}$. Then the map in (3.1.2)(ii) corresponds to an isomorphism $\mathcal{L}'_{\mathcal{O}} \to \mathcal{L}_{\mathcal{O}}$ of irreducible G-equivariant local systems on the G-orbit $\mathcal{O}$. Since we have already replaced the category $\mathbf{S}_L$ by its skeleton, this is the identity, proving the anti-symmetry property of the relation $\leq$ on $\mathbf{S}_L$ in Definition 3.9.

These prove all but the last statement regarding finiteness of the set $\mathbf{S}_L$. Observe that there are only finitely many orbits by our assumption. Each orbit has only finitely many distinct irreducible G-equivariant local systems, since these correspond to irreducible representations of the finite group $\mathrm{G}_x/\mathrm{G}^o_x$, for a fixed point x in a G-orbit. These prove the last statement. □

Remark 3.11. If the stabilizers at all points on X are connected, then the G-equivariant local systems are trivial and therefore the corresponding irreducible G-equivariant local systems are the one dimensional trivial local systems. In this case, the above partial order reduces to the partial order on the G-orbits, given by $\mathcal{O}' \leq \mathcal{O}$ if $\mathcal{O}' \subseteq \bar{\mathcal{O}}$. We will see below that there are indeed many examples where this is what occurs: see Proposition 3.14.

Theorem 3.12. *Let* G *denote a not necessarily connected linear algebraic group acting on the scheme* X *with finitely many orbits, all defined over an algebraically closed base field k. Let $\mathcal{S}$ denote the corresponding G-stable stratification of* X *defined by the orbits.*

(i) Then the category of G*-equivariant perverse sheaves $\mathcal{P}_\mathrm{G}(\mathrm{X}, \mathcal{S}_\mathrm{G})$ has enough projectives. Moreover, if S denotes a* G*-orbit on* X*, $\bar{S}$ denotes its closure in* X *and $\mathcal{L}_S$ denotes an irreducible* G*-equivariant local system on S, then the equivariant perverse*

sheaf $\Delta(\mathcal{L}_S)$ *is a projective cover of* $\mathrm{IC}^{\mathrm{G}}(\mathcal{L}_S[dim(S)])$ *in* $\mathcal{P}_{\mathrm{G}}(\bar{S}, \mathcal{S}_{\mathrm{G}})$*, which denotes the full subcategory of* $\mathcal{P}_{\mathrm{G}}(\mathrm{X}, \mathcal{S}_{\mathrm{G}})$ *with supports in* $\bar{S}$*.*

(ii) Assume, in addition, that the strata defined by the G*-orbits satisfy the following hypotheses:*

(a) the inclusion of each stratum $S \to X$ *is affine, and*

(b) $\mathrm{H}^2(\mathrm{EG}\underset{\mathrm{G}}{\times}(\mathrm{G}/\mathrm{G}_x), \mathbb{Q}_\ell) \cong \mathrm{H}^2(\mathrm{BG}_x, \mathbb{Q}_\ell) = 0$ *in case* $char(k) = p > 0$ *and* $p \neq \ell$ *(*$\mathrm{H}^2(\mathrm{EG}\underset{\mathrm{G}}{\times}(\mathrm{G}/\mathrm{G}_x), \mathbb{Q}) \cong \mathrm{H}^2(\mathrm{BG}_x, \mathbb{Q}) = 0$*, in case* $char(k) = 0$*).*

Then every G*-equivariant perverse sheaf has a bounded projective resolution. In particular, if* G *is the trivial group and* $\mathrm{H}^2(\mathrm{S}, \mathbb{Q}_\ell) = 0$ *in positive characteristic* p *(*$\mathrm{H}^2(\mathrm{S}, \mathbb{Q}) = 0$*, in case* $char(k) = 0$*) for all strata* S*, then every perverse sheaf on* X *has a bounded projective resolution by projective objects in the category of perverse sheaves.*

Proof. We will discuss the proof only in positive characteristics, as the proof in characteristic 0 follows along the same lines. We begin by observing that there are enough projective objects in the category $\mathcal{P}_{\mathrm{G}}(S, \mathcal{S}_{\mathrm{G}})$ of G-equivariant perverse sheaves on a fixed stratum S. Let x denote a fixed (closed) on the stratum S, so that we may assume $\pi_1(\mathrm{EG} \times_{\mathrm{G}} S, x)\widehat{\ }_\ell \cong \pi_1(\mathrm{BG}_x)\widehat{\ }_\ell \cong (\mathrm{G}_x/\mathrm{G}_x^0)\widehat{\ }_\ell$. Therefore, the ℓ-adic G-equivariant local systems on $\mathrm{EG} \times_{\mathrm{G}} S$ correspond to finite dimensional left modules over the group-algebra $\mathbb{Q}_\ell{-}((\mathrm{G}_x/\mathrm{G}_x^0)\widehat{\ }_\ell)$. Since $(\mathrm{G}_x/\mathrm{G}_x^0)$ is finite, so is $(\mathrm{G}_x/\mathrm{G}_x^0)\widehat{\ }_\ell$, and therefore $\mathbb{Q}_\ell{-}((\mathrm{G}_x/\mathrm{G}_x^0)\widehat{\ }_\ell)$ is a finite dimensional $\mathbb{Q}_\ell$-algebra. Moreover, this category is semi-simple and therefore every module is projective. Therefore, the category of finite dimensional left modules over the group-algebra $\mathbb{Q}_\ell - ((\mathrm{G}_x/\mathrm{G}_x^0)\widehat{\ }_\ell)$ has enough projectives. A corresponding result holds with $\mathbb{Q}$ in the place of $\mathbb{Q}_\ell$ when we consider complex varieties.

It suffices to verify the five conditions in Proposition 3.8 to prove (i). The verification of the conditions (i) through (iii) and (v) are clear. One may argue as follows to see the condition (iv) there is true. Let S denote a G-orbit, $\bar{S}$ denote its closure and let $j_S : S \to \bar{S}$ denote the corresponding open immersion.

Taking $\Delta(\mathcal{L}_S)$ ($\nabla(\mathcal{L}_S)$) to denote the standard (co-standard) equivariant perverse sheaves associated to a stratum S and an irreducible representation L_S of $\pi_1(\mathrm{EG}\underset{\mathrm{G}}{\times}S, s)$, with $\mathcal{L}_S$ denoting the corresponding G-equivariant local system on $\mathrm{EG}\underset{\mathrm{G}}{\times}S$, one needs to show that the canonical map $\alpha : \Delta(\mathcal{L}_S) \to \mathrm{IC}^{\mathrm{G}}(\mathcal{L}_S[dim(S)])$ is a projective cover. The required arguments are exactly as in the non-equivariant case: however, we will provide sufficient details, mainly for completeness.

Recall $\Delta(\mathcal{L}_S) = \tau^p_{\geq 0} j_{S,G!}(\mathcal{L}_S[dim(S)])$. Now it suffices to prove the following:

$$\mathrm{Hom}^1_{\mathrm{D^b_G}(\mathrm{X}, \mathcal{S}_{\mathrm{G}})}(\Delta(\mathcal{L}_{\mathrm{S}}), \mathrm{IC}^{\mathrm{G}}(\mathcal{L}'_{\mathrm{T}})) = 0$$

$$\text{for all G-equivariant local systems } \mathcal{L}'_{\mathrm{T}} \text{ on a G-orbit } \mathrm{T} \subseteq \bar{\mathrm{S}} \text{ and} \tag{3.1.3}$$

$$\mathrm{Hom}^0_{\mathrm{D^b_G}(\mathrm{X}, \mathcal{S}_{\mathrm{G}})}(\Delta(\mathcal{L}_{\mathrm{S}}), \mathrm{IC}^{\mathrm{G}}(\mathcal{L}'_{\mathrm{T}})) = \mathbb{Q}_\ell \text{ if } T = S \text{ and } \mathcal{L}_S = \mathcal{L}'_T \tag{3.1.4}$$

$$= 0 \text{ for } T \subseteq \bar{S}, T \neq S.$$

Then (3.1.3) will prove that $\Delta(\mathcal{L}_S)$ is a projective object in the abelian category $\mathcal{P}_{\mathrm{G}}(\bar{S}, \mathcal{S}_{\mathrm{G}})$ while (3.1.4) will show it is a cover of $\mathrm{IC}^{\mathrm{G}}(\mathcal{L}_S)$.

In view of Proposition 3.6(iii'), it suffices to prove the corresponding isomorphisms after replacing $\Delta(\mathcal{L}_S)$ by $j_{S,G!}(\mathcal{L}_S[dim(S)])$. Then the left-hand-side of (3.1.3) identifies with

$$\mathrm{Hom}^1_{\mathrm{D}^b_{\mathrm{G}}(\mathrm{S},\mathcal{S}_{\mathrm{G}})}(\mathcal{L}_{\mathrm{S}}[\mathrm{d}_{\mathrm{S}}], \mathrm{j}^*_{\mathrm{S}}\mathrm{IC}^{\mathrm{G}}(\mathcal{L}'_{\mathrm{T}})). \tag{3.1.5}$$

At this point we need to consider two cases. The first case is when $\mathrm{T} \subseteq \bar{\mathrm{S}}$ and $\mathrm{T} \neq \mathrm{S}$. In this case $j^*_S\mathrm{IC}^{\mathrm{G}}(\mathcal{L}'_T) = 0$, so that the group in (3.1.5) is trivial. The next case is when $\mathrm{T} = \mathrm{S}$. In this case the group in (3.1.5) identifies with $\mathrm{H}^1(\mathrm{BG}_x, \mathcal{L}^\vee_{\mathrm{S}} \otimes \mathcal{L}'_{\mathrm{S}}) = 0$, where x denotes a point of S: see the proof of Proposition 4.1 for additional details. (One may also simply observe that the functor j^*_S is exact and that $\mathcal{L}_S[d_S]$ is a projective object in the category of G-equivariant perverse sheaves on the stratum S: see [14, Lemma 4.4].) In a similar way, one sees that the left-hand-side of (3.1.4) identifies with

$$\mathrm{Hom}^0_{\mathrm{D}^b_{\mathrm{G}}(\mathrm{S},\mathcal{S}_{\mathrm{G}})}(\mathcal{L}_{\mathrm{S}}[\mathrm{d}_{\mathrm{S}}], \mathrm{j}^*_{\mathrm{S}}\mathrm{IC}^{\mathrm{G}}(\mathcal{L}'_{\mathrm{T}})). \tag{3.1.6}$$

If $\mathrm{T} \subseteq \bar{\mathrm{S}}$ and $\mathrm{T} \neq \mathrm{S}$, then $j^*_S\mathrm{IC}^{\mathrm{G}}(\mathcal{L}_T) = 0$, so that the group in (3.1.6) is trivial. If $T = S$, then it follows that

$$\begin{aligned}\mathrm{Hom}^0_{\mathrm{D}^b_{\mathrm{G}}(\mathrm{X},\mathcal{S}_{\mathrm{G}})}(\mathrm{j}^{\mathrm{p}}_{\mathrm{S,G!}}(\mathcal{L}_{\mathrm{S}}[\mathrm{d}_{\mathrm{S}}]), \mathrm{j}^{\mathrm{p}}_{\mathrm{T,G!}}(\mathcal{L}'_{\mathrm{T}}[\mathrm{d}_{\mathrm{T}}])) &\cong \mathrm{Hom}^0_{\mathrm{D}^b_{\mathrm{G}}(\mathrm{S})}(\mathcal{L}_{\mathrm{S}}[\mathrm{d}_{\mathrm{S}}], \mathrm{j}^*_{\mathrm{S,G}}\mathrm{j}_{\mathrm{T,G!}}(\mathcal{L}'_{\mathrm{T}}[\mathrm{d}_{\mathrm{T}}]))\\ &\cong \mathrm{H}^0(\mathrm{EG}\times_{\mathrm{G}}\mathrm{S}, \mathcal{H}\mathrm{om}(\mathcal{L}, \mathcal{L}')),\\ &\cong \mathrm{H}^0(\mathrm{BG}^0_x, \mathcal{H}\mathrm{om}(\mathcal{L}, \mathcal{L}')^{\mathrm{G}_x/\mathrm{G}^0_x})\end{aligned} \tag{3.1.7}$$

since each stratum S is a G-orbit and $\mathrm{EG}\times_{\mathrm{G}} S \cong \mathrm{BG}_x$, where G_x denotes the stabilizer at a point $x \in S$. Moreover, $\mathcal{H}om$ denotes the internal Hom in the category of G_x-equivariant local systems. Clearly $\mathcal{H}om(\mathcal{L}_S, \mathcal{L}'_S)^{\mathrm{G}_x/\mathrm{G}^0_x} \cong \mathbb{Q}_\ell$ if $\mathcal{L}_S \cong \mathcal{L}'_S$ and 0 otherwise as $\mathcal{L}_S$ and $\mathcal{L}'_S$ are both irreducible G-equivariant local systems on $\mathcal{S}$. Therefore the group in (3.1.6) is 0 if $\mathcal{L}'_S \neq \mathcal{L}_S$ and it is $\mathbb{Q}_\ell$ if $\mathcal{L}'_S = \mathcal{L}_S$, as $\mathcal{L}_S$ and $\mathcal{L}'_S$ are both irreducible G-equivariant local systems on $\mathcal{S}$. These complete the proof of the statement that $\Delta(\mathcal{L}_S)$ is a projective cover of $\mathrm{IC}^{\mathrm{G}}(\mathcal{L}_S)$ in $\mathcal{P}_{\mathrm{G}}(\bar{S}, \mathcal{S}_{\mathrm{G}})$. By taking Verdier duals, one may prove the corresponding statement that $\nabla(\mathcal{L}_S)$ is an injective hull of $\mathrm{IC}^{\mathrm{G}}(\mathcal{L}_S)$ in $\mathcal{P}_{\mathrm{G}}(\bar{S}, \mathcal{S}_{\mathrm{G}})$. Therefore, these complete the proof of the first statement in the Theorem.

Recall that there is an additional condition (namely (6)) in [3, (3.2)] that says $\mathrm{Ext}^2_{\mathbf{A}}(\Delta(s), \nabla(t)) = 0$, for all $s, t \in S$. This condition translates to $\mathrm{Hom}^2_{\mathcal{P}_{\mathrm{G}}(\mathrm{X},\mathcal{S}_{\mathrm{G}})}(\mathrm{j}^{\mathrm{p}}_{\mathrm{S,G!}}(\mathcal{L}_{\mathrm{S}}[\mathrm{dim(S)}]), \mathrm{j}^{\mathrm{p}}_{\mathrm{T,G*}}(\mathcal{L}'_{\mathrm{T}}[\mathrm{dim(T)}])) = 0$ for all irreducible G-equivariant local systems $\mathcal{L}_S$ on S and $\mathcal{L}'_T$ on T. That the conditions in statement (ii) of the Theorem imply the above vanishing condition may be seen by the following argument. By Lemma 3.7, the canonical map

$$\mathrm{Hom}^2_{\mathcal{P}_{\mathrm{G}}(\mathrm{X},\mathcal{S}_{\mathrm{G}})}(\mathrm{j}^{\mathrm{p}}_{\mathrm{S},\mathrm{G}!}(\mathcal{L}_{\mathrm{S}}[\dim(\mathrm{S})]), \mathrm{j}^{\mathrm{p}}_{\mathrm{T},\mathrm{G}*}(\mathcal{L}'_{\mathrm{T}}[\dim(\mathrm{T})])) \rightarrow$$
$$\mathrm{Hom}^2_{\mathrm{D}^{\mathrm{b}}_{\mathrm{G}}(\mathrm{X},\mathcal{S}_{\mathrm{G}})}(\mathrm{j}^{\mathrm{p}}_{\mathrm{S},\mathrm{G}!}(\mathcal{L}_{\mathrm{S}}[\dim(\mathrm{S})]), \mathrm{j}^{\mathrm{p}}_{\mathrm{T},\mathrm{G}*}(\mathcal{L}'_{\mathrm{T}}[\dim(\mathrm{T})]))$$

is an injection. Therefore, it suffices to prove the groups

$$\mathrm{Hom}^2_{\mathrm{D}^{\mathrm{b}}_{\mathrm{G}}(\mathrm{X},\mathcal{S}_{\mathrm{G}})}(\mathrm{j}^{\mathrm{p}}_{\mathrm{S},\mathrm{G}!}(\mathcal{L}_{\mathrm{S}}[\dim(\mathrm{S})]), \mathrm{j}^{\mathrm{p}}_{\mathrm{T},\mathrm{G}*}(\mathcal{L}'_{\mathrm{T}}[\dim(\mathrm{T})])) \tag{3.1.8}$$

are trivial. Making use of the assumption that the inclusion maps $j : S \rightarrow X$ are affine, Proposition 3.6(iv), shows the above groups are trivial for $S \neq T$.

Therefore, for $S = T$, again making use of the assumption that the inclusion maps $j : S \rightarrow X$ are affine, the vanishing of the groups in (3.1.8) reduces to the vanishing of

$$\begin{aligned}\mathrm{H}^2(\mathrm{EG}\underset{\mathrm{G}}{\times}(\mathrm{G}/\mathrm{G}_x), \mathcal{H}\mathrm{om}(\mathcal{L}_{\mathrm{S}}, \mathcal{L}'_{\mathrm{S}})) &\cong \mathrm{H}^2(\mathrm{BG}_x, \mathcal{H}\mathrm{om}(\mathcal{L}_{\mathrm{S}}, \mathcal{L}'_{\mathrm{S}}))\\ &\cong \mathrm{H}^2(\mathrm{BG}^0_x, \mathcal{H}\mathrm{om}(\mathcal{L}_{\mathrm{S}}, \mathcal{L}'_{\mathrm{S}})^{\mathrm{G}_x/\mathrm{G}^0_x}),\end{aligned}$$

where $\mathcal{H}om$ denotes the internal *hom* in the category of G-equivariant local systems on S. If $\mathcal{L}_S \neq \mathcal{L}'_S$, then $\mathcal{H}om(\mathcal{L}_S, \mathcal{L}'_S)^{\mathrm{G}_x/\mathrm{G}^0_x} \cong 0$ and otherwise it is isomorphic to $\mathbb{Q}_\ell$. This completes the proof of (ii). □

Remark 3.13. Observe that the vanishing condition in Proposition 3.12(ii) is not satisfied in general: this would need $\mathrm{H}^2(\mathrm{BG}_x, \mathbb{Q}_\ell)$ in positive characteristic p (and $\mathrm{H}^2(\mathrm{BG}_x, \mathbb{Q})$, in characteristic 0) to be trivial, which is usually not the case. Therefore, one cannot conclude in general, based on the above arguments, that one has a bounded projective resolution in the category of G-equivariant perverse sheaves, for any G-equivariant perverse sheaf.

Proposition 3.14. (***Examples.***) *(i) Assume the base field k is perfect and of arbitrary characteristic. If* X *is a toric variety for a split torus* T *both defined over k, then there are only finitely many* T*-orbits on* X*. Similarly, if* X *is a spherical variety associated to the connected reductive groups* G*, both defined over an algebraically closed field k of arbitrary characteristic, there are only finitely many* G*-orbits on* X*. (In addition, if* B *is a Borel subgroup of* G*, it also has only finitely many orbits on* X*.)*

(ii) The stabilizer groups are products of a connected subgroup and a finite abelian subgroup for all toric varieties defined over any algebraically closed field k.

(iii) Assume the base field is $\mathbb{C}$. Then the stabilizer groups are all connected for all reductive varieties, and more generally for what are called spherical simply connected (scs) varieties (in the sense of [10], sections 1 and 5), associated to connected reductive groups G*, so that the* G*-equivariant local systems on the orbits are all constant.*

(iv) Assume the base field is $\mathbb{C}$. Then the stabilizer groups are all extensions of connected groups by finite abelian groups for all spherical varieties. Therefore, the G*-equivariant irreducible local systems on the orbits are* 1*-dimensional.*

(v) The immersions $j : \mathcal{O} \to X$ corresponding to the various G*-orbits are affine for any toric variety defined with respect to a split torus over any field. The same holds for all toroidal imbeddings* X *of spherical imbeddings of homogeneous spaces with respect to complex reductive groups.*[2]

(vi) Let G *denote a connected reductive group and* B *a Borel subgroup in* G*. Let* X *denote a Schubert variety on the flag variety* G/B *provided with the left action by* B *on* X *(induced by the Bruhat decomposition of* G*). Then the* B*-orbits on* X *form a* B*-stable stratification of* X*, with each stratum being an affine space.*

Proof. The statements in (i) are well-known: see [31, 1.2] for toric varieties and [7, Chapter 2] or [38, p. 26] for spherical varieties. Let X denote a toric variety for the split torus $T = \mathbb{G}_m^n$. Then clearly the stabilizers are all of the form $\mathbb{G}_m^k \times F$, where F is a finite subgroup of T, and $k \leq n$. This proves (ii). For any imbeddings of reductive groups G, or more generally for reductive varieties, as well as for *spherical simply connected (scs) varieties* the stabilizer groups are all connected, so that G-equivariant local systems on the orbits are in fact constant. (See [10, section 5].) This proves (iii).

In (iv), G_x^0 is then a spherical subgroup of G_x, and hence its normalizer $N_G(G_x^0)/G_x^0$ is diagonalizable (in particular, abelian). So the subgroup G_x/G_x^0 is abelian. Therefore, the corresponding G-equivariant local systems are 1-dimensional. This proves (iv).

For toroidal imbeddings, the local structure (see [7, 2.4], [38, Theorem 29.1] or [6, 6.2.2]) shows the immersions $j : \mathcal{O} \to X$, corresponding to the G-orbits $\mathcal{O}$ are all affine. These observations then prove the statements in (v). The last statement on Schubert varieties is well-known. □

Proof of Theorem 1.1. Proposition 3.3 establishes all the statements except for the existence of projectives, projective covers, and their properties. Theorem 3.12 establishes the existence of projectives and projective covers for the simple objects. To see that every object has a projective cover, one needs to make use of the fact that the abelian category $\mathcal{P}_G(X, \mathcal{S}_G)$ is Artinian, so that every object has finite length, and therefore is what is called a *length category*: then one invokes Proposition 3.8 and an argument as in [19, 8.2] to show such a length category is equivalent to the category of all finite length left-modules over an Artinian ring. (One may also readily adapt the arguments in [14, Theorem 4.6] which establishes a corresponding result for perverse sheaves on a stratified topological space, so that the fundamental group of each stratum is finite. Note that the above cited Theorem itself is based on the arguments in [3, Theorem 3.2.1].) This will show the existence of projective covers in general. □

[2] The last statement in fact extends to toroidal imbeddings over algebraically closed fields of positive characteristics: this is in fact proven in 5.1.4.

4. Highest weight categories and quasi-hereditary algebras from equivariant perverse sheaves: proof of Theorem 1.2

Let k denote an algebraically closed field, ℓ a prime $\neq char(k)$, G a not necessarily connected linear algebraic group and X a scheme of finite type so that G acts on X with finitely many G-orbits. Let $\mathcal{S}$ denote the stratification of X by the G-orbits.

Proposition 4.1. *Assume the above situation.*

Then the following hold, where $\mathcal{L}_S$ ($\mathcal{L}_T$) is an irreducible G*-equivariant local system on the orbit S (T, respectively) and assuming the characteristic of the base field is $p > 0$:*

$$\begin{aligned}\mathrm{Hom}^{\mathrm{n}}_{\mathrm{D^b_G(X,\mathcal{S}_G)}}(\mathrm{j^p_{S,G!}}(\mathcal{L}_\mathrm{S}[\mathrm{d_S}]),\mathrm{j^p_{T,G!}}(\mathcal{L}'_\mathrm{T}[\mathrm{d_T}])) &= 0,\ \textit{if}\ n \leq -1, \\ &= 0,\ \textit{if}\ n = 0,\ \textit{unless}\ S \subseteq \bar{T}, \\ &= 0,\ \textit{if}\ n = 1,\ \textit{unless}\ S \subseteq \bar{T}\ \textit{and}\ S \neq T, \\ &= \mathbb{Q}_\ell,\ \textit{if}\ n = 0, S = T,\ \textit{and}\ \mathcal{L}_S = \mathcal{L}'_T \\ &= V,\ \textit{if}\ n = 0\ \textit{or}\ n = 1\end{aligned} \tag{4.0.1}$$

where V is a finite dimensional vector space over $\mathbb{Q}_\ell$.

In case the base field is the complex numbers, the corresponding result holds with $\mathbb{Q}_\ell$ replaced by $\mathbb{Q}$ everywhere.

The corresponding results with $\mathbb{Q}$ ($\mathbb{Q}_\ell$) replaced by $\mathbb{C}$ ($\bar{\mathbb{Q}}_\ell$, respectively) hold when the above equivariant derived categories are replaced by the ones in (1.0.2).

Proof. The first equality follows from Proposition 3.6(i’). To obtain the second equality, first one invokes Proposition 3.6(iii’) with P there replaced by $j^p_{T,G!}(\mathcal{L}_\mathrm{T}[d_\mathrm{T}])$. Then we obtain (just as in (3.1.4)), for $i = 0$ and $i = 1$:

$$\begin{aligned}\mathrm{Hom}^{\mathrm{i}}_{\mathrm{D^b_G(X,\mathcal{S}_G)}}(\mathrm{j^p_{S,G!}}(\mathcal{L}_\mathrm{S}[\mathrm{d_S}]),\mathrm{j^p_{T,G!}}(\mathcal{L}'_\mathrm{T}[\mathrm{d_T}])) &\cong \mathrm{Hom}^{\mathrm{i}}_{\mathrm{D^b_G(X,\mathcal{S}_G)}}(\mathrm{j_{S,G!}}(\mathcal{L}_\mathrm{S}[\mathrm{d_S}]),\mathrm{j_{T,G!}}(\mathcal{L}'_\mathrm{T}[\mathrm{d_T}])), \quad (4.0.2)\\ &\cong \mathrm{Hom}^{\mathrm{i}}_{\mathrm{D^b_G(S)}}(\mathcal{L}_\mathrm{S}[\mathrm{d_S}],\mathrm{Rj^!_{S,G}j_{T,G!}}(\mathcal{L}'_\mathrm{T}[\mathrm{d_T}])), \\ &\cong 0 \text{ unless } S \subset \bar{T}\end{aligned}$$

since $Rj^!_{S,\mathrm{G}}j_{T,G!}(\mathcal{L}'_\mathrm{T}[d_\mathrm{T}]) = 0$, unless $S \subseteq \bar{T}$. Taking $i = 0$, this proves the second equality on the right.

The remaining statements are proven exactly as in the proof of Theorem 3.12. In case $S = T$, for $i = 0, 1$, it follows that

$$\mathrm{Hom}^{\mathrm{i}}_{\mathrm{D^b_G(X,\mathcal{S}_G)}}(\mathrm{j^p_{S,G!}}(\mathcal{L}_\mathrm{S}[\mathrm{d_S}]),\mathrm{j^p_{T,G!}}(\mathcal{L}'_\mathrm{T}[\mathrm{d_T}])) \cong \mathrm{Hom}^{\mathrm{i}}_{\mathrm{D^b_G(S)}}(\mathcal{L}_\mathrm{S}[\mathrm{d_S}],\mathrm{j^*_{S,G}j_{T,G!}}(\mathcal{L}'_\mathrm{T}[\mathrm{d_T}])) \tag{4.0.3}$$

$$\cong \mathrm{H^i}(\mathrm{EG} \times_{\mathrm{G}} \mathrm{S}, \mathcal{H}\mathrm{om}(\mathcal{L}_{\mathrm{S}}, \mathcal{L}'_{\mathrm{S}})),$$
$$\cong \mathrm{H^i}(\mathrm{BG}^0_x, \mathcal{H}\mathrm{om}(\mathcal{L}_{\mathrm{S}}, \mathcal{L}'_{\mathrm{S}})^{\mathrm{G}_x/\mathrm{G}^0_x}),$$

since each stratum S is a G-orbit and $\mathrm{EG} \times_{\mathrm{G}} S \cong \mathrm{BG}_x$, where G_x denotes the stabilizer at a point $x \,\varepsilon\, S$. Moreover, $\mathcal{H}om$ denotes the internal Hom in the category of G_x-equivariant local systems. Clearly $\mathcal{H}om(\mathcal{L}_S, \mathcal{L}'_S)^{\mathrm{G}_x/\mathrm{G}^0_x} \cong \mathbb{Q}_\ell$ if $\mathcal{L}_S \cong \mathcal{L}'_S$ and 0 otherwise. Taking $i = 0$, this proves the fourth equality.

The fifth equality follows by identifying the right hand side of (4.0.2) for $i = 1$ with $\mathrm{H}^1(\mathrm{EG} \times_{\mathrm{G}} \mathrm{S}, \mathcal{R}\mathrm{Hom}(\mathcal{L}_{\mathcal{S}}, \mathrm{Rj}^!_{\mathrm{S,G}}(\mathrm{j}_{\mathrm{T,G!}}(\mathcal{L}'_{\mathrm{S}}[\mathrm{d_S}][-\mathrm{d_T}]))$, which is clearly a finite dimensional vector space over $\mathbb{Q}_\ell$ ($\mathbb{Q}$) in positive characteristic (over $k = \mathbb{C}$, respectively).

Next we consider the third equality. Observe that if $S = T$, then $(S, \mathcal{L}_S) \le (T, \mathcal{L}'_T)$ only if $\mathcal{L}_S = \mathcal{L}'_T$. Then the right hand side (4.0.2) with $i = 1$ identifies with

$$\mathrm{H}^1(\mathrm{BG}_x, \mathcal{H}\mathrm{om}(\mathcal{L}_{\mathrm{S}}, \mathcal{L}'_{\mathrm{T}})) \cong \mathrm{H}^1(\mathrm{BG}^0_x, \mathcal{H}\mathrm{om}(\mathcal{L}_{\mathrm{S}}, \mathcal{L}_{\mathrm{S}})^{\mathrm{G}_x/\mathrm{G}^0_x})$$
$$\cong \mathrm{H}^1(\mathrm{BG}^0_x, \mathbb{Q}_\ell) = \mathrm{H}^1(\mathrm{BT}_x, \mathbb{Q}_\ell)^{\mathrm{W}_x} = 0,$$

where T_x denotes a maximal torus in G^0_x with W_x denoting the corresponding Weyl group. Moreover, as observed above, $Rj^!_{S,\mathrm{G}} j_{T,G!}(\mathcal{L}_{\mathrm{T}}[d_{\mathrm{T}}]) = 0$, unless $S \subseteq \bar{T}$. Therefore, this proves the third equality as well, thereby completing the proof of all but the last statement in the Proposition. We skip the proofs of the last two statements as they are clear. □

Proof of Theorem 1.2. Let $\mathrm{D}^b_{\mathrm{G,c}}(\mathrm{X})$ denote the G-equivariant derived category of X, provided with the t-structure, whose heart is the category of equivariant perverse sheaves. Let $\bar{\mathrm{D}}^b_{\mathrm{G},c}(\mathrm{X})$ denote the corresponding derived category considered in (1.0.2).

Recall that we defined a partial order on the (skeletal) category $\mathbf{S}_L$ of pairs $\{(\mathcal{O}, \mathcal{L}_{\mathcal{O}}) | \mathcal{O} \,\varepsilon\, \mathcal{S}, \mathcal{L}_{\mathcal{O}}$ a G − equivariant irreducible local system on $\mathcal{O}\}$. (In case the stabilizers on each G-orbit are connected, then the irreducible G-equivariant local systems reduce to 1-dimensional trivial local systems, and the above partial order reduces to the partial order on the orbits, where $\mathcal{O}' \le \mathcal{O}$, if $\mathcal{O}' \subseteq \bar{\mathcal{O}}$.)

Let $\mathcal{P}_{\mathrm{G}}(\mathrm{X}, \mathcal{S}_{\mathrm{G}})$ denote the category of equivariant perverse sheaves on X with respect to the stratification $\mathcal{S}$. Then, Propositions 3.10 and 4.1 show that the hypotheses of [32, (5.9) Theorem] hold, so that [32, (5.9) Theorem] readily provides a proof of both statements in Theorem 1.2. The following statements describe the associated quasi-hereditary algebra and the highest weight category. □

4.1. The associated quasi-hereditary algebra and the highest weight category

We will denote the standard objects $j^p_{\mathcal{O},G!}(\mathcal{L}_{\mathcal{O}})$ by $V(\mathcal{L}_{\mathcal{O}})$. Now each $V(\mathcal{L}_{\mathcal{O}})$ has a projective cover by Theorem 1.1(iii), which we will denote by $\mathrm{P}(\mathcal{L}_{\mathcal{O}})$. Then we let $\mathrm{T} = \oplus_{\mathcal{L}_{\mathcal{O}}} \mathrm{P}(\mathcal{L}_{\mathcal{O}})$ and $\mathrm{A} = Hom_{\bar{\mathrm{D}}^b_{\mathrm{G,c}}(\mathrm{X})}(\mathrm{T}, \mathrm{T})$. Propositions 3.10 and 4.1 verify that the hypotheses of [32, (5.9) Theorem] are satisfied, so that by [32, (5.9) Theorem], this is a quasi-hereditary algebra. The highest weight category $\bar{\mathcal{C}}$ in the above theorem is given by the category of all finitely generated modules over A. (One may also want to consult the discussion on this in the introduction.)

Remark 4.2. It is worth pointing out that, unless the group G is trivial (or finite), it is far from clear that the category of equivariant perverse sheaves is a highest weight category. One may see this as follows. Observe from Proposition 3.10 that the set $\mathbf{S}_L$ is a partially ordered finite set. The co-standard objects are the equivariant perverse sheaves defined as $j^p_{\mathcal{O},G!}(\mathcal{L}_{\mathcal{O}}[d_{\mathcal{O}}])$ for an irreducible local-system $\mathcal{L}_{\mathcal{O}}$ on the orbit $\mathcal{O}$ and the standard objects are their duals, defined as $j^p_{\mathcal{O},G*}(\mathcal{L}_{\mathcal{O}}[d_{\mathcal{O}}])$. These objects are thus indexed by the partially ordered set $\mathbf{S}_L$, and belong to the category of equivariant perverse sheaves $\mathcal{P}_{\rm G}({\rm X}, \mathcal{S}_{\rm G})$. We may further assume that the stabilizer groups on each orbit is connected, so that the G-equivariant fundamental group on each orbit is trivial, and the G-equivariant local systems on each orbit are the trivial 1-dimensional local systems. Therefore, the weight poset identifies with the set of orbits partially ordered by inclusion. Then, [32, (5.17)(b)] applies to show that $\mathcal{P}_{\rm G}({\rm X}, \mathcal{S}_{\rm G})$ is a highest weight category if and only if the group

$$\mathrm{Hom}^2_{\mathcal{P}_{\rm G}({\rm X},\mathcal{S}_{\rm G})}(\mathrm{j}^{\rm p}_{\mathcal{O},{\rm G}!}(\mathcal{L}_{\mathcal{O}}[\mathrm{d}_{\mathcal{O}}]), \mathrm{j}^{\rm p}_{\mathcal{O}',{\rm G}!}(\mathcal{L}_{\mathcal{O}'}[\mathrm{d}_{\mathcal{O}'}])) = 0 \tag{4.1.1}$$

for all pairs of orbits $\mathcal{O}$, $\mathcal{O}'$ and irreducible G-equivariant local systems $\mathcal{L}_{\mathcal{O}}$ and $\mathcal{L}_{\mathcal{O}'}$. By Lemma 3.7, the canonical map

$$\begin{aligned} &\mathrm{Hom}^2_{\mathcal{P}_{\rm G}({\rm X},\mathcal{S}_{\rm G})}(\mathrm{j}^{\rm p}_{\mathcal{O},{\rm G}!}(\mathcal{L}_{\mathcal{O}}[\dim(\mathcal{O})]), \mathrm{j}^{\rm p}_{\mathcal{O}',{\rm G}*}(\mathcal{L}'_{\mathcal{O}'}[\dim(\mathcal{O}')])) \\ \to\ &\mathrm{Hom}^2_{\mathrm{D}^{\rm b}_{\rm G}({\rm X},\mathcal{S}_{\rm G})}(\mathrm{j}^{\rm p}_{\mathcal{O},{\rm G}!}(\mathcal{L}_{\mathcal{O}}[\dim(\mathcal{O})]), \mathrm{j}^{\rm p}_{\mathcal{O}',{\rm G}*}(\mathcal{L}'_{\mathcal{O}'}[\dim(\mathcal{O}')])) \end{aligned}$$

is an injection. Therefore, it suffices to prove the groups

$$\mathrm{Hom}^2_{\mathrm{D}^{\rm b}_{\rm G}({\rm X},\mathcal{S}_{\rm G})}(\mathrm{j}^{\rm p}_{\mathcal{O},{\rm G}!}(\mathcal{L}_{\mathcal{O}}[\dim(\mathcal{O})]), \mathrm{j}^{\rm p}_{\mathcal{O}',{\rm G}*}(\mathcal{L}'_{\mathcal{O}'}[\dim(\mathcal{O}')])) \tag{4.1.2}$$

are trivial.

We may assume (for the sake of simplicity), that the immersions $j_{\mathcal{O}} : \mathcal{O} \to X$ associated to each orbit $\mathcal{O}$ is affine. Then the above groups are trivial for $\mathcal{O} \neq \mathcal{O}'$ by Proposition 3.6(iv). Then $j_{\mathcal{O},G!}(\mathcal{L}_{\mathcal{O}})$ and $Rj_{\mathcal{O},G*}(\mathcal{L}_{\mathcal{O}})$ are G-equivariant perverse sheaves, so that a necessary and sufficient condition that the category $\mathcal{P}_{\rm G}({\rm X}, \mathcal{S}_{\rm G})$ is a highest weight category is that $\mathrm{Hom}^2_{\mathcal{P}_{\rm G}({\rm X},\mathcal{S}_{\rm G})}(\mathrm{j}_{\mathcal{O},{\rm G}!}(\mathcal{L}_{\mathcal{O}}), \mathrm{Rj}_{\mathcal{O},{\rm G}*}(\mathcal{L}_{\mathcal{O}})) = 0$ for all the orbits $\mathcal{O}$. Now, the last group injects into $\mathrm{Hom}^2_{\mathrm{D}_{\rm G}({\rm X})}(\mathrm{j}_{\mathcal{O},{\rm G}!}(\mathcal{L}_{\mathcal{O}}), \mathrm{Rj}_{\mathcal{O},{\rm G}*}(\mathcal{L}_{\mathcal{O}})) \cong \mathrm{H}^2_{\rm G}(\mathcal{O}, \mathcal{L}_{\mathcal{O}})$, by Lemma 3.7 again. However, the condition that $\mathrm{H}^2_{\rm G}(\mathcal{O}, \mathcal{L}_{\mathcal{O}}))$ is trivial is almost never satisfied.

In fact, $\mathrm{H}^2_{\rm G}(\mathcal{O}, \mathcal{L}_{\mathcal{O}}) \cong \mathrm{H}^2(\mathrm{BG}_x, \mathbb{Q}_\ell)$, which is hardly ever trivial, unless G_x is trivial or finite. Since this condition should hold for all the G-orbits, it is hardly ever satisfied unless the group G itself is trivial or finite.

5. Vanishing of odd dimensional intersection cohomology for spherical varieties in positive characteristics: proof of Theorem 1.6

Throughout this section we will restrict to schemes of finite type over a base field k that is algebraically closed and of positive characteristic p. Throughout this section, we will adopt the convention where perverse sheaves on a smooth scheme with a single stratum, will be just local systems, without any dimension shift: that is, if X is smooth and $\mathcal{L}$ is an ℓ-adic local system on X, then $\mathcal{L}$ is a perverse sheaf on X. (Note that this differs from the convention in [2, 4.0, p. 102], where $\mathcal{L}[dim\,(\mathrm{X})]$ will be the corresponding perverse sheaf.)

We begin with the following result.

Theorem 5.1. *(Degeneration of the spectral sequence in equivariant intersection cohomology). Let* X *be a projective equi-dimensional* G*-scheme, where* G *is connected. Let* $\mathcal{L}$ *denote a* G*-equivariant local system on an open dense smooth sub-variety of* X *such that* $\mathcal{L}$ *is semi-simple as a* G*-equivariant local system. Let* $\mathrm{IC}^{\mathrm{G}}(\mathrm{X};\mathcal{L})$ *denote the corresponding equivariant intersection cohomology complex. Then the spectral sequence:*

$$E_2^{s,t} = \mathrm{H}^{\mathrm{s}}(\mathrm{BG}; \mathrm{R}^{\mathrm{t}}\pi_*(\mathrm{IC}^{\mathrm{G}}(\mathrm{X};\mathcal{L}))) \Rightarrow \mathrm{IH}_{\mathrm{G}}^{\mathrm{s+t}}(\mathrm{X};\mathcal{L})$$

degenerates, where $\pi : \mathrm{EG}\underset{\mathrm{G}}{\times}\mathrm{X} \to \mathrm{BG}$ *is the obvious map. Thus,* $\mathrm{IH}^*_{\mathrm{G}}(\mathrm{X};\mathcal{L}) \cong \mathrm{H}^*(\mathrm{BG}) \otimes \mathrm{IH}^*(\mathrm{X};\mathcal{L})$*.*

Proof. This is essentially the same as in [23, Proposition (13)] where only the case G is a one dimensional torus is considered. That G be connected is necessary to ensure that all local systems on BG are in fact constant. Let U denote an open smooth G-stable sub-variety of X on which $\mathcal{L}$ is a local system. Since X is equi-dimensional, U is the disjoint union of its connected components $\mathrm{U_i}$ all of which are of the same dimension. Since G is connected the $\mathrm{U_i}$ are stable under the group action. Let $\mathcal{L}_i$ denote the G-equivariant local system on U defined by $\mathcal{L}_{i_{|\mathrm{U_j}}} = \mathcal{L}_{|\mathrm{U_i}}$ if $j = i$, and $= 0$ otherwise. Then one may see that $\mathrm{IC}^{\mathrm{G}}(\mathrm{X};\mathcal{L}) = \oplus_{\mathrm{i}}\mathrm{IC}^{\mathrm{G}}(\mathrm{X};\mathcal{L}_{\mathrm{i}})$. Clearly each $\mathcal{L}_i$ is a semi-simple G-equivariant local system. Invoking, Proposition 6.1, we see that each $\mathrm{IC}^{\mathrm{G}}(\mathrm{X};\mathcal{L}_{\mathrm{i}})$ and hence $\mathrm{IC}^{\mathrm{G}}(\mathrm{X};\mathcal{L})$ is a pure perverse sheaf. Therefore the Hard Lefschetz theorem holds for $\mathrm{IH}^*(\mathrm{X};\mathcal{L})$ and the same proof as in [23, Proposition (13)] applies. □

Theorem 5.2. *Let* X *denote a projective equi-dimensional variety provided with the action of a torus* T *and let* $\mathcal{L}$ *denote a* T*-equivariant local system on an open smooth* T*-stable sub-variety of* X*. Assume that* $\mathcal{L}$ *is semi-simple as a* G*-equivariant local system. Let* $i : X^{\mathrm{T}} \to X$ *denote the inclusion of the fixed point sub-scheme. Then one obtains the following isomorphisms after inverting all non zero elements of* $\mathrm{H}^*(\mathrm{BT})$ *(that is, on localization at the prime ideal* (0)*):*

$$\mathrm{IH}^*_{\mathrm{T}}(\mathrm{X};\mathcal{L})_{(0)} \cong \mathrm{H}^*(\mathrm{BT})_{(0)} \otimes \mathrm{IH}^*(\mathrm{X};\mathcal{L}) \cong \mathrm{H}^*(\mathrm{BT})_{(0)} \otimes \mathbb{H}^*(\mathrm{X}^{\mathrm{T}}; \mathrm{R}i^!\mathrm{IC}(\mathrm{X};\mathcal{L})). \qquad (5.0.1)$$

In particular, if $\mathrm{IH}^i(\mathrm{X};\mathcal{L}) = 0$ *for all odd i and x is an isolated fixed point of* T*, then* $\mathrm{IH}^i_x(\mathrm{X};\mathcal{L}) = \mathcal{H}^i(\mathrm{IC}(\mathrm{X};\mathcal{L}))_x = 0$ *for all odd i. (Here* $\mathcal{H}^i(\mathrm{IC}(\mathrm{X};\mathcal{L}))_x$ *denotes the stalk of the sheaf* $\mathcal{H}^i(\mathrm{IC}(\mathrm{X};\mathcal{L}))$ *at x.)*

Proof. The first isomorphism follows from Theorem 5.1 by localizing at (0). By the localization theorem (see [23, Theorem (17)]), one has the isomorphism:

$$\mathrm{IH}^*_{\mathrm{T}}(\mathrm{X};\mathcal{L})_{(0)} \cong \mathrm{H}^*_{\mathrm{T}}(\mathrm{X}^{\mathrm{T}}; \mathrm{Ri}^! IC^{\mathrm{T}}(\mathrm{X};\mathcal{L}))_{(0)}. \tag{5.0.2}$$

Then $\mathrm{H}^*_{\mathrm{T}}(\mathrm{X}^{\mathrm{T}}; \mathrm{Ri}^!\mathrm{IC}^{\mathrm{T}}(\mathrm{X};\mathcal{L}))_{(0)} \cong \mathrm{H}^*(\mathrm{BT})_{(0)} \otimes \mathrm{H}^*(\mathrm{X}^{\mathrm{T}}, \mathrm{Ri}^!\mathrm{IC}^{\mathrm{T}}(\mathrm{X};\mathcal{L}))$.

Next let $i_x : x \to X^{\mathrm{T}}$ be the inclusion of an isolated fixed point of T. Then $Ri^!\mathrm{IC}^{\mathrm{T}}(\mathrm{X};\mathcal{L})$ breaks up into the sum of complexes one of which is

$$Ri^!_x\mathrm{IC}^{\mathrm{T}}(\mathrm{X};\mathcal{L}) \cong \mathrm{Di}^*_x\mathrm{D}(\mathrm{IC}^{\mathrm{T}}(\mathrm{X};\mathcal{L})) \simeq (\mathrm{i}^*_x\mathrm{IC}^{\mathrm{T}}(\mathrm{X};\mathcal{L}^\vee))^\vee[-2\mathrm{d}]$$

where d is the dimension of X. This proves the last assertion of the theorem. □

Next we consider the following proposition which will be used repeatedly in the paper.

Proposition 5.3. *Let* $\pi : \mathrm{Y} \to \mathrm{X}$ *denote a finite* G*-equivariant surjective map of* G*-varieties. Let* X_0 *denote a* G*-stable open subscheme of* X *and let* $\mathrm{Y}_0 = \mathrm{X}_0 \underset{\mathrm{X}}{\times} \mathrm{Y}$ *be its inverse image under* π*. If* $\mathcal{L}$ *is a* G*-equivariant local system on* Y_0 *and* $\pi_0 : \mathrm{Y}_0 \to \mathrm{X}_0$ *is the map induced by* π*, after possibly shrinking* X_0*, we may assume that* $\pi_{0*}(\mathcal{L})$ *is a* G*-equivariant local system on* X_0*. Moreover,* $\pi_*(\mathrm{IC}^{\mathrm{G}}(\mathrm{Y};\mathcal{L})) \simeq \mathrm{IC}^{\mathrm{G}}(\mathrm{X};\pi_{0*}(\mathcal{L}))$*.*

Proof. Since the map π is finite, one may readily show that $\pi_*(\mathrm{IC}^{\mathrm{G}}(\mathrm{Y};\mathcal{L}))$ satisfies all the axioms of an intersection cohomology complex on X except possibly for the axiom that says there exists a dense smooth open G-stable subscheme V of X so that $\pi_*(\mathrm{IC}^{\mathrm{G}}(\mathrm{Y};\mathcal{L}))_{|\mathrm{V}}$ is a local system. We proceed to show this presently.

Since intersection cohomology is invariant under normalization (see [23, (A-17)]), we will normalize all the varieties and assume the varieties are integral. Let y_0 and x_0 be the generic points of Y and X. Assume the characteristic of $k(x_0)$ is p. Then either $k(y_0)$ is separable over $k(x_0)$ or there exists some positive integer N so that $k(y_0)^{p^{\mathrm{N}}}.k(x_0)$ is separable over $k(x_0)$. If $k(y_0)$ is separable over $k(x_0)$, we take $\mathrm{N} = 0$, $p^{\mathrm{N}} = 1$; otherwise p^{N} is the inseparable degree of $k(y_0)$ over $k(x_0)$. Let $\mathrm{Y}^{(\mathrm{N})}$ be the pull-back of $\pi : \mathrm{Y} \to \mathrm{X}$ along the Frobenius $\mathrm{Fr}^{\mathrm{N}} : \mathrm{X} \to \mathrm{X}$ and let $\mathrm{Y} \to \mathrm{Y}^{(\mathrm{N})}$ be the map induced by $\mathrm{Fr}^{\mathrm{N}} : \mathrm{Y} \to \mathrm{Y}$ and $\pi : \mathrm{Y} \to \mathrm{X}$. Then the function field of $\mathrm{Y}^{(\mathrm{N})}$ is $k(y_0)^{p^{\mathrm{N}}}.k(x_0)$ which is separable over $k(x_0)$. Thus the projection $\mathrm{Y}^{(\mathrm{N})} \xrightarrow{\bar{\pi}} \mathrm{X}$ induces an étale map of the generic point of $\mathrm{Y}^{(\mathrm{N})}$ to the generic point of X; therefore there exist open subsets U of $\mathrm{Y}^{(\mathrm{N})}$ and V of X so that the map $\bar{\pi}_{|\mathrm{U}} : \mathrm{U} \to \mathrm{V}$ is étale. We proceed to show that we may take $\mathrm{U} = \bar{\pi}^{-1}(\mathrm{V}) = \mathrm{V} \underset{\mathrm{X}}{\times} \mathrm{Y}^{(\mathrm{N})}$.

Let $F = \{z \,\varepsilon\, \mathrm{Y}^{(\mathrm{N})} | \bar{\pi} \text{ is not étale at } z\}$. This is a proper closed subset of $\mathrm{Y}^{(\mathrm{N})}$. Since $\bar{\pi}$ is induced by base-change from π, it is also finite. Therefore $\bar{\pi}(F)$ is a proper closed subset of X. Let V $=\mathrm{X}-\bar{\pi}(\mathrm{F})$ and $\mathrm{U}=\bar{\pi}^{-1}(\mathrm{V})$. Then $\bar{\pi}_{|\mathrm{U}}$ is étale. Moreover, since $\bar{\pi}_{|\mathrm{U}}$ is also obtained by base-change from π, it is also a *finite* map. Finally let the inverse image of U in Y by the induced map $\mathrm{Y} \to \mathrm{Y}^{(\mathrm{N})}$ be W, that is, $\mathrm{W}=\mathrm{Y} \underset{\mathrm{Y}^{(\mathrm{N})}}{\times} \mathrm{U}$. The map π restricted to W factors as the composition of the purely inseparable map $\mathrm{W} \to \mathrm{U}$ and the finite étale surjective map $\mathrm{U} \to \mathrm{V}$. Therefore, if $\mathcal{L}$ is a local system on W, the direct image $\pi_{|\mathrm{W}*}(\mathcal{L})$ is a local system on V. The G-equivariance is clear from the above argument; therefore $\pi_*(\mathrm{IC}^{\mathrm{G}}(\mathrm{Y};\mathcal{L})) \simeq \mathrm{IC}^{\mathrm{G}}(\mathrm{X};\pi_{|\mathrm{W}*}(\mathcal{L}))$. This completes the proof. □

5.1. Spherical varieties in positive characteristics: local structure and weak-resolution of singularities

5.1.1. In positive characteristic, the *local structure of spherical varieties* is due to Knop (see [27]), which we recall presently. Let G denote a connected reductive group, X a G-spherical variety and let $x \,\varepsilon\, \mathrm{X}$. After (possibly) replacing X by an open G-stable sub-scheme we may assume $\mathrm{G}x$ is the unique closed G-orbit in X and that X is quasi-projective. Then one can find a G-linearized ample line bundle $\mathcal{L}$, and a global section s of $\mathcal{L}$ which is an eigenvector of a Borel subgroup B of G. Let $\mathrm{X_s}$ be the open subset of X where s is non zero. One needs to choose the line bundle $\mathcal{L}$ and the section s as in [27, Korollar 2.3] so that the section s vanishes everywhere on $\mathrm{G}x-\mathrm{B}x$, but not everywhere on $\mathrm{G}x$. Then one can show $\mathrm{X_s}=\mathrm{X}_{\mathrm{B}x}=\{\mathrm{B}y | y \,\varepsilon\, \mathrm{X}, \overline{\mathrm{B}y} \supseteq \mathrm{B}x\}$.

Let P denote the subgroup of G which stabilizes $\mathrm{X_s}$. Then P is a parabolic subgroup of G and $\mathrm{G}x \cap \mathrm{X}_{\mathrm{B}x}=\mathrm{B}x$. The following is then shown in [27]: the unipotent radical P_u of P acts properly on $\mathrm{X_s}$, the quotient $\mathrm{X_s}/\mathrm{P}_u$ exists and there exists a sub-variety Z of $\mathrm{X_s}$ such that the natural maps $\rho: \mathrm{P}_u \times \mathrm{Z} \to \mathrm{X_s}$ and $\bar{\rho}: \mathrm{Z} \to \mathrm{X_s}/\mathrm{P}_u$ are finite and surjective. Moreover, Z is stable under a maximal torus T of P. We may further assume that T is contained in B. This provides the following commutative square:

$$\begin{array}{ccc} \mathrm{P}_u \times \mathrm{Z} & \xrightarrow{\rho} & \mathrm{X_s} \\ \downarrow{\scriptstyle \tilde{\pi}} & & \downarrow{\scriptstyle \pi} \\ \mathrm{Z} & \xrightarrow{\bar{\rho}} & \mathrm{X_s}/\mathrm{P}_u \end{array} \tag{5.1.1}$$

The map ρ is equivariant for the left-action of P_u. It is also T equivariant when T acts on $\mathrm{P}_u \times \mathrm{Z}$ by $t.(p,z)=(t.p.t^{-1}, t.z)$.

When the base field is $\mathbb{C}$, there exists a Z as in (5.1.1) such that the maps ρ and $\bar{\rho}$ are isomorphisms, and so that Z is stable under a Levi subgroup L of P. This may fail in positive characteristics, but $\mathrm{X_s}/\mathrm{P}_u$ still has an action of P/P_u and the latter is isomorphic to a Levi subgroup L. Moreover, since X is spherical, $\mathrm{X_s}$ contains a dense B-orbit and hence $\mathrm{X_s}/\mathrm{P}_u$ contains a dense orbit of B/P_u so that $\mathrm{X_s}/\mathrm{P}_u$ is an affine spherical L-variety. Moreover the image of $\mathrm{L}x$ is closed in $\mathrm{X_s}/\mathrm{P}_u$.

5.1.2. This closed orbit $\mathrm{L}x$ is a torus ($\cong \mathbb{G}_m^c$, for some $c > 0$) by the choice of L. This follows from the observation that $\mathrm{P}x = \mathrm{B}x$ (since $\mathrm{G}x \cap \mathrm{X}_{\mathrm{B}x} = \mathrm{B}x$), and that therefore, the stabilizer of x in L contains the derived group of L. Let $f_1, ..., f_c \,\varepsilon\, k[\mathrm{L}x]$ be the eigen-vectors of L (actually of L/L_x) that provide the isomorphism $\mathrm{L}x \cong \mathbb{G}_{\mathrm{m}}^{\mathrm{c}}$. By standard arguments from geometric invariant theory, see for example [MFK] p. 195, there exists a large enough q, with q a power of p, so that f_i^q, $i = 1, ..., c$ extend to maps $\phi_i : \mathrm{X_s}/\mathrm{P}_u \to \mathbb{G}_{\mathrm{m}}$ which are also eigen-vectors of L/L_x. Let $\phi = (\phi_1, ..., \phi_c) : \mathrm{X_s}/\mathrm{P}_u \to \mathbb{G}_{\mathrm{m}}^{\mathrm{c}}$ be the corresponding induced map. The composition $\mathrm{L}x \to \mathrm{X_s}/\mathrm{P}_u \to \mathbb{G}_{\mathrm{m}}^{\mathrm{c}}$ is the identity map raised to the q-th power. Let $\mathcal{S} = \phi^{-1}(1)$, $1 = (1, ..., 1) \,\varepsilon\, \mathbb{G}_m^c$. The dimension of $\mathcal{S}$ = dimension of $\mathrm{X_s}/\mathrm{P}_u - \mathrm{c}$ = the codimension of the G-orbit of x.

The observation that the $\{\phi_i|i\}$ are eigen-vectors of L/L_x shows that $\mathcal{S}$ is stable under the action of L_x. Then one obtains an induced L-equivariant map $r : \mathrm{L} \times \mathcal{S} \to \mathrm{X_s}/\mathrm{P}_u$. The same observation that the $\{\phi_i|i\}$ are eigen-vectors of L/L_x shows that the resulting induced map

$$\bar{r} : \mathrm{L} \underset{\mathrm{L}_x}{\times} \mathcal{S} \to \mathrm{X_s}/\mathrm{P}_u \tag{5.1.3}$$

is bijective and therefore purely inseparable.

5.1.4. Next assume that X is a *toroidal imbedding* of a spherical homogeneous space G/H. Then one may take Z to be a toric variety for a maximal torus in L_x. We proceed to show that then the immersion of any G-orbit $\mathcal{O}$ into X is affine. We begin with the commutative square:

$$\begin{array}{ccc} \mathrm{P}_u \times (\mathrm{Z} \cap \mathcal{O}) & \xrightarrow{\rho'} & \mathrm{X_s} \cap \mathcal{O} \\ \downarrow{\scriptstyle \tilde{\pi}} & & \downarrow{\scriptstyle \pi} \\ \mathrm{P}_u \times \mathrm{Z} & \xrightarrow{\rho} & \mathrm{X_s} \end{array} \tag{5.1.4}$$

The fact that the map ρ is equivariant for the left-action of P_u shows that the above square is in fact a cartesian square. Since the map ρ is a finite surjective map, it follows that the map $\rho' : \mathrm{P}_u \times (\mathrm{Z} \cap \mathcal{O}) \to \mathrm{X_s} \cap \mathcal{O}$ is also finite and surjective. Clearly $\mathrm{P}_u \times (\mathrm{Z} \cap \mathcal{O})$ is affine, as Z is now a toric variety and $\mathrm{Z} \cap \mathcal{O}$ corresponds to an orbit for the corresponding torus on Z. Now [36, Proposition 32.11.1] shows that the scheme $\mathrm{X_s} \cap \mathcal{O}$ is affine, thereby proving the required assertion.

5.1.5. Weak resolution of singularities

Let X denote a G-spherical variety. Using the embedding theory of spherical homogeneous spaces (which also works in positive characteristics), one can construct a spherical variety $\tilde{\mathrm{X}}$ along with a proper G-equivariant bi-rational map $\pi : \tilde{\mathrm{X}} \to \mathrm{X}$ such that $\tilde{\mathrm{X}}$

is covered by open subsets $\tilde{X}_s$ as above, where the $\tilde{X}_s/P_u$ are affine toric varieties with quotient singularities. In fact, one can take $\tilde{X}$ to be toroidal.

In characteristic zero, this gives a resolution of singularities. In the general case, one obtains a rationally smooth $\tilde{X}$. To see this, consider the finite surjective map $\rho : P_u \times Z \to \tilde{X}_s$ as in (5.1.1). The variety Z is rationally smooth since it is a simplicial toric variety (that is, a toric variety whose fan is simplicial). Therefore $P_u \times Z$ is rationally smooth and the local cohomology groups of $\tilde{X}_s$ with supports in any fixed geometric point are trivial in all degrees except the top degree. Since $\tilde{X}_s$ is irreducible, these local cohomology groups may be identified with the cohomology of the stalks of the dualizing complex. This, in turn, may be identified with the dual of the sheaves $U \to H^{2d}_c(U; \mathbb{Q}_\ell)$, where $d = dim(\tilde{X}_s)$ and $H^{2d}_c(U; \mathbb{Q}_\ell)$ denotes the ℓ-adic cohomology of U with compact supports. (See [8, Proposition (A.1)].) Therefore these are equal to $\mathbb{Q}_\ell$ proving $\tilde{X}_s$ is rationally smooth.

5.1.6. Vanishing of odd dimensional intersection cohomology

Lemma 5.4. *Let* X *denote a not-necessarily normal spherical variety and let* $\mathcal{L}$ *denote a* G*-equivariant local system on the open* G*-orbit. Let* $\pi : \tilde{X} \to X$ *denote the normalization and let* $\pi^*(\mathcal{L}) = \tilde{\mathcal{L}}$*. Let* $IC^G(X; \mathcal{L})$ *(*$IC^G(\tilde{X}; \tilde{\mathcal{L}})$*) denote the intersection cohomology complex of* X *with respect to* $\mathcal{L}$ *(of* $\tilde{X}$ *with respect to* $\tilde{\mathcal{L}}$*, respectively). Then the intersection cohomology sheaves* $\mathcal{H}^i(IC^G(X; \mathcal{L}))$ *vanish for all odd* i*, if and only if the sheaves* $\mathcal{H}^i(IC^G(\tilde{X}; \tilde{\mathcal{L}}))$ *vanish for all odd* i*.*

Proof. First observe that the map π is an isomorphism on the dense G-orbit. Therefore one may readily show that $R\pi_*(IC^G(\tilde{X}; \tilde{\mathcal{L}})) \cong IC^G(X; \mathcal{L})$. Now consider the Leray spectral sequence:

$$E_2^{s,t} = R^s\pi_*\mathcal{H}^t(IC^G(\tilde{X}; \tilde{\mathcal{L}}))) \Rightarrow R^{s+t}\pi_*(IC^G(\tilde{X}; \tilde{\mathcal{L}})) \cong \mathcal{H}^{s+t}(R\pi_*(IC^G(\tilde{X}; \tilde{\mathcal{L}})))$$
$$\cong \mathcal{H}^{s+t}(IC^G(X; \mathcal{L})).$$

Since π is a finite map, $E_2^{s,t} = 0$ for all $s > 0$ in this spectral sequence; therefore one obtains the isomorphism $\pi_*\mathcal{H}^t(IC^G(\tilde{X}; \tilde{\mathcal{L}})) \cong E_2^{0,t} \cong E_\infty^{0,t} = \mathcal{H}^t(IC^G(X; \tilde{\mathcal{L}}))$. Now the lemma follows readily. □

Proposition 5.5. *Let* X *denote a projective* G*-spherical variety and let* $\mathcal{L}$ *denote a* G*-equivariant local system on the open* G*-orbit. Then* $IH^i(X; \mathcal{L}) = 0$ *for all odd* i*. The same holds for all* T*-equivariant local systems on the open* G*-orbit.*

Proof. Exactly the same proof as in characteristic 0 applies here in view of the discussion on local systems in positive characteristics as in Proposition 2.4. However, we sketch the details, for the sake of completeness. We will first use the following technique to reduce to the case of the constant local system. We may first assume that X is normal by

Lemma 5.4. Next let $Gx_o \cong G/G_{x_o}$ denote the open G-orbit in X. Let $\tilde{X}$ denote the normalization of X in the function field $k(G/G^0_{x_o})$. Then we obtain the cartesian square

$$\begin{array}{ccc} G/G^0_{x_o} & \longrightarrow & \tilde{X} \\ \downarrow{\scriptstyle \pi_o} & & \downarrow{\scriptstyle \pi} \\ G/G_{x_o} & \longrightarrow & X \end{array} \tag{5.1.5}$$

where the maps π_o and π are finite. Note that G acts on $\tilde{X}$ and that $\tilde{X}$ is a spherical G-variety. Let $\underline{\mathbb{Q}}_\ell$ denote the constant G-equivariant local system on $G/G^0_{x_o}$. Now $R\pi_{o*}(\underline{\mathbb{Q}}_\ell) = \pi_{o*}(\underline{\mathbb{Q}}_\ell)$. The stalk of this sheaf at x_o is the ℓ-adic regular representation of the finite group $G_{x_o}/G^0_{x_o}$. Therefore, by Proposition 2.4, the G-equivariant local system $\pi_{o*}(\underline{\mathbb{Q}}_\ell)$ can be written as a sum $\underset{\chi}{\oplus} \dim(\chi)\mathcal{L}_\chi$, where $\mathcal{L}_\chi$ is the local system corresponding to the irreducible character χ of the finite group $G_{x_o}/G^0_{x_o}$ and the sum varies over all such characters. Therefore,

$$\pi_* IC(\tilde{X}, \underline{\mathbb{Q}}_\ell) = \underset{\chi}{\oplus} \dim(\chi) IC(X, \mathcal{L}_\chi).$$

Taking the hyper-cohomology, it follows that

$$IH^i(\tilde{X}) = \underset{\chi}{\oplus} \dim(\chi) IH^i(X, \mathcal{L}_\chi)$$

for all i. Thus, it suffices to consider X with the constant local system.

Next, let $\pi : \tilde{X} \to X$ denote a G-equivariant weak-resolution of singularities as in 5.1.5. Then $\tilde{X}$ is a projective rationally smooth spherical variety and T acts on $\tilde{X}$ with only finitely many fixed points. The fixed point formula for torus actions (that is, Theorem 5.2) now gives the isomorphism:

$$H^*_T(\tilde{X}; \underline{\mathbb{Q}}_\ell)_{(0)} \simeq H^*(BT)_{(0)} \otimes H^*(\tilde{X}^T; Ri^!(\underline{\mathbb{Q}}_\ell))$$

Since $\tilde{X}$ is rationally smooth, it follows that $Ri^!(\underline{\mathbb{Q}}_\ell) \simeq \underline{\mathbb{Q}}_\ell[-2n]$ where n is the dimension of $\tilde{X}$. Since $\tilde{X}^T$ is finite, it follows that $H^*_T(\tilde{X}; \underline{\mathbb{Q}}_\ell)_{(0)}$ is trivial in *odd degrees*. Now it follows as in Theorem 5.2 that $H^n(\tilde{X}; \underline{\mathbb{Q}}_\ell) = 0$ for all odd n. The decomposition theorem in intersection cohomology shows that $IH^i(X; \underline{\mathbb{Q}}_\ell)$ is a split summand of $H^i(\tilde{X}; \underline{\mathbb{Q}}_\ell)$ for any i. Since the latter is trivial for all odd i, this completes the proof of the Proposition for all G-equivariant local systems. The assertion about the T-equivariant local systems follows from Proposition 5.7 below. □

Lemma 5.6. *Let* G *denote a linear algebraic group acting on a scheme* X *and let* H *denote a closed linear algebraic subgroup of* G*. Let x denote a chosen point of* X*. Then:*

(i) $\pi_1(\mathrm{EG} \times_{\mathrm{G}} \mathrm{X}, x)\widehat{\ }_\ell$ *classifies* G*-equivariant* ℓ*-adic local systems on* $\mathrm{EG} \times_{\mathrm{G}} \mathrm{X}$.

(ii) *If* $\pi_1(\mathrm{G/H}, *)\widehat{\ }_\ell = \pi_0(\mathrm{G/H}, *)\widehat{\ }_\ell = 0$*, where* $*$ *denotes the base point of* G/H *corresponding to* H*, then the map* $\mathrm{EH}\underset{\mathrm{H}}{\times}\mathrm{X} \simeq \mathrm{EG} \times_{\mathrm{G}} (\mathrm{G} \times_{\mathrm{H}} \mathrm{X}) \to \mathrm{EG} \times_{\mathrm{G}} \mathrm{X}$ *induces an isomorphism on* $\pi_1(\quad)\widehat{\ }_\ell$*. Here,* $\widehat{\ }_\ell$ *denotes the completion at the prime* ℓ.

Proof. The proof of (i) reduces to the corresponding statement for schemes, by making using of the comparison between the Borel construction defined simplicially and the corresponding Borel construction defined using the geometric classifying spaces as in Theorem 7.2. The proof of (ii) reduces to the existence of a long-exact sequence:

$$\cdots \pi_1(\mathrm{G/H}, *)\widehat{\ }_\ell \to \pi_1(\mathrm{EH}\underset{\mathrm{H}}{\times}\mathrm{X}, x)\widehat{\ }_\ell \to \pi_1(\mathrm{EG}\times_{\mathrm{G}}\mathrm{X}, x)\widehat{\ }_\ell \to \pi_0(\mathrm{G/H}, *)\widehat{\ }_\ell \cdots$$

The existence of the above long exact sequence often makes use of etale homotopy theory: see [1]. We skip the details as they are generally well-known. □

Proposition 5.7. *Let* G *denote a connected reductive group,* T *a fixed maximal torus,* B *a Borel subgroup containing* T *and* P *a parabolic subgroup containing* B*. Let* X *denote a connected scheme with an action by* P*. Let* X *denote a* G*-scheme. Then the following hold:*

(i) *The restriction functor from the category of* G*-equivariant* ℓ*-adic local systems on* X *to the category of* B *(or* T*)-equivariant* ℓ*-adic local systems is also an equivalence of categories.*

(ii) *the restriction functor from the category of* P*-equivariant* ℓ*-adic local systems on* X *to the category of* T*-equivariant* ℓ*-adic local systems is an equivalence of categories.*

Proof. Throughout the proof, $*$ will denote a chosen fixed base point for the schemes considered. The first assertion follows from Lemma 5.6 and the observation that

$$\pi_1(\mathrm{G/B}, *)\widehat{\ }_\ell \cong \pi_0(\mathrm{G/B}, *)\widehat{\ }_\ell = 0 \text{ and } \pi_1(\mathrm{B/T}, *)\widehat{\ }_\ell \cong \pi_0(\mathrm{B/T}, *)\widehat{\ }_\ell = 0.$$

To see the triviality of the completed π_1 of G/B, one may consider a lifting of G and B to over $\mathbb{C}$, where such a result is well-known.

We will next consider the second assertion. Let L denote a Levi subgroup of P. Then L∩B is a Borel subgroup of L. Moreover, P/B ≅L/(L∩B). Therefore, $\pi_1(\mathrm{P/B}, *)\widehat{\ }_\ell \cong \pi_1(\mathrm{L/(L\cap B)}, *)\widehat{\ }_\ell \cong 0$ and $\pi_0(\mathrm{P/B})\widehat{\ }_\ell \cong \pi_0(\mathrm{L/L\cap B})\widehat{\ }_\ell \cong 0$. Therefore, the second assertion follows. □

Remarks 5.8. In view of the above discussion, it suffices to consider T-equivariant local systems in the rest of the paper: this is important for us, since the variety Z appearing in (5.1.1) is only stable by T and not by G.

Proposition 5.9. *Let* X *denote a* G*-spherical variety and let* $x \,\varepsilon\, \mathrm{X}$ *be a fixed point for* T*. Then* $\mathcal{H}^i(\mathrm{IC}(\mathrm{X};\mathcal{L}))_x = 0$ *for all odd* i *and all* T*-equivariant (or equivalently all* G*-equivariant) local systems* $\mathcal{L}$ *on the open* G*-orbit.*

Proof. By Lemma 5.4, we may assume that X is normal. Then x admits an open G-stable quasi-projective neighborhood U_x (see [37]). Thus, we may replace X by the closure of U_x, and assume that X is projective. Now we conclude by Theorem 5.2 together with Proposition 5.5. (Observe that the fundamental group of the open orbit, $\pi_1(\mathrm{G}x_o)$ acts on the stalks of $\mathcal{L}$ through its image in $\pi_1(\mathrm{EG}\times_{\mathrm{G}}\mathrm{G}x_{\mathrm{o}})$ which is finite. Therefore $\mathcal{L}$ is semi-simple as a local system, and Theorem 5.2 applies.) □

Theorem 5.10. *Let* X *denote a* G*-spherical variety and let* $\mathcal{L}$ *denote a* G*-equivariant local system on the open dense orbit. Then* $\mathcal{H}^i(\mathrm{IC}(\mathrm{X};\mathcal{L})) = 0$ *for all odd* i*.*

Proof. The proof proceeds by ascending induction on the dimension of the G-spherical variety for any connected reductive group G. Since a spherical variety of dimension 1 may be assumed to be normal and hence non-singular, we may start the induction with spherical varieties of dimension 1. Observe that the conclusion is that the stalks of the intersection cohomology sheaves vanish in odd dimensions at all points x on the given scheme X, and that the case x is a fixed point of G is handled by Proposition 5.9. Therefore, we may assume x is a chosen point on a G-orbit of positive dimension: then the strategy of our proof is to reduce to showing the corresponding vanishing when the scheme X is replaced by a transverse slice $\mathcal{S}$ to the G-orbit at x, as the corresponding slice is also a spherical variety of lower dimension for a smaller reductive group. This is the same strategy used in the proof of the corresponding statement for complex spherical varieties (see [9, Theorem 4]): the main difference now is that working with the transverse slice takes more effort in positive characteristics.

We recall the commutative square (see (5.1.7)):

$$\begin{array}{ccc} \mathrm{P}_u \times \mathrm{Z} & \xrightarrow{\rho} & \mathrm{X_s} \\ \downarrow{\scriptstyle \tilde{\pi}} & & \downarrow{\scriptstyle \pi} \\ \mathrm{Z} & \xrightarrow{\bar{\rho}} & \mathrm{X_s}/\mathrm{P}_u \end{array} \tag{5.1.7}$$

We may replace the varieties Z and $\mathrm{X_s}/\mathrm{P}_u$ by their normalizations, if necessary and assume all the varieties in the diagram in (5.1.7) are normal. Further, we may assume X is irreducible, that $\mathrm{G}x_0$ denotes the open G-orbit on X, that $x_0 \,\varepsilon\, \mathrm{X_s}$ and that $\mathcal{L}$ is a G-equivariant local system on $\mathrm{G}x_0$; observe that $\mathrm{X_s}\cap \mathrm{G}x_0$ is a union of the open P-orbits on $\mathrm{X_s}$. We will denote $\mathrm{X_s}\cap \mathrm{G}x_0$ by $\mathrm{X_s}(x_0)$. Now $\mathcal{L}_0 = \mathcal{L}_{|\mathrm{X_s}(x_0)}$ a P-equivariant local system. Recall that the map ρ is T equivariant when T acts on $\mathrm{P}_u\times \mathrm{Z}$ by $t.(p,z) = (t.p.t^{-1}, t.z)$. Moreover all the other maps in (5.1.7) are T-equivariant and the same map ρ is also P_u-equivariant when P_u acts on $\mathrm{P}_u\times \mathrm{Z}$ by left-translation on the factor P_u. Therefore, the inverse image of $X_s(x_0)$ by the map ρ will be of the form $P_u \times Z_0$ for a T-stable open subscheme Z_0 of Z. Let $\rho_0 = \rho_{|P_u \times Z_0}$ and $\bar{\rho}_0 = \bar{\rho}_{|Z_0}$.

Now, the pull-back $\rho_0^*(\mathcal{L}_0)$ is both a P_u-equivariant and T-equivariant local system on $P_u \times Z_0$. $\rho_0^*(\mathcal{L}_0)$ clearly descends to a T-equivariant local system on Z_0. Call this local system $\mathcal{L}_1$.

Next recall that the maps ρ and $\bar{\rho}$ are finite surjective maps between normal varieties. Therefore, one may invoke Proposition 5.3 to show that $\rho_{0*}(\rho_0^*\mathcal{L}_0)$ is a T-equivariant local system on a T-stable open subscheme of X_s; we will denote this by $\mathcal{L}_0'$. Similarly $\bar{\rho}_{0*}(\mathcal{L}_1)$ is a T-equivariant local system on a T-stable open subscheme of X_s/P_u and

$$\rho_*(\mathrm{IC}^T(P_u \times Z; \rho_0^*(\mathcal{L}_0))) \simeq \mathrm{IC}^T(X_s, \mathcal{L}_0') \text{ while,} \tag{5.1.8}$$

$$\bar{\rho}_*(\mathrm{IC}^T(Z; \mathcal{L}_1)) \simeq \mathrm{IC}^T(X_s/P_u, \bar{\rho}_{0*}(\mathcal{L}_1)). \tag{5.1.9}$$

Moreover one may also observe that $\mathcal{L}_0$ is a split summand of the local system $\mathcal{L}_0' = \rho_{0*}(\rho_0^*(\mathcal{L}_0))$. Therefore it suffices to show the odd dimensional cohomology sheaves of $\mathrm{IC}^T(X_s, \mathcal{L}_0')$ are trivial. Since the map ρ is finite, by (5.1.8) one may identify $\mathcal{H}^i(\mathrm{IC}^T(X_s, \mathcal{L}_0'))_{\bar{x}}$ with $\underset{\bar{y} \in \rho^{-1}(\bar{x})}{\oplus} \mathcal{H}^i(\mathrm{IC}^T(P_u \times Z, \rho_0^*(\mathcal{L}_0)))_{\bar{y}}$. Now observe that the map $\tilde{\pi}$ in (5.1.7) is evidently smooth with fibers $= P_u$; therefore $(\mathrm{IC}^T(P_u \times Z; \rho_0^*(\mathcal{L}_0)) \simeq \tilde{\pi}^*(\mathrm{IC}^T(Z; \mathcal{L}_1))$ and one reduces to showing $\mathcal{H}^i(\mathrm{IC}^T(Z; \mathcal{L}_1)) = 0$ for all odd i. Now the finiteness of the map $\bar{\rho}$ and (5.1.9) show:

$$\mathcal{H}^i(\mathrm{IC}^T(X_s/P_u; \bar{\rho}_{0*}(\mathcal{L}_1)))_{\bar{x}} \cong \underset{\bar{y} \in \bar{\rho}^{-1}(\bar{x})}{\oplus} \mathcal{H}^i(\mathrm{IC}^T(Z, \mathcal{L}_1))_{\bar{y}}.$$

Therefore, it suffices to show $\mathcal{H}^i(\mathrm{IC}(X_s/P_u; \bar{\rho}_{0*}(\mathcal{L}_1))) = 0$ for all odd i.

Let Gx denote the unique closed G-orbit on X. Let $\bar{r} : L \underset{L_x}{\times} \mathcal{S} \rightarrow X_s/P_u$ denote the purely inseparable map as in (5.1.3) and let $(X_s/P_u)_0$ denote an open smooth sub-variety of X_s/P_u one which the local system $\bar{\rho}_{0*}(\mathcal{L}_1)$ is defined. Let $\bar{r}_0 : \bar{r}^{-1}((X_s/P_u)_0) \rightarrow (X_s/P_u)_0$ denote the map induced by $\bar{r}$. Then $\mathrm{IC}(X_s/P_u; \bar{\rho}_{0*}(\mathcal{L}_1))_x \simeq \mathrm{IC}(\mathcal{S}; \bar{r}_0^*(\bar{\rho}_{0*}(\mathcal{L}_1)_{|\mathcal{S}}))_x$. This follows from Proposition 5.3 applied to the purely inseparable map $\bar{r} : L \underset{L_x}{\times} \mathcal{S} \rightarrow X_s/P_u$. Therefore it suffices to prove that $\mathcal{H}^i(\mathrm{IC}(\mathcal{S}; \mathcal{L})) = 0$ for any $T_x^o = T \cap L_x^o$-equivariant local system on the open L_x-orbit in $\mathcal{S}$. (Observe that T_x is a maximal torus in L_x by the choice of L and T.) Since the dimension of $\mathcal{S}$ is the codimension of the G-orbit at x, the inductive hypothesis and the correspondence between L_x^o-equivariant and T_x^o-equivariant local systems as in Proposition 5.7 apply to complete the proof. (Recall that the dimension of $\mathcal{S}$ = the codimension of the G-orbit at x. The case x is a fixed point of G is handled by Proposition 5.9.) □

6. The weight filtration on the equivariant derived category: proof of Theorem 1.8

We next discuss the weight filtration on the equivariant derived category. (See [24, section 4] for a discussion of these in the setting of the ℓ-adic derived category on algebraic stacks.) We will henceforth assume that the base field $k = \mathbb{F}_q$, $\bar{k} = \bar{\mathbb{F}}_q$ is its algebraic closure and that G_o is a linear algebraic group defined over $\mathbb{F}_q$ acting on the scheme $\mathrm{X_o}$ also defined over $\mathbb{F}_q$. We will also assume that a G_o-stable stratification, $\mathcal{S}_o$ of $\mathrm{X_o}$ is given. Any object in the derived category $\mathrm{D}^b(\mathrm{X_o}, \bar{\mathbb{Q}}_\ell)$ will be denoted with a subscript o as in K_o. The corresponding objects defined over $\bar{\mathbb{F}}_q$ will be denoted without the subscript o. One may make this more precise as follows. Let $\epsilon : Spec\,\bar{\mathbb{F}}_q \rightarrow Spec\,\mathbb{F}_q$ denote the obvious map. Then, given an object K_o over $Spec\,\mathbb{F}_q$, $K = \epsilon^*(K_o)$ will denote its pull-back. Similarly, schemes that are defined over $Spec\,\bar{\mathbb{F}}_q$ will be denoted without the subscript o, while those defined over $Spec\,\mathbb{F}_q$ will be denoted with the subscript o.

Let $\mathrm{EG}_o^{gm,m}$ denote a finite degree approximation to the principal G_o-bundle over the finite degree approximation $\mathrm{BG}_o^{gm,m}$ to the classifying space $\mathrm{BG_o}$: see Definition 7.1. (Comparison with the simplicial construction of EG and the resulting equivariant derived categories is discussed in the Appendix.) A complex of $\bar{\mathbb{Q}}_\ell$-sheaves $K_o\, \varepsilon\, \mathrm{D^b}(\mathrm{EG_o^{gm,m}} \underset{\mathrm{G_o}}{\times} \mathrm{X_o}, \bar{\mathbb{Q}}_\ell)$ will be said to be *(exactly) pure of weight of w* if it satisfies the condition as on [2, p. 126]: that is, the eigen-values of the geometric Frobenius Fr at each stalk of each cohomology sheaf $\mathcal{H}^i(K)$ over an $\mathbb{F}_q$-rational point of $\mathrm{EG}_o^{gm,m} \underset{\mathrm{G}_o}{\times} \mathrm{X_o}$ are of absolute value $q^{(i+w)/2}$.

A complex K_o is *mixed of weight $\leq w$ ($\geq w$)* if each cohomology sheaf $\mathcal{H}^i(K_o)$ has a finite ascending filtration by $\bar{\mathbb{Q}}_\ell$-sheaves so that the associated graded pieces are all pure of weight $\leq w$ ($\geq w$, respectively).

Lisse $\bar{\mathbb{Q}}_\ell$-adic sheaves (as in [16, I.-Pureté]) will be more often called ℓ-adic local systems.

Proposition 6.1. *(i) Let $\mathcal{L}_{S_o}$ denote a $\bar{\mathbb{Q}}_\ell$-adic G_o-equivariant local system on the G_o-orbit $\mathcal{O}_o$. Assume $\mathcal{L}_{S_o}$ corresponds to an irreducible $\bar{\mathbb{Q}}_\ell$-representation of $\pi_1(\mathrm{EG_o} \underset{G_o}{\times} \mathcal{O}_o, x_o)\widehat{\ }_\ell$ for some point $x_o\, \varepsilon\, \mathcal{O}_o$. Then the corresponding equivariant intersection cohomology complex $\mathrm{IC}^{\mathrm{G}_o}(\mathcal{L}_{S_o})$ is pure.*

(ii) Every G_o-equivariant perverse sheaf that is mixed and simple is pure.

Proof. We show in Theorem 7.2 (see also [26, Theorem 1.6]) that

$$\pi_1(\mathrm{EG_o} \underset{G_o}{\times} \mathcal{O}_o, x_o)\widehat{\ }_\ell \cong \pi_1(\mathrm{EG}_o^{gm,m} \underset{\mathrm{G_o}}{\times} \mathcal{O}_o, x_o)\widehat{\ }_\ell$$

for m large enough. Therefore, by taking an m large enough, the first statement follows from the corresponding statement in the non-equivariant case applied to the scheme $\mathrm{EG}_o^{gm,m} \underset{\mathrm{G_o}}{\times} \mathcal{O}_o$.

Next observe that all mixed and simple G_o-equivariant perverse sheaves are the equivariant intersection cohomology complexes on the orbit-closures associated to irreducible G-equivariant ℓ-adic local systems on the corresponding orbits. Therefore, (ii) follows. □

One may observe from our definition above, and the corresponding property for $\bar{\mathbb{Q}}_\ell$-sheaves on schemes of finite type over k, that the category of $\bar{\mathbb{Q}}_\ell$-sheaves of exact weight w *is closed under extensions as well as sub- and quotient objects.*

Next let K_o and L_o denote two mixed bounded complexes of $\bar{\mathbb{Q}}_\ell$-sheaves with constructible G_o-equivariant cohomology sheaves on $\mathrm{X_o}$. One may now readily verify the following (local assertions):

(1) If K_o has weights $\le w'$ and L_o has weights $\ge w$, then $\mathcal{RH}om(K_o, L_o)$ has weights $\ge w - w'$.
(2) If instead K_o has weights $\ge w'$ and L_o has weights $\le w$, then $\mathcal{RH}om(K_o, L_o)$ has weights $\le w - w'$.
(3) If K_o has weights $\le w'$ and L_o has weights $\le w$, then $K_o \otimes L_o$ has weights $\le w + w'$.
(4) Let $f_o : \mathrm{X_o} \to \mathrm{Y_o}$ denote a G_o-equivariant map and let K_o (L_o) denote a mixed bounded complex of $\bar{\mathbb{Q}}_\ell$-sheaves on $E\mathrm{G}_o \underset{\mathrm{G}_o}{\times} \mathrm{X_o}$ with weights $\ge w$ ($\le w$, respectively). Then $Rf_{o*}K_o$ ($Rf_{o!}L_o$, when it is defined) is also mixed and has weights $\ge w$ ($\le w$, respectively). Here $Rf_{o!}$ is defined by finding a G_o-equivariant compactification of f_o. Therefore, if in addition f_o is also proper, and K_o is pure of weight w, then so is $Rf_{o*}K_o$.
(5) Let Fr_{q^n} denote the geometric Frobenius *raising the coordinates to the* q^n*-th power.* One verifies that this induces a G_o-equivariant map $\mathrm{X_o} \longrightarrow \mathrm{X_o}$. Let F_o denote a $\bar{\mathbb{Q}}_\ell$-sheaf on $\mathrm{X_o}$; if F denotes the induced sheaf on X, then one may readily verify that there exists an isomorphism $(Fr_{q^n})^*F \to F$. (See [2, (5.1.1)].) Then the functor $F_o \longrightarrow (F, (Fr_q)^*)$ from the category of G_o-equivariant perverse sheaves on $\mathrm{X_o}$ to the category of G-equivariant perverse sheaves F on X provided with an isomorphism $(Fr_q)^*F \overset{\cong}{\to} F$ is fully-faithful. Moreover the *image* of the above functor is a subcategory that is closed under extensions and sub-quotients.

Let $\sigma_o : \mathrm{EG_o}^{gm,m} \underset{\mathrm{G_o}}{\times} \mathrm{X_o} \longrightarrow \mathrm{Spec}\,\mathbb{F}_\mathrm{q}$ denote the obvious structure map for some fixed m large enough. If $M_o \,\varepsilon\, D^b_{\mathrm{G}_o}(\mathrm{EG_o}^{gm,m} \underset{\mathrm{G_o}}{\times} \mathrm{X_o}, \mathcal{S}_\mathrm{o}^{\mathrm{G_o}}, \bar{\mathbb{Q}}_\ell)$, one obtains the spectral sequence:

$$\mathrm{E}_2^{p,q} = \mathrm{H}^\mathrm{p}_{\mathrm{et}}((\mathrm{Spec}\,\mathbb{F}_\mathrm{q}); \mathcal{H}^\mathrm{q}(\mathrm{M_o})) = \mathrm{R}^\mathrm{p}\sigma_{0*}(\mathcal{H}^\mathrm{q}(\mathrm{M_o})) \Rightarrow \mathrm{H}^{\mathrm{p+q}}\mathrm{R}\Gamma(\mathrm{EG}_\mathrm{o}^{\mathrm{gm,m}} \underset{\mathrm{G_o}}{\times} \mathrm{X_o}, \mathrm{M_o}) \tag{6.0.1}$$

$$= R^{p+q}\Gamma(Spec\,\mathbb{F}_q, R\sigma_{0*}(M_o)). \tag{6.0.2}$$

Since $\mathbb{F}_q$ is a finite field with q-elements, $Gal(\bar{\mathbb{F}}_q/\mathbb{F}_q) \cong \hat{\mathbb{Z}}$; therefore $\mathrm{E}_2^{p,q} = 0$ if $p \neq 0$ or 1. Hence one obtains a short-exact sequence (see [12, Chapter XV, Proposition 5.5]):

$$0 \to \mathrm{E}_\infty^{1,n-1} \to \mathrm{H}^\mathrm{n}\mathrm{R}\Gamma(\mathrm{EG}_\mathrm{o}^{\mathrm{gm,m}} \underset{\mathrm{G_o}}{\times} \mathrm{X_o}, \mathrm{M_o}) \to \mathrm{E}_\infty^{0,\mathrm{n}} \to 0. \tag{6.0.3}$$

Next, let K_o and L_o denote two bounded complexes of $\bar{\mathbb{Q}}_\ell$-sheaves with constructible G_o-equivariant cohomology sheaves on $\mathrm{EG}_o^{gm,m} \underset{\mathrm{G}_o}{\times} \mathrm{X_o}$, and let $M_o = \mathcal{RH}om(K_o, L_o)$. Let $Rhom(K, L)$ denote the complex $\mathcal{RH}om(K_o, L_o)$ pulled back to $Spec\,\bar{\mathbb{F}}_q$, and let $\mathrm{Hom}(K_o, L_o) = \mathrm{H}^0 Rhom(K_o, L_o)$ as in [2, section 5.1]. The short-exact sequence in (6.0.3) now becomes:

$$0 \to (\mathrm{Ext}^{n-1}(K, L))_{Fr} \to \mathrm{Ext}^n(K_o, L_o) \to \mathrm{Ext}^n(K, L)^{Fr} \to 0, \tag{6.0.4}$$

where $(\mathrm{Ext}^{n-1}(K, L))_{Fr}$ $(\mathrm{Ext}^n(K, L)^{Fr})$ denotes the co-invariants (the invariants, respectively) under the action of the Galois group $Gal(\bar{\mathbb{F}}_q/\mathbb{F}_q)$, or equivalently under the Frobenius Fr_q.

One may now observe from Proposition 3.6(i') that if K and L are G-equivariant perverse sheaves then $\mathrm{Ext}^i(K, L) = 0$ if $i < 0$. Therefore, taking $n = 0$, the short-exact sequence in (6.0.4) now provides the isomorphism

$$\mathrm{Hom}(K_\mathrm{o}, \mathrm{L_o}) \cong \mathrm{Hom}(K, L)^{Fr}, \tag{6.0.5}$$

provided K_o and L_o are G_o-equivariant perverse sheaves. Taking $n = 1$, the short-exact sequence (6.0.4) also provides the short-exact sequence

$$0 \to (\mathrm{Hom}(K, L))_{Fr} \to \mathrm{Ext}^{\mathit{1}}(K_o, L_o) \to \mathrm{Ext}^{\mathit{1}}(K, L)^{Fr} \to \mathit{0}. \tag{6.0.6}$$

Finally we make the following important observation: *assume in addition that* K_o *and* L_o *are pure with the weight of* K_o *less than or equal to the weight of* L_o. Then

$$\mathrm{Ext}^1(K, L)^{Fr} = 0. \tag{6.0.7}$$

To see this, first observe that $\mathrm{Ext}^1(K, L) = \mathcal{H}^1(\epsilon^* \mathcal{RH}om(K_o, L_o))$. This has strictly positive weights. Therefore, the conclusion in (6.0.7) follows.

Next we recall the definition of a *mixed category* from [3, Definition 4.1.1]. Let $\mathbf{M}$ denote an abelian category which is Artinian. Let $Irr(\mathbf{M})$ denote the isomorphism classes of irreducible objects in $\mathbf{M}$. Such a category $\mathbf{M}$ is a *mixed category* if there is given a map

$$w : Irr(\mathbf{M}) \to \mathbb{Z}$$

called *weight*, so that for any two objects $M, N \,\varepsilon\, Irr(\mathbf{M})$,

$$\mathrm{Ext}^1_{\mathbf{M}}(M, N) = 0 \text{ if } w(M) \leq w(N).$$

Let $(\mathbf{M}, w)$ denote a mixed category. An object $M \in \mathbf{M}$ is *pure of weight* i, if all its irreducible components have weight i. Any such object is *semi-simple*. It is known (see [3, Lemma 4.1.2]) that any object L in $\mathbf{M}$ has a unique finite increasing filtration $W_\bullet$, so that $gr_i^W(L) = W_i L / W_{i-1} L$ is pure of weight i, for all $i \,\varepsilon\, \mathbb{Z}$.

Proposition 6.2. *Let* G_o *denote a linear algebraic group acting on a scheme* $\mathrm{X_o}$ *(of finite type over* $\mathbb{F}_q$*). Then, the category of mixed* G_o*-equivariant perverse sheaves* P_o *(with respect to the given* G_o*-stable stratification,* $\mathcal{S}_{\mathrm{G}_o}$*), so that each* $grd_i^{\mathrm{W}}(\mathrm{P_o})$ *is semi-simple, is a mixed category.*

Proof. Let $\mathrm{P}_o, \mathrm{Q_o}$ denote two simple G_o-equivariant perverse sheaves with the weight of $\mathrm{P}_o <$ the weight of $\mathrm{Q_o}$. Then it is easy to see that both $\mathrm{Hom(P,Q)_{Fr}}$ and $\mathrm{Ext}^1\mathrm{(P,Q)^{Fr}}$ are trivial so that $\mathrm{Ext}^1(\mathrm{P_o}, \mathrm{Q_o}) = 0$. Next suppose P_o and $\mathrm{Q_o}$ have the same weights. Since we are considering extensions in the above category, an extension $0 \to \mathrm{Q_o} \to \mathrm{R_o} \to \mathrm{P_o} \to 0$ will be such that $\mathrm{R_o}$ is also pure of the same weight as P_o and $\mathrm{Q_o}$, so that the above extension splits. Therefore, for two simple G_o-equivariant perverse sheaves P_o and $\mathrm{Q_o}$ with the weight of $\mathrm{P}_o \leq$ the weight of $\mathrm{Q_o}$, $\mathrm{Ext}^1(\mathrm{P_o}, \mathrm{Q_o}) = 0$ proving the proposition. □

Remark 6.3. The above mixed categories come equipped with the usual *Tate-twist* (see [3, Definition 4.1.4]) as well as the notion of gradings and a degrading functor: see [3, 4.3]. The Tate-twit (d) has the property that $w(M(d)) = w(M) + d$, for each simple object M.

Henceforth, we will impose the following strong condition:

$$\mathcal{H}^i(\mathrm{IC}^{\mathrm{G}_o}(\mathcal{L}_o[d_{S_o}])) = 0 \text{ if } i + d_{S_o} \text{ is odd and } \mathrm{IC}^{\mathrm{G}_o}(\mathcal{L}_{S_o}[d_{S_o}]) \text{ is pure of weight } d_{S_o} = dim(S_o) \tag{6.0.8}$$

Here $\mathrm{IC}^{\mathrm{G}_o}(\mathcal{L}_{S_o}[d_{S_o}])$ denotes the G_o-equivariant intersection cohomology complex on the closure of the stratum S_o of dimension d_{S_o}, and obtained by starting with the G_o-equivariant local system $\mathcal{L}_{S_o}$ on S.

The above conditions have been verified for large classes of complex spherical varieties in [9] and [10] and extended in Theorem 1.6 to positive characteristics.

Remark 6.4. In view of our assumptions that the base field is perfect, the stabilizer groups are all defined over the same base field, and so are their connected components: see [35, 12.1.1 and 12.1.2].

Definition 6.5. We will let $\tilde{\mathcal{P}}^{\mathrm{G}_o}_{mixed}$ denote the full subcategory of the above category of G_o-equivariant perverse sheaves consisting of those P_o so that for each $j \,\varepsilon\, \mathbb{Z}$, $grd^j_W(P_o)$, is a finite sum of equivariant intersection cohomology complexes $\mathrm{IC}^{\mathrm{G}_o}(\mathcal{L}_{S_o}[d_{S_o}])((d_{S_o} - j)/2)$ if $(d_{S_o} - j)$ is even and trivial otherwise.

Remark 6.6. Now the functor sending a mixed G_o-equivariant perverse sheaf to the underlying G_o-equivariant perverse sheaf is a degrading functor $v : \tilde{\mathcal{P}}^{\mathrm{G}_o}_{mixed} \to \mathcal{P}^{\mathrm{G}_o}$ in the sense of [3, 4.3]. All of these readily follow from the well-known results for schemes, since we are using the finite degree approximations of EG to define equivariant derived categories.

Proposition 6.7. *We will assume as before that* G_o *is a linear algebraic group acting on the given scheme* $\mathrm{X_o}$ *with finitely many orbits and provided with the* G_o*-stable stratification by orbits.*

Let $\mathrm{IC}^{\mathrm{G}_o}(\mathcal{L}_\alpha[d_\alpha])$, $\mathrm{IC}^{\mathrm{G}_o}(\mathcal{L}_\beta[d_\beta])$ *denote the* G_o*-equivariant intersection cohomology complexes on the strata* $S_{\alpha,o}$ *and* $S_{\beta,o}$ *associated to the irreducible* G*-equivariant local systems* $\mathcal{L}_\alpha$ *and* $\mathcal{L}_\beta$*, respectively. Denoting by* $\mathrm{IC}^{\mathrm{G}}(\mathcal{L}_\alpha[d_\alpha])$ *and* $\mathrm{IC}^{G}(\mathcal{L}_\beta[d_\beta])$*, the corresponding equivariant -intersection cohomology complexes on* $\mathrm{EG}\times_{\mathrm{G}}\mathrm{X}$*, we obtain:*

$$\mathrm{Ext}^1(\mathrm{IC}^{\mathrm{G}}(\mathcal{L}_\alpha[d_\alpha]),\mathrm{IC}^{\mathrm{G}}(\mathcal{L}_\beta[d_\beta]) = 0 \textit{ if } d_\alpha - d_\beta \textit{ is even and}$$
$$\mathrm{Ext}^1(\mathrm{IC}^{\mathrm{G}}(\mathcal{L}_\alpha[d_\alpha]),\mathrm{IC}^{\mathrm{G}}(\mathcal{L}_\beta[d_\beta]) \textit{ is pure of weight } d_\beta - d_\alpha + 1 \textit{ if } d_\alpha - d_\beta \textit{ is odd.}$$

(Here the last statement means that, if $d_\alpha - d_\beta$ *is odd, the geometric Frobenius acts on* $\mathrm{Ext}^1(\mathrm{IC}^{\mathrm{G}}(\mathcal{L}_\alpha[d_\alpha]),\mathrm{IC}^{\mathrm{G}}(\mathcal{L}_\beta[d_\beta]))$ *with the eigen-values having absolute value* $q^{(d_\beta-d_\alpha+1)/2}$*.)*

Proof. Suppose the stratum $S_{\beta,o}\subseteq \bar S_{\alpha,o}-S_{\alpha,o}$. In this case, observe that

$$\begin{aligned}\mathrm{D}(\mathrm{IC}^{\mathrm{G}_o}(\mathcal{L}_\alpha[d_\alpha])) &\simeq \mathrm{IC}^{\mathrm{G}_o}(\mathcal{L}_\alpha^\vee[d_\alpha])(d_\alpha) \text{ and similarly } \mathrm{D}(\mathrm{IC}^{\mathrm{G}_o}(\mathcal{L}_\beta[d_\beta]))\\ &\simeq \mathrm{IC}^{\mathrm{G}_o}(\mathcal{L}_\beta^\vee[d_\beta])(d_\beta).\end{aligned}$$

Therefore,

$$\begin{aligned}&\mathrm{Ext}^1(\mathrm{IC}^{\mathrm{G}_o}(\mathcal{L}_\alpha[d_\alpha]),\mathrm{IC}^{\mathrm{G}_o}(\mathcal{L}_\beta[d_\beta]) = \mathrm{Ext}^1(\mathrm{D}(\mathrm{IC}^{\mathrm{G}_o}(\mathcal{L}_\beta[d_\beta])),\mathrm{D}(\mathrm{IC}^{\mathrm{G}_o}(\mathcal{L}_\alpha[d_\alpha])))\\ &\quad= \mathrm{Ext}^1(\mathrm{IC}^{\mathrm{G}_o}(\mathcal{L}_\beta^\vee[d_\beta])(d_\beta),\mathrm{IC}^{\mathrm{G}_o}(\mathcal{L}_\alpha^\vee[d_\alpha])(d_\alpha)\\ &\quad= \mathrm{Ext}^1(\mathrm{IC}^{\mathrm{G}_o}(\mathcal{L}_\beta^\vee[d_\beta]),\mathrm{IC}^{\mathrm{G}_o}(\mathcal{L}_\alpha^\vee[d_\alpha]))(d_\alpha-d_\beta).\end{aligned}$$

Therefore, in case $S_{\beta,o}\subseteq \bar S_{\alpha,o}-S_{\alpha,o}$, one may interchange $S_{\alpha,o}$ and $S_{\beta,o}$ and assume that $S_{\beta,o}$ is not contained in $\bar S_{\alpha,o}-S_{\alpha,o}$.

Next one may readily prove as in the non-equivariant case that the natural map $j^p_{S_{\alpha,\mathrm{G}_o}!}(\mathcal{L}[d_\alpha])\to \mathrm{IC}^{\mathrm{G}_o}(\mathcal{L}_\alpha[d_\alpha])$ is an epimorphism in the category $\mathcal{P}_{\mathrm{G_o}}(\mathrm{X_o},\mathcal{S}_{\mathrm{o},\mathrm{G_o}})$. Let $K_{\alpha,o} = kernel(j^p_{S_{\alpha,\mathrm{G}_o}!}(\mathcal{L}[d_\alpha])\to \mathrm{IC}^{\mathrm{G}_o}(\mathcal{L}_\alpha[d_\alpha]))$. Then the short exact sequence

$$K_{\alpha,o}\to j^p_{S_{\alpha,\mathrm{G}_o}!}(\mathcal{L}[d_\alpha])\to \mathrm{IC}^{\mathrm{G}_o}(\mathcal{L}_\alpha[d_\alpha]))$$

in the abelian category $\mathcal{P}_{\mathrm{G_o}}(\mathrm{X_o},\mathcal{S}_{\mathrm{o},\mathrm{G_o}})$ is a distinguished triangle in $\mathrm{D}_{\mathrm{G_o}}(\mathrm{X_o},\mathcal{S}_{\mathrm{o},\mathrm{G_o}})$. Let $i_{\alpha,o}:\bar S_{\alpha,o}-S_{\alpha,o}\to \bar S_{\alpha,o}$ denote the obvious closed immersion. Then $K_{\alpha,o}$ has supports in $\bar S_{\alpha,o}-S_{\alpha,o}$, so that one may write $K_{\alpha,o}=i_{\alpha*}(K'_{\alpha,o})$. Denoting by i_α the corresponding immersion $\bar S_\alpha - S_\alpha\to \bar S_\alpha$, and by K_α the corresponding complex on $\bar S_\alpha$, we will prove:

$$\mathrm{Hom}(K_\alpha,\mathrm{IC}^{\mathrm{G}}(\mathcal{L}_\beta[d_\beta]) = \mathrm{Hom}(\mathrm{i}_{\alpha*}(K'_\alpha),\mathrm{IC}^{\mathrm{G}}(\mathcal{L}_\beta[d_\beta]) = \mathrm{Hom}(K'_\alpha, Ri^!_\alpha(\mathrm{IC}^{\mathrm{G}}(\mathcal{L}_\beta[d_\beta]))) = 0. \tag{6.0.9}$$

To see this, first observe that K_α is a perverse sheaf and so that if $i_\gamma : S_\gamma \to \bar{S}_\alpha - S_\alpha$ is the inclusion of a stratum, then $K'_\gamma = i^*_\gamma(K_\alpha) \,\varepsilon\, \mathrm{D(S_\gamma)}^{\leq \mathrm{m(S_\gamma)}}$. On the other hand, $Ri^!_\gamma(\mathrm{IC^G}(\mathcal{L}_\beta[d_\beta])) \,\varepsilon\, \mathrm{D(S_\gamma)}^{\geq \mathrm{m(S_\gamma)}+1}$, if $S_\gamma \subseteq \bar{S}_\beta$. The other case to consider is when $S_\gamma \cap \bar{S}_\beta$ is empty, in which case $Ri^!_\gamma(\mathrm{IC^G}(\mathcal{L}_\beta[d_\beta])) = 0$. Therefore, if $\bar{S}_{\alpha,o} - S_{\alpha,o} = \mathcal{S}_{\gamma,o}$ is a single stratum, we have proven (6.0.9). Since the stratum $S_{\beta,o}$ is not contained in $\bar{S}_{\alpha,o} - S_{\alpha,o}$, the only possibilities are that $\bar{S}_{\beta,o} \cap (\bar{S}_{\alpha,o} - S_{\alpha,o})$ is either empty or is a union of strata contained in both.

In general, one uses induction on the number of strata contained in $\bar{S}_{\alpha,o} - S_{\alpha,o}$. Let $j_\gamma : S_\gamma \to \bar{S}_\alpha - S_\alpha$ denote the open immersion of a stratum and let $i_\delta : S_\delta \to \bar{S}_\alpha - S_\alpha$ denote the closed immersion of its complement. Then one obtains the following exact sequence (as part of a long exact sequence):

$$\begin{aligned}\mathrm{Hom}(K'_\alpha, i_{\delta *} Ri^!_\delta Ri^!_\alpha(\mathrm{IC^G}(\mathcal{L}_\beta[d_\beta]))) &\to \mathrm{Hom}(K'_\alpha, Ri^!_\alpha(\mathrm{IC^G}(\mathcal{L}_\beta[d_\beta])))\\ &\to \mathit{Hom}(K'_\alpha, Rj_{\gamma *} j^*_\gamma Ri^!_\alpha(\mathrm{IC^G}(\mathcal{L}_\beta[d_\beta]))).\end{aligned}$$

Then one obtains the identifications:

$$\begin{gathered}\mathrm{Hom}(K'_\alpha, i_{\delta *} Ri^!_\delta Ri^!_\alpha(\mathrm{IC^G}(\mathcal{L}_\beta[d_\beta]))) \cong \mathrm{Hom}(\mathrm{i}^*_\delta K'_\alpha, Ri^!_\delta Ri^!_\alpha(\mathrm{IC^G}(\mathcal{L}_\beta[d_\beta]))) \text{ and}\\ \mathrm{Hom}(K'_\alpha, Rj_{\gamma *} j^*_\gamma Ri^!_\alpha(\mathrm{IC^G}(\mathcal{L}_\beta[d_\beta]))) \cong \mathrm{Hom}(\mathrm{j}^*_\gamma K'_\alpha, j^*_\gamma Ri^!_\alpha(\mathrm{IC^G}(\mathcal{L}_\beta[d_\beta]))).\end{gathered}$$

Now $j^*_\gamma(K'_\alpha) = j^*_\gamma(i^*_\alpha(K_\alpha)) \,\varepsilon\, D(S_\gamma)^{\leq m(S_\gamma)}$ while $j^*_\gamma Ri^!_\alpha(\mathrm{IC^{G_o}}(\mathcal{L}_\beta[d_\beta])) \,\varepsilon\, D(S_\gamma)^{\geq m(S_\gamma)+1}$ so that the last Hom will be trivial. (One may observe here that $R(i_\alpha \circ j_\gamma)^! = j^*_\gamma Ri^!_\alpha$.) One may prove by ascending induction on the number of strata in $\bar{S}_{\alpha,o} - S_{\alpha,o}$ that

$$\mathrm{Hom}(\mathrm{i}^*_\delta K'_\alpha, Ri^!_\delta Ri^!_\alpha(\mathrm{IC^{G_o}}(\mathcal{L}_\beta[d_\beta]))) = 0.$$

Therefore, the middle term, $\mathrm{Hom}(K'_\alpha, Ri^!_\alpha(\mathrm{IC^{G_o}}(\mathcal{L}_\beta[d_\beta]))) = 0$ as well.

Therefore, one observes that the induced map

$$\mathrm{Ext}^1(\mathrm{IC^G}(\mathcal{L}_\alpha[d_\alpha]), \mathrm{IC^G}(\mathcal{L}_\beta[d_\beta])) \to \mathrm{Ext}^1(j^p_{S_\alpha !}(\mathcal{L}_\alpha[d_\alpha]), \mathrm{IC^G}(\mathcal{L}_\beta[d_\beta]))$$

is an injection. Now Proposition 3.6(iii') shows that there is a natural *injective* map from the last group to $\mathrm{Ext}^1(j_{S_\alpha !}(\mathcal{L}_\alpha[d_\alpha], \mathrm{IC^G}(\mathcal{L}_\beta[d_\beta])))$: observe that $\mathrm{IC^G}(\mathcal{L}_\beta[d_\beta]))$ is already a perverse sheaf. Now one may make use of the adjunction between $j_{S_\alpha !}$ and $Rj^!_{S_\alpha}$ to obtain the identification:

$$\begin{aligned}\mathrm{Ext}^1(j_{S_\alpha !}(\mathcal{L}_\alpha[d_\alpha]), \mathrm{IC^G}(\mathcal{L}_\beta[d_\beta])) &\cong \mathrm{Ext}^1(\mathcal{L}_\alpha[d_\alpha], Rj^!_{S_\alpha}(\mathrm{IC^G}(\mathcal{L}_\beta[d_\beta]))) \qquad (6.0.10)\\ &\cong \mathrm{H}^1(\mathrm{BG}^0_x, i^*_x \mathcal{H}\mathrm{om}(\mathcal{L}_\alpha, \mathrm{Rj}^!_{\mathrm{S}_\alpha}(\mathrm{IC^G}(\mathcal{L}_\beta[\mathrm{d}_\beta - \mathrm{d}_\alpha])))^{\mathrm{G}_x/\mathrm{G}^0_x})\end{aligned}$$

where $i_x : x \to S_\alpha$ is the immersion of a closed point. Observe that $Rj^!_{S_\alpha}(\mathrm{IC^G}(\mathcal{L}_\beta[d_\beta - d_\alpha])) \cong Dj^*_{S_\alpha} D(\mathrm{IC^G}(\mathcal{L}_\beta[d_\beta - d_\alpha]))$ and therefore $\mathcal{H}^v(Rj^!_{S_\alpha}(\mathrm{IC^G}(\mathcal{L}_\beta[d_\beta - d_\alpha])))$ is pure of weight $d_\beta - d_\alpha + v$ unless it is trivial. Since taking the $\mathrm{G}_x/\mathrm{G}^0_x$-invariants is an exact functor, and $\mathcal{L}_\alpha[d_\alpha]$ is assumed to be pure of weight d_α, it follows that $\mathcal{H}^v(i_x^*\mathcal{H}om(\mathcal{L}_\alpha, Rj^!_{S_\alpha}(\mathrm{IC^G}(\mathcal{L}_\beta[d_\beta - d_\alpha])))^{\mathrm{G}_x/\mathrm{G}_x^0})$ is also pure of weight $d_\beta - d_\alpha + v$ unless it is trivial.

Next we consider the spectral sequence

$$\begin{aligned}\mathrm{E}_2^{u,v} &= \mathrm{H}^u(\mathrm{BG}_x^0, \mathcal{H}^v(i_x^*\mathcal{H}\mathrm{om}(\mathcal{L}_\alpha, \mathrm{Rj}^!_{\mathrm{S}_\alpha}(\mathrm{IC^G}(\mathcal{L}_\beta[\mathrm{d}_\beta - \mathrm{d}_\alpha])))^{\mathrm{G}_x/\mathrm{G}_x^0})) \qquad (6.0.11)\\ &\Rightarrow \mathrm{H}^{u+v}(\mathrm{BG}_x^0, i_x^*\mathcal{H}\mathrm{om}(\mathcal{L}_\alpha, \mathrm{Rj}^!_{\mathrm{S}_\alpha}(\mathrm{IC^G}(\mathcal{L}_\beta[\mathrm{d}_\beta - \mathrm{d}_\alpha])))^{\mathrm{G}_x/\mathrm{G}_x^0}).\end{aligned}$$

As G_x^0 is connected, BG_x^0 is simply connected, and therefore,

$$\mathrm{E}_2^{u,v} \cong \mathrm{H}^u(\mathrm{BG}_x^0, \mathbb{Q}_\ell) \otimes \mathcal{H}^v(\mathrm{i}_x^*\mathcal{H}\mathrm{om}(\mathcal{L}_\alpha, \mathrm{Rj}^!_{\mathrm{S}_\alpha}(\mathrm{IC^G}(\mathcal{L}_\beta[\mathrm{d}_\beta - \mathrm{d}_\alpha])))^{\mathrm{G}_x/\mathrm{G}_x^0}).$$

From this one may derive the following consequences:

(i) Since the differentials in the above spectral sequence $d^r : \mathrm{E}_\mathrm{r}^{u,v} \to \mathrm{E}_2^{u+r,v-r+1}$, while $\mathrm{H^i}(\mathrm{BG}_x^0, \mathbb{Q}_\ell) = 0$ for all odd i, and the assumption (6.0.8) holds, the spectral degenerates at E_2.
(ii) As shown in [11, Theorem 1.15], $\mathrm{H^i}(\mathrm{BG}_x^0, \mathbb{Q}_\ell)$ is pure of weight i, for i even.

Therefore, it follows that the $\mathrm{E}_2^{u,v}$-terms for $u+v = 1$ are all pure of weight $d_\beta - d_\alpha + 1$ if $d_\beta - d_\alpha$ is odd. Therefore, the same conclusion holds for $\mathrm{Ext}^1(j_{S_\alpha !}(\mathcal{L}_\alpha[d_\alpha]), \mathrm{IC^G}(\mathcal{L}_\beta[d_\beta]))$. The assertion on the vanishing of the Ext^1 if $d_\beta - d_\alpha$ is even, now follows from the hypothesis (6.0.8). □

Lemma 6.8. *(See [3, Lemma 4.2.1].) Let C be any Artinian category. Denote by $\{L\}$ a collection of representatives of isomorphism classes of irreducible objects in C, and let $F_L = End(L)$. Let $\mathrm{P}\varepsilon C$ denote an object equipped with a filtration $\mathrm{P} = \mathrm{P}^0 \supseteq \mathrm{P}^1 \supseteq \cdots \supseteq \mathrm{P}^\mathrm{N} = 0$ for $N \geq 0$, with semi-simple successive quotients. Let $Irr(C)$ denote the set of irreducible objects in C. Then the following properties are equivalent:*

(i) P *is a projective object and the filtration is the radical filtration. (This means that for each i, $\mathrm{P^i}/\mathrm{P^{i+1}}$ is the largest semi-simple quotient of* $\mathrm{P^i}$ *in C: see [3, 2.4].)*
(ii) *For any $i \geq 1$ and any semi-simple object $M\varepsilon C$ the map* $\mathrm{Hom}(\mathrm{P^i}/\mathrm{P^{i+1}}, \mathrm{M}) \to \mathrm{Ext}^1(\mathrm{P}/\mathrm{P^i}, \mathrm{M})$ *coming from the short exact sequence* $\mathrm{P^i}/\mathrm{P^{i+1}} \to \mathrm{P}/\mathrm{P^{i+1}} \to \mathrm{P}/\mathrm{P^i}$ *is bijective.*
(iii) *For any $i \geq 1$ and L an irreducible object in C, the group $E^i(L) = \mathrm{Ext}^1(\mathrm{P}/\mathrm{P^i}, \mathrm{L})$ is a finitely generated F_L -module, $E^i(L) = 0$ for all but finitely many L, and there exists an isomorphism*

$$\mathrm{P^i}/\mathrm{P^{i+1}} \to \oplus_{\mathrm{L}\varepsilon\mathrm{Irr(C)}}\mathrm{E^i(L)}^* \otimes_{\mathrm{F_L}} \mathrm{L}, \ \textit{where}\ \mathrm{E^i(L)}^* := \mathrm{Hom}_{\mathrm{F_L}}(\mathrm{E^i(L)}, \mathrm{F_L}),$$

such that under the chain of isomorphisms $\mathrm{Ext}^1(\mathrm{P}/\mathrm{P}^{\mathrm{i}}, \mathrm{P}^{\mathrm{i}}/\mathrm{P}^{\mathrm{i}+1}) \to \mathrm{Ext}^1(\mathrm{P}/\mathrm{P}^{\mathrm{i}}, \oplus_{\mathrm{L}}\mathrm{E}^{\mathrm{i}}(\mathrm{L})^* \otimes_{\mathrm{F_L}} \mathrm{L}) = \oplus_{\mathrm{L}}\mathrm{E}^{\mathrm{i}}(\mathrm{L})^* \otimes_{\mathrm{F_L}} \mathrm{E}^{\mathrm{i}}(\mathrm{L}) = \oplus_{\mathrm{L}}\mathrm{End}_{\mathrm{F_L}}(\mathrm{E}^{\mathrm{i}}(\mathrm{L}))$*, the class of the extension* $\mathrm{P}^i/\mathrm{P}^{\mathrm{i}+1} \to \mathrm{P}/\mathrm{P}^{\mathrm{i}+1} \to \mathrm{P}/\mathrm{P}^{\mathrm{i}}$ *goes to* $\Sigma_L id_{E^i(L)}$.

Proof of Theorem 1.8. The proof makes use of Lemma 6.8, Theorem 3.12 and Proposition 6.7. We start with the short-exact sequence (6.0.6):

$$0 \to (\mathrm{Hom}(K, L))_{Fr} \to \mathrm{Ext}^1(K_o, L_o) \to \mathrm{Ext}^1(K, L)^{Fr} \to 0. \tag{6.0.12}$$

Let $\mathrm{P}_o \,\varepsilon\, \mathcal{P}^{\mathrm{G}_o}$ denote an object so that the associated object $\mathrm{P} \,\varepsilon\, \mathcal{P}^G$ is an indecomposable projective. Let $\mathrm{P}_o = \mathrm{P}^0_{\mathrm{o}} \supseteq \mathrm{P}^1_{\mathrm{o}} \supseteq \mathrm{P}^2_{\mathrm{o}} \supseteq \cdots \supseteq$ denote a descending filtration on P_o so that the associated graded terms are semi-simple. The goal is to find a lift $\tilde{\mathrm{P}}_o$ of P_o so that P^i_o is the underlying perverse sheaf associated to $\tilde{\mathrm{P}}^i_o$ which belongs to $\tilde{\mathcal{P}}^{\mathrm{G}_o}_{mixed}$. We do this by inductively lifting $\mathrm{P}_o/\mathrm{P}^{\mathrm{i}}_{\mathrm{o}}$ to $\widetilde{\mathrm{P_o}/\mathrm{P}^{\mathrm{i}}_{\mathrm{o}}} \,\varepsilon\, \tilde{\mathcal{P}}^{\mathrm{G}_o}_{mixed}$ so that

(a) the weights of $\widetilde{\mathrm{P_o}/\mathrm{P}^{\mathrm{i}}_{\mathrm{o}}}$ are $\geq w - i + 1$ for some $w \,\varepsilon\, \mathbb{Z}$ and so that

(b) $\widetilde{\mathrm{P_o}/\mathrm{P}^{\mathrm{i}+1}_{\mathrm{o}}}$ is an extension of $\widetilde{\mathrm{P_o}/\mathrm{P}^{\mathrm{i}}_{\mathrm{o}}}$ by a pure perverse sheaf $\widetilde{\mathrm{P}^{\mathrm{i}}_{\mathrm{o}}/\mathrm{P}^{\mathrm{i}+1}_{\mathrm{o}}}$.

We may start the induction, since $\mathrm{P}_o/\mathrm{P}^1_{\mathrm{o}}$ is irreducible (that is, simple) and hence admits a lifting to $\widetilde{\mathrm{P_o}/\mathrm{P}^1_{\mathrm{o}}} \,\varepsilon\, \mathcal{P}^{\mathrm{G}_o}_{mixed}$: see Proposition 6.1 (ii). Let w denote the weight of $\widetilde{\mathrm{P_o}/\mathrm{P}^1_{\mathrm{o}}}$ and assume we have found a lifting $\widetilde{\mathrm{P_o}/\mathrm{P}^{\mathrm{i}}_{\mathrm{o}}}$ so that its weights are $\geq w - i + 1$.

Recall that we omit subscript o to denote an object defined on $\mathrm{EG} \times_{\mathrm{G}} \mathrm{X}$. Next we consider the isomorphism

$$\mathrm{P}^{\mathrm{i}}/\mathrm{P}^{\mathrm{i}+1} \to \oplus_\alpha \mathrm{Ext}^1(\mathrm{P}/\mathrm{P}^{\mathrm{i}}, \mathrm{IC}^{\mathrm{G}}(\mathcal{L}_\alpha[\mathrm{d}_\alpha]))^* \otimes_{\bar{\mathbb{Q}}_\ell} \mathrm{IC}^{\mathrm{G}}(\mathcal{L}_\alpha[\mathrm{d}_\alpha]) \tag{6.0.13}$$

from Lemma 6.8(iii). It follows from Lemma 6.8(ii) that the map

$$\mathrm{Hom}(\mathrm{P}^{\mathrm{i}-1}/\mathrm{P}^{\mathrm{i}}, \mathrm{IC}^{\mathrm{G}}(\mathcal{L}_\alpha[\mathrm{d}_\alpha])) \to \mathrm{Ext}^1(\mathrm{P}/\mathrm{P}^{\mathrm{i}-1}, \mathrm{IC}^{\mathrm{G}}(\mathcal{L}_\alpha[\mathrm{d}_\alpha]))$$

coming from the exact sequence $\mathrm{P}^{i-1}/\mathrm{P}^{\mathrm{i}} \to \mathrm{P}/\mathrm{P}^{\mathrm{i}} \to \mathrm{P}/\mathrm{P}^{\mathrm{i}-1}$ is an isomorphism. Therefore, one has a long-exact sequence:

$$\begin{aligned}\mathrm{Hom}(\mathrm{P}^{\mathrm{i}-1}/\mathrm{P}^{\mathrm{i}}, \mathrm{IC}^{\mathrm{G}}(\mathcal{L}_\alpha[\mathrm{d}_\alpha]) \xrightarrow{\cong} \mathrm{Ext}^1(\mathrm{P}/\mathrm{P}^{\mathrm{i}-1}, \mathrm{IC}^{\mathrm{G}}(\mathcal{L}_\alpha[\mathrm{d}_\alpha])) \to \mathrm{Ext}^1(\mathrm{P}/\mathrm{P}^{\mathrm{i}}, \mathrm{IC}^{\mathrm{G}}(\mathcal{L}_\alpha[\mathrm{d}_\alpha]))\\ \to \mathrm{Ext}^1(\mathrm{P}^{\mathrm{i}-1}/\mathrm{P}^{\mathrm{i}}, \mathrm{IC}^{\mathrm{G}}(\mathcal{L}_\alpha[\mathrm{d}_\alpha])).\end{aligned} \tag{6.0.14}$$

Since the first map is an isomorphism, the second map must be the zero map and therefore, the last map must be an injection.

By our inductive hypotheses, $\widetilde{\mathrm{P}^{\mathrm{i}-1}_{\mathrm{o}}/\mathrm{P}^{\mathrm{i}}_{\mathrm{o}}}$ is pure and therefore semi-simple as an object of $\tilde{\mathcal{P}}^{\mathrm{G}_o}_{mixed}$ and of weight $w-i+1$. By Proposition 6.7, $\mathrm{Ext}^1(\mathrm{P}^{\mathrm{i}-1}/\mathrm{P}^{\mathrm{i}}, \mathrm{IC}^{\mathrm{G}}(\mathcal{L}_\alpha[\mathrm{d}_\alpha]))$ is zero if $d_\alpha - w + i$ is odd and is pure of weight $d_\alpha - w + i$ if this is even. In view of the injection in (6.0.14), it follows that the same conclusion holds for $\mathrm{Ext}^1(\mathrm{P}/\mathrm{P}^{\mathrm{i}}, \mathrm{IC}^{\mathrm{G}}(\mathcal{L}_\alpha[\mathrm{d}_\alpha]))$. Therefore, $\oplus_\alpha \mathrm{Ext}^1(\mathrm{P}/\mathrm{P}^{\mathrm{i}}, \mathrm{IC}^{\mathrm{G}}(\mathcal{L}_\alpha[\mathrm{d}_\alpha]))^* \otimes_{\mathbb{Q}_\ell} \mathrm{IC}^{\mathrm{G}}(\mathcal{L}_\alpha[\mathrm{d}_\alpha])$ defines a pure object in $\widetilde{\tilde{\mathcal{P}}^{\mathrm{G}_o}_{mixed}}$ of weight $w-i$. Using the identification in (6.0.13), this defines a pure structure $\mathrm{P}^{\mathrm{i}}_{\mathrm{o}}/\mathrm{P}^{\mathrm{i+1}}_{\mathrm{o}}$ on $\mathrm{P}^{i}_{o}/\mathrm{P}^{\mathrm{i+1}}_{\mathrm{o}}$ of weight $w-i$. Finally, one concludes using Lemma (6.8)(iii) that the class of the extension $\mathrm{P}^{i}/\mathrm{P}^{\mathrm{i+1}} \to \mathrm{P}/\mathrm{P}^{\mathrm{i+1}} \to \mathrm{P}/\mathrm{P}^{\mathrm{i}}$ is invariant under the action of the Frobenius on $\mathrm{Ext}^1(\mathrm{P}/\mathrm{P}^{\mathrm{i}}, \mathrm{P}^{\mathrm{i}}/\mathrm{P}^{\mathrm{i+1}})$ defined by the mixed structures on $\widetilde{\mathrm{P}^{\mathrm{i}}/\mathrm{P}^{\mathrm{i+1}}}$ and $\widetilde{\mathrm{P}/\mathrm{P}^{\mathrm{i}}}$ and therefore comes from an extension $\widetilde{\mathrm{P}^{\mathrm{i}}_{\mathrm{o}}/\mathrm{P}^{\mathrm{i+1}}_{\mathrm{o}}} \to \widetilde{\mathrm{P}_{\mathrm{o}}/\mathrm{P}^{\mathrm{i+1}}_{\mathrm{o}}} \to \widetilde{\mathrm{P}_{\mathrm{o}}/\mathrm{P}^{\mathrm{i}}_{\mathrm{o}}}$. (See, for example, the proof of [3, Lemma 4.4.8].) Moreover, by our assumption, the weight of $\widetilde{\mathrm{P}_{\mathrm{o}}/\mathrm{P}^{\mathrm{i}}_{\mathrm{o}}} \geq w-i+1$, while $\widetilde{\mathrm{P}^{\mathrm{i}}_{\mathrm{o}}/\mathrm{P}^{\mathrm{i+1}}_{\mathrm{o}}}$ is pure of weight $w-i$. Therefore, $\widetilde{\mathrm{P}_{\mathrm{o}}/\mathrm{P}^{\mathrm{i+1}}_{\mathrm{o}}} \,\varepsilon\, \tilde{\mathcal{P}}^{\mathrm{G}_o}_{mixed}$, so that P_o has been lifted to $\tilde{\mathrm{P}}_o \in \tilde{\mathcal{P}}^{\mathrm{G}_o}_{mixed}$ and this completes the inductive step and therefore the proof of (i) in Theorem 1.8.

We will next prove the second statement in Theorem 1.8, that is, we prove $\tilde{\mathrm{P}}_o \in \tilde{\mathcal{P}}^{\mathrm{G}_o}_{mixed}$ is a projective object. Recall that $\tilde{\mathcal{P}}^{\mathrm{G}_o}_{mixed}$ denotes the subcategory of mixed G-equivariant perverse sheaves so that the objects are mixed objects and so that the associated graded terms are semi-simple. Therefore $\mathrm{Ext}^1_{\tilde{\mathcal{P}}^{\mathrm{G}_o}_{mixed}}(\tilde{\mathrm{P}}_o, \mathrm{IC}^{\mathrm{G}_o}(\mathcal{L}_\alpha[d_\alpha]))$ consists of extensions which are also mixed perverse sheaves whose associated graded terms are also semi-simple. Therefore,

$$\mathrm{Ext}^1_{\tilde{\mathcal{P}}^{\mathrm{G}_o}_{mixed}}(\tilde{\mathrm{P}}_o, \mathrm{IC}^{\mathrm{G}_o}(\mathcal{L}_\alpha[d_\alpha])) \subseteq \mathrm{Ext}^1(\tilde{\mathrm{P}}_o, \mathrm{IC}^{\mathrm{G}_o}(\mathcal{L}_\alpha[d_\alpha]))$$

where the last Ext is taken in the category of all equivariant perverse sheaves.

Observe that it suffices to prove the first group is trivial for all intersection cohomology complexes $\mathrm{IC}^{\mathrm{G}_o}(L_\alpha[d_\alpha])$. In view of the above injection it suffices to prove that the last Ext group above is trivial. Clearly Ext^1 computed in the abelian category of equivariant perverse sheaves on $\mathrm{EG}\underset{\mathrm{G}}{\times}\mathrm{X}$ identifies with the Ext^1-computed in the G-equivariant derived category on $\mathrm{EG}\underset{\mathrm{G}}{\times}\mathrm{X}$: see [2, Remarque 3.1.7(ii)]. Since P is assumed to a projective object in $\mathcal{P}^G$, it follows that $\mathrm{Ext}^1(\mathrm{P}, \mathrm{IC}^{\mathrm{G}}(\mathcal{L}_\alpha[\mathrm{d}_\alpha])) = 0$ and therefore, $\mathrm{Ext}^1(\mathrm{P}, \mathrm{IC}^{\mathrm{G}}(\mathcal{L}_\alpha[\mathrm{d}_\alpha]))^{\mathrm{Fr}} = 0$. Now the short-exact sequence (6.0.12) shows that one has:

$$\mathrm{Ext}^1_{\tilde{P}^{G_o}_{mixed}}(\tilde{\mathrm{P}}_o, \mathrm{IC}^{\mathrm{G}_o}(\mathcal{L}_\alpha[d_\alpha])) \subseteq \mathrm{Ext}^1(\mathrm{P}_{\mathrm{o}}, \mathrm{IC}^{\mathrm{G}_{\mathrm{o}}}(\mathcal{L}_\alpha[\mathrm{d}_\alpha])) \cong \mathrm{Hom}(\mathrm{P}, \mathrm{IC}^{\mathrm{G}}(\mathcal{L}_\alpha[\mathrm{d}_\alpha]))_{\mathrm{Fr}}$$

But one observes that $\mathrm{Hom}(\mathrm{P}, \mathrm{IC}^{\mathrm{G}}(\mathcal{L}_\alpha[\mathrm{d}_\alpha]))_{\mathrm{Fr}} \cong \mathrm{Hom}(\mathrm{P}/\mathrm{P}^1, \mathrm{IC}^{\mathrm{G}}(\mathcal{L}_\alpha[\mathrm{d}_\alpha]))_{\mathrm{Fr}}$ since $\mathrm{IC}^{G}(\mathcal{L}_\alpha[d_\alpha])$ is a simple perverse sheaf and any map from $\mathrm{P} \to \mathrm{IC}^{\mathrm{G}}(\mathcal{L}_\alpha[d_\alpha])$ factors through its largest semi-simple quotient P/P^1. But then the above Hom is nonzero if and only if $\widetilde{\mathrm{P}_{\mathrm{o}}/\mathrm{P}^1_{\mathrm{o}}} = \mathrm{P}_{\mathrm{o}}/\mathrm{P}^1_{\mathrm{o}} \cong \mathrm{IC}^{\mathrm{G}_{\mathrm{o}}}(\mathcal{L}_\alpha[\mathrm{d}_\alpha])$, and in this case, we have an extension

$$0 \to \mathrm{IC}^{\mathrm{G}_o}(\mathcal{L}_\alpha[d_\alpha]) \to \tilde{\mathrm{Q}}_o \to \tilde{\mathrm{P}}_o \to 0.$$

The condition that this extension is taken in $\tilde{P}^{\mathrm{G}_o}_{mixed}$ means that $grd^{\mathrm{W}}(\tilde{\mathrm{Q}}_o) \cong \mathrm{IC}^{\mathrm{G}_o}(\mathcal{L}_\alpha[d_\alpha]) \oplus \widetilde{\mathrm{P}_{\mathrm{o}}/\mathrm{P}^1_{\mathrm{o}}}$. Let w denote the weight of $\mathrm{IC}^{\mathrm{G}_o}(\mathcal{L}_\alpha[d_\alpha])$. Then $\tilde{\mathrm{Q}}_o \to grd^{W}(\tilde{\mathrm{Q}}_o)$

is a surjection, since w is the highest weight appearing in the mixed perverse sheaf Q. The fact that $grd^W(\tilde{\mathrm{Q}}_o) \cong \mathrm{IC}^{\mathrm{G}_o}(\mathcal{L}_\alpha[d_\alpha]) \oplus \widetilde{\mathrm{P_o/P_o^1}}$ implies that the composite map $\mathrm{IC}^{\mathrm{G}_o}(\mathcal{L}_\alpha[d_\alpha]) \to \tilde{\mathrm{Q}}_o \to grd^W(\tilde{\mathrm{Q}}_o)$ is a split monomorphism, so that the map $\mathrm{IC}^{\mathrm{G}_o}(\mathcal{L}_\alpha[d_\alpha]) \to \tilde{\mathrm{Q}}_o$ is also a split monomorphism. This shows that the extension $0 \to \mathrm{IC}^{\mathrm{G}_o}(\mathcal{L}_\alpha[d_\alpha]) \to \tilde{\mathrm{Q}}_o \to \tilde{\mathrm{P}}_o \to 0$ is a split extension, and hence trivial in $\mathrm{Ext}^1_{\tilde{P}^{\mathrm{G}_o}_{mixed}}(\tilde{\mathrm{P}}_o, \mathrm{IC}^{\mathrm{G}_o}(\mathcal{L}_\alpha[d_\alpha]))$. It follows that $\mathrm{Ext}^1_{\tilde{P}^{\mathrm{G}_o}_{mixed}}(\tilde{\mathrm{P}}_o, \mathrm{IC}^{\mathrm{G}_o}(\mathcal{L}_\alpha[d_\alpha]))$ is trivial, thereby proving that $\tilde{\mathrm{P}}_o$ is a projective object in $\tilde{P}^{\mathrm{G}_o}_{mixed}$. □

Remark 6.9. It is important to point out that the above proof holds without requiring that the strata are acyclic and also in the equivariant context. Observe that the corresponding non-equivariant version is in fact [3, Lemma 4.4.8] (see also [3, Lemma 4.4.7]), which is proven only for the case the strata are indeed affine spaces. Clearly such an assumption is not possible in our situation, since one of the key examples of a G-equivariant stratification is where the strata are the G-orbits, and such orbits are seldom acyclic, except in very special cases such as that of Schubert varieties.

7. Appendix: comparison of equivariant derived categories

We summarize the following comparison results proved in detail in [26, section 5]. The equivariant derived categories associated to the action of a group G on a space are usually defined as certain full subcategories of the derived category on the Borel construction associated to the group action. Different models for the Borel construction, therefore provide one with different models for the equivariant derived categories. The geometric model which has been introduced in [5], and in the scheme-theoretic framework in [39] and [30], complements the simplicial model which was discussed in [15], [18] and [24], each with its own advantages and dis-advantages.

However, the geometric model is perhaps more suited for handling properties like the weight filtration as in section 4, and also more commonly used in the literature dealing with equivariant derived categories, whereas the simplicial model is more functorial and suited for comparison between derived categories defined with respect to different group actions.

7.1. Equivariant derived categories: version I

Presently we proceed to discuss briefly a model for the equivariant derived category that is valid in all characteristics, making use of a geometric model for the classifying space BG for a linear algebraic group due to Totaro: see [39]. (There is a similar construction also due to [30].)

Definition 7.1. We will often use $\mathrm{EG}^{gm,m}$ to denote the m-th term of an admissible gadget $\{\mathrm{U_m}|\mathrm{m}\}$: the superscript gm stands for *geometric*. This is discussed below in 7.4.

Making use of $\mathrm{EG}^{gm,m}$ we may now define a characteristic free algebraic model for the equivariant derived category.

7.1.1. Convention

Throughout the rest of the paper, we will adopt the following conventions. If X is a scheme defined over k, we will consider ℓ-adic sheaves (that is either $\mathbb{Q}_\ell$-adic or $\bar{\mathbb{Q}}_\ell$-adic sheaves as discussed in Definition 2.1 on X_{et}, which denotes the étale site of X.) If X is a scheme defined over the complex numbers, we may consider sheaves of $\mathbb{Q}$-vector spaces or $\mathbb{C}$-vector spaces on the transcendental site of $\mathrm{X}(\mathbb{C})$ or ℓ-adic sheaves on X_{et}. We will denote by $\mathrm{D}(\mathrm{X})$ ($\mathrm{D}^+(\mathrm{X})$, $\mathrm{D}^b(\mathrm{X})$) the unbounded derived category (the bounded below derived category, the bounded derived category, respectively) of complexes of sheaves of $\mathbb{Q}$-vector spaces or $\mathbb{C}$-vector spaces on $\mathrm{X}(\mathbb{C})$ or ℓ-adic sheaves on X_{et} depending on the context.

Observe that, if k is algebraically closed,

$$\mathrm{H}^{\mathrm{i}}_{\mathrm{et}}(\mathrm{EG}^{\mathrm{gm,m}}, \mathbb{Q}_\ell) = 0 \text{ for all } 0 < \mathrm{i} \leq 2\mathrm{m} - 2 \text{ and } \mathrm{H}^0_{\mathrm{et}}(\mathrm{EG}^{\mathrm{gm,m}}, \mathbb{Q}_\ell) = \mathbb{Q}_\ell.$$

This follows from the fact that $\mathrm{EG}^{gm,m} = \mathrm{U}_{\mathrm{m}}$, which is an open G-stable subscheme of a G-representation of codimension at least $m > 1$. (It may be deduced from the hypothesis in the definition of the admissible gadgets in (7.4) that $\mathrm{codim}_{\mathrm{W_m}}(\mathrm{W_m} \backslash \mathrm{U_m}) = \mathrm{m}(\mathrm{codim_W}(\mathrm{Z}))$.) The corresponding results also hold with Z or $\mathbb{Q}$-coefficients over an algebraically closed field of characteristic 0. (Here we apply Lemma 7.3 with $c = m$.) Therefore, for each fixed finite interval $\mathrm{I} = [\mathrm{a}, \mathrm{b}]$ of the integers, $a \leq b$, and each integer $m \geq 0$, we now define

$$\mathrm{D}^{\mathrm{I}}(\mathrm{EG}^{\mathrm{gm,m}} \underset{\mathrm{G}}{\times} \mathrm{X}) = \{\mathrm{K} \,\varepsilon\, \mathrm{D}(\mathrm{EG}^{\mathrm{gm,m}} \underset{\mathrm{G}}{\times} \mathrm{X})) | \mathcal{H}^{\mathrm{i}}(\mathrm{K}) = 0, \mathrm{i} \notin \mathrm{I}\}.$$

For each I, with $2m - 2 \geq |I| = b - a$, we then let

$$\begin{aligned}\mathrm{D}^{\mathrm{I},gm}_{\mathrm{G,m}}(\mathrm{X}) = &\text{ the full subcategory of } \mathrm{D}^{\mathrm{I}}(\mathrm{EG}^{\mathrm{gm,m}} \underset{\mathrm{G}}{\times} \mathrm{X}) \text{ consisting of those K} \\ &\text{such that there exists an } \mathrm{L} \,\varepsilon\, D(\mathrm{X}) \text{ so that } \pi^*_{\mathrm{m}}(\mathrm{K}) \overset{\simeq}{\leftarrow} \mathrm{p}^*_{2,\mathrm{m}}(\mathrm{L}).\end{aligned} \tag{7.1.2}$$

Here $\pi_m : \mathrm{EG}^{gm,m} \times \mathrm{X} \to \mathrm{EG}^{\mathrm{gm,m}} \underset{\mathrm{G}}{\times} \mathrm{X}$ is the quotient map and $p_{2,m} : \mathrm{EG}^{gm,m} \times \mathrm{X} \to \mathrm{X}$ is the projection. In case we need to clarify the choice of the geometric classifying spaces, we will denote $\mathrm{D}^{\mathrm{I,gm}}_{\mathrm{G},m}(\mathrm{X})$ by $\mathrm{D}^{\mathrm{I}}_{\mathrm{G}}(\mathrm{EG}^{gm,m} \underset{\mathrm{G}}{\times} \mathrm{X})$. One observes that if $\mathrm{I} \subseteq \mathrm{J}$, then one obtains a fully faithful imbedding $\mathrm{D}^{\mathrm{I,gm}}_{\mathrm{G},m}(\mathrm{X}) \to \mathrm{D}^{\mathrm{J,gm}}_{\mathrm{G,m}}(\mathrm{X})$, so that varying I, one obtains a filtration of $\mathrm{D}^{b,gm}_{\mathrm{G},m}(\mathrm{X})$, which is defined similarly, except that the vanishing of the cohomology sheaves $\mathcal{H}^i(K)$ is for all i outside of some finite interval I depending on K. One may then take the $2 - \mathit{limit}$ over m as $m \to \infty$, to define an equivariant derived category that is independent on the choice of the finite degree approximation.

7.2. Equivariant derived categories: version II

This is already discussed in section 2 and this model corresponds to the derived category of the corresponding quotient stack.

Then we have the following comparison theorem.

Theorem 7.2. *(See [26, Theorem 1.6].) For each fixed* $m \geq 0$*, we obtain the diagram of simplicial schemes (where* p_1 *is induced by the projection* $\mathrm{EG}^{gm,m} \times \mathrm{X} \to \mathrm{X}$ *and* p_2 *is induced by the projection* $\mathrm{EG} \times (\mathrm{EG}^{gm,m} \times \mathrm{X}) \to \mathrm{EG}^{\mathrm{gm,m}} \times \mathrm{X}$*):*

$$\begin{array}{ccc} & \mathrm{EG} \times_{\mathrm{G}} (\mathrm{EG}^{\mathrm{gm,m}} \times \mathrm{X}) & \\ \swarrow^{p_1} & & \searrow^{p_2} \\ \mathrm{EG}\underset{\mathrm{G}}{\times}\mathrm{X} & & \mathrm{EG}^{gm,m}\underset{\mathrm{G}}{\times}\mathrm{X} \end{array} \tag{7.2.1}$$

(i) For each interval $\mathrm{I} = [\mathrm{a}, \mathrm{b}]$ *of the integers, with* $2m - 2 \geq b - a$*,* $p_1^* : \mathrm{D}^{\mathrm{I}}_{\mathrm{G}}(\mathrm{X}) \to \mathrm{D}^{\mathrm{I}}_{\mathrm{G}}(\mathrm{EG}^{\mathrm{gm,m}} \times \mathrm{X})$ *and* $p_2^* : \mathrm{D}^{\mathrm{I},gm}_{\mathrm{G,m}}(\mathrm{X}) \to \mathrm{D}^{\mathrm{I}}_{\mathrm{G}}(\mathrm{EG}^{\mathrm{gm,m}} \times \mathrm{X})$ *are equivalences of categories. Moreover, in positive characteristics, both the functors* p_1^* *and* p_2^* *send complexes that are mixed and pure to complexes that are mixed and pure. There exists an equivalence of derived categories:*

$$\mathrm{D}^{\mathrm{b,gm}}_{\mathrm{G}}(\mathrm{X}) \simeq \mathrm{D}^{\mathrm{b}}_{\mathrm{G}}(\mathrm{X})$$

which is natural in X *and* G*. The above equivalences hold in all characteristics with the derived categories of complexes of* ℓ*-adic sheaves on the étale site and hold in characteristic* 0 *with the derived categories of complexes of sheaves of* $\mathbb{Q}$*-vector spaces or* $\mathbb{C}$*-vector spaces.*

(ii) Moreover, both the maps p_i*,* $i = 1, 2$*, induce isomorphisms on the fundamental groups completed away from the characteristic.*

7.3. Admissible gadgets

Next we consider the following background material needed for the definition of the geometric classifying spaces. We start with the following lemma.

Lemma 7.3. *Let* V *denote a representation of the linear algebraic group* G*, all defined over a perfect field* k *of finite* ℓ*-cohomological dimension for some prime* $\ell \neq char(k)$*. Let* $\mathrm{U} \subseteq \mathrm{V}$ *denote an open* G*-stable subscheme so that the complement* $\mathrm{V} - \mathrm{U}$ *has codimension* $c > 1$ *in* V*.*

(i) Then denoting by $\bar{\mathrm{k}}$ *the algebraic closure of* k*,*

$$\mathrm{H}^{\mathrm{n}}_{\mathrm{et}}(\mathrm{U} \underset{\mathrm{Spec\,k}}{\times} \mathrm{Spec}\,\bar{\mathrm{k}}, \mathbb{Z}/\ell^{\nu}) = 0 \text{ for all } 0 < \mathrm{n} < 2\mathrm{c} - 1 \text{ and } \mathrm{H}^0_{\mathrm{et}}(\mathrm{U} \underset{\mathrm{Spec\,k}}{\times} \mathrm{Spec}\,\bar{\mathrm{k}}, \mathbb{Z}/\ell^{\nu}) = \mathbb{Z}/\ell^{\nu}.$$

(ii) For any scheme X*,*

$$R^n f_*(\mathbb{Z}/\ell^\nu) = 0 \textit{ for all } 0 < n < 2c-1, \textit{ and } R^0 f_*(\mathbb{Z}/\ell^\nu) = \mathbb{Z}/\ell^\nu$$

where $f : \mathrm{U}\times\mathrm{X} \to \mathrm{X}$ *denotes the projection.*

(iii) In case the field $k = \mathbb{C}$*, the corresponding results also hold for* $\mathbb{Z}$ *and* $\mathbb{Q}$ *in the place of* $\mathbb{Z}/\ell^\nu$*.*

Proof. (i) It suffices to consider the case k is algebraically closed. Then (i) follows from the long-exact sequence

$$\cdots \to \mathrm{H}^{\mathrm{n}}_{\mathrm{et,V-U}}(\mathrm{V}, \mathbb{Z}/\ell^\nu) \to \mathrm{H}^{\mathrm{n}}_{\mathrm{et}}(\mathrm{V}, \mathbb{Z}/\ell^\nu) \to \mathrm{H}^{\mathrm{n}}_{\mathrm{et}}(\mathrm{U}, \mathbb{Z}/\ell^\nu) \to \mathrm{H}^{\mathrm{n+1}}_{\mathrm{et,V-U}}(\mathrm{V}, \mathbb{Z}/\ell^\nu) \cdots \to$$

and the fact that $\mathrm{H}^{\mathrm{i}}_{\mathrm{et,V-U}}(\mathrm{V}, \mathbb{Z}/\ell^\nu) = 0$ for all $i < 2c$ while $\mathrm{H}^{\mathrm{i}}_{\mathrm{et}}(\mathrm{V}, \mathbb{Z}/\ell^\nu) = 0$ for all $i > 0$, $\mathrm{H}^0_{\mathrm{et}}(\mathrm{V}, \mathbb{Z}/\ell^\nu) = \mathbb{Z}/\ell^\nu$. These complete the proof of (i).

The assertion $\mathrm{H}^{\mathrm{i}}_{\mathrm{et,V-U}}(\mathrm{V}, \mathbb{Z}/\ell^\nu) = 0$ for all $i < 2c$ is a cohomological semi-purity statement. (See [26, Proof of Lemma 6.1].)

(ii) follows readily from (i). We skip the proof of (iii) which follows along the same lines as the proofs of (i) and (ii). □

Since different choices are possible for such geometric classifying spaces, we proceed to consider this in the more general framework of *admissible gadgets* as defined in [30, section 4.2]. The following definition is a variation of the above definition in [30].

7.4. Admissible gadgets associated to a given G*-scheme*

We shall say that a pair (W, U) of smooth schemes over k is a *good pair* for G if W is a k-rational representation of G and $\mathrm{U} \subsetneq \mathrm{W}$ is a G-invariant non-empty open subset on which G acts freely and so that U/G is a scheme. It is known (*cf.* [39, Remark 1.4]) that a good pair for G always exists.

Definition 7.4. A sequence of pairs $\{(\mathrm{W_m}, \mathrm{U_m}) | \mathrm{m} \geq 1\}$ of smooth schemes over k is called an *admissible gadget* for G, if there exists a good pair (W, U) for G such that $\mathrm{W_m} = \mathrm{W}^{\times^{\mathrm{m}}}$ and $\mathrm{U_m} \subsetneq \mathrm{W_m}$ is a G-invariant open subset such that the following hold for each $m \geq 1$.

1. $(\mathrm{U_m}\times\mathrm{W}) \cup (\mathrm{W}\times\mathrm{U_m}) \subseteq \mathrm{U_{m+1}}$ as G-invariant open subschemes.
2. $\{\mathrm{codim}_{\mathrm{U_{m+1}}}(\mathrm{U_{m+1}} \setminus (\mathrm{U_m}\times\mathrm{W})) | \mathrm{m}\}$ is a strictly increasing sequence, that is, $\mathrm{codim}_{\mathrm{U_{m+2}}}(\mathrm{U_{m+2}} \setminus (\mathrm{U_{m+1}}\times\mathrm{W})) > \mathrm{codim}_{\mathrm{U_{m+1}}}(\mathrm{U_{m+1}} \setminus (\mathrm{U_m}\times\mathrm{W}))$.
3. $\{\mathrm{codim}_{\mathrm{W_m}}(\mathrm{W_m} \setminus \mathrm{U_m}) | \mathrm{m}\}$ is a strictly increasing sequence, that is, $\mathrm{codim}_{\mathrm{W_{m+1}}}(\mathrm{W_{m+1}} \setminus \mathrm{U_{m+1}}) > \mathrm{codim}_{\mathrm{W_m}}(\mathrm{W_m} \setminus \mathrm{U_m})$.
4. $\mathrm{U_m}$ has a free G-action, the quotient $\mathrm{U_m}/\mathrm{G}$ is a smooth quasi-projective scheme over k and $\mathrm{U_m} \to \mathrm{U_m}/\mathrm{G}$ is a principal G-bundle.

Lemma 7.5. *Let* U *denote a smooth quasi-projective scheme over a field* K *with a free action by the linear algebraic group* G *so that the quotient* U/G *exists as a smooth quasi-projective scheme over* K*. Then if* X *is any locally linear scheme over* K*, the quotient* $\mathrm{U}\underset{\mathrm{G}}{\times}\mathrm{X} \cong (\mathrm{U}\underset{\mathrm{Spec\,K}}{\times}\mathrm{X})/\mathrm{G}$ *(for the diagonal action of* G*) exists as a scheme over* K*.*

Proof. This follows, for example, from [29, Proposition 7.1]. □

An *example* of an admissible gadget for G can be constructed as follows, starting with a good pair (W, U) for G. The choice of such a good pair will vary depending on G, but may be made as follows. Choose a faithful k-rational representation R of G of dimension n, that is, G admits a closed immersion into GL(R). Then G acts freely on an open subset U of $\mathrm{W} = \mathrm{R}^{\oplus \mathrm{n}} = \mathrm{End(R)}$ so that U/G is a scheme. (For example, U = GL(R).) Let $\mathrm{Z} = \mathrm{W}\backslash\mathrm{U}$.

Given a good pair (W, U), we now let

$$\mathrm{W_m} = \mathrm{W}^{\times \mathrm{m}}, \mathrm{U_1} = \mathrm{U} \text{ and } \mathrm{U_{m+1}} = (\mathrm{U_m}\times\mathrm{W}) \cup (\mathrm{W}\times\mathrm{U_m}) \text{ for } \mathrm{m} \geq 1. \tag{7.4.1}$$

Setting $\mathrm{Z_1} = \mathrm{Z}$ and $\mathrm{Z_{m+1}} = \mathrm{U_{m+1}} \setminus (\mathrm{U_m}\times\mathrm{W})$ for $m \geq 1$, one checks that $\mathrm{W_m}\backslash\mathrm{U_m} = \mathrm{Z^m}$ and $\mathrm{Z_{m+1}} = \mathrm{Z^m}\times\mathrm{U}$. In particular, $\mathrm{codim}_{\mathrm{W_m}}(\mathrm{W_m}\backslash\mathrm{U_m}) = \mathrm{m}(\mathrm{codim_W(Z)})$ and $\mathrm{codim}_{\mathrm{U_{m+1}}}(\mathrm{Z_{m+1}}) = (\mathrm{m}+1)\mathrm{d} - \mathrm{m}(\dim(\mathrm{Z})) - \mathrm{d} = \mathrm{m}(\mathrm{codim_W(Z)})$, where $d = \dim(\mathrm{W})$. Moreover, $\mathrm{U_m} \to \mathrm{U_m}/\mathrm{G}$ is a principal G-bundle and the quotient $\mathrm{V_m} = \mathrm{U_m}/\mathrm{G}$ exists as a smooth quasi-projective scheme.

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