Mod-l Topological G-Theory for Algebraic Stacks

Outline

- Mod-l topological G-theory of algebraic stacks: the definition

- Thomason's theorem and some of its applications for schemes

- Counter-example to the obvious extension of Thomason's theorem to algebraic stacks

- The motivating example of quotient stacks (for the correct solution)

- The iso-variant étale site

- The descent spectral sequence and applications

Mod-l topological G-theory of algebraic stacks: the definition

• Topological K-theory and G-theory for ordinary spaces

• Mod-l topological K-theory and G-theory for schemes:

using étale homotopy types

using a localized version of (algebraic) K and G-theories

• Moore spectra, the Bott-element

The definition of mod-l topological K and G-theories for algebraic stacks

- Henceforth all objects are of finite type and over an algebraically closed field. $l \neq char(k)$. $\nu >> 0$.

- $K^{top}(\mathcal{S})/l^{\nu} = K(\mathcal{S})/l^{\nu}[\beta^{-1}]$

- $G^{top}(\mathcal{S})/l^{\nu} = G(\mathcal{S})/l^{\nu}[\beta^{-1}]$

Thomason's Theorem for schemes

- Hypercohomology on a site with enough points

- **Thomason's theorem** X of finite type over k, $char(k) \neq l$, k algebraically closed (for simplicity). Then the augmentation:

$$G^{top}/l^{\nu}(X) \rightarrow \mathbb{H}_{et}(X, \mathbf{G}^{top}/l^{\nu}(\quad))$$

is a weak-equivalence. Therefore, there is a spectral sequence:

$$E_2^{s,t} = H^s_{et}(X, \pi_t \mathbf{G}^{top}/l^{\nu}(\quad)/l^{\nu}) \Rightarrow \pi_{t-s}(G^{top}/l^{\nu}(X).$$

Among the *applications*:

• a general Riemann-Roch theorem, i.e. the square

$$\begin{array}{ccc} G/l^\nu(X) & \longrightarrow & G^{top}/l^\nu(X) \\ {\scriptstyle f_*}\Big\downarrow & & \Big\downarrow{\scriptstyle f_*^{top}} \\ G/l^\nu(Y) & \longrightarrow & G^{top}/l^\nu(Y) \end{array}$$

homotopy commutes for any proper map $f : X \to Y$.

• (trivial application) The E_2-terms of the above spectral sequence provide a definition of étale cohomology of X when X is smooth.

• A simple counter-example to the obvious extension of Thomason's theorem to algebraic stacks

The Isovariant étale site

- The motivating example of quotient stacks

- The inertia stack:

$$\begin{array}{ccc} I_{\mathcal{S}} & \longrightarrow & \mathcal{S} \\ \downarrow & & \downarrow \Delta \\ \mathcal{S} & \xrightarrow{\Delta} & \mathcal{S} \times \mathcal{S} \end{array}$$

- The Isovariant étale site:Definition

- Coarse moduli spaces:definition

- Gerbes

Key Theorem

Assume the stack $\mathcal{S}$ is a gerbe over its coarse-moduli space $\mathcal{M}$. Let $p : \mathcal{S} \to \mathcal{M}$ be the obvious map. Then p^* induces an equivalence :$\mathcal{M}_{et} \to \mathcal{S}_{iso.et}$

Outline of Proof

Theorem:gluing of iso-variant étale sites

Given $\mathcal{S}$, there exists a finite filtration $\mathcal{S}_0 \subseteq \mathcal{S}_1 \cdots \subseteq \mathcal{S}_n = \mathcal{S}$ by locally closed algebraic sub-stacks with $\mathcal{S}_i - \mathcal{S}_{i-1}$ a gerbe over its coarse-moduli space.

The isovariant étale topos of $\mathcal{S}$ is obtained by gluing the isovariant étale topoi of $\mathcal{S}_i - \mathcal{S}_{i-1}$.

$\mathcal{S}_{iso.et}$ has enough points and they correspond to the geometric points of the coarse-moduli spaces of each $\mathcal{S}_i - \mathcal{S}_{i-1}$.

The isovariant étale site has finite l-cohomological dimension, $l \neq char(k)$.

Outline of proof

The main theorem: the isovariant descent spectral sequence

The obvious augmentation:

$$G^{top}(\mathcal{S})/l^{\nu} \rightarrow \mathbb{H}_{iso.et}(\mathcal{S}, G^{top}(\quad)/l^{\nu})$$

is a weak-equivalence.

There exists a strongly convergent spectral sequence ($l \neq char(k)$):

$$E_2^{s,t} = H^s_{iso.et}(\mathcal{S}, \pi_t G^{top}(\quad)/l^{\nu}) \Rightarrow \pi_{-s+t}(G^{top}(\mathcal{S})/l^{\nu})$$

Outline of proof

Applications

- A general Riemann-Roch theorem for proper maps of finite cohomological dimension between Artin stacks

- New homology theories for Artin stacks that have finite l-cohomological dimension

- Further remarks on such theories.