RIEMANN-ROCH FOR ALGEBRAIC STACKS:III VIRTUAL STRUCTURE SHEAVES AND VIRTUAL FUNDAMENTAL CLASSES

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Abstract. In this paper we apply the Riemann-Roch and Lefschetz-Riemann-Roch theorems proved in our earlier papers to define virtual fundamental classes for the moduli stacks of stable curves in great generality and establish various formulae for them.

Table of Contents

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1. **Introduction**

This is the last in a series of papers on the Riemann-Roch problem for algebraic stacks. The first part (see [J-4]) presented a solution to this problem in general for the natural transformation between the G-theory and topological G-theory of algebraic stacks. It also introduced a new site associated to algebraic stacks called the isovariant étale site using which we proved a descent theorem for the topological G-theory of algebraic stacks extending Thomason's basic results to algebraic stacks. Continuing along the same direction, we defined and studied Bredon style homology theories for algebraic stacks in [J-5]. We also established Riemann-Roch theorems as natural transformations between the G-theory of dg-stacks and these Bredon-style homology theories. These are only for algebraic stacks that admit coarse-moduli spaces which are quasi-projective schemes over a Noetherian excellent base scheme (for example, a field k). It is important to observe that these already include Artin stacks. One may recall that applications to virtual fundamental classes, dictated that we work out all these papers in the setting of dg-stacks.

In the present paper, we indeed establish various formulae for the virtual structure sheaves on dg-stacks associated to obstruction theories at the level of the G-theory of dg-stacks. Using Riemann-Roch and Lefschetz-Riemann-Roch theorems, these provide push-forward and localization formulae for virtual fundamental classes taking values in Bredon style homology theories. Making use of the relationships between Bredon style homology and homology computed on the smooth and étale sites, one extends these formulae to more traditional homology theories for algebraic stacks. In fact we show that it is possible to derive most formulae for virtual fundamental classes (some not known before) , by first proving an appropriate formula at the level of virtual structure sheaves and then by applying Riemann-Roch (as in [J-5] section 8) to it. For example, we prove a general push-forward formula for virtual structure sheaves; then by applying Riemann-Roch to it we show it is possible to derive a general push-forward formula for virtual fundamental classes, special cases of which provide a proof of the conjecture of Cox, Katz and Lee as well as a strong form of the localization formula for virtual fundamental classes, both proven elsewhere by distinct and separate methods at the level of virtual fundamental classes. All of these seem to validate the idea, we believe due to Yuri Manin (and passed onto me by Bertrand Toen), that Riemann-Roch techniques could be used to derive most formulae for virtual fundamental classes, once the corresponding formulae for virtual structure sheaves are obtained. The latter seem more manageable and, as we show here, could be studied by standard techniques in G-theory, suitably modified to handle virtual objects. The dg-stacks considered in this paper are all dg-stacks in the sense of [J-5], i.e. they are algebraic stacks provided with a perfect obstruction theory. The restriction to such dg-stacks (and not the more general derived moduli algebraic stacks) is mainly because a comprehensive theory of derived moduli stacks is either still under development (or is only emerging) at present.

Throughout the paper we will assume that for each of the algebraic stacks $\mathcal{S}$ considered, a coarse moduli space $\mathfrak{M}$ exists which is quasi-projective as a scheme and that the obvious map $p : \mathcal{S} \rightarrow \mathfrak{M}$ is finite and has finite cohomological dimension. (The last assumption will be satisfied if the orders of the stabilizer groups are prime to the characteristics, for example in characteristic 0.) In addition we will freely adopt the terminology from [J-5]. Recall that, there, we start with Bloch-Ogus style homology/cohomology theories defined on algebraic spaces with respect to complexes of sheaves denoted $\Gamma(\bullet)$ and $\Gamma^h(\bullet)$. An equivariant form of these are discussed briefly in section 5 and the reader may consult that section to recall this theory. We begin section 2 by defining first virtual structure sheaves and then virtual fundamental classes in great generality. This makes intrinsic use of the Riemann-Roch transformation and the Bredon-style homology theories defined in [J-5]. We show that our definition reduces to the more traditional cycle-theoretic definition (or definition in terms of homology classes) - see Corollary 2.10 and Theorem 1.2. The following is one of the main theorems proved in section 2.

Definition 1.1. (The resolution property) We say that a stack $\mathcal{S}$ has the *resolution property* if every coherent sheaf on the stack $\mathcal{S}$ is the quotient of a vector bundle.

Theorem 1.2. *(See Corollary 2.10). Let $\mathcal{S}$ denote a Deligne-Mumford stack provided with a perfect obstruction theory $E^\bullet$ in the sense of section 2.*

(i) Then the virtual fundamental class of $(\mathcal{S}, E^\bullet)$ is defined without any further assumptions on $\mathcal{S}$ or $E^\bullet$ except those assumed in 1.1.7 taking values both in Bredon-style homology theories as in [J-5] and also in homology theories defined on the étale site of the stack $\mathcal{S}$.

(ii) Moreover, assume in addition to the above situation that the the resolution property holds. Then the image of the class $[\mathcal{S}]^{virt}_{Br}$ in the étale homology of the stack with respect to $\Gamma(*)$, agrees with the virtual fundamental classes defined cycle theoretically in the latter.

We review the definition of the virtual structure sheaves defined with respect to a given obstruction theory and establish the above theorem in section 2. We also develop techniques for relating formulae in Bredon homology with formulae in the homology theories of algebraic stacks defined on the smooth (and étale sites of algebraic stacks. We discuss Gysin maps in the context of G-theory in section 3. This is done so that we obtain more convenient expressions for the virtual structure sheaves considered in section 2. Section 4 is devoted to a thorough study of push-forward for virtual structure sheaves and virtual fundamental classes for algebraic stacks. In section 5, we extend the Bredon homology and cohomology in [J-5] to the equivariant setting and prove a Lefschetz-Riemann-Roch in this setting: these are used in Theorems 1.8 and 1.9. Sections 6 and 7 summarize some of techniques used elsewhere: for example the key properties of K-theory and G-theory for algebraic stacks, equivariant cohomology for algebraic stacks provided with a group action and operational Chern classes form vector bundles on Deligne-Mumford stacks.

We begin section 4, by obtaining convenient expressions for the virtual structure sheaves: given a Deligne-Mumford stack $\mathcal{S}$ and an obstruction theory $E^\bullet$ for it, we obtain several expressions for $\mathcal{O}_{\mathcal{S}}^{virt}$ as a class in $\pi_0(G(\mathcal{S}))$. One may assume one of these for the following discussion: the theorems below form some of the remaining main results of the paper.

1.0.1. *We make the standing assumption that for all Deligne-Mumford stacks we consider, the stabilizer groups have order prime to the residue characteristics: observe this holds automatically in characteristic* 0.

1.1. Next we consider push-forward for closed immersions of Deligne-Mumford stacks provided with compatible obstruction theories. The appropriate context for all of these is the following: assume that $u : \mathcal{T} \to \mathcal{S}$ and $v : \tilde{\mathcal{T}} \to \tilde{\mathcal{S}}$ are closed immersions and that the square

$$\begin{array}{ccc} \mathcal{T} & \xrightarrow{u} & \mathcal{S} \\ \downarrow{\scriptstyle i_{\mathcal{T}}} & & \downarrow{\scriptstyle i} \\ \tilde{\mathcal{T}} & \xrightarrow{v} & \tilde{\mathcal{S}} \end{array}$$

is cartesian, with both $\tilde{\mathcal{S}}$ and $\tilde{\mathcal{T}}$ smooth Deligne-Mumford stacks and where the the vertical maps are *also closed immersions*. To handle the equivariant case, we may assume that all these stacks are provided with the action of a smooth group scheme G and the morphisms above are all G-equivariant. We will assume that (i) one is provided with a perfect obstruction theory $E^\bullet$ ($F^\bullet$) for $\mathcal{S} \to \tilde{\mathcal{S}}$ ($\mathcal{T} \to \tilde{\mathcal{T}}$, respectively) (ii) that $E^\bullet$ ($F^\bullet$) has a global resolution by a complex of vector bundles and (iii) that these are *weakly compatible* in the following sense: there is given a G-equivariant map $\phi : u^*(E^\bullet) \to F^\bullet$ of complexes so that there exists a distinguished triangle $K^\bullet \to u^*(E^\bullet) \to F^\bullet$ and $K^\bullet$ is of perfect amplitude contained in $[-1, 0]$. (For example, the two obstruction theories are weakly compatible if $E^\bullet$ and $F^\bullet$ may be replaced (upto G-equivariant quasi-isomorphism) by complexes of G-equivariant vector bundles which will be still denoted $E^\bullet$ and $F^\bullet$ and the given map $\phi : u^*(E^\bullet) \to F^\bullet$ is an *epimorphism* in each degree. It follows that, in this case, the kernel, $K^\bullet = ker(\phi)$ is a complex of vector bundles.)

We will assume henceforth one of the following hypotheses:

• there exists a class (which we denote) $\lambda_{-1}(\hat{K}^0)$ in $\pi_0(G_{\mathcal{T}\times\mathbb{A}^1}(\hat{\mathcal{S}}))$ so that for each $t\varepsilon\mathbb{A}^1$, $i_t^*(\lambda_{-1}(\hat{K}^0)\varepsilon\pi_0(G_{\mathcal{T}\times t}((\hat{\mathcal{S}})_t))$ identifies with the class of $\lambda_{-1}(K^0)$ in $\pi_0(G(\mathcal{T}))$ or

• we are in the equivariant case.

Observe that in the latter case, we let T denote a fixed torus acting on a Deligne-Mumford stack X all defined over an algebraically closed field k. T' will denote a sub-torus with associated prime ideal $\mathfrak{p}$ in $R(T)$. $G(X,T)$ denote the T-equivariant G-theory of a stack X provided with the action of the torus T. ($K(X,T)$ will denote the corresponding T-equivariant K-theory and if X admits a closed immersion into $\tilde{X}$ onto which the T-action extends, $K_X(\tilde{X},T)$ will denote the T-equivariant K-theory of $\tilde{X}$ with supports in X.) Now observe that one has the isomorphism $\pi_0(G(\mathcal{T}\times\mathbb{A}^1,T))_{(\mathfrak{p})} = \pi_0(G_{\mathcal{T}\times\mathbb{A}^1}(\hat{\mathcal{S}},T))_{(\mathfrak{p})} \cong \pi_0 G(\hat{\mathcal{S}},T)_{(\mathfrak{p})}$ and hence the class $\lambda_{-1}(K^0)$ in the first group (i.e. in $\pi_0(G(\mathcal{T},T))_{(\mathfrak{p})}$) lifts to a class in $\pi_0 G(\hat{\mathcal{S}},T)_{(\mathfrak{p})}$ which we denote by $\lambda_{-1}(\hat{K}^0)$ and a class $\lambda_{-1}(K^0_{\mathcal{S}})\varepsilon\pi_0(G(\mathcal{S},T))_{(\mathfrak{p})}$. Observe also that in either case one may identify $\lambda_{-1}(K^0_{\mathcal{S}})$ with a class in $\pi_0(K_{\mathcal{S}}(\tilde{\mathcal{S}},T))$ (or a localization of the latter in the equivariant case) so that tensor product with this class is well-defined and one may take its Chern-character (as a local Chern character). We let $\mathcal{O}_{\mathcal{S}}^{virt}$ ($\mathcal{O}_{\mathcal{T}}^{virt}$) denote the virtual structure sheaf associated to $\mathcal{S}$ ($\mathcal{T}$, respectively).

Definition 1.3. We define the *virtual Todd class* of the obstruction theory $E^\bullet$ with values in $H^*_{smt}(\mathcal{S}, \Gamma(*))$ as $Td(E_0).Td(E_1)^{-1}$ where $E_i = (E^i)^\vee$. This will be denoted $Td(T\mathcal{S})^{virt}$. Then we will define the *virtual fundamental class* in $H^{smt}_*(\mathcal{S}, \Gamma(*))$, $[\mathcal{S}]^{virt}$, to be $\sigma_*([\mathcal{S}]^{virt}_{Br})$ where $[\mathcal{S}]^{virt}_{Br}$ denotes the virtual fundamental class in Bredon homology as defined in Definition 2.3 and σ_* is the map from Bredon homology to étale homology.

Remark 1.4. One may observe readily that since the stacks are all Deligne-Mumford, the Todd-classes considered above are invertible in the étale l-adic cohomology of the stack.

Theorem 1.5. *(Push forward of virtual structure sheaves and virtual fundamental classes) Assume the above situation. Then* $\lambda_{-1}(\hat{K}^0)$ *defines a class in* $\pi_0(G_{\mathcal{T}}(\mathcal{S}))$ *and one obtains the formulae:*

(1.1.1) $$u_*(\mathcal{O}^{virt}_{\mathcal{T}}.\lambda_{-1}(K^{-1})) = \mathcal{O}^{virt}_{\mathcal{S}}.\lambda_{-1}(K^0_{\mathcal{S}})$$

in $\pi_0(G_{\mathcal{T}}(\mathcal{S}))$ *and*

(1.1.2) $$u_*(\tau(\mathcal{O}^{virt}_{\mathcal{T}}))ch(\lambda_{-1}(K^{-1})) = \tau(\mathcal{O}^{virt}_{\mathcal{S}}).ch(\lambda_{-1}(K^0_{\mathcal{S}})$$

in $H^{Br}_*(\mathcal{S}, \Gamma(*))$ *which is the Bredon-style homology defined in [J-5].*

Next we obtain the formula

(1.1.3) $$u_*(([\mathcal{T}]^{virt}).Eu(K_1)) = [\mathcal{S}]^{virt}.Eu((K^0_{\mathcal{S}})^\vee)$$

in $H^{et}_*(\mathcal{S}, \Gamma(*))$. *Here* $K_1 = (K^{-1})^\vee$ *while* $Eu(V^\vee) = Td(V^\vee).Ch(\lambda_{-1}(V))$ *(= the bottom term of* $Ch(\lambda_{-1}(V))$ *in the sense of Definition 2.3) for a vector bundle* V *where* Ch *denotes the Chern character with values in étale cohomology and* Eu *denotes the Euler class.*

Remarks 1.6. 1. Observe that the notion of compatibility of obstruction theories adopted above is indeed weaker than the usual notion of compatibility as in [B-F] or [KKP]. Hence the adjective weak-compatibility is used in our situation. There seem to be obstruction theories that are weakly compatible and not compatible: for example, the obstruction theories as in the theorems below associated to the closed immersion of the fixed point stack for the action of a given torus on an algebraic stack.

2. The above theorem provides many useful formulae for virtual fundamental classes and virtual structure sheaves, some of which are considered next. For example, we answer the following strong form of the conjecture of Cox, Katz and Lee (see [CKL]).

1.1.4. Let X denote a smooth projective variety. Let $\beta \varepsilon CH_1(X)$ denote a class and let $\mathcal{M}_{0,n}(X, \beta)$ denote the moduli stack of n-pointed genus 0 stable maps to X of class β. Let V denote a vector bundle over X so that it is *convex.* i.e. $H^1(C, f^*(V)) = 0$ for all genus 0-stable maps $f : C \to X$. Let $e_{n+1} : \mathcal{M}_{0,n+1}(X, \beta) \to X$ send $(f, C, p_1, ..., p_{n+1})$ to $f(p_{n+1})$ and $\pi_{n+1} : \mathcal{M}_{0,n+1}(X, \beta) \to \mathcal{M}_{0,n}(X, \beta)$ denote the map forgetting the point p_{n+1}. Let $\mathcal{V}_{\beta,n} = \pi_{n+1*}e^*_{n+1}(V)$; this is a vector bundle on $\mathcal{M}_{o,n}(X, \beta)$ in view of the convexity of V. Let $i : Y \to X$ denote the inclusion of the zero locus of a regular section of V and for each $\gamma \varepsilon H_2(Y, \mathbb{Z})$ with $i_*(\gamma) = \beta$, let $i_\gamma : \mathcal{M}_{0,n}(Y, \gamma) \to \mathcal{M}_{0,n}(X, \beta)$ denote the induced closed immersion.

Theorem 1.7. *(Conjecture of Cox, Katz and Lee: see [CKL] and also [CK] p. 386) Assuming the above situation*

$$\Sigma_{i_*(\gamma)=\beta} i_{\gamma*}(\mathcal{O}^{virt}_{\mathcal{M}_{0,n}(Y,\gamma)}) = \lambda_{-1}(\Gamma(\mathcal{V}_{\beta,n})).\mathcal{O}^{virt}_{\mathcal{M}_{0,n}(X,\beta)} \text{ in } \pi_0(G(\mathcal{M}_{0,n}(X, \beta), \mathcal{O}_{\mathcal{M}_{0,n}(X,\beta)})).$$

(Here $\Gamma(\mathcal{V}_{\beta,n})$ *denotes the sheaf of sections of the vector bundle* $\mathcal{V}_{\beta,n}$*.) In particular,*

$$\Sigma_{i_*(\gamma)=\beta} i_{\gamma*}([\mathcal{M}_{0,n}(Y, \gamma)]^{virt}) = [\mathcal{M}_{0,n}(X, \beta)]^{virt} \circ Eu(\Gamma(\mathcal{V}_{\beta,n})) \text{ in } H^{Br}_*(\mathcal{M}_{0,n}(X, \beta); \Gamma^h(*))$$

holds in $H^{Br}_*(\mathcal{M}_{0,n}(X, \beta), \Gamma(*))_{\mathbb{Q}}$ *for any choice of homology theories* $\Gamma^h(*)$ *as above. Here* $Eu(\mathcal{V}_{\beta,n})$ *denotes an Euler class, which is defined as the term of appropriate weight and degree in* $ch(\lambda_{-1}(\Gamma(\mathcal{V}_{\beta,n})))$.

We also obtain:

$$\Sigma_{i_*(\gamma)=\beta} i_{\gamma*}([\mathcal{M}_{0,n}(Y, \gamma)]^{virt}) = [\mathcal{M}_{0,n}(X, \beta)]^{virt} \circ Eu(\Gamma(\mathcal{V}_{\beta,n})) \text{ in } H^{smt}_*(\mathcal{M}_{0,n}(X, \beta); \Gamma(\bullet))$$

Here $Eu(\Gamma(\mathcal{V}_{\beta,n}))$ *denotes the usual Euler class in étale cohomology.*

We conclude by considering localization formulae for virtual structure sheaves and virtual fundamental classes.

Theorem 1.8. *Assume in addition to the hypotheses in 1.1 that the base scheme is an algebraically closed field, the stacks $\mathcal{S}$ and $\tilde{S}$ are provided with actions by a torus T, T' is a given sub-torus with the associated prime ideal in $R(T)$ being $\mathfrak{p}$. Moreover, we require that $\mathcal{T} = \mathcal{S}^{T'}$ and $\tilde{T} = (\tilde{S})^{T'}$. (Here the fixed point stacks are defined in [J-4] section 6.) We let the obstruction theory $F^\bullet$ be defined as $u^*(E^\bullet)^{T'}$. Then the class $\lambda_{-1}(K^0)\varepsilon\pi_0(K(\mathcal{T},T))$ lifts to a class $\lambda_{-1}(K^0_{\mathcal{S}})\varepsilon\pi_0(G(\mathcal{S},T))_{(\mathfrak{p})} \cong \pi_0(K_{\mathcal{S}}(\tilde{S},T))_{(\mathfrak{p})}$ and one obtains the formula:*

$$u_*(\mathcal{O}^{virt}_{\mathcal{T}}.\lambda_{-1}(K^{-1})) = \mathcal{O}^{virt}_{\mathcal{S}}.\lambda_{-1}(K^0_{\mathcal{S}}) \tag{1.1.5}$$

in $\pi_0(G(\mathcal{S}))_{\mathfrak{p}}$. This implies the formula

$$u_*([\mathcal{T}]^{virt}.Eu(K_1)) = [\mathcal{S}]^{virt}.Eu((K^0_{\mathcal{S}})^\vee) \tag{1.1.6}$$

in T-equivariant étale homology of $\mathcal{S}$ localized at the prime ideal $\mathfrak{p}$. Here $K_1 = (K^{-1})^\vee$ and Eu denotes the Euler class.

Theorem 1.9. *Assume the hypotheses of the last theorem. Then one has a Gysin map $u_* : \pi_0(K_{\mathcal{T}}(\tilde{T},\mathcal{O}^{virt}_{\tilde{\mathcal{T}}},T))_{(\mathfrak{p})} \to \pi_0(K_{\mathcal{S}}(\tilde{S},\mathcal{O}^{virt}_{\tilde{\mathcal{T}}},\mathcal{T}))_{(\mathfrak{p})}$ defined where the relative K-groups above are the Grothendieck groups of the category of perfect complexes of modules over dg-stacks defined as in appendix A. This has the property that*

$$u_*(\mathcal{O}^{virt}_{\tilde{\mathcal{T}}} \underset{\mathcal{O}_{\tilde{\mathcal{T}}}}{\otimes} \lambda_{-1}(K^{-1})) = \mathcal{O}^{virt}_{\tilde{\mathcal{S}}} \underset{\mathcal{O}_{\tilde{\mathcal{S}}}}{\otimes} \lambda_{-1}(K^0_{\mathcal{S}})$$

where $K^0_{\mathcal{S}}$ is viewed as a class in $\pi_0(K_{\mathcal{S}}(\tilde{S},T))_{(\mathfrak{p})}$. Consequently one obtains

$$u^*u_*(\mathcal{F}) = \mathcal{F} \otimes \lambda_{-1}(K^0) \otimes \lambda_{-1}(K^{-1})^{-1},\ \mathcal{F}\varepsilon\pi_0(K(\mathcal{T},\mathcal{O}^{virt}_{\mathcal{T}},T)).$$

Moreover, a pull-back is defined on étale l-adic homology (under our hypotheses) and we obtain:

$$u_*([\mathcal{T}]^{virt}.Eu(K_1).Eu(K_0)^{-1}) = [\mathcal{S}]^{virt}$$

in $H^T_*(\mathcal{T},\Gamma(*))_{(\mathfrak{p})}$. Here $H^T_*(\mathcal{S},\Gamma(*))$ denotes the homology of the stack $[\mathcal{T}/T]$ computed on the étale site with respect to the complex $\Gamma(*)$ and $\mathfrak{p}$ is the prime ideal in $R(T)$ corresponding to the sub-torus T'. Moreover, $K_0 = (K^0)^\vee$, $K_1 = (K^{-1})^\vee$ and $Eu(K_i)$ is the corresponding Euler class in in $H^*_T(\mathcal{T},\Gamma(*))_{(\mathfrak{p})}$.

Remark 1.10. If we let the *Euler class of the virtual normal bundle* be defined by $Eu(K_1)^{-1}.Eu(K_0)$ we recover the main result in [GP] proven there by other means. Observe that the use of *dg*-stacks and Riemann-Roch simplifies the proof considerably. Moreover the formula in (1.1.3) and (1.1.6) seems to be not known before.

Acknowledgments. We would like to thank Bertrand Toen and Angelo Vistoli on several discussions over the years on algebraic stacks. As one can see a key role is played by the push-forward formula in Proposition 3.2 originally proved by Vistoli in the context of intersection theory on algebraic stacks: see [Vi-1]. The relevance of dg-stacks and the possibility of defining push-forward and other formulae for the virtual fundamental classes using Riemann-Roch theorems on stacks, became clear to the author at the MSRI program on algebraic stacks in 2001 and especially during many conversations with Bertrand Toen while they were both supported by the MSRI.

After this paper was written up, we learned from David Cox that an alternate solution of the conjecture of Cox, Katz and Lee appears in the recent paper [KKP]. However, as one can see, there are several important differences in the proofs. The most important of course is that we prove an analogue of this formula for virtual structure sheaves first as a corollary to our more general push-forward formula in Theorem 4.9, making use of methods from K-theory and deformation to the normal cone. The conjectured formula of virtual fundamental classes then follows by applying our Riemann-Roch to the formula at the level of virtual structure sheaves. Another difference that seems worth mentioning is that our formula holds in all possible homology theories defined with respect to the complexes $\Gamma^h(*)$ satisfying the basic hypotheses in [J-5] section 3. Moreover, the formulae in (1.1.3) and (1.1.6) seem to be not known before.

1.1.7. Basic frame work. We will adopt the terminology and conventions from [J-5] throughout the paper. For the sake of completeness we will recall these here. We will adopt the following terminology throughout the paper. Let S denote an excellent Noetherian separated scheme which will serve as the base scheme. All objects we consider will be *locally finitely presented over S, and locally Noetherian.* In particular, all objects we consider are locally quasi-compact. However, our results are valid, for the most part only for objects that are finitely presented over the base scheme S or for disjoint unions of such objects. Since we consider mostly dg-stacks, G-theory and K-theory will always mean the theory associated to the dg-stack as in appendix A. i.e. If $\mathcal{S}$ is an algebraic stack provided with a dg-structure sheaf $\mathcal{A}$ and an action by a smooth group scheme G, we will let $\mathbf{G}(\mathcal{S},\mathcal{A},G)$ ($\mathbf{K}(\mathcal{S},\mathcal{A},G)$, respectively) denote the G-theory spectrum (the K-theory spectrum, respectively) of the category of coherent G-equivariant $\mathcal{A}$-modules on $\mathcal{S}$, (perfect G-equivariant $\mathcal{A}$-modules, respectively) as defined in Definition 6.2.

As pointed out earlier, the dg-stacks considered in this paper are all dg-stacks in the sense of [J-5], i.e. they are algebraic stacks provided with a perfect obstruction theory. The restriction to such dg-stacks (and not the more general derived moduli algebraic stacks) is mainly because a comprehensive theory of derived moduli stacks is either still under development (or is only emerging) at present. In view of the fact that the Bredon style theories in [J-5] are defined for all dg-stacks, makes it possible to extend the results of this paper to the setting of derived moduli stacks, if the reader so desires: the only key results that need to be extended to this setting would be those of sections 3 and 4 on the pushforward formulae for the virtual structure sheaves.

We will adopt the following conventions regarding moduli spaces. A *coarse moduli-space* for an algebraic stack $\mathcal{S}$ will be a *proper map* $p : \mathcal{S} \to \mathfrak{M}_{\mathcal{S}}$ (with $\mathfrak{M}_{\mathcal{S}}$ an algebraic space) which is a uniform categorical quotient and a uniform geometric quotient in the sense of [KM] 1.1 Theorem. Moreover, for purposes of Riemann-Roch, we will assume that p always has *finite cohomological dimension*. (Observe that this hypothesis is satisfied if the order of the residual gerbes are prime to the residue characteristics, for example in characteristic 0 for all Deligne-Mumford stacks. Observe also that the notion of coarse moduli space above may be a bit different from the notion adopted in [Vi-1].) It is shown in [KM] that if the stack $\mathcal{S}$ is separated and Deligne-Mumford, of finite type over k and the obvious map $I_{\mathcal{S}} \to \mathcal{S}$ is finite, then a coarse moduli space exists with all of the above properties.

Convention 1.11. Henceforth a stack will mean a DG-stack. DG-stacks whose associated underlying stack is of Deligne-Mumford type will be referred to as Deligne-Mumford DG-stacks. We will assume that all coarse-moduli spaces that we consider are quasi-projective schemes. In the presence of an action by a smooth affine group scheme, we will assume these are G-quasi-projective in the sense that they admit G-equivariant locally closed immersion into a projective space on which the group G acts linearly. Given a presheaf of spectra P, $P_{\mathbb{Q}}$ will denote its localization at $\mathbb{Q}$. (Observe that then $\pi_*(P_{\mathbb{Q}}) = \pi_*(P) \otimes \mathbb{Q}$.)

2. Virtual structure sheaves and virtual fundamental classes: definitions and basic properties

We will begin by defining *virtual structure sheaves* and then as an application of Riemann-Roch theorems *virtual fundamental classes*.

2.0.8. *Virtual structure sheaves and virtual fundamental classes.* Presently we will define virtual structure sheaves and virtual fundamental classes associated to perfect obstruction theories: *our approach using the Riemann-Roch makes it possible to define virtual fundamental classes even when global resolutions of coherent sheaves by vector bundles do not exist.*

Throughout this discussion we will fix a base object B which will be in general any *smooth* Artin stack of finite type over the given base scheme. (The base scheme may be assumed to be a field or a general Noetherian excellent scheme of finite type over a field.) Let $b = dim(B)$. All objects and morphisms we consider in this section will be over B and therefore we will often omit the adjective *relative*. We begin by recalling briefly the definition of the *intrinsic normal cone* from [B-F] section 3. (See also [CK] pp. 178-179). *Convention: in what follows we will ignore the fact the base is a smooth stack and not a field. Since this stack is smooth, all this does is to necessitate modifying the dimensions by adding b to them.*

First we proceed to define *virtual structure sheaves* associated to perfect obstruction theories, following [B-F]. Let $\mathcal{S}$ denote a Deligne-Mumford stack with $u : U \to \mathcal{S}$ an atlas and let $i : U \to M$ denote a closed immersion into a smooth scheme. Let $C_{U/M}$ ($N_{U/M}$) denote the normal cone (normal bundle, respectively) associated to the closed immersion i. (Recall that if $\mathcal{I}$ denotes the sheaf of ideals associated to the closed immersion i, $C_{U/M} = Spec\underset{n}{\oplus}\mathcal{I}^n/\mathcal{I}^{n+1}$ and $N_{U/M} = SpecSym(\mathcal{I}/\mathcal{I}^2)$. Now $[C_{U/M}/i^*(T_M)]$ ($[N_{U/M}/i^*(T_M)]$) denotes the *intrinsic normal cone* denoted $\mathcal{C}_{\mathcal{S}}$ (the intrinsic abelian normal cone denoted $\mathcal{N}_{\mathcal{S}}$, respectively). In case the algebraic stack $\mathcal{S}$ is provided with the action of a smooth group scheme G, we will assume that this action lifts to an action on the intrinsic normal cone and the intrinsic abelian normal cone. This hypothesis is satisfied, for example, if the stack $\mathcal{S}$ admits a closed immersion into a smooth Deligne-Mumford stack onto which the action of G extends making the above closed immersion G-equivariant.

Let $E^\bullet$ denote a complex of $\mathcal{O}_{\mathcal{S}}$-modules so that it is trivial in positive degrees and whose cohomology sheaves in degrees 0 and -1 are coherent. Let $L^\bullet_{\mathcal{S}}$ denote the *cotangent complex* of the stack $\mathcal{S}$ over the base B. A morphism $\phi : E^\bullet \to L^\bullet_{\mathcal{S}}$ in the derived category of complexes of $\mathcal{O}_{\mathcal{S}}$-modules is called an *obstruction theory* if ϕ induces an isomorphism (surjection) on taking the cohomology sheaves in degree 0 (in degree -1, respectively). In case $\mathcal{S}$ is provided with the action of a smooth group scheme G, we will assume that $E^\bullet$ is a complex of G-equivariant sheaves of $\mathcal{O}_{\mathcal{S}}$-modules and that the homomorphism ϕ is G-equivariant. (Observe that, in this case, the cotangent complex $L^\bullet_{\mathcal{S}}$ is automatically a complex of G-equivariant $\mathcal{O}_{\mathcal{S}}$-modules.) As in [B-F] section 5, we call the obstruction theory $E^\bullet$ *perfect* if $E^\bullet$ is of perfect amplitude contained in $[-1, 0]$ (i.e. locally on the étale site of the stack, it is quasi-isomorphic to a complex of vector bundles concentrated in degrees 0 and -1). In this case, one may define the *virtual dimension* of $\mathcal{S}$ with respect to the obstruction theory $E^\bullet$ as $rank(E^0) - rank(E^1) + b$. Moreover, in this case, we let $\mathcal{E}_{\mathcal{S}} = h^1/h^0(E^\bullet) = [\mathcal{E}_1/\mathcal{E}_0]$ where $\mathcal{E}_i = C(E^{-i})$ where C denotes the corresponding *abelian cone stack* as in [B-F] section 1.

Now the morphism ϕ defines a closed immersion $\phi^\vee : \mathcal{N}_{\mathcal{S}} \to \mathcal{E}_{\mathcal{S}}$. Composing with the closed immersion $\mathcal{C}_{\mathcal{S}} \to \mathcal{N}_{\mathcal{S}}$ one observes that $\mathcal{C}_{\mathcal{S}}$ is a closed cone sub-stack of $\mathcal{E}_{\mathcal{S}}$. Let the corresponding closed immersion be denoted $i_{\mathcal{C}_{\mathcal{S}}}$. Let $0_{\mathcal{E}_{\mathcal{S}}} : \mathcal{S} \to \mathcal{E}_{\mathcal{S}}$ denote the vertex of the cone stack $\mathcal{E}_{\mathcal{S}}$.

Definition 2.1. (Virtual structure sheaf) We let $\mathcal{O}^{virt}_{\mathcal{S}} = L0^*_{\mathcal{E}_{\mathcal{S}}}(\mathcal{O}_{C_{\mathcal{S}}}) = \mathcal{O}_{\mathcal{S}} \overset{L}{\underset{0^{-1}_{\mathcal{E}_{\mathcal{S}}}(\mathcal{O}_{\mathcal{E}_{\mathcal{S}}})}{\otimes}} 0^{-1}_{\mathcal{E}_{\mathcal{S}}}(\mathcal{O}_{C_{\mathcal{S}}})$ and call it the *virtual structure sheaf* of the stack $\mathcal{S}$. (Observe that in the G-equivariant case this defines a complex of G-equivariant $\mathcal{O}_{\mathcal{S}}$-modules.)

If one assumes *the resolution property*, one may obtain the following alternate description of $\mathcal{O}^{virt}_{\mathcal{S}}$. Clearly such a description also holds locally on the stack $\mathcal{S}$ *always.* Let $\mathcal{E}_1 = C(E^{-1})$ and let $\mathcal{O}_{\mathcal{E}_1} : \mathcal{S} \to \mathcal{E}_1$ denote the vertex of the cone stack $\mathcal{E}_1$. We let $C(E^\bullet)$ be defined by the cartesian square:

(2.0.9)
$$\begin{array}{ccc} C(E^\bullet) & \xrightarrow{i_{C(E^\bullet)}} & E^{-1\vee} \\ \downarrow & & \downarrow \\ \mathcal{C}_{\mathcal{S}} & \xrightarrow{i_{\mathcal{C}_{\mathcal{S}}}} & \mathcal{E}_{\mathcal{S}} \end{array}$$

In view of our hypotheses, $C(E^\bullet)$ has an induced action by the smooth group scheme G in the G-equivariant situation. Let $\mathcal{E}_1 = C(E^{-1})$ and let $\mathcal{O}_{\mathcal{E}_1} : \mathcal{S} \to \mathcal{E}_1$ denote the vertex of the cone stack $\mathcal{E}_1$. Now $\mathcal{O}^{virt}_{\mathcal{S}} \cong L0^*_{\mathcal{E}_1}(\mathcal{O}_{C(E^\bullet)}) = \mathcal{O}_{\mathcal{S}} \overset{L}{\underset{0^{-1}_{\mathcal{E}_1}(\mathcal{O}_{\mathcal{E}_1})}{\otimes}} 0^{-1}_{\mathcal{E}_1}(\mathcal{O}_{C(E^\bullet)})$ the equality holding locally on $\mathcal{S}_{smt}$ in general and globally on $\mathcal{S}$ if one has the resolution property.

One may now observe that $(\mathcal{S}, \mathcal{O}^{virt}_{\mathcal{S}})$ is a *DG*-stack in the sense of the appendix as follows. Recall that the sheaf $\mathcal{O}_{C_{\mathcal{S}}}$ is defined by a coherent sheaf of ideals in $\mathcal{O}_{\mathcal{E}_{\mathcal{S}}}$; locally on the étale site of the stack $\mathcal{S}$, one may find a resolution of $\mathcal{O}_{C_{\mathcal{S}}}$ by a complex of the form $\mathcal{O}_{\mathcal{E}_{\mathcal{S}}} \leftarrow P^{-1} \leftarrow \cdots P^{-n} \leftarrow \cdots$ with each P^{-i} a locally free coherent sheaf on $\mathcal{E}_{\mathcal{S}}$. Therefore, on applying $L0^*_{\mathcal{E}_{\mathcal{S}}}$ to the above complex (where $0_{E_{\mathcal{S}}}$ is the zero section $\mathcal{S} \to \mathcal{E}_{\mathcal{S}}$), one gets a complex of locally free coherent $\mathcal{O}_{\mathcal{S}}$-modules, again locally on the étale site. Therefore the cohomology sheaves of $\mathcal{O}^{virt,i}_{\mathcal{S}}$ are all coherent $\mathcal{O}_{\mathcal{S}}$-modules. Proposition 2.2 below shows that $\mathcal{H}^i(\mathcal{O}^{virt}_{\mathcal{S}}) = 0$ for $i << 0$. Making use of the hypothesis that the stack is Noetherian, one may now replace $\mathcal{O}^{virt}_{\mathcal{S}}$ upto quasi-isomorphism by a bounded complex of coherent $\mathcal{O}_{\mathcal{S}}$-modules: see [J-5] Example 2.11 for more details. Therefore the hypotheses in the Definition 6.1 are satisfied. We will denote $(\mathcal{S}, \mathcal{O}^{virt}_{\mathcal{S}})$ for simplicity by $\mathcal{S}^{virt}$.

Often in the literature, one uses a Gysin map $0^!_{\mathcal{E}_1}$ in the place of $L0^*_{\mathcal{E}_1}$. Therefore, we next proceed to define such *Gysin maps at the level of G-theory of algebraic stacks* and show that one could use it in the place of $L0^!_{\mathcal{E}_1}$. (Further properties of Gysin maps are discussed in the next section.)

Consider a cartesian square

(2.0.10)
$$\begin{array}{ccc} X' & \xrightarrow{x} & X \\ \downarrow g & & \downarrow f \\ Y' & \xrightarrow{y} & Y \end{array}$$

of Deligne-Mumford stacks where y is a *regular local immersion* of *smooth* algebraic stacks. We may assume all the stacks are provided with the action of a smooth group scheme G and that all the maps above are G-equivariant. We will assume that these are all non-dg stacks, or stacks in the usual sense. We will now define the refined *Gysin-map* (or often what will be simply called the Gysin map)

$$y^! : G(X,G) \rightarrow G(X',G) \tag{2.0.11}$$

If $\mathcal{O}_{Y'}$ is the structure sheaf of Y', $y_*(\mathcal{O}'_Y)\varepsilon\pi_0(K_{Y'}(Y,G)) \simeq \pi_0(G(Y,G))$. (Recall $K_{Y'}(Y,G)$ is the Waldhausen K-theory spectrum of perfect complexes on Y with supports in Y'.) Now pull-back of this class by f defines the class $f^*(y_*(\mathcal{O}_{Y'}))\varepsilon\pi_0(K_{X'}(X,G))$. Next observe the natural pairing $\circ : \pi_*K_{X'}(X,G) \otimes \pi_*G(X,G) \rightarrow \pi_*G_{X'}(X,G) \overset{\simeq}{\rightarrow} \pi_*G(X',G)$. Therefore, we define for any $F\varepsilon\pi_*G(X,G)$, $y^!(F)$ = the class of $F \circ f^*(y_*(\mathcal{O}_{Y'}))$ in $\pi_*G(X',\mathcal{O}_{X'})$. (In case f and g are the identity maps, one may verify, that $y^!(F)$ identifies with $y^*(F)$).)

In the above case, one may define a refined Gysin map

$$y^! : D_-(Mod(X,G)) \rightarrow D_{-,X'}(Mod(X,G)) \tag{2.0.12}$$

where $Mod(X,G)$ ($Mod(X',G)$) denotes the category of G-equivariant coherent $\mathcal{O}_X$ ($\mathcal{O}_{X'}$) modules. $D_-(Mod(X,G))$ ($D_{-,X'}(Mod(X,G))$) will denote the derived category of complexes in $Mod(X,G)$ that are bounded above (complexes in $Mod(X,G)$ that are bounded above and whose cohomology sheaves have support in X', respectively). We let $y^!(M) = M \overset{L}{\underset{\mathcal{O}_X}{\otimes}} Lf^*(y_*(\mathcal{O}_{Y'}))$.

Proposition 2.2. *Assume the situation in (2.0.10). Then $\mathcal{O}^{virt}_{\mathcal{S}}$ and $0^!_{\mathcal{E}_{\mathcal{S}}}(\mathcal{O}_{C_{\mathcal{S}}})$ define the same class in $\pi_0(G(\mathcal{S},\mathcal{O}_{\mathcal{S}}))$.*

Proof. Observe that $0_{\mathcal{E}_{\mathcal{S}},*}(L0^*_{\mathcal{E}_{\mathcal{S}}}(\mathcal{O}_{C(E_{\mathcal{S}})})) = \mathcal{O}_{\mathcal{E}_{\mathcal{S}},*}(\mathcal{O}_{\mathcal{S}}) \underset{\mathcal{O}_{\mathcal{E}_{\mathcal{S}}}}{\otimes} P^\bullet$ while $0^!_{\mathcal{E}_{\mathcal{S}}}(\mathcal{O}_{C(E_{\mathcal{S}})}) = Q^\bullet \underset{\mathcal{O}_{\mathcal{E}_{\mathcal{S}}}}{\otimes} \mathcal{O}_{C_{\mathcal{S}}}$ where $P^\bullet \rightarrow \mathcal{O}_{C_{\mathcal{S}}}$ and $Q^\bullet \rightarrow \mathcal{O}_{\mathcal{S}}$ are resolutions by complexes of locally free coherent $\mathcal{O}_{\mathcal{E}_{\mathcal{S}}}$-modules. Since $\mathcal{O}_{\mathcal{E}_{\mathcal{S}}}$ is a sheaf of commutative rings, it is clear that the two complexes $0_{\mathcal{E}_{\mathcal{S}},*}(L0^*_{\mathcal{E}_{\mathcal{S}}}(\mathcal{O}_{C_{\mathcal{S}}}))$ and $0^!_{\mathcal{E}_{\mathcal{S}}}(\mathcal{O}_{C_{\mathcal{S}}})$ are quasi-isomorphic as $\mathcal{O}_{\mathcal{E}_{\mathcal{S}}}$-modules. $0^!_{\mathcal{E}_{\mathcal{S}}}(\mathcal{O}_{C_{\mathcal{S}}})$ has supports contained in $\mathcal{S}$ and the push-forward map $0_{\mathcal{E}_{\mathcal{S}},*}$ is inverse to the isomorphism $\pi_*G_{\mathcal{S}}(\mathcal{E}_{\mathcal{S}}) \overset{\simeq}{\rightarrow} \pi_*G(\mathcal{S})$ defined by devissage in G-theory. Therefore the identification of $\mathcal{O}^{virt}_{\mathcal{S}}$ with $0^!_{\mathcal{E}_{\mathcal{S}}}(\mathcal{O}_{C_{\mathcal{S}}})$ as classes in $\pi_0(G(\mathcal{S}))$ is clear. ☐

We let $G(\mathcal{S}^{virt},G)$ denote the G-equivariant G-theory of the DG-stack $\mathcal{S}^{virt}$. Let $\mathbb{H}^*_{Br}(\mathcal{S}^{virt},G;\Gamma^h(*))$ denote a Bredon-style G-equivariant homology theory associated to the DG-stack $\mathcal{S}^{virt}$. Let $\tau = \tau^G_{\mathcal{S}^{virt}} : \pi_*(G(\mathcal{S}^{virt},G)) \rightarrow H^*_{Br}(\mathcal{S}^{virt},G;\Gamma^h(*))$ denote the Riemann-Roch transformation considered in [J-5] section 8.

2.0.13. It will be important to use the relationship between the Riemann-Roch transformation and the local Chern character to be able to define the *virtual fundamental classes*. We will do this presently. Let $\mathcal{S}$ denote a separated Deligne-Mumford stack with coarse moduli space $\mathfrak{M}$ and let $p : \mathcal{S} \rightarrow \mathfrak{M}$ denote the obvious map. Let $\bar{F}$ denote a perfect complex on $\mathfrak{M}$ and let $F = p^*(\bar{F})$ denote its inverse image on $\mathcal{S}$. Then $\tau(F)$ corresponds to the map that sends a perfect complex $\mathcal{E}$ on the stack $\mathcal{S}$ to $\tau_{\mathfrak{M}}(p_*(p^*(\bar{F})) \otimes \mathcal{E}) = \tau_{\mathfrak{M}}(\bar{F} \otimes p_*(\mathcal{E})) = \tau_{\mathfrak{M}}(\bar{F}).ch^{\tilde{\mathfrak{M}}|\mathfrak{M}}(i_*p_*(\mathcal{E}))$, where $i : \mathfrak{M} \rightarrow \tilde{\mathfrak{M}}$ is a closed immersion of $\mathfrak{M}$ into a smooth scheme, and $ch^{\tilde{\mathfrak{M}}|\mathfrak{M}}$ is the local Chern character. In particular, if $\mathcal{E}$ has supports in a closed algebraic sub-stack $\mathcal{S}_0$ of $\mathcal{S}$ with pure codimension c, $i_*p_*(\mathcal{E})$ has supports in a closed sub-scheme of $\mathfrak{M}$ of pure codimension c. Therefore, in this case, (in view of our cohomological semi-purity hypothesis - see [J-5] section 3)) $ch^{\tilde{\mathfrak{M}}|\mathfrak{M}}(i_*p_*(\mathcal{E}))(j)$ is trivial in $H^i_{\mathfrak{M}}(\tilde{\mathfrak{M}};\Gamma(j))$ for $j < c$. If, in addition, $ch^{\tilde{\mathfrak{M}}|\mathfrak{M}}(i_*p_*(\mathcal{E}))(c) \neq 0$ as well, it follows that, in this case the *non-trivial term of highest weight* in $\tau(F)$ is in $d - c$, where d= the weight of the non trivial term in $\tau_{\mathfrak{M}}(\bar{F})$ of highest weight. Moreover if the non-trivial term in $\tau_{\mathfrak{M}}(\bar{F})$ of highest weight is in weight d and degree $2d$, the non-trivial term of highest weight in $\tau(F)$ is in weight $d - c$ and degree $2d - 2c$.

Definition 2.3. Let d denote the virtual dimension of the stack $\mathcal{S}$ with respect to the given obstruction theory. We define *the virtual fundamental class* of the stack $\mathcal{S}$ in Bredon homology to be $\tau(\mathcal{O}_{\mathcal{S}^{virt}})_{2d}(d)$, i.e. the part of $\tau(\mathcal{O}_{\mathcal{S}^{virt}})$ in degree $2d$ and weight d. This will be denoted $[\mathcal{S}]^{virt}_{Br}$. If $p : \mathcal{S} \rightarrow \mathfrak{M}$ denotes the obvious map from the stack to its moduli space, we will also let $[\mathcal{S}]^{virt}_{Br}$ denote $p_*(\tau(\mathcal{O}_{\mathcal{S}^{virt}})_{2d}(d))\varepsilon H^{Br}_{2d}(\mathfrak{M},\Gamma(d))$.

The term of highest weight i and degree $2i$ in $\tau(\mathcal{O}^{virt}_{\mathcal{S}})$ that is non-trivial will be called the *the leading term of* $\tau(\mathcal{O}^{virt}_{\mathcal{S}})$. We may define this more generally as follows. Let X denote any scheme of finite type over the base scheme S and let $\alpha\varepsilon H^{et}_*(X,\Gamma(\bullet))$ where $\Gamma(\bullet)$ is a complex as in [J-5] ( ). Then the *leading term (bottom term)* of

α will be defined to be the non-zero term of highest weight (lowest weight, respectively). Let $\beta \varepsilon H_*^{Br}(\mathcal{S}, \Gamma(\bullet))$ or a class in $H_*^{et}(\mathcal{S}, \Gamma(\bullet)) \cong H_*^{et}(\mathfrak{M}, \Gamma(\bullet))$ where $\mathcal{S}$ is any separated Deligne-Mumford stack as in (2.0.13) and $\mathfrak{M}$ is its coarse moduli space. Then the *leading term* of β denotes the term of highest weight in β. It follows from [J-5] Proposition 6.14 that any class $\beta \varepsilon H_*^{Br}(\mathcal{S}, \Gamma(\bullet))$ has a leading term; the hypotheses in [J-5] (3.0.4) imply the same conclusion for classes $\beta \varepsilon H_*^{et}(\mathcal{S}, \Gamma(\bullet))_{\mathbb{Q}} \cong H_*^{et}(\mathfrak{M}, \Gamma(\bullet))_Q$.

We proceed to compare the virtual fundamental class defined above with the virtual fundamental class defined cycle theoretically elsewhere. The following theorems Theorem 2.6 will be the key to this. First we make the following observation.

Proposition 2.4. *Assume the homology theories satisfy the hypotheses as in [J-5] section 3; in particular they come equipped with localization sequence for algebraic spaces and hence simplicial algebraic spaces.*

If $p : \mathcal{S} \rightarrow \mathfrak{M}$ is the finite map from a Deligne-Mumford stack to its coarse-moduli space and $x : X \rightarrow \mathcal{S}$ is an atlas with $B_x\mathcal{S}$ its associated simplicial classifying space, the induced map $p_* : H_*^{et}(\mathcal{S}, \Gamma(\bullet)) \otimes \mathbb{Q} \cong \mathbb{H}_{et}^*(B_x\mathcal{S}, \Gamma^h(\bullet)) \otimes \mathbb{Q} \rightarrow H_*^{et}(\mathfrak{M}, \Gamma(\bullet)) \otimes \mathbb{Q}$ is an isomorphism.

Proof. We may assume the proposition holds for all closed algebraic sub-stacks of $\mathcal{S}$ whose moduli spaces are of dimension less than that of $\mathfrak{M}$. Therefore, the existence of localization sequences reduce to proving the proposition for finite quotient stacks $[X/G]$ for the action of a finite constant group-scheme on a scheme X and where the map p in the proposition is also flat. In this case, the isomorphism in the proposition is clear. □

Next we make the following observations.

2.0.14.

- Let X denote a scheme of finite type over a field k. Let $Z_*(X, 0)$ denote the cycle complex of X. Now the *support* of a class $\alpha \varepsilon Z_*(X)$ is the union of the irreducible closed sub-schemes of X appearing in the cycle α. Next observe that if $[\alpha] \varepsilon CH_n(X)$, with $[\alpha] \neq 0$, is represented by the algebraic cycle α, the dimension of the support of α equals n. For any homology theory for schemes considered in [J-5] sections 3 and 4 defined as the étale hypercohomology of X with respect to a complex $\Gamma^h(\bullet)$, the weight of the of the cycle class $cycl([\alpha]) \varepsilon \underset{n}{\oplus} H_{2n}^{et}(X, \Gamma(n))$ is also $n =$ the dimension of support of α.
- Next let $\mathcal{S}$ denote a separated Deligne-Mumford stack of finite type over a field k with coarse moduli space $\mathfrak{M}$ and $p : \mathcal{S} \rightarrow \mathfrak{M}$ the obvious projection. Let $CH_*(\mathcal{S})$ denote the Chow groups of the stack $\mathcal{S}$ defined in any of the equivalent ways as in [Vi-1] or [J-2]. The weight of any class $[\alpha] \varepsilon CH_n(\mathcal{S})$ is n and if $[\alpha] \neq 0$ this equals the dimension of the support of any representative of the algebraic cycle $p_*([\alpha]) \varepsilon CH_n(\mathfrak{M})$. Therefore, if $\beta = cycl([\alpha]) \varepsilon H_{2n}^{et}(\mathcal{S}, \Gamma(n))$ for any homology theory defined as above with respect to the complex $\Gamma^h(\bullet)$, then the weight of β is the dimension of the support for any representative for the algebraic cycle $p_*([\alpha])$.

Corollary 2.5. *Let $\mathcal{S}$ denote a separated Deligne-Mumford stack of finite type over a field k. (i) Assume that $[\alpha], [\beta] \varepsilon \underset{n}{\oplus} CH_n(\mathcal{S})$ are represented by the cycles α and β. Assume that each irreducible component of the support of α and the support of β coincide on an open nonempty sub-stack $\mathcal{S}_0$ of $\mathcal{S}$ and $dim(\mathcal{S} - \mathcal{S}_0) < dim(\mathcal{S})$ and that the restrictions of $[\alpha]$ and $[\beta]$ to $CH_n(\mathcal{S}_0)$ are equal. Then the leading terms of $cycl([\alpha])$ and $cycl([\beta])$ have the same weight where cycl denotes the cycle map into any homology theory defined as above.*

(ii) (ii) Let $\gamma \varepsilon \underset{i>c}{\oplus} H_{et}^{2i}(\mathcal{S}, \Gamma(i))$ for some $c \geq 0$. Then the leading term of $\gamma \circ \alpha$ has weight at most $a - c$ where a denotes the dimension of support of α.

Proof. Clearly it suffices to assume $\mathcal{S} = X$ is a scheme of finite type over k. We may also assume that $[\alpha]$ and $[\beta]$ belong to $CH_n(X)$ for some fixed integer n. Let α (β) be a representative for $[\alpha]$ ($[\beta]$, respectively). Let the support of α (β) be denoted A (B, respectively). These are closed sub-schemes of X. Let $Y = A \cup B$. Observe that $dim(A) = dim(B) = dim(Y) = n$. Now the localization sequence in Chow-groups for Y, X and $X - Y$ will show that the class $[\alpha] = i_*([\bar{\alpha}])$ and $[\beta] = i_*([\bar{\beta}])$ for classes $\bar{\alpha}$, $\bar{\beta} \varepsilon CH_*(Y)$ where $i : Y \rightarrow X$ is the obvious closed immersion. Let X_0 denote the open sub-scheme of X which intersects each component of Y non-trivially. Now another localization sequence argument for $Y - Y \cap X_0$, Y and $Y \cap X_0$ shows that the difference $[\bar{\alpha}] - [\bar{\beta}] = i_*([\gamma])$ for some class $[\gamma] \varepsilon CH_*(Y - Y \cap X_0)$. However, the dimension of $Y - Y \cap X_0$ is strictly lower than the dimension of $Y =$ the weights of $[\bar{\alpha}]$ and $[\bar{\beta}]$ so that we obtain the equality $[\bar{\alpha}] = [\bar{\beta}]$ in $CH_n(Y)$. Therefore $cycl([\alpha]) = i_*cycl([\bar{\alpha}]) = i_*cycl([\bar{\beta}]) = cycl([\beta])$ in any homology theory considered above.

The proof of (i) follows readily from the above observations. (ii) is clear from the observation that the pairing between cohomology and homology sends $H^{2j}_{et}(\mathcal{S},\Gamma(j)) \otimes H^{et}_{2i}(\mathcal{S},\Gamma(i))$ to $H^{et}_{2i-2j}(\mathcal{S},\Gamma(j-i))$. □

In what follows, we will let $\Gamma(\bullet) = \{\Gamma(r)|r\}$ denote either one of the following: (i) a collection of complexes as in [J-5] section 3, defining a Bloch-Ogus style cohomology theory on all algebraic spaces and which extends to the étale sites of all Deligne-Mumford stacks *or* (ii) the higher cycle complex. In the first situation, we defined a Chern-character map $Ch : \pi_0 K(\mathcal{S}) \rightarrow \mathbb{H}^*_{et}(\mathcal{S},\Gamma(\bullet)) \otimes \mathbb{Q}$ in [J-2] section 4. In the second situation we define an (operational) Chern character map Ch on $\pi_0 K(\mathcal{S})$: this is discussed in appendix B. In the following discussion, Ch will denote either one of these depending on the situation.

Theorem 2.6. *Assume in addition to the above that the following hold:*

$\mathcal{S} = [X/G]$ is a finite quotient stack with the map $X \rightarrow \mathcal{S}$ finite and that the map $p_* : H^{et}_*(\mathcal{S},\Gamma(\bullet)) \otimes \mathbb{Q} \rightarrow H^{et}_*(\mathfrak{M},\Gamma(\bullet)) \otimes \mathbb{Q}$ is an isomorphism.

Let $\mathcal{G}\varepsilon\pi_0(G(\mathcal{S}))$, $\mathcal{F}\varepsilon\pi_0(K(\mathcal{S}))$ and $\alpha\varepsilon\mathbb{H}^n_{et}(\mathcal{S},\Gamma(\bullet)) \otimes \mathbb{Q}$ for a fixed integer n. Then

$$\alpha \circ \sigma_*(\tau_{\mathcal{S},\mathfrak{M}}(\mathcal{F} \circ \mathcal{G})) = \alpha \circ \sigma_*(ch(\mathcal{F}) \circ \tau_{\mathcal{S},\mathfrak{M}}(\mathcal{G})) \\ = \alpha \circ Ch(\mathcal{F}) \circ \sigma_*(\tau_{\mathcal{S},\mathfrak{M}}(\mathcal{G})) \tag{2.0.15}$$

Moreover $\sigma_* : H^{Br}(\mathcal{S},\Gamma(\bullet)) \rightarrow H^{et}_*(\mathcal{S},\Gamma(\bullet)) \otimes \mathbb{Q}$ is the map from Bredon homology to étale homology considered in [J-5] Theorem 6.15, Ch denotes the Chern-character as defined above, ch denotes the Chern-character with values in Bredon cohomology and $\circ$ denotes pairings between homology and cohomology.

Next let $\mathcal{E}$ denote a vector bundle on the stack $\mathcal{S}$ and let $\mathcal{G}\varepsilon\pi_0(G(\mathcal{E}))$, $\mathcal{F}\varepsilon\pi_0(K(\mathcal{E}))$ and $\alpha\varepsilon\mathbb{H}^n_{et}(\mathcal{E},\Gamma(\bullet))$ for a fixed integer n. Then

$$\alpha \circ \sigma_*(\tau_{\mathcal{E},\bar{\mathcal{E}}}(\mathcal{F} \circ \mathcal{G})) = \alpha \circ \sigma_*(ch(\mathcal{F}) \circ \tau_{\mathcal{E},\bar{\mathcal{E}}}(\mathcal{G})) \\ = \alpha \circ Ch(\mathcal{F}) \circ \sigma_*(\tau_{\mathcal{E},\bar{\mathcal{E}}}(\mathcal{G})) \tag{2.0.16}$$

where $\bar{\mathcal{E}}$ denotes a coarse moduli space for the vector bundle $\mathcal{E}$. σ_* (Ch) now denotes the corresponding maps associated to the stack $\mathcal{E}$

Proof. We will prove the statement in (2.0.15) first. Recall that the map $p_*(\tau_{\mathcal{S},\mathfrak{M}}(\mathcal{E}))$ is the map $\mathcal{G} \mapsto \tau_{\mathfrak{M}}(p_*(p^*(\mathcal{G}) \circ \mathcal{E})) = \tau_{\mathfrak{M}}(\mathcal{G} \circ p_*(\mathcal{E})) = ch^{Br}(\mathcal{G}) \circ \tau_{\mathfrak{M}}(p_*(\mathcal{E}))$, for $\mathcal{G}\varepsilon\Gamma(\mathfrak{M}, i^{-1}\pi_*(K(\quad)_{\tilde{\mathfrak{M}}}))$. (Here $\mathfrak{M} \rightarrow \tilde{\mathfrak{M}}$ is a closed immersion into a smooth scheme and ch^{Br} denotes the Chern-character with values in the cohomology of $\mathfrak{M}$.) Therefore, one identifies the class $p_*(\tau_{\mathcal{S},\mathfrak{M}}(\mathcal{E}))$ with the class $\tau_{\mathfrak{M}}(p_*(\mathcal{E}))$. Let $p^*_{et} : \mathbb{H}^*_{et}(\mathfrak{M},\Gamma^h(\bullet)) \otimes \mathbb{Q} \rightarrow \mathbb{H}^*_{et}(\mathcal{S}, p^*\Gamma^h(\bullet)) \otimes \mathbb{Q} \cong$ denote the obvious map. The definition of the map σ_* from Bredon homology to étale homology shows that one obtains:

$$\sigma_*(\tau_{\mathcal{S},\mathfrak{M}}(\mathcal{E})) = p^*_{et}(\tau_{\mathfrak{M}}(p_*(\mathcal{E}))) \tag{2.0.17}$$

(In view of our hypothesis the map p^*_{et} is also an isomorphism with inverse defined by $(1/d)p_{et*}$, where d is the degree of the map p.)

Let n be the degree of the map $x : X \rightarrow \mathcal{S}$. Now we obtain

$n\tau_{\mathfrak{M}}(p_*(\mathcal{F} \circ \mathcal{E})) = \tau_{\mathfrak{M}}(p_* x_* x^*(\mathcal{F} \circ \mathcal{E}))$

$= p_* x_* \tau_X(x^*(\mathcal{F} \circ \mathcal{E})) = p_* x_* \tau_X(x^*(\mathcal{F}) \circ x^*(\mathcal{E}))$

The first equality holds since the map x is finite étale of degree n. The second equality holds by Riemann-Roch applied to the composite finite map $X \rightarrow \mathcal{S} \rightarrow \mathfrak{M}$. In view of our hypothesis, it follows that

$\alpha \circ np^*_{et}(\tau_{\mathfrak{M}}(p_*(\mathcal{F} \circ \mathcal{E}))) = \alpha \circ dx_* \tau_X(x^*(\mathcal{F}) \circ x^*(\mathcal{E}))$

$= \alpha \circ dx_*(Ch(x^*(\mathcal{F}) \circ \tau_X(x^*\mathcal{E}))) = \alpha \circ d(Ch(\mathcal{F}) \circ x_*(\tau_X(x^*\mathcal{E})))$

The last but one equality holds by the formula (see [Ful] Theorem 18.2) :$\tau(\mathcal{F} \circ \mathcal{E}) = Ch(\mathcal{F}) \circ \tau(\mathcal{E})$ for a class $\mathcal{F}\varepsilon\pi_0(K(X))$ and $\mathcal{E}\varepsilon\pi_0(G(X))$. If we take $\mathcal{F} = \mathcal{O}_{\mathcal{S}}$ and $\alpha = 1$ in the above formula, we obtain: $dx_*\tau_X(x^*(\mathcal{E})) = np^*_{et}(\tau_{\mathfrak{M}}(p_*(\mathcal{E})))$. Substituting this into the formula for $np^*_{et}(\tau_{\mathfrak{M}}(p_*(\mathcal{F} \circ \mathcal{E})))$, we obtain the formula

$$\alpha \circ p^*_{et}\tau_{\mathfrak{M}}(p_*(\mathcal{F} \circ \mathcal{E})) = \alpha \circ Ch(\mathcal{F}) \circ p^*_{et}(\tau_{\mathfrak{M}}(p_*(\mathcal{E}))) \tag{2.0.18}$$

In view of the identification in (2.0.17), this proves the assertion in (2.0.15).

Next we consider the assertion in (2.0.16). To keep our discussion brief, we will assume that $\alpha = 1$. For this first observe that the vector bundle $\mathcal{E}$ is also a separated Deligne-Mumford stack so that it has a coarse moduli space $\bar{\mathcal{E}}$; by our standing hypothesis in 1.0.1, this will be a vector bundle over $\mathfrak{M}$ and hence quasi-projective as a scheme. Since the obvious induced map $\mathcal{E} \to \bar{\mathcal{E}} \underset{\mathfrak{M}}{\times} \mathcal{S}$ is purely inseparable and surjective, and all cohomology we consider is with rational coefficients, we will identify $\mathcal{E}$ with $\bar{\mathcal{E}} \underset{\mathfrak{M}}{\times} \mathcal{S}$. Let $\mathcal{E}_X = X \underset{\mathcal{S}}{\times} \mathcal{E}$ and let $x_{\mathcal{E}} : \mathcal{E}_X \to \mathcal{E}$ and $p_{\mathcal{E}} : \mathcal{E} \to \bar{\mathcal{E}}$ denote the obvious induced maps. Now we obtain the following identifications:

$$np_{\mathcal{E}*}\tau_{\mathcal{E},\bar{\mathcal{E}}}(\mathcal{F} \circ \mathcal{G}) = n\tau_{\bar{\mathcal{E}}}(p_{\mathcal{E}*}(\mathcal{F} \circ \mathcal{G})) = \tau_{\bar{\mathcal{E}}}(p_{\mathcal{E}*}(x_{\mathcal{E}*}x_{\mathcal{E}}^*(\mathcal{F} \circ \mathcal{G})))$$

$$= p_{\mathcal{E}*}x_{\mathcal{E}*}\tau_{\mathcal{E}_X}(x_{\mathcal{E}}^*(\mathcal{F} \circ \mathcal{G})) = p_{\mathcal{E}*}x_{\mathcal{E}*}\tau_{\mathcal{E}_X}(x_{\mathcal{E}}^*(\mathcal{F}) \circ x_{\mathcal{E}}^*(\mathcal{G}))$$

$$= p_{\mathcal{E}*}x_{\mathcal{E}*}(Ch(x_{\mathcal{E}}^*(\mathcal{F})) \circ \tau_{\mathcal{E}_X}(x_{\mathcal{E}}^*(\mathcal{G}))) = p_{\mathcal{E}*}(Ch(\mathcal{F}) \circ x_{\mathcal{E}*}\tau_{\mathcal{E}_X}(x_{\mathcal{E}}^*(\mathcal{G})))$$

The first equality above is from the definition of the Riemann-Roch transformation as in [J-5] and the second equality is from the observation $x_{\mathcal{E}}$ is finite étale of degree n and the fifth is from [Ful] Theorem 18.2 as above. The remaining identifications are clear. Next observe that $p_{\mathcal{E}*} : H_*^{et}(\mathcal{E}, \Gamma(\bullet)) \otimes \mathbb{Q} \to H_*^{et}(\bar{\mathcal{E}}, \Gamma(\bullet)) \otimes \mathbb{Q}$ is an isomorphism with inverse provided by the map $(1/d)p_{\mathcal{E}}^*$. Therefore, one obtains the identification

$$np_{\mathcal{E}}^*(\tau_{\bar{\mathcal{E}}}(p_{\mathcal{E}*}(\mathcal{F} \circ \mathcal{G}))) = Ch(\mathcal{F}) \circ dx_{\mathcal{E}*}(\tau_{\mathcal{E}_X}(x_{\mathcal{E}}^*(\mathcal{G})))$$

Taking $\mathcal{F} = \mathcal{O}_{\mathcal{E}}$, one obtains $np_{\mathcal{E}}^*\tau_{\bar{\mathcal{E}}}(p_{\mathcal{E}*}(\mathcal{G})) = dx_{\mathcal{E}*}(\tau_{\mathcal{E}_X}(x_{\mathcal{E}}^*(\mathcal{G})))$. Substituting this into the last formula proves the assertion in (2.0.16). □

Definition 2.7. (Euler class). We define the *Euler class* of a vector bundle F to be the bottom term of $Ch(\lambda_{-1}(F^\vee))$.

Remark 2.8. See [F-L] Chapter 1 for more details on this definition in the setting of vector bundles on schemes.

We begin with the following observation:

2.0.19. Next observe from [J-5] Proposition 6.5, that the Riemann-Roch transformation localizes on $\tilde{\mathfrak{M}}_{et}$ and hence on $\mathfrak{M}_{et}$. Locally on $\mathfrak{M}_{et}$ one knows that the Deligne-Mumford stack $\mathcal{S}$ is a quotient stack. This implies that for each irreducible component of $\mathfrak{M}$, one may find a nonempty Zariski open sub-scheme U and a finite étale surjective map $\tilde{U} \to U$ so that the stack $\tilde{\mathcal{S}}_U = \mathcal{S} \underset{\mathfrak{M}}{\times} U \underset{U}{\times} \tilde{U} \cong \mathcal{S} \underset{\mathfrak{M}}{\times} \tilde{U}$ is a finite quotient stack. Let $\pi : \tilde{\mathcal{S}}_U \to \mathcal{S} \underset{\mathfrak{M}}{\times} U = \mathcal{S}_U$ denote the induced map. Then the induced map $p^* : \mathbb{H}_{et}^*(\mathcal{S} \underset{\mathfrak{M}}{\times} U, \Gamma^h(\bullet))_{\mathbb{Q}} \to \mathbb{H}_{et}^*(\mathcal{S} \underset{\mathfrak{M}}{\times} \tilde{U}, \Gamma^h(\bullet))_{\mathbb{Q}}$ is a split injection. Therefore if one has equality of two classes α, $\beta \varepsilon \mathbb{H}_{smt}^*(\mathcal{S}, \Gamma^h(\bullet))$ after restriction to $\mathcal{S}_U$ and pull back by p^*, then one has equality of α and β on restriction to the stack $\mathcal{S}_U$.

Corollary 2.9. *Let $\mathcal{S}$ denote a separated Deligne-Mumford stack.*

(i) Assume $\mathcal{G}M$ is a bounded complex of coherent $\mathcal{O}_{\mathcal{E}}$-modules and $\mathcal{F}$ is a bounded complex of vector bundles on $\mathcal{S}$. Then the leading term of $\sigma_*(\tau_{\mathcal{S},\mathfrak{M}}(\mathcal{F} \circ \mathcal{G}))$ = the leading term of $\sigma_*(ch(\mathcal{F}) \circ \tau_{\mathcal{S},\mathfrak{M}}(\mathcal{G}))$ = (the bottom term of $Ch(\mathcal{F})) \circ$ (the leading term of $(\sigma_*(\tau_{\mathcal{S},\mathfrak{M}}(\mathcal{G})))$. In particular, if $\mathcal{F}$ denotes the class $\lambda_{-1}(F)$ of a vector bundle F on $\mathcal{S}$, one obtains the formula:

leading term of $\sigma_*(\tau_{\mathcal{S},\mathfrak{M}}(\mathcal{F} \circ \mathcal{G})) = Eu(F^\vee)) \circ$ (the leading term of $(\sigma_*(\tau_{\mathcal{S},\mathfrak{M}}(\mathcal{G})))$.

(ii) Next assume that $\mathcal{E}$ a vector bundle on $\mathcal{S}$, $\mathcal{G}$ a bounded complex of coherent $\mathcal{O}_{\mathcal{E}}$-modules and $\mathcal{F}$ a bounded complex of vector bundles on $\mathcal{E}$. Assume that the support of $\mathcal{F} \otimes \mathcal{G}$ is contained in $\mathcal{S}$.

Then the leading term of $\sigma_*(\tau_{\mathcal{E},\bar{\mathcal{E}}}(\mathcal{F} \circ \mathcal{G}))$ = the leading term of $\sigma_*(ch(\mathcal{F}) \circ \tau_{\mathcal{E},\bar{\mathcal{E}}}(\mathcal{G}))$ = (the bottom term of $Ch(\mathcal{F})) \circ$ (the leading term of $(\sigma_*(\tau_{\mathcal{E},\bar{\mathcal{E}}}(\mathcal{G})))$. In particular, if $\mathcal{F}$ denotes the class $\lambda_{-1}(F)$ of a vector bundle F on $\mathcal{E}$, one obtains the formula:

leading term of $\sigma_*(\tau_{\mathcal{E},\bar{\mathcal{E}}}(\mathcal{F} \circ \mathcal{G})) = Eu(F^\vee)) \circ$ (the leading term of $(\sigma_*(\tau_{\mathcal{E},\bar{\mathcal{E}}}(\mathcal{G})))$.

Proof. Since the proof of (i) is entirely similar to that of (ii) we will only consider (ii). The first equality in (ii) is clear in view of the formula $\tau_{\mathcal{E},\bar{\mathcal{E}}}(\mathcal{F} \circ G) = ch(\mathcal{G}) \circ \tau_{\mathcal{E},\bar{\mathcal{E}}}(\mathcal{G})$, where ch denotes the Chern character map defined on Bredon cohomology: see [J-5] Remark 6.10.

To prove the remaining assertions, first we consider the Riemann-Roch transformation and the Chern character with values in the Chow-groups of the stack $\mathcal{E}$ where the support of classes in the Chow groups is defined as in 2.0.14. For each irreducible component S_i of maximal dimension of $Support(\mathcal{F} \otimes \mathcal{G})$, choose a Zariski open neighborhood U *open* in the irreducible component of $\mathfrak{M}$ that contains S_i so that there is a finite étale surjective map $\pi : \tilde{U} \to U$ and the induced stack $\tilde{\mathcal{S}}_U = \mathcal{S} \underset{\mathfrak{M}}{\times} \tilde{U}$ is a finite quotient stack. Let $\mathcal{E}_0$ denote the pull-back of $\mathcal{E}$ to $\tilde{\mathcal{S}}$ and let $\mathcal{F}_0$ ($\mathcal{G}_0$) denote the pull-back of $\mathcal{F}$ ($\mathcal{G}$) to $\mathcal{E}_0$. On $\tilde{\mathcal{S}}_U$ and hence on $\mathcal{S}_U$ one has strict equality:

$$\sigma_*(\tau_{\mathcal{E}_0, \bar{\mathcal{E}}_0}(\mathcal{F}_0 \circ \mathcal{G}_0)) = Ch(\lambda_{-1}(\mathcal{F}_0)) \circ \sigma_*(\tau_{\mathcal{E}_0, \bar{\mathcal{E}}_0}(\mathcal{G}_0)) \tag{2.0.20}$$

It follows that U intersects non-trivially with an irreducible component T_i of the support of $Ch(\lambda_{-1}(\mathcal{F})) \circ \sigma_*(\tau_{\mathcal{E}, \bar{\mathcal{E}}}(\mathcal{G}))$. In particular the leading terms of the expressions on either side identify on $\mathcal{S}_U$. It follows (see 2.0.14) that T_i is a component of maximal dimension in the support of $Ch(\lambda_{-1}(F)) \circ \sigma_*(\tau_{\mathcal{E}, \bar{E}}(\mathcal{G}))$. Since U is chosen to intersect non-trivially with the component S_i, it follows that the $U \cap S_i$ is open and dense in S_i and similarly $U \cap T_i$ is open and dense in T_i. Since one has the strict equality $\sigma_*(\tau_{\mathcal{E}_0, \bar{\mathcal{E}}_0}(\mathcal{F}_0 \circ \mathcal{G}_0)) = Ch(\lambda_{-1}(\mathcal{F}_0)) \circ \sigma_*(\tau_{\mathcal{E}_0, \bar{\mathcal{E}}_0}(\mathcal{G}_0))$ on $\mathcal{S}_U$, it follows that $dim(U \cap \mathcal{S}_i) = dim(U \cap T_i)$ and therefore, the dimension of T_i = the dimension of S_i.

Observe that one may cover $\mathfrak{M}$ by open sets of the form U as above. Therefore one may drop any components in a cycle representing $Ch(\lambda_{-1}(\mathcal{F})) \circ \tau_{\mathcal{E}, \bar{\mathcal{E}}}(\mathcal{G})$ on which the above class is zero and assume that each irreducible component of maximal dimension in the support of $Ch(\lambda_{-1}(\mathcal{F})) \circ \tau_{\mathcal{E}, \bar{\mathcal{E}}}(\mathcal{G})$ intersects non-trivially some open set U chosen as above (which intersects non-trivially with an irreducible component of the support of $\mathcal{F} \otimes \mathcal{G}$). It follows readily now that the dimension of supports of $\mathcal{F} \otimes \mathcal{G}$ and $Ch(\lambda_{-1}(\mathcal{F})) \circ \sigma_*(\tau_{\mathcal{E}, \bar{\mathcal{E}}}(\mathcal{G}))$ are the same.

Therefore one may apply Corollary 2.5(i) to derive the formula in the second statement of the Corollary when the Riemann-Roch transformation and the Chern character take values in the Chow groups of the stack. The second statement involving the Euler class follows immediately from its definition. The corresponding formulae in other cohomology theories then follow by applying a cycle map. □

Next we proceed to compare the virtual fundamental class defined as in (2.3) with the virtual fundamental class defined cycle theoretically assuming the resolution property holds. Let $C(E^\bullet)$ denote the cone in $\mathcal{E}_1$ defined as in (2.0.9). This is a closed sub-scheme of the vector bundle $\mathcal{E}_1$ and the support of $(\mathcal{O}_{C(E^\bullet)} \otimes \lambda_{-1}(pr^*(E^{-1})))$ is contained in $\mathcal{S}$. The cycle-theoretic definition of the virtual fundamental class is $0^!([C(E^\bullet)])$ where $[C(E^\bullet)]$ denotes the fundamental class of $C(E^\bullet)$ and $0^!$ is the *refined Gysin map* as in [Ful] Chapter 6. Recall we have already identified $[C(E^\bullet)]$ with the leading term of $\sigma_*(\tau_{\mathcal{E}, \bar{\mathcal{E}}}(\mathcal{O}_{C(E^\bullet)}))$: see [J-5] Theorem 1.1. Moreover the definition of Gysin maps as in section 3 (below) shows (since we only consider the term of top weight and degree) that one has the identification:

$$[\mathcal{S}]^{virt} = (\textit{the bottom term of } Ch(\lambda_{-1}(pr^*(\mathcal{E}^{-1})))) \circ (\textit{leading term of } \sigma_*(\tau_{\mathcal{E}, \bar{\mathcal{E}}}(\mathcal{O}_{C(E^\bullet)})) \tag{2.0.21}$$

Corollary 2.10. *Assume the above situation. Then* $[\mathcal{S}]^{virt}$ = *the leading term of* $\sigma_*(\tau_{\mathcal{E}, \bar{E}}(\mathcal{O}^{virt}_{\mathcal{S}}))$.

Moreover, $\sigma_*(\tau(\mathcal{O}_{\mathcal{S}^{virt}}))(i)_{2i} = 0$ *for all* $i > d$, *where* d = *the virtual dimension. If* $[C(E^\bullet)] \varepsilon H^{Br}_*(\mathcal{E}_1, \Gamma(*)) \cong \mathbb{H}^{et}_*(\mathcal{E}_1, \Gamma(*))$ *is nonzero, the leading term of* $\tau(\mathcal{O}_{\mathcal{S}^{virt}})$ *is in weight* = d *and degree* $2d$, *where* d = *the virtual dimension.*

Proof. Recall (see 2.2) that $\mathcal{O}^{virt}_{\mathcal{S}} = \lambda_{-1}(pr^*(\mathcal{E}^{-1})) \otimes \mathcal{O}_{C(E^\bullet)}$ at least as classes in $\pi_0(G(\mathcal{S}))$. Moreover the support of this is contained in $\mathcal{S}$. Therefore, the hypotheses of Corollary 2.9 are satisfied. Therefore,

$$\text{the leading term of } \sigma_*(\tau_{\mathcal{E}, \bar{E}}(\mathcal{O}^{virt}_{\mathcal{S}})) = (\textit{the bottom term of } Ch(\lambda_{-1}(pr^*(\mathcal{E}^{-1})))) \circ (\textit{leading term of } \sigma_*(\tau_{\mathcal{E}, \bar{\mathcal{E}}}(\mathcal{O}_{C(E^\bullet)}))$$

$$= [\mathcal{S}]^{virt}.$$

The second assertion in the corollary now follows readily once we observe that the $\lambda_{-1}(pr^*(E^{-1}))$ is a Koszul resolution of $O_{\mathcal{S}}$ by $\mathcal{O}_{\mathcal{E}_1}$-modules, so that the complex $\lambda_{-1}(pr^*(E^{-1}))$ has supports in $\mathcal{S}$ which is of codimension = the rank of the vector bundle E^1 in $\mathcal{E}_1$. It is observed in (4.0.25) that the dimension of $C(E^\bullet)$ is $b + rank(E^0)$, where $b = dim(B)$, with B being the base-scheme. Therefore, we obtain the second assertion in the corollary. □

Corollary 2.11. *Assume the hypotheses of Theorem 1.5. Then the leading term of* $\sigma_*(\tau_{\mathcal{T}, \mathfrak{M}_\mathcal{T}}(\mathcal{O}^{virt}_{\mathcal{T}} \circ \lambda_{-1}(K^{-1}))) = Eu(K_1) \circ$*(the leading term of* $\sigma_*(\tau_{\mathcal{T}, \mathfrak{M}_\mathcal{T}}(\mathcal{O}^{virt}_{\mathcal{T}})))$ *and similarly the leading term of* $\sigma_*(\tau_{\mathcal{S}, \mathfrak{M}_\mathcal{S}}(\mathcal{O}^{virt}_{\mathcal{S}} \circ \lambda_{-1}(K^0_{\mathcal{S}}))) = Eu((K^0_{\mathcal{S}})^\vee) \circ$*(the leading term of* $\sigma_*(\tau_{\mathcal{S}, \mathfrak{M}_\mathcal{S}}(\mathcal{O}^{virt}_{\mathcal{S}})))$. *(Here* $\mathfrak{M}_T$ *(*$\mathfrak{M}_\mathcal{S}$*) denotes the coarse moduli space of the stack* $\mathcal{T}$ *(*$\mathcal{S}$*, respectively).)*

Proof. One may readily verify the hypotheses of Corollary 2.9(i) are satisfied in both the above situations. Therefore the conclusion follows. □

Corollary 2.12. *Assume in addition to the above situation that G denotes a smooth affine group scheme acting on the Deligne-Mumford stack $\mathcal{S}$. Then the results of the corollary 2.9 and the following ones extend to equivariant homology as defined in appendix B.*

Proof. In view of the observations made in (6.2.2), it suffices to show the equality of the classes as in the last corollaries in $\mathbb{H}^{et}_*(\mathcal{S}, \Gamma^h(\bullet))$. This is proven in the corollaries above. ☐

3. Gysin maps in G-theory

In this section we explore basic properties of Gysin maps at the level of G-theory with the goal of applying these in the next section. This roughly parallels the treatment in [Ful] where such Gysin maps are defined at the level of algebraic cycles.

Proposition 3.1. *Assume the situation in (2.0.10). Let $\alpha \varepsilon \pi_0(G(X))$, $\beta \varepsilon \pi_0(G(X'))$ so that $y^!(\alpha) = \beta$. Then $x_*(\beta) = \alpha \otimes Lf^*(y_*(\mathcal{O}_{Y'}))$ in $\pi_0(G_{X'}(X))$ and in $\pi_0(G(X))$..*

Proof. Observe that the map $x_* : \pi_0 G(X') \to \pi_0 G_{X'}(X)$ is an isomorphism with its inverse given by the devissage theorem in G-theory. The hypotheses imply that under this inverse isomorphism the class $\alpha \otimes Lf^*(y_*(\mathcal{O}_{Y'}))$ maps to the class β. Therefore, $x_*(\beta) = \alpha \otimes Lf^*(y_*(\mathcal{O}_{Y'}))$. ☐

Proposition 3.2. *Consider the commutative diagram*

$$\begin{array}{ccccc}
N \underset{Y}{\times} C & \longrightarrow & C' & \longrightarrow & C \\
\downarrow & & \downarrow & & \downarrow \\
N \underset{Y}{\times} X & \longrightarrow & X' & \xrightarrow{x} & X \\
\downarrow & & \downarrow g & & \downarrow f \\
N & \xrightarrow{\rho} & Y' & \xrightarrow{y} & Y
\end{array}$$

where the following hold: the bottom right square is as in (2.0.10) with f a local immersion, $C = C_X(Y)$ = the cone associated to this immersion and the rest of the diagram is defined so that all the squares are cartesian. Then

$$y^!(\mathcal{O}_C) = \mathcal{O}_{\mathcal{C}_{X'}(Y')} \text{ in } \pi_0(G(N \underset{Y}{\times} C)) \cong \pi_0(G(C'))$$

Proof. This is a rather well known result; the corresponding results for algebraic cycles appears in [Vi-1] and may be proved along similar lines by reducing to the case when Y' and X are divisors in Y. The key observation is that we define $y^! : G(C) \to G(C')$ by taking for the map f in (2.0.10) not the map f above but instead the composition of the two maps forming the right-most column. We skip the details. ☐

Proposition 3.3. *Assume the square*

$$\begin{array}{ccc}
X' & \xrightarrow{i'} & X \\
\downarrow g & & \downarrow f \\
Y' & \xrightarrow{i} & Y
\end{array}$$

is cartesian and that one is given maps $\pi : Y \to Y'$ and $s : Y' \to X'$ so that $g \circ s = id_{Y'}$ and $\pi \circ i = id_{Y'}$. Assume also, that both the maps i and s are regular local immersions. Then the composite map

$$\pi_0(G(X, \mathcal{O}_X)) \overset{i^!}{\to} \pi_0(G_{X'}(X, \mathcal{O}_X)) \overset{\simeq}{\to} \pi_0(G(X', \mathcal{O}_{X'})) \overset{s^!}{\to} \pi_0(G(Y', \mathcal{O}_{Y'}))$$

is also equal to the map induced by the map $M \mapsto \alpha \underset{\mathcal{O}_X}{\otimes} (f^* \pi^* (\lambda_{-1}(N_{Y'/Y}) \underset{\mathcal{O}_{Y'}}{\otimes} (\lambda_{-1}(N_{Y'/X'})))),\ M \varepsilon Coh(X, G) = Coh([X/G])$. Here $N_{Y'/Y}$ ($N_{Y'/X'}$) is the conormal sheaf associated to the closed immersion $Y' \to Y$ ($Y' \to X'$, respectively).

Proof. If M denotes a coherent $\mathcal{O}_X$-module, it follows from the definition that

$$i^!(M) = [M \underset{\mathcal{O}_X}{\otimes} f^*\pi^*\lambda_{-1}(N_{Y'/Y})]$$

which is the class of $M \underset{\mathcal{O}_X}{\otimes} f^*\pi^*\lambda_{-1}(N_{Y'/Y})\varepsilon\pi_0(G(X'))$.

Similarly, $g^*(\lambda_{-1}(N_{Y'/X'}))$ is a resolution of $s_*(\mathcal{O}_{Y'})$. It follows therefore, that for M a coherent $\mathcal{O}_X$-module,

$$s^!i^!(M) = [M \underset{\mathcal{O}_X}{\otimes} f^*\pi^*\lambda_{-1}(N_{Y'/Y})] \underset{\mathcal{O}_{X'}}{\otimes} g^*\lambda_{-1}(N_{Y'/X'})).$$

Observe, in view of our hypothesis that $N_{Y'/X'}$ is a locally free $\mathcal{O}_{Y'}$-module. Moreover $\pi^*(\lambda_{-1}(N_{Y'/Y}))$ is a resolution of $i_*(\mathcal{O}_{Y'})$. Therefore, each term of $F^\bullet = f^*\pi^*\lambda_{-1}(N_{Y'/X'})$ is a locally free $\mathcal{O}_X$-module. Moreover the commutativity of the square in the proposition shows that $i'^*(F^\bullet) = g^*(\lambda_{-1}(N_{Y'/X'}))$. Therefore

$$s^!i^!(M) = [M \underset{\mathcal{O}_X}{\otimes} f^*\pi^*\lambda_{-1}(N_{Y'/Y})] \underset{\mathcal{O}_{X'}}{\otimes} i'^*(F^\bullet).$$

Since $M \underset{\mathcal{O}_X}{\otimes} f^*\pi^*\lambda_{-1}(N_{Y'/Y})$ has supports contained in X' and each term of the complex $F^\bullet$ is a locally free $\mathcal{O}_X$-module, one obtains the identification

$$[M \underset{\mathcal{O}_X}{\otimes} f^*\pi^*\lambda_{-1}(N_{Y'/Y})] \underset{\mathcal{O}_{X'}}{\otimes} i'^*(F^\bullet) = [M \underset{\mathcal{O}_X}{\otimes} f^*\pi^*\lambda_{-1}(N_{Y'/Y}) \underset{\mathcal{O}_X}{\otimes} F^\bullet]$$

as classes in $\pi_0(G_{Y'}(Y)) \cong \pi_0(G(Y'))$. □

Remark 3.4. The above proposition enables us to obtain a convenient reformulation of the virtual structure sheaves as in Definition 4.1.

Later on in this section, we will need the following alternate definition of the refined Gysin maps defined using deformation to the normal cone. We begin by defining the specialization map to the normal cone at the level of G-theory. If $X' \to X$ is a closed immersion of Deligne-Mumford stacks, one performs the blow-up of $X \times \mathbb{P}^1$ along $X \times \{\infty\}$; let this be denoted M and let $\tilde{X}$ be the blow-up of X along X'. Let M^o denote the complement of $\tilde{X}$ in M. Now $j : X \times \mathbb{A}^1$ imbeds as an open sub-stack of M^o with complement $C = C_{X'}X =$ the normal cone to X' in X. Therefore one obtains the localization sequence: $G(C)\overset{i_*}{\to}G(M^o)\overset{j^*}{\to}G(X \times \mathbb{A}^1)$ where $i : C \to M^o$ is the obvious closed immersion. Since C is a divisor in M^o, it follows that i is a regular closed immersion of codimension 1 and therefore that one has a pull-back $i^* : G(M^o) \to G(C)$. Moreover the composition $i^* \circ i_* : G(C) \to G(C)$ is null-homotopic, since the normal bundle to the immersion i is trivial. Therefore, the map $i^* : G(M^o) \to G(C)$ factors through j^*. The induced map $G(X \times \mathbb{A}^1) \to G(C)$ will be denoted sp'. We define the *specialization map* $sp : G(X) \to G(C)$ as the composition $sp' \circ pr_1^*$, where $pr_1 : X \times \mathbb{A}^1 \to X$ is the obvious projection.

Given a diagram as in (2.0.10), one may first replace it with the diagram:

$$\begin{array}{ccc} X' & \xrightarrow{x_0} & C_{X'}X \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f_0} \\ Y' & \xrightarrow{y_0} & N_{Y'}Y \end{array}$$

One has a refined Gysin map $y_0{}^! : G(C_{X'}X) \to G(X')$. We may pre-compose this with the specialization map $sp : G(X) \to G(C_{X'}X)$ to define the *alternate refined Gysin map* $y^!_{alt} : G(X) \to G(X')$.

Proposition 3.5. $y^! = y^!_{alt} : \pi_0(G(X)) \to \pi_0(G(X'))$

Proof. First observe by the localization sequence that the restriction $j^* : \pi_0(G(M^o)) \to \pi_0(G(X \times \mathbb{A}^1)$ is surjective. (See [Qu] section 5, Theorem 5: observe that this is stated for abelian categories and therefore applies to algebraic (non-dg) stacks as well.) Therefore the specialization map on the Grothendieck groups is simply defined by starting with a class α in $\pi_0(G(X))$, pulling it back to $\pi_0(G(X \times \mathbb{A}^1))$ by pr_1^*, lifting this to a class in $\pi_0(G(M^o))$ and then applying i^*. Therefore, the specialization map at the level of Grothendieck groups is compatible with pairings in the following sense: assume the situation of 2.0.10. Now the specializations $sp : G_{Y'}(Y) \to G_{Y'}(N_{Y'}Y)$ and $sp : G(X) \to G(C_{X'}X)$ are compatible in the sense the following square commutes:

$$\begin{array}{ccc} \pi_0(G_{Y'}(Y)) \otimes \pi_0(G(X)) & \longrightarrow & \pi_0 G(X') \\ \Big\downarrow{\scriptstyle sp\otimes sp} & & \Big\downarrow{\scriptstyle id} \\ \pi_0(G_{Y'}(N_{Y'}Y)) \otimes \pi_0(G(C_{X'}X)) & \longrightarrow & \pi_0 G(X') \end{array}$$

For this, it suffices to show that the Koszul-Thom class of Y' in Y specializes to the Koszul-Thom class of Y' in $N_{Y'}Y$. We skip this verification to the reader. □

4. Push-forward and localization formulae for virtual structure sheaves and virtual fundamental classes

Next we proceed to establish a push-forward formula for the virtual fundamental classes. Using Lefschetz-Riemann-Roch, it suffices to establish a push-forward formula for the virtual structure sheaves instead. For this, we will first find another more convenient alternate definition of the virtual structure sheaf. We will assume henceforth that the given stack $\mathcal{S}$ admits a *G-equivariant closed immersion* into a smooth Deligne-Mumford stack $\tilde{\mathcal{S}}$ onto which the G-action extends. Assuming this closed immersion is denoted i and is defined locally by the sheaf of ideals $\mathcal{I}$, the cotangent complex of $\mathcal{S}$ truncated outside the interval $[-1, 0]$ can be identified with the complex:

(4.0.22) $$\tau_{\ge -1} L^{\bullet}\mathcal{S} : \mathcal{I}/\mathcal{I}^2 \rightarrow i^*(\Omega_{\tilde{\mathcal{S}}})$$

4.0.23. *Basic push-forward hypothesis. We will also assume henceforth that the obstruction theory is given by a strict map of complexes $E^{\bullet} \rightarrow \tau_{\ge -1}L^{\bullet}_{\mathcal{S}}$ and that E^i, $i = -1, 0$ are vector bundles.* As observed in [GP], the hypotheses that every coherent sheaf on the stack is a quotient of a vector bundle, implies one may make the above assumption without further loss of generality.

One may show that our hypothesis that $E^{\bullet}$ is an obstruction theory associated to the immersion i (in the above sense) implies that the sequence of sheaves $E^{-1} \rightarrow E^0 \oplus \mathcal{I}/\mathcal{I}^2 \xrightarrow{\gamma} \Omega_{\tilde{\mathcal{S}}|\mathcal{S}} \rightarrow 0$ is *exact*. Then one obtains the associated exact sequence of abelian cones:

(4.0.24) $$0 \rightarrow T\tilde{\mathcal{S}}_{|\mathcal{S}} \rightarrow C(\mathcal{I}/\mathcal{I}^2)\underset{\mathcal{S}}{\times}\mathcal{E}_0 \rightarrow C(Q) \rightarrow 0$$

where $C(Q)$ is the cone associated to $Q = ker(\gamma)$ and $\mathcal{E}_0 = C(E_0^{\vee})$. Since Q is a quotient of E^{-1}, $C(Q)$ imbeds in $\mathcal{E}_1$. The normal cone to $\mathcal{S}$ in $\tilde{\mathcal{S}}$, $C_{\mathcal{S}}(\tilde{\mathcal{S}})$ is a closed sub-stack of $C(\mathcal{I}/\mathcal{I}^2)$. We let $C\mathcal{E}_0 = C_{\mathcal{S}}(\tilde{\mathcal{S}})\underset{\mathcal{S}}{\times}\mathcal{E}_0$, which is a $T\tilde{\mathcal{S}}_{|\mathcal{S}}$-cone.

If Q' denotes the kernel of $D \rightarrow E^0 \oplus \mathcal{I}/\mathcal{I}^2 \xrightarrow{\gamma} \Omega_{\tilde{\mathcal{S}}|\mathcal{S}}$, we obtain the short exact sequence $0 \rightarrow Q' \rightarrow E^0 \oplus C_{\mathcal{S}}(\tilde{\mathcal{S}}) \rightarrow \Omega_{\tilde{\mathcal{S}}|\mathcal{S}} \rightarrow 0$ and therefore the exact sequence $0 \rightarrow T\tilde{\mathcal{S}}_{|\mathcal{S}} \rightarrow C_{\mathcal{S}}(\tilde{\mathcal{S}})\underset{\mathcal{S}}{\times}\mathcal{E}_0 \rightarrow C(Q') \rightarrow 0$.

Observe that $C(Q') = C(E^{\bullet})$ in the terminology used earlier. Viewing the above as an exact sequence of objects over $\mathcal{S}$, one may compute the dimension of $C(Q')$ as follows:

(4.0.25) $$dim(C(E^{\bullet})) = dim(C(Q')) = rank(E^0) + b$$

Moreover, $\mathcal{O}^{virt}_{\mathcal{S}} = 0^!_{\mathcal{E}_1}(\mathcal{O}_{C(Q')})$. Alternatively, one has the cartesian square

(4.0.26) $$\begin{array}{ccc} T\tilde{\mathcal{S}}_{|\mathcal{S}} & \longrightarrow & C\mathcal{E}_0 \\ \Big\downarrow{\scriptstyle p} & & \Big\downarrow{\scriptstyle f} \\ \mathcal{S} & \xrightarrow{0_{\mathcal{E}_1}} & \mathcal{E}_1 \end{array}$$

Here $p : T\tilde{\mathcal{S}}_{|\mathcal{S}} \rightarrow \mathcal{S}$ is the obvious projection; let $s_{T\tilde{\mathcal{S}}_{|\mathcal{S}}} : \mathcal{S} \rightarrow T\tilde{\mathcal{S}}_{|\mathcal{S}}$ denote the obvious zero-section. Now one obtains a quasi-isomorphism:

(4.0.27) $$\mathcal{O}^{virt}_{\mathcal{S}} \simeq s^!_{T\tilde{\mathcal{S}}_{|\mathcal{S}}} 0^!_{\mathcal{E}_1}(\mathcal{O}_{C\mathcal{E}_0})$$

(This follows from the observation: $s^!_{T\tilde{\mathcal{S}}_{|\mathcal{S}}} 0^!_{\mathcal{E}_1}(\mathcal{O}_{C\mathcal{E}_0}) \simeq s^*_{T\tilde{\mathcal{S}}_{|\mathcal{S}}} 0^!_{\mathcal{E}_1}(\mathcal{O}_{C\mathcal{E}_0}) \simeq 0^!_{\mathcal{E}_1}(s^*_{T\tilde{\mathcal{S}}_{|\mathcal{S}}}\mathcal{O}_{C\mathcal{E}_0}) = 0^!_{\mathcal{E}_1}(\mathcal{O}_{C(Q')})$. The last but one $\simeq$ follows from the observation that locally, $C\mathcal{E}_0$ is a product of $C(Q')$ and $T\tilde{\mathcal{S}}_{|\mathcal{S}}$.)

Proposition 3.3 shows that as classes in $\pi_0 G(\mathcal{S})$, one has the identification:

$[\mathcal{O}_{\mathcal{S}}^{virt}] = [f^*\pi_{\mathcal{E}}^*(\lambda_{-1}(E^{-1}) \underset{\mathcal{O}_{\mathcal{S}}}{\otimes} \lambda_{-1}(\Omega_{\tilde{\mathcal{S}}|\mathcal{S}}))]$.

Here $\pi_{\mathcal{E}} : \mathcal{E}_1 \to \mathcal{S}$ is the obvious projection. Observe that the right-hand-side is only a complex of quasi-coherent sheaves on $\mathcal{S}$: nevertheless it is a complex of *coherent sheaves* on the stack $C\mathcal{E}_0$ with supports in the closed sub-stack $\mathcal{S}$.

Definition 4.1. Henceforth we will let

$$\mathcal{O}_{\mathcal{S}}^{virt} = f^*\pi_{\mathcal{E}}^*(\lambda_{-1}(E^{-1}) \underset{\mathcal{O}_{\mathcal{S}}}{\otimes} \lambda_{-1}(\Omega_{\tilde{\mathcal{S}}|\mathcal{S}})) \tag{4.0.28}$$

viewed as a complex of sheaves on the stack $C\mathcal{E}_0$. Proposition 6.5 in Appendix A shows that if I is the sheaf of ideals defining $\mathcal{S}$ in $C\mathcal{E}_0$, then, $\Sigma_i \mathcal{RH}om_{\mathcal{O}_{C\mathcal{E}_0}}(I^{i-1}/I^i, f^*\pi_{\mathcal{E}}^*(\lambda_{-1}(E^{-1}) \underset{\mathcal{O}_{\mathcal{S}}}{\otimes} \lambda_{-1}(\Omega_{\tilde{\mathcal{S}}|\mathcal{S}})))[-i]$ is a complex of coherent $\mathcal{O}_{\mathcal{S}}$-modules and that as classes in $\pi_0 G(\mathcal{S})$ this identifies with the class $[\mathcal{O}_{\mathcal{S}}^{virt}]$. (In fact, the sum on the right is a finite sum.) Therefore, it is often convenient to use the following variant of the virtual structure sheaf:

$$\bar{\mathcal{O}}_{\mathcal{S}}^{virt} = \Sigma_i \mathcal{RH}om_{\mathcal{O}_{C\mathcal{E}_0}}(I^{i-1}/I^i, f^*\pi_{\mathcal{E}}^*(\lambda_{-1}(E^{-1}) \underset{\mathcal{O}_{\mathcal{S}}}{\otimes} \lambda_{-1}(\Omega_{\tilde{\mathcal{S}}|\mathcal{S}})))[-i] \tag{4.0.29}$$

4.0.30. Next assume that $i_0 : \mathcal{T} \to \tilde{\mathcal{T}}$ and $i : \mathcal{S} \to \tilde{S}$ are *closed immersions* and that the square

$$\begin{array}{ccc} \mathcal{T} & \xrightarrow{u} & \mathcal{S} \\ \downarrow{\scriptstyle i_0} & & \downarrow{\scriptstyle i} \\ \tilde{\mathcal{T}} & \xrightarrow{v} & \tilde{\mathcal{S}} \end{array}$$

is cartesian, with both $\tilde{\mathcal{S}}$ and $\tilde{\mathcal{T}}$ smooth Deligne-Mumford stacks and where the maps u and v are closed immersions.

4.0.31. *Weak compatibility of obstruction theories.* We will assume that one is provided with a perfect obstruction theory $E^\bullet$ ($F^\bullet$) for $\mathcal{S} \to \tilde{\mathcal{S}}$ ($\mathcal{T} \to \tilde{\mathcal{T}}$, respectively) satisfying the hypotheses as in 4.0.23 and that these are *weakly compatible* in the following sense: there is given a G-equivariant map $\phi : u^*(E^\bullet) \to F^\bullet$ of complexes so that there exists a distinguished triangle $K^\bullet \to u^*(E^\bullet) \to F^\bullet$ and $K^\bullet$ is of perfect amplitude contained in $[-1, 0]$. For example, the two obstruction theories are weakly compatible if one has G-equivariant resolutions of coherent sheaves by vector bundles, $E^\bullet$ and $F^\bullet$ may be replaced by complexes of vector bundles and the given map $\phi : u^*(E^\bullet) \to F^\bullet$ is an *epimorphism*. It follows that, in this case, the kernel, $K^\bullet = ker(\phi)$ is a complex of vector bundles.

Lemma 4.2. *$E^\bullet$ and $F^\bullet$ are weakly compatible if and only if there exists a distinguished triangle $K'^\bullet \to E'^\bullet \to \pi^*(F^\bullet) \to K'^\bullet[1]$ of complexes of $\mathcal{O}_{C_{\mathcal{T}}(\mathcal{S})}$-modules so that (i) $K'^\bullet$ and $E'^\bullet$ are complexes of perfect amplitude contained in $[-1, 0]$ and (ii) $0^*(E'^\bullet) = u^*(E^\bullet)$. Here $\pi : C_{\mathcal{T}}(\mathcal{S}) \to \mathcal{T}$ is the obvious projection while $0 : \mathcal{T} \to C_{\mathcal{T}}(\mathcal{S})$ is the obvious closed immersion of the vertex of the cone.*

Proof. Assume that one is given a distinguished triangle $K'^\bullet \to E'^\bullet \to \pi^*(F^\bullet) \to K'^\bullet[1]$ satisfying the above hypotheses. Taking $K^\bullet = 0^*(K'^\bullet)$ provides a distinguished triangle $K^\bullet \to u^*(E^\bullet) \to F^\bullet \to K^\bullet[1]$ showing the weak compatibility of the obstruction theories. Conversely given a distinguished triangle, $K^\bullet \to u^*(E^\bullet) \to F^\bullet \to K^\bullet[1]$ with $K^\bullet$ a complex of perfect amplitude contained in $[-1, 0]$, one may take $E'^\bullet = \pi^*(u^*(E^\bullet))$ and $K'^\bullet = \pi^*(K^\bullet)$. □

4.0.32. *The deformed virtual structure sheaf.* Let $C_{\mathcal{T}}(\mathcal{E}_0)$ denote the normal cone associated to the composite closed immersion $\mathcal{T} \to \mathcal{S} \to \mathcal{E}_0$. Now $C_{\mathcal{T}}(\mathcal{S})$ is a closed sub-scheme of $C_{\mathcal{T}}(\mathcal{E}_0)$: moreover the obvious projection $\mathcal{E}_0 \to \mathcal{S}$ induces a splitting to the above map so that $C_{\mathcal{T}}(\mathcal{S})$ is a factor of the cone $C_{\mathcal{T}}(\mathcal{E}_0)$. Moreover $C_{\mathcal{T}}(\mathcal{E}_{0|\mathcal{T}})$ is also a sub-scheme of $C_{\mathcal{T}}(\mathcal{E}_0)$. Now a local computation will show that the obvious map $C_{\mathcal{T}}(\mathcal{S}) \underset{\mathcal{T}}{\times} C_{\mathcal{T}}(\mathcal{E}_{0|\mathcal{T}}) \to C_{\mathcal{T}}(\mathcal{E}_0)$ is an isomorphism. In addition, one readily obtains the isomorphism $C_{\mathcal{T}}(\mathcal{E}_{0|\mathcal{T}}) \cong C_{\mathcal{F}_0}(\mathcal{E}_{0|\mathcal{T}}) \underset{\mathcal{T}}{\times} \mathcal{F}_0$. Therefore, one obtains the isomorphism

$$C_{\mathcal{T}}(\mathcal{E}_0) \cong C_{\mathcal{T}}(\mathcal{S}) \underset{\mathcal{T}}{\times} C_{\mathcal{F}_0}(\mathcal{E}_{0|\mathcal{T}}) \underset{\mathcal{T}}{\times} \mathcal{F}_0 \tag{4.0.33}$$

We consider the commutative diagram:

(4.0.34)
$$\begin{array}{ccccc} C_{C_{\mathcal{T}}(\mathcal{S})}(C_{\tilde{\mathcal{T}}}(\tilde{\mathcal{S}})) \underset{C_{\mathcal{T}}(\mathcal{S})}{\times} C_{\mathcal{T}}(\mathcal{E}_0) & \xrightarrow{\pi_0} & C_{\mathcal{T}}(\tilde{\mathcal{T}}) \underset{\mathcal{T}}{\times} \mathcal{F}_0 & \xrightarrow{\beta} & C_{\mathcal{T}}(\tilde{\mathcal{T}}) \\ \downarrow{\scriptstyle \pi_1} & & & & \downarrow{\scriptstyle \alpha} \\ C_{C_{\mathcal{T}}(\mathcal{S})}(C_{\tilde{\mathcal{T}}}(\tilde{\mathcal{S}})) & \xrightarrow{\phi_1} & C_{\mathcal{T}}(\mathcal{S}) & \xrightarrow{\pi} & \mathcal{T} \end{array}$$

where π_0 is the obvious projection induced by the projections $C_{\mathcal{T}}(\mathcal{S}) \rightarrow \mathcal{T}$, $C_{C_{\mathcal{T}}(\mathcal{S})}(C_{\tilde{\mathcal{T}}}(\tilde{\mathcal{S}})) \rightarrow C_{\mathcal{T}}(\tilde{\mathcal{T}})$ and $C_{\mathcal{T}}(\mathcal{E}_0) \rightarrow \mathcal{F}_0$. Moreover π_1 denotes the projection to the first factor. We let the composition of maps $\alpha \circ \beta$ by f_0 and let $C = C_{C_{\mathcal{T}}(\mathcal{S})}(C_{\tilde{\mathcal{T}}}(\tilde{\mathcal{S}}))$. Let $0 : C_{\mathcal{T}}(\tilde{\mathcal{T}}) \underset{\mathcal{T}}{\times} \mathcal{F}_0 \rightarrow C \underset{C_{\mathcal{T}}(\mathcal{S})}{\times} C_{\mathcal{T}}(\mathcal{E}_0)$ denote the obvious closed immersion. (Observe that this is a section to the map π_0.) Then we provide the following definition of the deformed virtual structure sheaf:

Definition 4.3. $\mathcal{O}_C^{virt} = \pi_0^* f_0^*(\lambda_{-1}(E_{|\mathcal{T}}^{-1}) \otimes \lambda_{-1}(\Omega_{C_{\tilde{\mathcal{T}}}(\tilde{\mathcal{S}})|_{\mathcal{T}}}))$ and call this the *deformed virtual structure sheaf.*

Remark 4.4. Having replaced $\mathcal{S}$ by the cone $C_{\mathcal{T}}(\mathcal{S})$ and the virtual structure sheaf $\mathcal{O}_{\mathcal{S}}^{virt}$ by its deformation, $\mathcal{O}_{C_{\mathcal{T}}(\mathcal{S})}$, we have greater flexibility: the main advantage is the presence of the morphism $\pi : C_{\mathcal{T}}(\mathcal{S}) \rightarrow \mathcal{T}$ so that $\pi \circ 0 = id_{\mathcal{T}}$, where $0 : \mathcal{T} \rightarrow C_{\mathcal{T}}(\mathcal{S})$ is the obvious zero-section imbedding. See 4.0.50 below for more details on this deformation. Throughout the following theorem we will let $C_{C_{\mathcal{T}}(\mathcal{S})}(C_{\tilde{\mathcal{T}}}(\tilde{\mathcal{S}})) \underset{C_{\mathcal{T}}(\mathcal{S})}{\times} C_{\mathcal{T}}(\mathcal{E}_0)$ ($C_{\mathcal{T}}(\tilde{\mathcal{T}}) \underset{\mathcal{T}}{\times} \mathcal{F}_0$) be denoted by $C\mathcal{E}_0$ ($C\mathcal{F}_0$, respectively).

Theorem 4.5. *Assume the above situation. Now one obtains the formula*

$$\pi_0^* f_0^*(\lambda_{-1}(K^{-1})) \otimes 0_*(\mathcal{O}_{\mathcal{T}}^{virt}) = 0_*(f_0^*(\lambda_{-1}(K^{-1})) \underset{\mathcal{O}_{C_{\mathcal{T}}(\tilde{\mathcal{T}}) \underset{\mathcal{T}}{\times} \mathcal{F}_0}}{\otimes} \mathcal{O}_{\mathcal{T}}^{virt}) = \lambda_{-1}(\pi_0^* f_0^* K^0) \otimes \mathcal{O}_C^{virt}$$

in $\pi_0(G_{\mathcal{T}}(C\pi^*(\mathcal{E}_{0|\mathcal{T}})))$. *In case* $\mathcal{S}$, $\mathcal{T}$ *are provided with a compatible action by a smooth group scheme* G *and the obstruction theories are* G*-equivariant, the last formula holds in* $\pi_0(G_{\mathcal{T}}(C(\pi^*(\mathcal{E}_{0|\mathcal{T}})), G))$. *(Here* $\otimes$ *denotes the tensor product over* $\mathcal{O}_{C_{C_{\mathcal{T}}(\mathcal{S})}(C_{\tilde{\mathcal{T}}}(\tilde{\mathcal{S}})) \underset{C_{\mathcal{T}}(\mathcal{S})}{\times} C_{\mathcal{T}}(\mathcal{E}_0)}$.*)*

Proof. Let $\Omega_{C_{\tilde{\mathcal{T}}}(\tilde{\mathcal{S}})|\mathcal{T}}$ ($\Omega_{\tilde{\mathcal{S}}_0|\mathcal{S}_0}$) denote the restriction of $\Omega_{C_{\tilde{\mathcal{T}}}(\tilde{\mathcal{S}})}$ ($\Omega_{\tilde{\mathcal{T}}}$, respectively) to $\mathcal{T}$. Let g and π_F be defined by the following obvious diagram:

(4.0.35)
$$\begin{array}{ccccc} C_{\mathcal{T}}\mathcal{S} & & & & C_{\mathcal{T}}(\mathcal{E}_0) \\ \downarrow{\scriptstyle \pi} & & & & \downarrow{\scriptstyle \pi_0} \\ \mathcal{T} & \xleftarrow[\pi_F]{} & \mathcal{F}_1 & \xleftarrow[g]{} & C\mathcal{F}_0 \end{array}$$

Step 1. By definition, the right-hand-side identifies with

(4.0.36)
$$\lambda_{-1}(\pi_0^* f_0^* \Omega_{C_{\tilde{\mathcal{T}}}(\tilde{\mathcal{S}})|\mathcal{T}}) \underset{\mathcal{O}_{C\mathcal{E}_0}}{\otimes} \lambda_{-1}(\pi_0^* f_0^*(E_{|\mathcal{T}}^{-1})) \underset{\mathcal{O}_{C\mathcal{E}_0}}{\otimes} \lambda_{-1}(\pi_0^* f_0^* K^0)$$

Definition 4.1 applied to the cartesian square

(4.0.37)
$$\begin{array}{ccc} T\tilde{\mathcal{T}}_{|\mathcal{T}} & \longrightarrow & C_{\mathcal{T}}(\tilde{\mathcal{T}}) \underset{\mathcal{T}}{\times} \mathcal{F}_0 \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\ \mathcal{T} & \xrightarrow[0_{\mathcal{F}_1}]{} & \mathcal{F}_1 \end{array}$$

shows that the left-hand-side identifies with

(4.0.38)
$$0_*(\lambda_{-1}(g^* \pi_F^* \Omega_{\tilde{\mathcal{T}}|\mathcal{T}}) \underset{\mathcal{O}_{C\mathcal{F}_0}}{\otimes} \lambda_{-1}(g^* \pi_F^*(F^{-1})) \underset{\mathcal{O}_{C\mathcal{F}_0}}{\otimes} \lambda_{-1}(\pi_0^* f_0^* K^{-1}))$$

Henceforth $\otimes$ *will denote* $\underset{\mathcal{O}_{C\mathcal{E}_0}}{\otimes}$.

Step 2. Next, one considers the obvious immersion $0 : C_{\mathcal{T}}(\tilde{\mathcal{T}})\underset{\mathcal{T}}{\times}\mathcal{F}_0$ in $C_{C_{\mathcal{T}}(\mathcal{S})}(C_{\tilde{\mathcal{T}}}(\tilde{\mathcal{S}}))\underset{C_{\mathcal{T}}(\mathcal{S})}{\times}C_{\mathcal{T}}(\mathcal{E}_0)$ (which we denoted 0). This factors as

$$C_{\mathcal{T}}(\tilde{\mathcal{T}})\underset{\mathcal{T}}{\times}\mathcal{F}_0\overset{\alpha}{\to}C_{C_{\mathcal{T}}(\mathcal{S})}(C_{\tilde{\mathcal{T}}}(\tilde{\mathcal{S}}))\underset{C_{\mathcal{T}}(\mathcal{S})}{\times}\pi^*(\mathcal{F}_0)\overset{v}{\to}C_{C_{\mathcal{T}}(\mathcal{S})}(C_{\tilde{\mathcal{T}}}(\tilde{\mathcal{S}}))\underset{C_{\mathcal{T}}(\mathcal{S})}{\times}C_{\mathcal{T}}(\mathcal{E}_0).$$

The first observation here is that, in this situation, one obtains the identifications:

$$C_{\mathcal{T}}(\mathcal{E}_0) = C_{\mathcal{T}}(\mathcal{S})\underset{\mathcal{T}}{\times}C_{\mathcal{F}_0}(\mathcal{E}_{0|\mathcal{T}}), \quad \pi^*(\mathcal{F}_0) = C_{\mathcal{T}}(\mathcal{S})\underset{\mathcal{T}}{\times}\mathcal{F}_0 \tag{4.0.39}$$

4.0.40. Therefore, the map α identifies with the map $C_{\mathcal{T}}(\tilde{\mathcal{T}})\underset{\mathcal{T}}{\times}\mathcal{F}_0\overset{i\times id}{\to}C_{C_{\mathcal{T}}(\mathcal{S})}(C_{\tilde{\mathcal{T}}}(\tilde{\mathcal{S}}))\underset{\mathcal{T}}{\times}\mathcal{F}_0$ where $i : C_{\mathcal{T}}(\tilde{\mathcal{T}}) \to C_{C_{\mathcal{T}}(\mathcal{S})}(C_{\tilde{\mathcal{T}}}(\tilde{\mathcal{S}}))$ is the obvious immersion.

Next we apply the Proposition 3.2 to the bottom square of the following diagram (i.e. the bottom square corresponds to the bottom right square in Proposition 3.2):

$$\begin{array}{ccc} C_{\mathcal{T}}(\tilde{\mathcal{T}}) & \overset{x}{\longrightarrow} & C_{C_{\mathcal{T}}(\mathcal{S})}(C_{\tilde{\mathcal{T}}}(\tilde{\mathcal{S}})) \\ \downarrow & & \downarrow \phi_1 \\ \mathcal{T} & \longrightarrow & C_{\mathcal{T}}(\mathcal{S}) \\ \downarrow & & \downarrow \phi_0 \\ \tilde{\mathcal{T}} & \overset{y}{\longrightarrow} & N_{\tilde{\mathcal{T}}}(\tilde{\mathcal{S}}) \end{array} \tag{4.0.41}$$

Therefore, it follows first that $y^!(\mathcal{O}_{C_{C_{\mathcal{T}}(\mathcal{S})}(C_{\tilde{\mathcal{T}}}(\tilde{\mathcal{S}}))}) = \mathcal{O}_{C_{\mathcal{T}}(\tilde{\mathcal{T}})}$ and then by invoking Proposition 3.1 that

$$x_*(\mathcal{O}_{C_{\mathcal{T}}(\tilde{\mathcal{T}})}) = \mathcal{O}_{C_{C_{\mathcal{T}}(\mathcal{S})}(C_{\tilde{\mathcal{T}}}(\tilde{\mathcal{S}}))} \otimes \lambda_{-1}(\phi^*\tilde{\pi}^*(N_{\tilde{\mathcal{T}}}(\tilde{\mathcal{S}})))$$

in $\pi_0(G(C_{C_{\mathcal{T}}(\mathcal{S})}(C_{\tilde{\mathcal{T}}}(\tilde{\mathcal{S}})), \mathcal{O}_{C_{C_{\mathcal{T}}(\mathcal{S})}(C_{\tilde{\mathcal{T}}}(\tilde{\mathcal{S}}))}))$.

Here $\phi = \phi_0 \circ \phi_1$ ($\tilde{\pi}$) is the map forming the right vertical column in the above square (is the projection $N_{\tilde{\mathcal{T}}}(\tilde{\mathcal{S}}) \to \tilde{S}_0$, respectively). Moreover, it is clear that $\lambda_{-1}(\phi^*\tilde{\pi}^*(N_{\tilde{\mathcal{T}}}(\tilde{\mathcal{S}}))) = \lambda_{-1}(\phi_1^*\pi^*(N_{\tilde{\mathcal{T}}}(\tilde{\mathcal{S}}))_{\mathcal{T}})$. It follows from the observation about the map α in 4.0.40 above that

$$\alpha_*(\mathcal{O}_{C\mathcal{F}_0}) = \mathcal{O}_{C_{C_{\mathcal{T}}(\mathcal{S})}(C_{\tilde{\mathcal{T}}}(\tilde{\mathcal{S}}))\underset{C_{\mathcal{T}}(\tilde{\mathcal{S}})}{\times}\pi^*(\mathcal{F})} \otimes \lambda_{-1}(\phi^*\tilde{\pi}^*(N_{\tilde{\mathcal{T}}}(\tilde{\mathcal{S}}))) \tag{4.0.42}$$

in $\pi_0(G_{C\mathcal{F}_0}(C_{C_{\mathcal{T}}(\mathcal{S})}(C_{\tilde{\mathcal{T}}}(\tilde{\mathcal{S}}))\underset{C_{\mathcal{T}}(\tilde{\mathcal{S}})}{\times}\pi^*(\mathcal{F}), \mathcal{O}_{C_{C_{\mathcal{T}}(\mathcal{S})}(C_{\tilde{\mathcal{T}}}(\tilde{\mathcal{S}}))\underset{C_{\mathcal{T}}(\tilde{\mathcal{S}})}{\times}\pi^*(\mathcal{F}_{0|\mathcal{T}})}))$.

The short exact sequence $0 \to {K'}^0 \to {E'}^0 \to \pi^*(F^0) \to 0$ shows on taking the symmetric algebras associated to ${E'}^0$ and $\pi^*(F^0)$ that the kernel of the obvious surjection $Sym({E'}^0) \to Sym(\pi^*(F^0))$ is the ideal ${K'}^0 \otimes Sym({E'}^0)$. One may identify $({K'}^0 \otimes Sym({E'}^0)/({K'}^0 \otimes Sym({E'}^0))^2$ with $({K'}^0 \otimes \pi^*(Sym(F^0)))/({K'}^0 \otimes \pi^*(Sym(F^0)))^2 = {K'}^0/({K'}^0)^2 \otimes \pi^*(Sym(F^0))$. Clearly one obtains a natural map of the last term to $({K'}^0 \otimes Sym({E'}^0)/({K'}^0 \otimes Sym({E'}^0))^2$; by working locally one may show this is an isomorphism. Therefore one gets the formula:

$$v_*(\mathcal{O}_{C_{C_{\mathcal{T}}(\mathcal{S})}(C_{\tilde{\mathcal{T}}}(\tilde{\mathcal{S}}))}\underset{C_{\mathcal{T}}(\tilde{\mathcal{S}})}{\times}\pi^*(\mathcal{F}_0)) = \mathcal{O}_{C\mathcal{E}_0} \otimes \lambda_{-1}(\pi_0^* f_0^* K^0) \tag{4.0.43}$$

$$= \mathcal{O}_{C\mathcal{E}_0} \otimes \lambda_{-1}(\pi_1^*\phi_1^*\pi^*(K^0)) \tag{4.0.44}$$

in the Grothendieck group $\pi_0(G(C\mathcal{E}_0, \mathcal{O}_{C\mathcal{E}_0}))$. (Recall $K^{\bullet\prime} = \pi^*(K^\bullet)$. Therefore the commutative diagram in (4.0.34) shows that $\pi_0^* f_0^*(\lambda_{-1}(K^0)) = \pi_1^*(\phi_1^*(\pi^*(\lambda_{-1}(K^0))))$.) Combining these provides the identification

$$0_*(\mathcal{O}_{C\mathcal{F}_0}) = \mathcal{O}_{C\mathcal{E}_0} \otimes \lambda_{-1}(\pi_1^*\phi_1^*\pi^* N_{\tilde{\mathcal{T}}}(\tilde{\mathcal{S}})) \otimes \lambda_{-1}(\pi_0^* f_0^* K^0) \\ = \mathcal{O}_{C\mathcal{E}_0} \otimes \lambda_{-1}(\pi_0^* f_0^*(N_{\tilde{\mathcal{T}}}(\tilde{\mathcal{S}})_{|\mathcal{T}})) \otimes \lambda_{-1}(\pi_0^* f_0^* K^0) \tag{4.0.45}$$

in $\pi_0 G_{C\mathcal{F}_0}(C\mathcal{E}_0, \mathcal{O}_{C\mathcal{E}_0})$.

Step 3. Here we show that

$$\lambda_{-1}(\pi_0^* f_0^* \Omega_{C_{\tilde{\mathcal{T}}}(\tilde{\mathcal{S}})|\mathcal{T}}) = \lambda_{-1}(\pi_0^* g^* \pi_F^* \Omega_{\tilde{\mathcal{T}}|\mathcal{T}}) \otimes \lambda_{-1}(\pi_0^* g^* \pi_F^* N_{\tilde{\mathcal{T}}}(\tilde{\mathcal{S}})) \\ = \lambda_{-1}(\pi_0^* f_0^* \Omega_{\tilde{\mathcal{T}}|\mathcal{T}}) \otimes \lambda_{-1}(\pi_0^* f_0^* N_{\tilde{\mathcal{T}}}(\tilde{\mathcal{S}})_{|\mathcal{T}}) \tag{4.0.46}$$

Observe that the normal cone to the immersion $\tilde{\mathcal{T}}$ in $C_{\tilde{\mathcal{T}}}(\tilde{\mathcal{S}})$ identifies with the normal bundle $N_{\tilde{\mathcal{T}}}(\tilde{\mathcal{S}})$. We begin with the the *split short exact sequence* $0 \to N_{\tilde{\mathcal{T}}}(\tilde{\mathcal{S}}) \to \tilde{0}^*(\Omega_{C_{\tilde{\mathcal{T}}}(\tilde{\mathcal{S}})}) \to \Omega_{\tilde{\mathcal{T}}} \to 0$. Here $\tilde{0} : \tilde{\mathcal{T}} \to C_{\tilde{\mathcal{T}}}(\tilde{\mathcal{S}})$ is the obvious map. Let $\tilde{\pi} : C_{\tilde{\mathcal{T}}}(\tilde{\mathcal{S}}) \to \tilde{\mathcal{T}}$ denote the obvious projection. We apply the pull-back by $\tilde{\pi}^*$ and restriction to $C_{\mathcal{S}_0}(\mathcal{S})$ (= restriction to $\mathcal{T}$ and pull-back by π^*) to obtain:

$$\lambda_{-1}(\pi^* \Omega_{C_{\tilde{\mathcal{T}}}(\tilde{\mathcal{S}})|\mathcal{T}}) = \lambda_{-1}(\pi^*(\Omega_{\tilde{\mathcal{T}}|\mathcal{T}})) \otimes \lambda_{-1}(\pi^* N_{\tilde{\mathcal{T}}}(\tilde{\mathcal{S}})_{|\mathcal{T}}) \tag{4.0.47}$$

of perfect complexes. Recall from the commutative diagram (4.0.34) that $\pi \circ \phi_1 \circ \pi_1 = f_0 \circ \pi_0$. Therefore, the pull-back of this by $\pi_1^* \circ \phi_1^*$ then provides the identification

$$\lambda_{-1}(\pi_0^* f_0^* \Omega_{C_{\tilde{\mathcal{T}}}(\tilde{\mathcal{S}})|\mathcal{T}}) = \lambda_{-1}(\pi_0^* f_0^* \Omega_{\tilde{\mathcal{T}}|\mathcal{T}}) \otimes \lambda_{-1}(\pi_0^* f_0^* N_{\tilde{\mathcal{T}}}(\tilde{\mathcal{S}})_{|\mathcal{T}})$$

Finally observe that the map $\pi_F \circ g : C_{\mathcal{T}}(\tilde{\mathcal{T}}) \underset{\mathcal{T}}{\times} \mathcal{F}_0 \to \mathcal{T}$ also identifies with the map f_0 defined in (4.0.34). See also (4.0.35). This provides the identification in (4.0.46).

Step 4. Next, using the observation that $0^* \pi_0^* = id$, the projection formula and the diagram (4.0.35), one may identify the term in (4.0.38) with

$$\lambda_{-1}(\pi_0^* g^* \pi_F^* \Omega_{\tilde{\mathcal{T}}|\mathcal{T}}) \otimes \lambda_{-1}(\pi_0^* g^* \pi_F^* F^{-1}) \otimes \lambda_{-1}(\pi_0^* f_0^* K^{-1}) \otimes 0_*(\mathcal{O}_{C_{\mathcal{T}}(\tilde{\mathcal{T}}) \underset{\mathcal{T}}{\times} \mathcal{F}_0}) \\ = \lambda_{-1}(\pi_0^* f_0^* \Omega_{\tilde{\mathcal{T}}|\mathcal{T}}) \otimes \lambda_{-1}(\pi_0^* f_0^* F^{-1}) \otimes \lambda_{-1}(\pi_0^* f_0^* K^{-1}) \otimes 0_*(\mathcal{O}_{C_{\mathcal{T}}(\tilde{\mathcal{T}}) \underset{\mathcal{T}}{\times} \mathcal{F}_0}) \tag{4.0.48}$$

Now we consider the term in (4.0.36). In view of 4.0.46, clearly this may be written as

$$\lambda_{-1}(\pi_0^* f_0^* \Omega_{\tilde{\mathcal{T}}|\mathcal{T}}) \otimes \lambda_{-1}(\pi_0^* f_0^* N_{\tilde{\mathcal{T}}|\mathcal{S}}(\tilde{\mathcal{S}})_{|\mathcal{T}})) \otimes \lambda_{-1}(\pi_0^* f_0^*(E_{|\mathcal{T}}^{-1})) \otimes \lambda_{-1}(\pi_0^* f_0^* K^0) \tag{4.0.49}$$

Therefore, a comparison of the terms in (4.0.48) with that in (4.0.49) (making use of (4.0.45)) shows that the left-hand-side (right-hand-side) of the equation we wish to establish in the theorem is obtained by tensoring the left-hand-side (right-hand-side, respectively) of (4.0.45) by $\lambda_{-1}(\pi_0^* f_0^* \Omega_{\tilde{\mathcal{T}}|\mathcal{T}}) \otimes \lambda_{-1}(\pi_0^* f_0^*(F_{-1})) \otimes \lambda_{-1}(\pi_0^* f_0^* K'^{-1})$. (Recall the short exact sequence $0 \to K^{-1} \to E_{|\mathcal{T}}^{-1} \to F^{-1} \to 0$, shows $\lambda_{-1}(E_{|\mathcal{T}}^{-1}) = \lambda_{-1}(K^{-1}) \otimes \lambda_{-1}(F^{-1})$.)

So far the arguments show that the required formula holds in the Grothendieck group of sheaves of modules over $\mathcal{O}_{C_{\mathcal{T}}(\mathcal{E}_0)}$ with supports contained in $C\mathcal{F}_0$. However, it is clear (see Definition 4.1 and Proposition 3.3) that the term $\mathcal{O}_{\mathcal{T}}^{virt}$ has supports in $\mathcal{T}$. Therefore, we obtain the required formula in the Grothendieck group with supports contained in $\mathcal{T}$. This completes the proof of the theorem. □

4.0.50. *Deformation to the normal cone.* We will presently define a deformation of the virtual structure sheaf making use of the deformation to the normal cone. This will produce the deformed virtual structure sheaf considered above. We begin with

$$\hat{D} = C_{Bl_{\mathcal{T} \times 0}(\mathcal{S} \times \mathbb{A}^1)}(Bl_{\tilde{\mathcal{T}} \times 0}(\tilde{S} \times \mathbb{A}^1)) \underset{Bl_{\mathcal{T} \times 0}(\mathcal{S} \times \mathbb{A}^1)}{\times} Bl_{\mathcal{T} \times 0}(\mathcal{E}_0 \times \mathbb{A}^1) \tag{4.0.51}$$

We begin with the composite map $\hat{D} \to C_{\mathcal{S}\times\mathbb{A}^1}(\tilde{\mathcal{S}} \times \mathbb{A}^1) \underset{\mathcal{S}\times\mathbb{A}^1}{\times} \mathcal{E}_0 \times \mathbb{A}^1 \cong (C_{\mathcal{S}}(\tilde{\mathcal{S}}) \times \mathbb{A}^1) \underset{\mathcal{S}\times\mathbb{A}^1}{\times} \mathcal{E}_0 \times \mathbb{A}^1 \to \mathcal{E}_1 \times \mathbb{A}^1$ where the last map is defined by the given obstruction theory on $\mathcal{S}$. The composition of this map with the obvious projection to $\mathcal{S}\times\mathbb{A}^1$ factors also as the projection of $\hat{D}$ to the factor $C_{Bl_{\mathcal{T}\times 0}(\mathcal{S}\times\mathbb{A}^1)}(Bl_{\tilde{\mathcal{T}}\times 0}(\tilde{S}\times\mathbb{A}^1))$ followed by the projection to the vertex of the cone given by $Bl_{\mathcal{T}\times 0}(\mathcal{S}\times\mathbb{A}^1)$ and the projection of the latter to $\mathcal{S}\times\mathbb{A}^1$. We will denote the composite map $\hat{D} \to C_{Bl_{\mathcal{T}\times 0}(\mathcal{S}\times\mathbb{A}^1)}(Bl_{\tilde{\mathcal{T}}\times 0}(\tilde{S}\times\mathbb{A}^1)) \to Bl_{\mathcal{T}\times 0}(\mathcal{S}\times\mathbb{A}^1)$ by $\hat{\pi}\hat{f}$: observe that this map is a map between schemes flat over $\mathbb{A}^1$. $\hat{\pi}\hat{f}_{t=1}$ identifies with the map $\pi\circ f: D \to \mathcal{E}_1 \to \mathcal{S}$ as in (4.0.26) and $\hat{\pi}\hat{f}_{t=0}$ identifies with the map $\phi_1\circ\pi_1$ as in (4.0.32).

Observe that $\pi^*(T_{C_{\tilde{\mathcal{T}}(\tilde{\mathcal{S}})}|\mathcal{T}})$ sits over $C_{\mathcal{T}}(\mathcal{S})$ so that we have the diagrams:

$$\begin{array}{ccc} \pi^*(T_{C_{\tilde{\mathcal{T}}(\tilde{\mathcal{S}})}|\mathcal{T}}) & & \\ \downarrow{\scriptstyle p_0} & & \\ C_{\mathcal{T}}(\mathcal{S}) & \xrightarrow{0} & C_{\mathcal{T}}(\mathcal{E}_1) \end{array} \tag{4.0.52}$$

(with the the obvious projection $p_0: \pi^*(\mathcal{E}_{1|\mathcal{T}}) \to C_{\mathcal{T}}(\mathcal{S})$) and

$$\begin{array}{ccc} T_{Bl_{\tilde{\mathcal{T}}\times 0}(\tilde{\mathcal{S}}\times\mathbb{A}^1)|Bl_{\mathcal{T}\times 0}(\mathcal{S}\times\mathbb{A}^1)} & & \\ \downarrow{\scriptstyle \hat{p}} & & \\ Bl_{\mathcal{T}\times 0}(\mathcal{S}\times\mathbb{A}^1) & \xrightarrow{0_{Bl}} & Bl_{\mathcal{T}\times 0}(\mathcal{E}_1\times\mathbb{A}^1) \end{array} \tag{4.0.53}$$

Let $\hat{p}: Bl_{\mathcal{T}\times 0}(\mathcal{E}_1\times\mathbb{A}^1) \to Bl_{\mathcal{T}\times 0}(\mathcal{S}\times\mathbb{A}^1)$ denote the obvious projection. Observe that the obvious maps $Bl_{\mathcal{T}\times 0}(\mathcal{S}\times\mathbb{A}^1)$ and $Bl_{\mathcal{T}\times 0}(\mathcal{S}\times\mathbb{A}^1)$ to $\mathbb{A}^1$ are flat. Therefore, to show $\hat{p}$ is smooth, it suffices to show that for each fiber of $\hat{p}$ over each point t of $\mathbb{A}^1$: see [AK] Chapter VII, Corollary (1.9). This assertion is clear since the fibers of $\hat{p}$ over each t may be identified with either $\mathcal{E}_1$ (if $t \neq 0$) and $Proj(E^{-1}\oplus 1)$ if $t = 0$. Since 0_{Bl} is a section to $\hat{p}$, it follows readily that 0_{Bl} is a regular immersion locally. (See, for example, [Ful], (B.7.3).) Let $\hat{E}^{-1}$ denote the conormal sheaf associated to the regular immersion 0_{Bl}. (Since $\hat{\mathcal{E}}_1 = Bl_{\mathcal{T}\times 0}(\mathcal{E}_1\times\mathbb{A}^1)$, $\hat{E}^{-1} = \Gamma(\hat{\mathcal{E}}_1)$ = the sheaf of sections of $\hat{\mathcal{E}}_1$.) Let $\hat{\tilde{\mathcal{S}}} = Bl_{\mathcal{T}\times 0}(\tilde{S}\times\mathbb{A}^1)$, $\hat{\mathcal{S}} = Bl_{\mathcal{T}\times 0}(\mathcal{S}\times\mathbb{A}^1)$. Observe that the obvious map $\hat{S} \to \mathbb{A}^1$ is *flat*. Clearly the sheaf of sections of $T_{Bl_{\tilde{\mathcal{T}}\times 0}(\tilde{\mathcal{S}}\times\mathbb{A}^1)|Bl_{\mathcal{T}\times 0}(\mathcal{S}\times\mathbb{A}^1)}$ is given by the relative sheaf of differentials $\Omega_{\hat{\tilde{\mathcal{S}}}/\mathbb{A}^1|\tilde{\mathcal{S}}}$.

We let

$$\mathcal{O}^{virt}_{\hat{\mathcal{S}}} = (\hat{p}\hat{f})^*(\lambda_{-1}(\Omega_{\hat{\tilde{\mathcal{S}}}/\mathbb{A}^1|\tilde{\mathcal{S}}} \otimes \lambda_{-1}(\hat{E}^{-1})) \tag{4.0.54}$$

This is a complex of coherent sheaves on $\hat{D}$, and for each $t\epsilon\mathbb{A}^1$, is a *perfect complex* on $\hat{D}_t$. (Observe that when $t = 1$, the corresponding complex is just the virtual structure sheaf $\mathcal{O}^{virt}_{\mathcal{S}}$ as in (4.0.28). When $t = 0$, the corresponding complex is the deformed virtual structure sheaf as in Definition 4.3.) We let $\bar{\mathcal{O}}^{virt}_{\hat{\mathcal{S}}}$ denote the corresponding complex of coherent sheaves on $\hat{\mathcal{S}}$ defined as in (4.0.29).

Recall that one has the isomorphisms $K_{\mathcal{T}}(\tilde{T}) \simeq G(T)$ and $K_{\mathcal{S}}(\tilde{\mathcal{S}}) \simeq G(\mathcal{S})$. Therefore, one has a restriction map $G(\mathcal{S}) \to G(\mathcal{T})$. Next we will also need to consider the equivariant case where a torus acts on the algebraic stacks $\mathcal{S}$ and $\tilde{\mathcal{S}}$. In this case we will assume the following :

- the base scheme is an algebraically closed field so that the results on the fixed point stacks as in [J-4] section 6 apply,
- $\mathcal{T} = \mathcal{S}^{T'}$ and $\tilde{\mathcal{T}} = \tilde{\mathcal{S}}^{T'}$ for a fixed sub-torus T' of T and
- $\mathfrak{p} \subseteq R(T)$ is the prime ideal corresponding to T'.

4.0.55. *Basic push-forward hypothesis:II.* We will assume henceforth that the vector bundle K^0 satisfies one of the following hypotheses:

- there exists a class (which we denote) $\lambda_{-1}(\hat{K}^0)$ in $\pi_0(G_{\mathcal{T}\times\mathbb{A}^1}(\hat{\mathcal{S}}))$ so that for each $t\epsilon\mathbb{A}^1$, $i_t^*(\lambda_{-1}(\hat{K}^0))\epsilon\pi_0(G_{\mathcal{T}\times t}((\hat{\mathcal{S}})_t))$ identifies with the class of $\lambda_{-1}(K^0)$ in $\pi_0(G(\mathcal{T}))$ *or*

- we are in the equivariant case.

Observe that in the latter case, one has the isomorphism

$$(\hat{\mathcal{S}})^T = \mathcal{T} \times \mathbb{A}^1 \tag{4.0.55}$$

To see this it suffices to observe that there are no fixed vectors in the normal cone $C_{\mathcal{T}}(\mathcal{S}) \subseteq C_{\tilde{\mathcal{T}}}(\tilde{\mathcal{S}})$. Since the fixed point stack $\mathcal{T} = \mathcal{S}^T$ ($\tilde{T} = (\tilde{\mathcal{S}})^T$) is defined as a closed sub-stack of $\mathcal{S}$ ($\tilde{\mathcal{S}}$, respectively) (see [J-4] section 6), one may reduce this assertion to the case of schemes where it is well-known. (See, for example, the proof of Proposition 6.8 in [J-4].) Therefore: $\pi_0(G(\mathcal{T} \times \mathbb{A}^1), T)_{(\mathfrak{p})} = \pi_0(G_{\mathcal{T}\times\mathbb{A}^1}(\hat{\mathcal{S}}); T)_{(\mathfrak{p})} \cong \pi_0 G(\hat{\mathcal{S}}; T)_{(\mathfrak{p})}$ and hence the class $\lambda_{-1}(K^0)$ in the first group lifts to a class in $\pi_0 G(\hat{\mathcal{S}}; T)_{(\mathfrak{p})}$. Observe also that in either case one may identify $\lambda_{-1}(\hat{K}^0)$ with a class in $\pi_0(K_{\hat{\mathcal{S}}}(\hat{\tilde{\mathcal{S}}}; T))$ (or a localization of the latter in the equivariant case) so that tensor product with this class is well-defined. A similar argument applies to show that the tensor product with the class $\lambda_{-1}(K^{-1})$ is well defined.

Definition 4.6. Observe that the class $i_1^*(\lambda_{-1}(\hat{K}^0))\varepsilon\pi_0(G_{\mathcal{T}\times 1}((\hat{\mathcal{S}})_1))$ ($\varepsilon\pi_0(G_{\mathcal{T}\times 1}((\hat{\mathcal{S}})_1), T)$ in the equivariant case) maps to a class in $\pi_0(G(\mathcal{S})) \cong \pi_0(K_{\mathcal{S}}(\tilde{\mathcal{S}}))$ (in $\pi_0(G(\mathcal{S}, T)) \cong \pi_0(K_{\mathcal{S}}(\tilde{\mathcal{S}}, T))$, respectively). (Recall $(\hat{\mathcal{S}})_{t=1} = \mathcal{S}$.) We will denote this class by $\lambda_{-1}(K^0_{\mathcal{S}})$.

Examples 4.7. There are various situations where the hypothesis 4.0.55 is satisfied. The simplest is where the stacks $\mathcal{T}$ and $\mathcal{S}$ are smooth so that the above K-groups identify with the corresponding homotopy groups of G-theory. In this case the required hypothesis is satisfied, by taking the obstruction theories to be $\Omega_{\mathcal{S}}[0]$ and $\Omega_{\mathcal{T}}[0]$. Observe that now K^0 identifies with the conormal sheaf. Using deformation to the normal cone, one may define a class as required.

An alternate situation is the following. Assume that there exists a vector bundle $\mathcal{K}^0_{\mathcal{S}}$ on $\mathcal{S}$ and a section s of $\mathcal{K}^0_{\mathcal{S}}$ so that $\mathcal{T}$ is defined as the sub-stack where s vanishes. Let $K^0_{\mathcal{S}} = \Gamma(\mathcal{K}^0_{\mathcal{S}}) =$ the sheaf of sections of $\mathcal{K}^0_{\mathcal{S}}$. Then $\lambda_{-1}(\mathcal{K}^0_{\mathcal{S}})$ is a perfect complex of $\mathcal{O}_{\mathcal{S}}$-modules which is a resolution of $u_*(\mathcal{O}_{\mathcal{T}})$. Let $\hat{\mathcal{K}}^0_{\mathcal{S}} = Bl_{\mathcal{T}\times 0}(\mathcal{K}^0_{\mathcal{S}} \times \mathbb{A}^1)$: this is a vector bundle on $\hat{S}$. *Observe that* $(\hat{\mathcal{K}}^0_{\mathcal{S}})_{t=t_0|\mathcal{T}} \cong K^0$ *where* $t_0\varepsilon\mathbb{A}^1$ *is any closed point and* K^0 *is defined as in 4.0.31.* Therefore, the class of $\lambda_{-1}(\hat{K}^0_{\mathcal{S}})\varepsilon\pi_0(G_{\mathcal{T}\times\mathbb{A}^1}(\hat{S}))$ satisfies the hypotheses in 4.0.55.

Henceforth we will denote $C_{\mathcal{T}}(\tilde{\mathcal{T}})\underset{\mathcal{T}}{\times}\mathcal{F}_0$ by $D_{\mathcal{T}}$ and the corresponding closed immersion $D_{\mathcal{T}} \to D = C_{\mathcal{S}}(\tilde{\mathcal{S}})\underset{\mathcal{S}}{\times}\mathcal{E}_0$ by w.

Proposition 4.8. *(Preliminary push-forward formula) Assume the above hypotheses. Now one obtains the formulae*

i) $w_*(\mathcal{O}^{virt}_{\mathcal{T}} \otimes g^*\pi^*_{\mathcal{F}}\lambda_{-1}(K^{-1})) = \mathcal{O}^{virt}_{\mathcal{S}} \otimes f^*\pi^*_{\mathcal{E}}(\lambda_{-1}(K^0_{\mathcal{S}}))$ *in* $\pi_0(G_{\mathcal{T}}(D)) \cong \pi_0(G(\mathcal{T}, \mathcal{O}_{\mathcal{T}}))$ *and hence in* $\pi_0(G_{\mathcal{S}}(D) \cong \pi_0(G(\mathcal{S}))$. *In the equivariant case, the corresponding formula holds in the above Grothendieck groups localized at the prime ideal* $\mathfrak{p}$.

ii) $w_*(\tau(\mathcal{O}^{virt}_{\mathcal{T}}) \otimes ch(g^*\pi^*_{\mathcal{F}}\lambda_{-1}(K^{-1}))) = \tau(\mathcal{O}^{virt}_{\mathcal{S}}) \otimes ch(f^*\pi^*_{\mathcal{E}}(\lambda_{-1}(K^0_{\mathcal{S}}))$ *in* $H^{Br}_*(\mathcal{S}, \Gamma(*))$.

(Here $\pi_{\mathcal{E}} : \mathcal{E}_1 \to \mathcal{S}$ *is the obvious projection and* $f : D \to \mathcal{E}_1$ *is the map considered in Definition 4.0.26.)*

Proof. One begins with the (homotopy) commutative diagram:

$$\begin{array}{ccccc}
G_{\mathcal{T}}(D_{\mathcal{T}}) & \xrightarrow{w_*} & G_{\mathcal{T}}(D) & \longrightarrow & G_{\mathcal{S}}(D) \\
\simeq \uparrow i_1^* & & \simeq \uparrow i_1^* & & \\
G_{\mathcal{T}\times\mathbb{A}^1}(D_{\mathcal{T}} \times \mathbb{A}^1) & \xrightarrow{w_*} & G_{\mathcal{T}\times\mathbb{A}^1}(\hat{D}) & & \\
i_0^* \downarrow \simeq & & i_0^* \downarrow \simeq & & \\
G_{\mathcal{T}}(D_{\mathcal{T}}) & \xrightarrow{0_*} & G_{\mathcal{T}}(C_{\mathcal{T}}(\mathcal{E}_0)) & &
\end{array}$$

Here we let $D = C\mathcal{E}_0 = C_{\mathcal{S}}(\tilde{\mathcal{S}})\underset{\mathcal{S}}{\times}\mathcal{E}_0$ and $D_{\mathcal{T}} = C\mathcal{F}_0 = C_{\mathcal{T}}(\tilde{\mathcal{T}})\underset{\mathcal{T}}{\times}\mathcal{F}_0$. The vertical maps in the first column are weak-equivalences provided by the homotopy property of G-theory and the maps in the rightmost column are the weak-equivalences provided by the usual devissage and the homotopy property in G-theory. By devissage, the G-theory with supports in $\mathcal{T}$ ($\mathcal{T} \times \mathbb{A}^1$) identifies with the G-theory of $\mathcal{T}$ (the G-theory of $\mathcal{T} \times \mathbb{A}^1$, respectively

). Therefore the horizontal maps in the diagram may be identified with the identity showing that the squares commute.

The image of $\mathcal{O}_{\mathcal{T}}^{virt} \otimes \pi_0^* f_0^* \pi^*(K^{-1})$ by the map 0_* in the bottom row is described by the last theorem. We next show that the class $\mathcal{O}_C^{virt}$ lifts to the class $\mathcal{O}_{\mathcal{S}}^{virt}$ under the isomorphisms forming the right vertical maps, i.e. the class of $\mathcal{O}_{\hat{\mathcal{S}}}^{virt}$ in $\pi_0(G_{\mathcal{T}\times \mathbb{A}^1}(\hat{D})$ maps under the map i_1^* (i_0^*) to the class of $\mathcal{O}_{\mathcal{S}}^{virt}$ in $\pi_0(G_{\mathcal{T}}(D))$ (the class of $\mathcal{O}_{C_{\mathcal{T}}(\mathcal{S})}^{virt}$ in $\pi_0(G_{\mathcal{T}}(C_{\mathcal{T}}(\mathcal{E}_0)))$, respectively).

For this recall first that

$$\mathcal{O}_{\hat{\mathcal{S}}}^{virt} = (\hat{\pi}\hat{f})^*(\lambda_{-1}(\Omega_{\hat{\tilde{\mathcal{S}}}|\tilde{\mathcal{S}}/\mathbb{A}^1}) \otimes \lambda_{-1}(\hat{E}^{-1})).$$

Therefore,

$$(\mathcal{O}_{\hat{\mathcal{S}}}^{virt})_{t=0} = (\hat{\pi}\hat{f})_0^*(\pi^*(\lambda_{-1}(\Omega_{C_{\tilde{\mathcal{T}}}(\tilde{\mathcal{S}})|\mathcal{T}}) \otimes \pi^*(\lambda_{-1}(E_{|\mathcal{T}}^{-1})))) = \pi_1^*\phi_1^*(\pi^*(\lambda_{-1}(\Omega_{C_{\tilde{\mathcal{T}}}(\tilde{\mathcal{S}})|\mathcal{T}})) \otimes \pi^*(\lambda_{-1}(E_{|\mathcal{T}}^{-1})))$$

$$= \pi_0^* f_0^*(\lambda_{-1}(\Omega_{C_{\tilde{\mathcal{T}}}(\tilde{\mathcal{S}})|\mathcal{T}} \otimes \lambda_{-1}(E_{|\mathcal{T}}^{-1})) = \mathcal{O}_C^{virt}$$

since $f_0 \circ \pi_0 = \pi \circ \phi_1 \circ \pi_1$. Clearly,

$$(\mathcal{O}_{\hat{\mathcal{S}}}^{virt})_{t=1} = f^*\pi^*(\lambda_{-1}(\Omega_{\tilde{\mathcal{S}}|\mathcal{S}}) \otimes \lambda_{-1}(E^{-1})) = \mathcal{O}_{\mathcal{S}}^{virt}$$

Observe that $\mathcal{O}_{\hat{\mathcal{S}}}^{virt}$ has supports in $\hat{\mathcal{S}}$ while $\mathcal{O}_C^{virt}$ has supports in $\mathcal{T}$ and $\mathcal{O}_{\mathcal{S}}^{virt}$ has supports in $\mathcal{S}$. Recall the class $\lambda_{-1}(\hat{K}^0)$ has supports in $\mathcal{T} \times \mathbb{A}^1$ and $\lambda_{-1}(K^0)$ has supports in $\mathcal{T}$. Therefore, $\mathcal{O}_{\hat{\mathcal{S}}}^{virt} \otimes (\hat{\pi}\hat{f})^*\lambda_{-1}(\hat{K}^0)$ has supports in $\mathcal{T} \times \mathbb{A}^1 \subseteq \hat{\mathcal{S}}$; similarly $\mathcal{O}_{\mathcal{S}}^{virt} \otimes (\hat{\pi}\hat{f})_{t=1}^* i_{t=1}^*(\lambda_{-1}(\hat{K}^0))$ has supports in $\mathcal{T} \times 1 \subseteq \mathcal{S} \times 1$ while $\mathcal{O}_C^{virt}$ has supports contained in $\mathcal{T} \times 0 \subseteq C_{\mathcal{T}}(\mathcal{E}_{0|\mathcal{T}})$. (Since $\lambda_{-1}(K_{\mathcal{S}}^0)$ lifts to a class in $\pi_0(K_{\mathcal{T}}(\mathcal{S}))$ it follows that one may take the product of the lifts of the classes $\lambda_{-1}(K^0)$ and $\mathcal{O}_C^{virt}$. A corresponding reasoning shows that the remaining tensor products above are also defined at the level of G-theory.) This completes the proof of the proposition in the non-equivariant case.

In the equivariant case the proof is exactly the same after localization; the key point is that after tensoring the above candidates for the virtual structure sheaves with the classes $(\hat{\pi}\hat{f})_t^*(\lambda_{-1}(K_{\hat{\mathcal{S}}}^0))$, the resulting complexes all live in the appropriate Grothendieck groups localized at the prime ideal $\mathfrak{p}$, and hence in the above localized Grothendieck groups with supports in $\mathcal{T} \times \mathbb{A}^1$; therefore they identify under the isomorphisms defined by i_0^* and i_1^*.

The formula ii) in the proposition follows from the first by applying the Riemann-Roch theorem and making use of the property (vii) in Theorem 1.1 of [J-5] (which relates the Riemann-Roch transformation and the Chern character with values in Bredon-style homology and cohomology, respectively). □

Theorem 4.9. *(Push-forward formula) Assume the above hypotheses. Now one obtains the formulae*

i) $u_*(\bar{\mathcal{O}}_{\mathcal{T}}^{virt} \otimes \lambda_{-1}(K^{-1})) = \bar{\mathcal{O}}_{\mathcal{S}}^{virt} \otimes \lambda_{-1}(K_{\mathcal{S}}^0)$ *in* $\pi_0(G_{\mathcal{T}}(\mathcal{S}))$ *and hence in* $\pi_0(G(\mathcal{S}))$ *in the non-equivariant case and in the above groups localized at the prime ideal* $\mathfrak{p}$ *in the equivariant case.*

ii) $u_*(\tau(\bar{\mathcal{O}}_{\mathcal{T}}^{virt}) \otimes ch(\lambda_{-1}(K^{-1}))) = \tau(\bar{\mathcal{O}}_{\mathcal{S}}^{virt}) \otimes ch(\lambda_{-1}(\lambda_{-1}(K_{\mathcal{S}}^0))$ *in* $H_*^{Br}(\mathcal{S}, \Gamma(*))$.

The last formula also holds in $H_{smt}^*(\mathcal{S}, \Gamma(*))$ *if the stack* $\mathcal{S}$ *is smooth and in equivariant forms of homology (and cohomology) (as in [J-5] Definition 5.12) in the equivariant case .*

Proof. It suffices to interpret the formula of the last theorem in the form stated. For that, we recall the cartesian squares:

(4.0.56)
$$\begin{CD} T\tilde{\mathcal{S}}_{|\mathcal{S}} @>>> D \\ @VpVV @VVfV \\ \mathcal{S} @>0_{\mathcal{E}_1}>> \mathcal{E}_1 \end{CD}$$

and

(4.0.57)
$$\begin{array}{ccc} T\tilde{\mathcal{T}}_{|\mathcal{T}} & \longrightarrow & D_{\mathcal{T}} \\ \downarrow{\scriptstyle p_0} & & \downarrow{\scriptstyle g} \\ \mathcal{T} & \xrightarrow{0_{\mathcal{F}_1}} & \mathcal{F}_1 \end{array}$$

Let $\pi_{\mathcal{E}}$ be the projection $\mathcal{E}_1 \to \mathcal{S}$, $z : \mathcal{S} \to T\tilde{\mathcal{S}}_{|\mathcal{S}}$ be the zero section and $i : T\tilde{\mathcal{S}}_{|\mathcal{S}} \to D$ the map in the above square. (Let $\pi_{\mathcal{F}}$ is the projection $\mathcal{F}_1 \to \mathcal{T}$, $z_{\mathcal{T}} : \mathcal{T} \to T\tilde{\mathcal{T}}_{|\mathcal{T}}$ be the zero section and $i_{\mathcal{T}} : T\tilde{\mathcal{T}}_{|\mathcal{T}} \to D_{\mathcal{T}}$ the map in the above square, respectively). Then one observes that the composition $i \circ z$ ($i_{\mathcal{T}} \circ z_{\mathcal{T}}$) is a section to the composite map $\pi_{\mathcal{E}} \circ f : D \to \mathcal{E}_1 \to S$ (to the composite map $\pi_{\mathcal{F}} \circ g : D_{\mathcal{T}} \to \mathcal{F}_1 \to \mathcal{T}$, respectively). Recall $\bar{\mathcal{O}}_{\mathcal{T}}^{virt} \varepsilon \pi_0 G(\mathcal{T}) \cong \pi_0 G_{\mathcal{T}}(D_{\mathcal{T}})$ and in fact $i_{\mathcal{T}*}(z_{\mathcal{T}*}(\bar{\mathcal{O}}_{\mathcal{T}}^{virt}))$ identifies with $\mathcal{O}_{\mathcal{T}}^{virt}$ under the above isomorphism. Similarly $i_* z_*(\bar{\mathcal{O}}_{\mathcal{S}}^{virt}) \varepsilon \pi_0 G(\mathcal{S})$ identifies with $\mathcal{O}_{\mathcal{S}}^{virt}$ under the isomorphism $\pi_0 G(\mathcal{S}) \cong \pi_0 G_{\mathcal{S}}(D)$. Therefore we obtain:

$$\mathcal{O}_{\mathcal{S}}^{virt} \otimes f^* \pi_{\mathcal{E}}^*(\lambda_{-1}(K_{\mathcal{S}}^0)) = i_* z_*(\bar{O}_{\mathcal{S}}^{virt}) \otimes f^* \pi_{\mathcal{E}}^*(K_{\mathcal{S}}^0)$$
$$= i_* z_*(\bar{O}_{\mathcal{S}}^{virt} \otimes z^* i^* f^* \pi_{\mathcal{E}}^*(K_{\mathcal{S}}^0)) = i_* z_*(\bar{O}_{\mathcal{S}}^{virt} \otimes K_{\mathcal{S}}^0).$$

Recall the isomorphism $\pi_0(K_{\mathcal{S}}(\tilde{\mathcal{S}})) \cong \pi_0(G(\mathcal{S}))$ and $\pi_0(K_{\mathcal{T}}(\tilde{\mathcal{T}})) \cong \pi_0(G(\mathcal{T}))$. Therefore, the above tensor products define well-defined classes in G-theory. This provides the required identification of the right-hand-side of the formula in Theorem 4.9 i) with the right-hand-side of the formula in Proposition 4.8 i).

We may identify the left-hand-side of the formula in i) using similar arguments applied to the second square above:

$$w_*(\mathcal{O}_{\mathcal{T}}^{virt} \otimes g^* \pi_{\mathcal{F}}^*(\lambda_{-1}(K^{-1}))) = w_*(i_{\mathcal{T}*} z_{\mathcal{T}*}(\bar{\mathcal{O}}_{\mathcal{T}}^{virt}) \otimes g^* \pi_{\mathcal{F}}^* \lambda_{-1}(K^{-1})) = w_*(i_{\mathcal{T}*} z_{\mathcal{T}*}(\bar{\mathcal{O}}_{\mathcal{T}}^{virt} \otimes \lambda_{-1}(K^{-1}))$$
$$= i_* z_* u_*(\bar{\mathcal{O}}_{\mathcal{T}}^{virt} \otimes \lambda_{-1}(K^{-1}))$$

The last identification uses $w \circ i_{\mathcal{T}} \circ z_{\mathcal{T}} = i \circ z \circ u$.

The second formula in the theorem follows from the first by applying Riemann-Roch. □

Remark 4.10. One would have liked to prove the equality in the first formula of the last theorem in $\pi_0 G_{\mathcal{T}}(\mathcal{S}, \mathcal{O}_{\mathcal{S}}^{virt})$; however, this does not seem to hold because the class of the virtual structure sheaf $\mathcal{O}_{\hat{\mathcal{S}}}^{virt}$ does not seem to specialize to the classes of the other virtual structure sheaves unless one uses G-theory in the usual sense.

4.0.58. Proofs of Theorems 1.5 and 1.8. The last theorem readily proves the first two formulae in Theorem 1.5 and the first formula in Theorem 1.8. Now corollary 2.11 completes the proof of the formula (1.1.3) in Theorem 1.5. The second formula in Theorem 1.8 follows by similar reasoning.

Examples 4.11. We already observed that the when the stacks $\mathcal{S}$ and $\mathcal{T}$ are smooth we recover the usual push-forward formula for the structure sheaves. All the remaining examples will fit into the second class of examples considered in 4.7.

a) Next we consider the following situation, in preparation for the general case of the setting of the conjecture of Cox, Katz and Lee as in Theorem 1.7. Accordingly X is a smooth projective variety and Y is a closed sub-variety. We will further assume that X is *convex*, for example, X is a flag variety. Let V be a *convex* vector bundle on X, so that $H^1(C, f^*(V)) = 0$ for all genus 0 stable maps $f : C \to X$ and let s be a section of V so that Y identifies with the zeros of s. $\beta \varepsilon CH_1(X, \mathbb{Z})$, $\gamma \varepsilon CH_1(Y, \mathbb{Z})$ are cycle classes so that γ maps to β under the map $Y \to X$. We consider the moduli stacks $\mathfrak{M}_{0,n}(X, \beta)$ and $\mathfrak{M}_{0,n}(Y, \gamma)$. Let $e_k : \mathfrak{M}_{0,n}(X, \beta) \to X$ be the obvious map sending the stable map $f : (C, p_1, ..., p_n) \to X$ to $f(p_k)$. The *universal stable curve* over $\mathfrak{M}_{0,n}(X, \beta)$ is $\pi_{n+1} : \mathfrak{M}_{0,n+1}(X, \beta)$ which ignores the last marked point and contracts any components which have become unstable. Let $\mathcal{V}_{\beta,n} = \pi_{n+1*} e_{n+1}^*(V)$ which is a vector bundle on $\mathfrak{M}_{0,n}(X, \beta)$ by the convexity of V. Observe that the section s defines a section σ of the bundle $\mathcal{V}_{\beta,n}$ such that $\sqcup_{i_*(\gamma)=\beta} \mathfrak{M}_{0,n}(Y, \gamma)$ identifies with the zeros of the section σ.

Now we obtain the cartesian square:

$$\begin{array}{ccc} \sqcup_{i_*(\gamma)=\beta}\mathfrak{M}_{0,n}(Y,\gamma) & \xrightarrow{s'} & \mathfrak{M}_{0,n}(X,\beta) \\ \downarrow & & \downarrow 0 \\ \mathfrak{M}_{0,n}(X,\beta) & \xrightarrow{\sigma} & \mathcal{V}_{\beta,n} \end{array} \tag{4.0.59}$$

Observe that this is a diagram as in 4.0.30, with $\tilde{\mathcal{S}} = \mathcal{V}_{\beta,n}$, $\tilde{\mathcal{T}} = \mathcal{S} = \mathfrak{M}_{0,n}(X,\beta)$ and $\mathcal{S}_0 = \sqcup_{i_*(\gamma)=\beta}\mathfrak{M}_{0,n}(Y,\gamma)$. Now we let $E^{-1} = \Gamma(\mathcal{V}_{\beta,n})$ = the sheaf of sections of $\mathcal{V}_{\beta,n}$, $E^0 = 0^*(\Omega_{\mathcal{V}_{\beta,n}})$ with the obvious map $E^{-1} \to E^0$. We also let $F^{-1} = s'^*(E^{-1})$ and $F^0 = s'^*\Omega_{\mathcal{S}}$ with the map $F^{-1} \to F^0$ defined as dual to the following map. The differential of the section σ defines a map $T\tilde{\mathcal{T}} \to T\mathcal{V}_{\beta,n}$: we compose this with the projection $T\mathcal{V}_{\beta,n} \to \mathcal{V}_{\beta,n}$ to obtain a map $T\tilde{\mathcal{T}} \to \mathcal{V}_{\beta,n}$. Now observe that the $F^{-1} = s'^*(E^{-1})$ so that $K^{-1} = 0$ and $K^0 = kernel(s'^*0^*(\Omega_{\mathcal{V}_{\beta,n}}) \to s'^*(\Omega_{\mathcal{S}})$ which identifies with $s'^*(E^{-1})$ again.

Theorem 4.9 shows that with the above obstruction theories, one obtains the formula:

$$\oplus_{i_*(\gamma)=\beta} i_{\gamma *}(\mathcal{O}_{\mathfrak{M}_{0,n}(Y,\gamma)}) = \lambda_{-1}(\Gamma(\mathcal{V}_{\beta,n})).\mathcal{O}_{\mathfrak{M}_{0,n}(X,\beta)}$$

2. Next we consider a generalization of the case in the previous example, where X is no longer required to be *convex*, but only smooth. We will also require that V satisfy the following hypotheses:

i) V is generated by global sections and ii) the exact sequence $\Gamma(X,V)\otimes\mathcal{O}_X \to V \to 0$ defines a closed immersion of X in the Grassmanian of r-planes in $\mathbb{A}^n$, where $n = dim(\Gamma(X,V))$.

In this situation we may first assume that X is imbedded in the Grassmanian, $G(r,k)$. Moreover, the section σ induces a section σ_Q of the universal quotient bundle Q on $G(r,k)$ via the tautological quotient mapping $H^0(X,V)\otimes \mathcal{O}_{G(r,k)} \to Q$. Let $G \subseteq G(r,k)$ be the zero locus of σ_Q. It follows that σ_Q is a regular section of Q, that $G \cong G(r,k-1)$ and that $Y = X \cap G$. Let $\beta \varepsilon H_2(X)$ be fixed and let β map to $d\varepsilon H_2(G(r,k)) \cong H_2(G) \cong \mathbb{Z}$. We let the vector bundle on $\mathfrak{M}_{0,n}(G(r,k),d)$ defined by Q be denoted $\mathcal{V}_{d,n}$. Therefore, we obtain the cartesian diagram as in 4.0.30 with $\mathcal{T} = \underset{i_*(\gamma)=\beta}{\sqcup}\mathfrak{M}_{0,n}(Y,\gamma)$, $\mathcal{S} = \mathfrak{M}_{0,n}(X,\beta)$, $\tilde{\mathcal{T}} = \mathfrak{M}_{0,n}(G,d)$ and $\tilde{\mathcal{S}} = \mathfrak{M}_{0,n}(G(r,k),d)$. Moreover $\tilde{\mathcal{T}}$ is defined by the vanishing of a section of $\mathcal{V}_{d,n}$.

In this case there is a proof of the required formula in [CKL] using prior work of [Gat]. However, we will show that Theorem 4.9 provides a quick independent proof. Let I define the sheaf of ideals defining $\mathcal{S}$ in $\tilde{\mathcal{S}}$. Since $\tilde{\mathcal{S}}$ is smooth, the complex $I/I^2 \to \Omega^1_{\tilde{\mathcal{S}}|\mathcal{S}}$ is an obstruction theory for $\mathcal{S}$. Now we claim, $F^{-1} = \Gamma(\mathcal{V}_{d,n|\mathcal{T}}) \oplus u^*(I/I^2) \to \Omega^1_{\tilde{\mathcal{S}}|\mathcal{T}} = u^*(\Omega^1_{\tilde{\mathcal{S}}|\mathcal{S}}) = F^0$ defines an obstruction theory for $\mathcal{T}$. First observe the short exact sequence:

$$\Gamma(C_{\tilde{\mathcal{T}}}(\tilde{\mathcal{S}})) \underset{\mathcal{O}_{\tilde{\mathcal{T}}}}{\otimes} \mathcal{O}_{\mathcal{T}} \to \Gamma(C_{\mathcal{T}}(\tilde{\mathcal{S}})) \to \Gamma(C_{\mathcal{T}}(\tilde{\mathcal{T}}) \to 0$$

where we have used $C_X(Y)$ = the normal cone of a closed sub-stack X in Y and $\Gamma(C_X(Y))$ denotes its sheaf of sections, which is the conormal sheaf. Next observe that $u^*(I/I^2)(= u^*\Gamma(C_{\mathcal{S}}(\tilde{\mathcal{S}})))$ maps to $\Gamma(C_{\mathcal{T}}(\tilde{\mathcal{S}}))$ so that the composition into $\Gamma(C_{\mathcal{T}}(\tilde{\mathcal{T}}))$ is a surjection. Moreover there is a natural surjection $\Gamma(\mathcal{V}_{d,n|\mathcal{T}}) = u^*(\Gamma(\mathcal{V}_{d,n}) \to u^*\Gamma(C_{\tilde{\mathcal{T}}}(\tilde{\mathcal{S}}))$. It follows that one has an induced surjection $F^{-1} \to \Gamma(C_{\mathcal{T}}(\tilde{\mathcal{S}}))$.

The differential $F^{-1} \to F^0$ is defined by the surjection $F^{-1} \to \Gamma(C_{\mathcal{T}}(\tilde{\mathcal{S}}))$ followed by the obvious map of the latter to $\Omega^1_{\tilde{\mathcal{S}}|\mathcal{T}}$. The fact that the map $F^{-1} \to \Gamma(C_{\mathcal{T}}(\tilde{\mathcal{S}}))$ is a surjection also shows that the sequence $F^{-1} \to \Omega^1_{\tilde{\mathcal{S}}|\mathcal{T}} \oplus \Gamma(C_{\mathcal{T}}\tilde{\mathcal{S}}) \to \Omega^1_{\tilde{\mathcal{S}}|\mathcal{T}} \to 0$ is *exact*. Therefore $F^{-1} \to F^0$ defines a perfect obstruction theory for $\mathcal{T}$. (See 4.0.23.)

Next one observes that there is a distinguished triangle $u^*(E^{-1}) \to F^{-1} \to \Gamma(\mathcal{V}_{d,n|\mathcal{T}})$ and that the map $u^*(E^0) \to F^0$ is an isomorphism. One views the map $u^*(E) \to F$ as double complex of sheaves and takes the total complex to obtain the mapping cone; one follows this by the shift $[-1]$ to obtain the homotopy fiber which is the complex K. These observations readily show that $K^{-1} = 0$ and that $K^0 = \Gamma(\mathcal{V}_{d,n|\mathcal{T}})$. Therefore, Theorem 4.9 provides the required formula directly.

4.1. Proof of the conjecture of Cox, Katz and Lee. (See Theorem 1.7.)

Finally we consider the most general case of the above examples, where X is still required to be smooth, but there are no other hypotheses on V except that it is convex. The required result will follow from the general push-forward formula and the examples 4.7 once we show that it is possible to choose weakly-compatible obstruction theories with $K^{-1} = 0$ and $\mathcal{K}^0_{\mathcal{S}} = \mathcal{V}_{\beta,0} = \pi_{n+1*}e^*_{n+1}(V)$ the vector bundle induced by V on $\mathcal{M}(X,\beta)_{0,n}$.

In this case let the base stack $B = \mathcal{M}_{0,n}$ = the stack of pre-stable curves with n-marked points. Clearly there is a forgetful map $F : \mathcal{M}_{0,n}(X,\beta) \to B$ which forgets the map but does not stabilize. Now one may make the following choice for a perfect relative obstruction theory for the stack $\mathcal{S} = \mathcal{M}_{0,n}(X,\beta)$: $E^\bullet = R\pi_{n+1*}e^*_{n+1}(\sigma_{\geq -1}L_X)[1]$ where L_X is the cotangent complex of X and $\sigma_{\geq -1}L_X$ its naive truncation to degrees ≥ -1. (Observe that the fibers of the map π_{n+1} are curves so that $R\pi_{n+1}$ has cohomological dimension at most 1.) In fact one has the following more explicit description of $\sigma_{\geq -1}L_X$: choose a closed immersion i of X into a smooth convex variety, $\mathbb{P}$, and let $\Gamma(C_X(\mathbb{P}))$ denote the corresponding co-normal sheaf. Then $\sigma_{\geq -1}L_X = \Gamma(C_X(\mathbb{P})) \to i^*(\Omega_{\mathbb{P}})$ as in 4.0.22.

This choice works even when X is not smooth, so that the same choice would be give us a relative obstruction theory for $\mathcal{T} = \mathcal{M}_{0,n}(Y,\gamma)$. However, to obtain a relative obstruction theory $F^\bullet$ weakly compatible with $E^\bullet$, one may make the following alternate choice: let $Ob^\bullet_Y$ be the two-term complex $V \oplus \Gamma(C_X(\mathbb{P}))_{|Y} \to \Omega_{\mathbb{P}|Y}$ in degrees -1 and 0 where the differential is defined as in the last example above. (As shown in the last example above, this in fact defines a perfect obstruction theory for Y.) Now a straight-forward spectral sequence computation will show that $F^\bullet = R\pi_{n+1*}ev^*_{n+1}(Ob^\bullet_Y)[1]$ is also a perfect obstruction theory for $\mathcal{M}_{0,n}(Y,\gamma)$.

To verify that these are weakly-compatible, one first needs to observe that the square

$$\begin{array}{ccc} M_{0,n+1}(Y,\gamma) & \xrightarrow{v} & M_{0,n+1}(X,\beta) \\ \downarrow{\scriptstyle \pi^Y_{n+1}} & & \downarrow{\scriptstyle \pi^X_{n+1}} \\ M_{0,n}(Y,\gamma) & \xrightarrow{u} & M_{0,n}(X,\beta) \end{array}$$

is cartesian. Moreover using the observation that π_{n+1} is flat of relative dimension 1, one may make use of Grothendieck duality and flat-base-change to conclude that the base-change map $u^*(R\pi^Y_{n+1*}) \to R\pi^X_{n+1*}v^*$ is an isomorphism of derived functors. Therefore, one observes that for the two obstruction theories, $E^\bullet$ and $F^\bullet$ defined above, $u^*(E^0) \simeq F^0$ and $F^{-1} = R\pi_{n+1*}ev^*_{n+1}(V) \oplus u^*(E^{-1})$. One may also observe using the convexity of the bundle V that $R^1\pi_{n+1*}ev^*_{n+1}(V) = 0$ so that $F^{-1} = \pi_{n+1*}ev^*_{n+1}(V) \oplus u^*(E^{-1})$. Now an argument as in the last two examples shows $K^{-1} = 0$ and $K^0_{\mathcal{S}} = \pi_{n+1*}ev^*_{n+1}(V)$. Therefore, Theorem 1.5 provides the required formula. and completes the proof of Theorem 1.7.

4.2. Proof of Theorem 1.9. Next assume the situation of Theorem 1.9. We first let $\mathcal{O}^{virt}_{\tilde{\mathcal{S}}}$ be the complex of sheaves of $\mathcal{O}_{\tilde{\mathcal{S}}}$ modules obtained as extension by zero of $\mathcal{O}^{virt}_{\mathcal{S}}$; similarly $\mathcal{O}^{virt}_{\tilde{\mathcal{T}}}$ will be the extension by zero of $\mathcal{O}^{virt}_{\mathcal{T}}$ to $\tilde{\mathcal{T}}$. We proceed to define a Gysin map $u_* : \pi_0(K_{\mathcal{T}}(\tilde{\mathcal{T}}, \mathcal{O}^{virt}_{\tilde{\mathcal{T}}}, T))_{(\mathfrak{p})} \to \pi_0(K_{\mathcal{S}}(\tilde{\mathcal{S}}, \mathcal{O}^{virt}_{\tilde{\mathcal{S}}}, T))_{\mathfrak{p}}$ where the Grothendieck groups are the Grothendieck groups of $Perf_{\mathcal{S}}(\tilde{\mathcal{S}}, \mathcal{O}^{virt}_{\tilde{\mathcal{S}}}, T)$ and of $Perf_{\mathcal{T}}(\tilde{\mathcal{T}}, \mathcal{O}^{virt}_{\tilde{\mathcal{T}}}, T)$.

Recall from (6.2) that an object $P\varepsilon Perf_{\mathcal{T}}(\tilde{\mathcal{T}}, \mathcal{O}^{virt}_{\tilde{\mathcal{T}}}, T)$ has a finite increasing filtration $F_0 \subseteq F_1 \subseteq \cdots \subseteq F_n$ so that for each $0 \leq i \leq n$, $F_i(P)/F_{i-1}(P) \simeq \mathcal{O}^{virt}_{\tilde{\mathcal{T}}} \underset{\mathcal{O}_{\tilde{\mathcal{T}}}}{\otimes} Q_i$, where $Q_i \varepsilon Perf_{\mathcal{T}}(\tilde{\mathcal{T}}, \mathcal{O}_{\tilde{\mathcal{T}}}, T)$ and is a complex of flat $\mathcal{O}_{\tilde{\mathcal{T}}}$-modules. Therefore, it suffices to define u_* on a class of the form $\mathcal{O}^{virt}_{\tilde{\mathcal{T}}} \underset{\mathcal{O}_{\tilde{\mathcal{T}}}}{\otimes} Q$, where Q is a complex as one of the Q_is above.

Next observe the isomorphism $u_* : \pi_0(K_{\mathcal{T}}(\tilde{\mathcal{T}}, T))_{(\mathfrak{p})} \to \pi_0(K_{\mathcal{S}}(\tilde{\mathcal{S}}, T))_{(\mathfrak{p})}$. Moreover u^* is also an isomorphism, though not the inverse of u_*. Therefore, for each class $Q_{\mathcal{T}} \varepsilon \pi_0(K_{\mathcal{T}}(\tilde{\mathcal{T}}, T))_{(\mathfrak{p})}$, there exists a unique class $Q_{\mathcal{S}} \varepsilon \pi_0(K_{\mathcal{S}}(\tilde{\mathcal{S}}, T))_{(\mathfrak{p})}$ such that $u^*(Q_{\mathcal{S}}) = Q_{\mathcal{T}}$. Observe that there is a natural pairing $\pi_0(K_{\mathcal{S}}(\tilde{\mathcal{S}}, T)) \otimes \pi_0(K_{\mathcal{S}}(\tilde{\mathcal{S}}, \mathcal{O}^{virt}_{\tilde{\mathcal{S}}}, T))$ induced by the tensor product. We define

$$u_*(\mathcal{O}^{virt}_{\tilde{\mathcal{T}}} \underset{\mathcal{O}_{\tilde{\mathcal{T}}}}{\otimes} Q_T \underset{\mathcal{O}_{\tilde{\mathcal{T}}}}{\otimes} \lambda_{-1}(K^{-1})) = \mathcal{O}^{virt}_{\tilde{\mathcal{S}}} \underset{\mathcal{O}_{\mathcal{S}}}{\otimes} \lambda_{-1}(K^0_{\mathcal{S}}) \otimes Q_{\mathcal{S}} \tag{4.2.1}$$

Observe also that the classes $\lambda_{-1}(K^0_S)\varepsilon\pi_0(K_{\mathcal{S}}(\tilde{S}, T))_{(\mathfrak{p})}$ and $\lambda_{-1}(K^{-1})\varepsilon\pi_0(K_{\mathcal{T}}(\tilde{T}, T))_{(\mathfrak{p})}$ are invertible. Therefore, the above formula defines u_* on $\pi_0(K_{\mathcal{T}}(\tilde{T}, \mathcal{O}^{virt}_{\tilde{T}}, T))_{(\mathfrak{p})}$.

Observe that a pull-back $u^* : \pi_0(K_{\mathcal{S}}(\tilde{S}, \mathcal{O}^{virt}_{\tilde{S}}, T)) \to \pi_0(K_{\mathcal{T}}(\tilde{T}, \mathcal{O}^{virt}_{\tilde{T}}, T))$ is always defined. In view of the formula for u_* above, we see that the composition $u^* \circ u_*$ is given by:

(4.2.2) $$u^* u_*(F) = F \underset{\mathcal{O}_{\tilde{T}}}{\otimes} \lambda_{-1}(K^0) \underset{\mathcal{O}_{\tilde{T}}}{\otimes} \lambda_{-1}(K^{-1})^{-1}, F \varepsilon \pi_0((K_{\mathcal{T}}(\tilde{T}, \mathcal{O}^{virt}_{\tilde{T}}, T)))_{(\mathfrak{p})}$$

We have thereby proven all but the last formula in Theorem 1.9.

We proceed to consider this next. By the hypotheses on the complexes $\Gamma^h(*)$ restricted to the étale sites of schemes, we may identify $H^{et,T}_*(\mathcal{S}, \Gamma(\bullet))$ with $\mathbb{H}^*_{et, ET\underset{T}{\times}\mathcal{S}}(ET \underset{T}{\times} \tilde{\mathcal{S}}, \Gamma(*))$. We will denote this by $H^T_*(\mathcal{S}, \Gamma(\bullet))$. In view of this we obtain a restriction map $u^* : H^T_*(\mathcal{S}, \Gamma(*)) \to H^*_T(\mathcal{T}, \Gamma(\bullet))$. Moreover one has a localization isomorphism $H^T_*(\mathcal{S}, \Gamma(*))_{(\mathfrak{p})} \cong H^T_*(\mathcal{T}, \Gamma(*))_{(\mathfrak{p})}$ induced by both u_* and hence u^*. Therefore this formula follows by multiplying both sides of the formula (1.1.5) in Theorem 1.8 by $Eu((K^0_{\mathcal{S}})^\vee)$ and applying projection formula to the resulting term on the left hand side. This completes the proof of Theorem 1.9.

5. Equivariant Bredon homology and cohomology:Lefschetz Riemann-Roch

First we observe from [J-6] that $\pi_*(K(X, G))$ is a λ-ring where X is a scheme provided with the action of a linear algebraic group. However the corresponding γ-filtration is not nilpotent. Therefore, we let $\pi_*(K(X, G))^{abs}_{\mathbb{Q}} = \underset{i}{\Pi} \pi_*(K(X, G))_Q(i)$, where $\pi_*(K(X, G))_{\mathbb{Q}}(i)$= the eigen-space for the Adams operation ψ_k with eigen value k^i. We will let the presheaf on $[X/G]_{iso.et}$ be defined by $U \mapsto \pi_*(K(U, G))^{abs}_{\mathbb{Q}}$ be denoted $\pi_*(K(\quad, G)_{[X/G]})_{\mathbb{Q}}$.

Let $\mathcal{S}$ denote an algebraic stack as before for which a coarse moduli space $\mathfrak{M}$ exists. Assume that a smooth group scheme G acts on $\mathcal{S}$: clearly G has an induced action on $\mathfrak{M}$. We will assume further that $\mathcal{S}$ is provided with a sheaf of dgas $\mathcal{A}$ so that $(\mathcal{S}, \mathcal{A})$ is a dg-stack in the sense of the next section. Taking $X = \tilde{\mathfrak{M}}$ (for a closed G-equivariant immersion $\mathfrak{M} \to \tilde{\mathfrak{M}}$), the above definition defines $\pi_*(K(\quad, G)^{abs}_{[\tilde{\mathfrak{M}}/G]})_{\mathbb{Q}}$. For each $i \geq 0$, let $\pi_*\mathbf{K}(\quad, \mathcal{A}, G)_{\mathcal{S}_{\mathbb{Q}}}(i)$ be defined by the co-cartesian square:

(5.0.3) $$\begin{array}{ccc} \pi_*(\mathbf{K}(\quad)_{[\mathfrak{M}/G]})_{\mathbb{Q}}(i) & \longrightarrow & \pi_*(\mathbf{K}(\quad, \mathcal{A}, G)_{\mathcal{S}})_{\mathbb{Q}}(i) \\ \uparrow & & \uparrow \\ \pi_*(\mathbf{K}(\quad)_{[\mathfrak{M}/G]})_{\mathbb{Q}} & \longrightarrow & \pi_*(\mathbf{K}(\quad, \mathcal{A}, G)_{\mathcal{S}})_{\mathbb{Q}} \end{array}$$

Then we let $\pi_*(\mathbf{K}(\quad, \mathcal{A}, G)_{\mathcal{S}})^{abs}_{\mathbb{Q}} = \underset{i}{\Pi} \pi_*(\mathbf{K}(\quad, \mathcal{A}, G)_{\mathcal{S}})_{\mathbb{Q}}(i)$

Now we will define the complexes of presheaves:

(5.0.4) $$K\Gamma^h_{[\mathcal{S}/G]}(\bullet) = \mathcal{H}om_{i^{-1}\pi_*(\mathbf{K}(\quad, G)^{abs}_{[\tilde{\mathfrak{M}}/G]_{\mathbb{Q}}})}(\pi_*(p^G_*\mathbf{K}(\quad, \mathcal{A}, G)^{abs}_{\mathcal{S}}{}_{\mathbb{Q}}), \pi_*(\mathbb{H}(\quad, G, Sp(\Gamma^h(\bullet)))_{\mathbb{Q}})) \quad and$$
$$K\Gamma_{[\mathcal{S}/G]}(\bullet) = \pi_*(p^G_*\mathbf{K}(\quad, \mathcal{A}, G)_{\mathcal{S}}{}^{abs}_{\mathbb{Q}}) \underset{i^{-1}\pi_*(\mathbf{K}(\quad, G)_{[\tilde{\mathfrak{M}}/G]}{}^{abs}_{\mathbb{Q}})}{\overset{L}{\otimes}} \pi_*(\mathbb{H}(\quad, G, Sp\Gamma(\bullet))_{\mathbb{Q}})$$

as *presheaves on* $[\mathfrak{M}/G]_{iso.et}$. $i : \mathfrak{M} \to \tilde{\mathfrak{M}}$ *is a* G*-equivariant closed immersion into a smooth scheme,* $\mathbb{H}(\quad, G, Sp(\Gamma^h(\bullet)))$ *(*$\mathbb{H}(\quad, G, Sp(\Gamma(\bullet)))$*) is the presheaf* $U \to \mathbb{H}_{et}(EG \underset{G}{\times} U, Sp(\Gamma^h(\bullet)))$
($U \to \mathbb{H}_{et}(EG \underset{G}{\times} U, Sp(\Gamma(\bullet)))$*),* $U \varepsilon [\mathfrak{M}/G]_{iso.et}$ *respectively) and the functor* $V \mapsto \mathbf{K}_{[\tilde{\mathfrak{M}}/G]}(V, G)$ *is the presheaf of spectra on* $[\mathfrak{M}/G]_{iso.et}$ *defined by* $\mathbf{K}_{[\tilde{\mathfrak{M}}/G]}(V, G) = \mathbf{K}(Perf([V/G]))$.

Remarks 5.1. 1. For $\mathbb{H}_{et}(EG \underset{G}{\times} U, Sp(\Gamma^h(\bullet)))$ to be defined, we will need to assume either that $\Gamma^h(\bullet)$ is a complex of sheaves on the big étale site of all algebraic spaces, or that the hyper-cohomology $\mathbb{H}_{et}(X, Sp(\Gamma^h(\bullet)))$ is contravariantly functorial in X for all smooth maps. In the former case, $\Gamma^h(\bullet)$ defines a complex on the étale site of the simplicial algebraic space $EG \underset{G}{\times} U$, so that $\mathbb{H}_{et}(EG \underset{G}{\times} U, Sp(\Gamma^h(\bullet)))$ is defined. In the latter case, one may adopt the technique in [J-2] section (3.6.4) to define this: i.e. one first takes $\{\mathbb{H}_{et}((EG \underset{G}{\times} U)_n, Sp(\Gamma^h(\bullet)))|n\}$. This forms a co-chain complex of abelian spectra; one applies a functor DN that produces a cosimplicial object from this and then takes its homotopy limit over Δ to define $\mathbb{H}_{et}(EG \underset{G}{\times} U, Sp(\Gamma^h(\bullet)))$.

2. If one is interested only in Riemann-Roch at the level of Grothendieck groups one may define

$$K\Gamma^h_{[\mathcal{S}/G]}(\bullet) = \mathcal{H}om_{\pi_0(\mathbf{K}(\ ,G)^{abs}_{[\mathfrak{M}/G]_{\mathbb{Q}}})}(\pi_0(p^G_*\mathbf{K}(\ ,\mathcal{A},G)^{abs}_{\mathcal{S}\ \mathbb{Q}}), \pi_*(\mathbb{H}(\ ,G,Sp(\Gamma^h(\bullet)))_{\mathbb{Q}})) \quad and$$

(5.0.4)
$$K\Gamma_{[\mathcal{S}/G]}(\bullet) = \pi_0(p^G_*\mathbf{K}(\ ,\mathcal{A},G)_{\mathcal{S}^{abs}_{\mathbb{Q}}}) \overset{L}{\underset{\pi_0(\mathbf{K}(\ ,G)_{[\mathfrak{M}/G]^{abs}_{\mathbb{Q}}})}{\otimes}} \pi_*(\mathbb{H}(\ ,G,Sp\Gamma(\bullet))_{\mathbb{Q}})$$

Here it is possible to ignore the imbedding $i : \mathfrak{M} \to \tilde{\mathfrak{M}}$ by making use of [Ful] Theorem 18.2.

Just as in [J-5] section 5, one may now provide the following definition of G-equivariant Bredon cohomology and homology for an algebraic stack $\mathcal{S}$ with a coarse moduli space $\mathfrak{M}$ and provided with the action by a smooth group scheme G. For the purposes of this paper, we only need a Lefschetz-Riemann-Roch theorem at the level of Grothendieck groups so that one may adopt the alternate definition of the above complexes given above.

Definition 5.2. (i) $H^s_{Br}(\mathcal{S}, G; \Gamma(t)) = Gr_{-s,t}(\Gamma([\mathfrak{M}/G], K\Gamma_{[\mathcal{S}/G]}(\bullet)))$

(ii) $H^{Br}_s(\mathcal{S}, G; \Gamma^h(t)) = Gr_{s,t}(\Gamma([\mathfrak{M}/G], K\Gamma_{[\mathcal{S}/G]}(\bullet)))$

Theorem 5.3. *(Existence of equivariant Bredon-style theories with good properties) Assume that for all algebraic stacks considered below, a coarse moduli space exists which is a quasi-projective scheme. Moreover assume that a fixed smooth group scheme G acts on the stack and hence on its moduli space.*

(i) Assume that $f : \mathcal{S}' \to \mathcal{S}$ is a G-equivariant map of algebraic stacks. Then f^* defines a map $H^s_{Br}(\mathcal{S}, G, \Gamma(t)) \to H^s_{Br}(\mathcal{S}', G, \Gamma(t))$ making Bredon style cohomology a contravariant functor $(alg.stacks/S \quad with \quad G-action) \to$ (graded rings). Both Bredon style cohomology and Bredon style local cohomology are provided with ring structures.

(ii) If, in addition f is proper, one obtains a map $f_* : H^{Br}_s(\mathcal{S}'; G, \Gamma(t)) \to H^{Br}_s(\mathcal{S}; G, \Gamma(t))$ making Bredon style homology a covariant functor for proper maps $(alg.stacks) \to (abelian \quad groups)$. In case $f : \mathcal{S}' \to \mathcal{S}$ is a flat map of relative dimension c, one also obtains a pull-back $f^* : H^{Br}_s(\mathcal{S}; G, \Gamma^h(t)) \to H^{Br}_{s+2c}(\mathcal{S}'; G, \Gamma^h(t+c))$ making Bredon style homology a contravariant functor for flat maps.

(iii) $H^{Br}_*(\mathcal{S}; G, \Gamma(\bullet))$ is a module over $H^*_{Br}(\mathcal{S}; G, \Gamma(\bullet))$ and the latter is a module over $\pi_*(K(\mathcal{S}, G, \mathcal{A}_{\mathcal{S}}))$.

(iv) Projection formula. Let $f : \mathcal{S}' \to \mathcal{S}$ denote a proper G-equivariant map of algebraic stacks. Now the following diagram commutes:

$$\begin{array}{ccccc}
H^*_{Br}(\mathcal{S}; G, \Gamma(s)) \otimes H^{Br}_*(\mathcal{S}'; G, \Gamma(t)) & \xrightarrow{f^*\otimes id} & H^*_{Br}(\mathcal{S}'; G, \Gamma(s)) \otimes H^{Br}_*(\mathcal{S}'; G, \Gamma(t)) & \longrightarrow & H^{Br}_*(\mathcal{S}'; G, \Gamma(t-s)) \\
\downarrow{\scriptstyle id\otimes f_*} & & & & \downarrow{\scriptstyle f_*} \\
H^*_{Br}(\mathcal{S}; G, \Gamma(s)) \otimes H^{Br}_*(\mathcal{S}; G, \Gamma(t)) & \longrightarrow & & & H^{Br}_*(\mathcal{S}; G, \Gamma(t-s))
\end{array}$$

(v) In case the algebraic stack $\mathcal{S}$ is a separated algebraic space of finite type over the base scheme, one obtains an isomorphism $H^*_{Br}(\mathcal{S}, G, \Gamma(\bullet)) \cong H^*_{et}(\mathcal{S}, G, \Gamma(\bullet))$ where the right hand side is the G-equivariant étale hypercohomology of $\mathcal{S}$ defined with respect to the complex $\Gamma(\bullet)$. Under the same hypothesis, one obtains an isomorphism $H^{Br}_*(\mathcal{S}, G, \Gamma(\bullet)) \cong H^{et}_*(\mathcal{S}, G, \Gamma(\bullet)) \cong \mathbb{H}^*_{et}(\mathcal{S}, G, \Gamma^h(\bullet))$. (The corresponding statements hold generically if the algebraic stack $\mathcal{S}$ is a separated Deligne-Mumford stack which generically is an algebraic space, i.e. if the stack $\mathcal{S}$ is an orbifold.)

(vi) There exists a multiplicative homomorphism $ch : \pi_*K(\mathcal{S}, G, \mathcal{A}) \to H^*_{Br}(\mathcal{S}; G, \Gamma(\bullet))$ called the Chern character

For the remaining statements we will need to require that the obvious map $p : \mathcal{S} \to \mathfrak{M}$ (from the stack to its coarse moduli space) is of finite cohomological dimension.

(vii) The Riemann-Roch transformation. In this case there exists a Riemann-Roch transformation:

$\tau^G : \pi_*G(\mathcal{S}, G, \mathcal{A}) \to H^{Br}_*(\mathcal{S}; G, \Gamma(\bullet))$

Moreover the Chern-character and τ are compatible in the usual sense:

i.e. $\tau(\alpha \circ \beta) = \tau(\alpha) \circ ch(\beta)$, where $\alpha \varepsilon \pi_0(G(\mathcal{S}, G, \mathcal{A}_{\mathcal{S}}))$ and $\beta \varepsilon \pi_0(K(\mathcal{S}, G, \mathcal{A}_{\mathcal{S}}))$.

(viii)Assume that the complex $\Gamma^h(\bullet)$ is defined on the smooth site of the stack $[\mathcal{S}/G]$. Then there exists a map σ_* from $H_*^{Br}(\mathcal{S}, G, \Gamma(\bullet))$ to the G-equivariant hypercohomology of the underlying (non-dg) stack computed on the smooth site, namely $\mathbb{H}^*_{smt}([\mathcal{S}/G], \Gamma^h(\bullet)) \otimes \mathbb{Q} \cong \mathbb{H}^*_{smt}(EG\underset{G}{\times}S, \Gamma^h(\bullet)) \otimes \mathbb{Q}$. This theory will be denoted $H_*^{smt}(EG\underset{G}{\times}\mathcal{S}, \Gamma(\bullet))$. In case $\mathcal{S}$ is a Deligne-Mumford stack provided with the action by a smooth affine group scheme G, and $\Gamma^h(\bullet)$ is any of the complexes considered in [J-5] section 4, one obtains a similar map $\sigma_* : H_*^{Br}(\mathcal{S}, G, \Gamma(\bullet)) \to \mathbb{H}^*_{et}(EG\underset{G}{\times}\mathcal{S}^+, \Gamma^h(\bullet)) \otimes \mathbb{Q}$ where $EG\underset{G}{\times}\mathcal{S}^+$ is the semi-simplicial algebraic stack obtained from $EG\underset{G}{\times}\mathcal{S}$ by forgetting the degeneracies: this type of cohomology is discussed in [J-5] Appendix B and also in [Ol]. This map is compatible with respect to push-forward associated to G-equivariant closed immersions, for all such algebraic stacks.

Proof. The proof of this theorem follows entirely along the same lines as the proof of Theorem 1.1 in [J-5] and is therefore skipped. □

Theorem 5.4. *(Lefschetz-Riemann-Roch: first form) Let $f : \mathcal{S}' \to \mathcal{S}$ denote a G-equivariant proper map strongly of finite cohomological dimension between algebraic dg-stacks provided with the action of a smooth group scheme G. Assume that a coarse moduli space $\mathfrak{M}'$ ($\mathfrak{M}$) exists for the stack $\mathcal{S}'$ ($\mathcal{S}$, respectively) as in [J-5] 1.0.2.*

Now one obtains the commutative square:

$$\begin{array}{ccc} \pi_*G(\mathcal{S}', G) & \xrightarrow{\tau_{[\mathcal{S}'/G]}} & H_*^{Br-G}([\mathcal{S}'/G], \Gamma^h(*)) \\ \downarrow f_* & & \downarrow f_* \\ \pi_*G(\mathcal{S}, G) & \xrightarrow{\tau_{[\mathcal{S}/G]}} & H_*^{Br-G}([\mathcal{S}/G], \Gamma^h(*)) \end{array}$$

(The notion of a map being strongly of finite cohomological dimension is defined in [J-5] section 8.)

Proof. Once again the proof the theorem follows along the same lines as the proof of Theorem 1.4 in [J-5] and is therefore skipped. □

6. Appendix A: G-theory and K-theory of DG-stacks, Equivariant homology for algebraic stacks

For the convenience of the reader, we summarize some of the key definitions and properties of dg-stacks and their G-theory and K-theory. Further details may be found in [J-6] and [J-7].

Definition 6.1. A DG-stack is an algebraic stack $\mathcal{S}$ of Artin type which is also Noetherian provided with a sheaf of commutative dgas, $\mathcal{A}$, on $\mathcal{S}_{smt}$, so that $\mathcal{A}^i = 0$ for $i > 0$ $\mathcal{H}^i(\mathcal{A}) = 0$ for $i << 0$ and each $\mathcal{A}^i$ is a coherent $\mathcal{O}_{\mathcal{S}}$-module. We will further assume that each $\mathcal{H}^*(\mathcal{A})$ is a sheaf of graded Noetherian rings. (The need to consider such stacks should be clear from from the applications to virtual structure sheaves and virtual fundamental classes considered in this paper. See [J-7] for a comprehensive study of such stacks from a K-theory point of view.) For the purposes of this paper, we will define a DG-stack $(\mathcal{S}, \mathcal{A})$ to have property P if the associated underlying stack $\mathcal{S}$ has property P: for example, $(\mathcal{S}, \mathcal{A})$ is *smooth* if $\mathcal{S}$ is smooth.Often it is convenient to also include disjoint unions of such algebraic stacks into consideration.

6.1. Morphisms of dg stacks. A 1-morphism $f : (\mathcal{S}', \mathcal{A}') \to (\mathcal{S}, \mathcal{A})$ of DG-stacks is a morphism of the underlying stacks $\mathcal{S}' \to \mathcal{S}$ together with a map $\mathcal{A} \to f_*(\mathcal{A}')$ compatible with the map $\mathcal{O}_{\mathcal{S}} \to f_*(\mathcal{O}_{\mathcal{S}'})$. Such a morphism will have property P if the associated underlying 1-morphism of algebraic stacks has property P. Clearly DG-stacks form a 2-category. If $(\mathcal{S}, \mathcal{A})$ and $(\mathcal{S}', \mathcal{A}')$ are two DG-stacks, one defines their *product* to be the product stack $\mathcal{S} \times \mathcal{S}'$ endowed with the sheaf of DGAs $\mathcal{A} \boxtimes \mathcal{A}'$. An *action* of a group scheme G on a DG-stack $(\mathcal{S}, \mathcal{A})$ will mean morphisms $\mu, pr_2 : (G \times \mathcal{S}, \mathcal{O}_G \boxtimes \mathcal{A}) \to (\mathcal{S}, \mathcal{A})$ and $e : (\mathcal{S}, \mathcal{A}) \to (G \times \mathcal{S}, \mathcal{O}_G \boxtimes \mathcal{A})$ satisfying the usual relations.

Let $i : \mathcal{S} \to \tilde{\mathcal{S}}$ denote a closed immersion of algebraic stacks. Assume $\mathcal{S}$ is provided with a sheaf of dgas $\mathcal{A}$ making $(\mathcal{S}, \mathcal{A})$ a dg-stack. One may now define a dg-structure sheaf $\tilde{\mathcal{A}} = i_*(\mathcal{A})$. For the following discussion we consider the category of modules over $\tilde{\mathcal{A}}$: clearly this discussion reduces to the case of modules over $\mathcal{A}$ by considering the case $i =$ the identity.

6.2. A left $\tilde{\mathcal{A}}$-module is a complex of sheaves M of $\mathcal{O}_{\tilde{\mathcal{S}}}$-modules, bounded above and so that M is a sheaf of left-modules over the sheaf of dgas $\tilde{\mathcal{A}}$. The category of all left $\tilde{\mathcal{A}}$-modules and morphisms will be denoted $Mod_l(\mathcal{S}, \tilde{\mathcal{A}})$. A diagram $M' \to M \to M'' \to M[1]$ in $Mod_l(\mathcal{S}, \tilde{\mathcal{A}})$ is a *distinguished triangle* if it is one in $Mod_l(\mathcal{S}, \mathcal{O}_{\tilde{\mathcal{S}}})$. We define a map $M' \to M$ in $Mod_l(\mathcal{S}, \tilde{\mathcal{A}})$ to be a quasi-isomorphism if it is a quasi-isomorphism in $Mod(\mathcal{S}, \mathcal{O}_{\tilde{\mathcal{S}}})$. Since we assume $\mathcal{A}$ is a sheaf of commutative dgas, there is an equivalence of categories between left and right modules; therefore, henceforth we will simply refer to $\tilde{\mathcal{A}}$-modules rather than left or right $\tilde{\mathcal{A}}$-modules. The derived category

$D(\tilde{\mathcal{S}}, \tilde{\mathcal{A}})$ is the localization of $Mod_l(\mathcal{S}, \tilde{\mathcal{A}})$ by inverting maps that are quasi-isomorphisms. An $\tilde{\mathcal{A}}$-module M is *perfect* if the following holds: there exists a non-negative integer n and distinguished triangles $F_iM \to F_{i+1}M \to \mathcal{A}\overset{L}{\underset{\mathcal{O}_S}{\otimes}} P_i \to F_iM[1]$ in $Mod(\mathcal{S}, \tilde{\mathcal{A}})$, for all $0 \le i \le n-1$ so that $F_0M \simeq \tilde{\mathcal{A}}\overset{L}{\underset{\mathcal{O}_{\tilde{\mathcal{S}}}}{\otimes}} P_0$ with each P_i a perfect complex of $\mathcal{O}_{\tilde{\mathcal{S}}}$-modules. (In the presence of a group-scheme action G on the stack, we define a $\tilde{\mathcal{A}}$-module M to be perfect if it has a similar filtration with each P_i a perfect complex of G-equivariant $\mathcal{O}_{\tilde{\mathcal{S}}}$-modules.) The morphisms between two such objects will be just morphisms of $\tilde{\mathcal{A}}$-modules. This category will be denoted $Perf(\tilde{\mathcal{S}}, \tilde{\mathcal{A}})$. One may similarly define the category $Perf_{\mathcal{S}}(\tilde{\mathcal{S}}, \tilde{\mathcal{A}})$ where the complexes P_i are required to be perfect complexes of $\mathcal{O}_{\tilde{\mathcal{S}}}$-modules with supports contained in $\mathcal{S}$. Let $Perf_{fl,\mathcal{S}}(\tilde{\mathcal{S}}, \tilde{\mathcal{A}})$ denote the full sub-category of $Perf_{\mathcal{S}}(\tilde{\mathcal{S}}, \tilde{\mathcal{A}})$ consisting of flat $\tilde{\mathcal{A}}$-modules. We will let $Coh(\mathcal{S}, \mathcal{A})$ ($Per(\mathcal{S}, \mathcal{A})$) denote the above category with this Waldhausen structure.

An $\tilde{\mathcal{A}}$-module M is coherent if $\mathcal{H}^*(M)$ is bounded and finitely generated as a sheaf of $\mathcal{H}^*(\tilde{\mathcal{A}})$-modules. Again morphisms between two such objects will be morphisms of $\tilde{\mathcal{A}}$-modules. This category will be denoted $Coh(\mathcal{S}, \tilde{\mathcal{A}})$.

Definition 6.2. The categories $Coh(\mathcal{S}, \tilde{\mathcal{A}})$, $Perf(\tilde{\mathcal{S}}, \tilde{\mathcal{A}})$ and $Perf_{\mathcal{S}}(\tilde{\mathcal{S}}, \tilde{\mathcal{A}})$ along with quasi-isomorphisms as $\tilde{\mathcal{A}}$-modules form Waldhausen categories with fibrations and weak-equivalences. The fibrations are maps of $\tilde{\mathcal{A}}$-modules that are degree-wise surjections (i.e. surjections of $\mathcal{O}_{\mathcal{S}}$-modules) and the weak-equivalences are maps of $\tilde{\mathcal{A}}$-modules that are quasi-isomorphisms. The K-theory (G-theory) spectra of $(\tilde{\mathcal{S}}, \tilde{\mathcal{A}})$ will be defined to be the K-theory of the Waldhausen category $Perf(\mathcal{S}, \tilde{\mathcal{A}})$ ($Coh(\mathcal{S}, \tilde{\mathcal{A}})$, respectively) and denoted $K(\mathcal{S}, \tilde{\mathcal{A}})$ ($G(\mathcal{S}, \tilde{\mathcal{A}})$, respectively). $K_{\mathcal{S}}(\tilde{\mathcal{S}}, \tilde{\mathcal{A}})$ will denote $K(Perf_{\mathcal{S}}(\tilde{\mathcal{S}}, \tilde{\mathcal{A}}))$. When $\mathcal{A} = \mathcal{O}_{\mathcal{S}}$, $K(\mathcal{S}, \mathcal{A})$ ($G(\mathcal{S}, \mathcal{A})$) will be denoted $K(\mathcal{S})$ ($G(\mathcal{S})$, respectively).

Proposition 6.3. *(i) There exists a natural tensor-product pairing* $Perf_{\mathcal{S}}(\tilde{\mathcal{S}}, \mathcal{O}_{\tilde{\mathcal{S}}}) \otimes Perf_{\mathcal{S}}(\tilde{\mathcal{S}}, \tilde{\mathcal{A}}) \to Perf_{\mathcal{S}}(\tilde{\mathcal{S}}, \tilde{\mathcal{A}})$ *making* $K(Perf_{\mathcal{S}}(\tilde{\mathcal{S}}, \tilde{\mathcal{A}}))$ *a module-spectrum over* $K(Perf_{\mathcal{S}}(\tilde{\mathcal{S}}, \mathcal{O}_{\tilde{\mathcal{S}}}))$.

(ii) Given a distinguished triangle $M' \to M \to M'' \to M'[1]$ *of* $\tilde{\mathcal{A}}$*-modules, with two of* M', M *and* M'' *in* $Perf_{\mathcal{S}}(\tilde{\mathcal{S}}, \tilde{\mathcal{A}})$*, the third also belongs to* $Perf_{\mathcal{S}}(\tilde{\mathcal{S}}, \tilde{\mathcal{A}})$.

(iii) Let $M \varepsilon Perf_{\mathcal{S}}(\tilde{\mathcal{S}}, \tilde{\mathcal{A}})$*. Then there exists a flat* $\tilde{\mathcal{A}}$*-module* $\tilde{M} \varepsilon Perf_{fl,\mathcal{S}}(\tilde{\mathcal{S}}, \tilde{\mathcal{A}})$ *together with a quasi-isomorphism* $\tilde{M} \to M$.

Proof. We skip the details here. One may consult [J-5] and [J-7] for details. □

We conclude this section with a brief discussion on the G-theory of DG-stacks. First we consider devissage as it relates to showing the G-theory with supports in a closed sub-stack is weakly-equivalent to the G-theory of the sub-stack. Let $i : (\mathcal{S}', \mathcal{A}') \to (\mathcal{S}, \mathcal{A})$ denote the closed immersion of an algebraic DG sub-stack defined by the sheaf of ideals $\mathcal{I}$ in $\mathcal{O}_{\mathcal{S}}$, i.e. $\mathcal{A}' = i^*(\mathcal{A})$. We say that a bounded complex of sheaves F of $\mathcal{A}$-modules on $\mathcal{S}$ has supports in $\mathcal{S}'$, if the cohomology sheaves, $\mathcal{H}^*(F)$ have supports in $\mathcal{S}'$. We let $G_{\mathcal{S}'}(\mathcal{S}, \mathcal{A})$ denote the K-theory of the Waldhausen category of bounded complexes of $\mathcal{A}$-modules with cohomology sheaves that have supports in $\mathcal{S}'$. (Observe that this implies the obvious map $\lim_{\infty \to k} \mathcal{RH}om_{\mathcal{O}_{\mathcal{S}}}(\mathcal{O}_{\mathcal{S}}/\mathcal{I}^k, F) \to F$ is a quasi-isomorphism.) If F' is a bounded complex of sheaves of $\mathcal{A}'$-modules on $\mathcal{S}'$, then clearly $i_*(F)$ is an $\mathcal{A}$-module on $\mathcal{S}$ with supports in $\mathcal{S}'$. We begin with the following lemma.

Lemma 6.4. *Let* F *denote a bounded complex of* $\mathcal{A}$*-modules with coherent cohomology sheaves that have supports in* $\mathcal{S}'$*. Then there is an integer* $k >> 0$ *so that,* $\mathcal{E}xt^n(\mathcal{O}_{\mathcal{S}}/\mathcal{I}^k, F) \cong \mathcal{H}^n(F)$ *for all* n*. i.e. There exists an integer* $k >> 0$ *so that the obvious map* $F \to \mathcal{RH}om_{\mathcal{O}_{\mathcal{S}}}(\mathcal{O}_{\mathcal{S}}/\mathcal{I}^k, F)$ *is a quasi-isomorphism.*

Proof. Since the cohomology sheaves of F have supports in $\mathcal{S}'$, the spectral sequence $E_2^{s,t} = \lim_{\infty \to k} \mathcal{E}xt^s_{\mathcal{O}_{\mathcal{S}}}(\mathcal{O}_{\mathcal{S}}/\mathcal{I}^k, \mathcal{H}^t(F)) = 0$ for all $s > 0$ and $\cong \mathcal{H}^t(F)$ for $s = 0$. Therefore, $\lim_{\infty \to k} \mathcal{E}xt^n_{\mathcal{O}_{\mathcal{S}}}(\mathcal{O}_{\mathcal{S}}/\mathcal{I}^k, F) \cong \lim_{\infty \to k} \mathcal{H}om_{\mathcal{O}_{\mathcal{S}}}(\mathcal{O}_{\mathcal{S}}/\mathcal{I}^k, \mathcal{H}^n(F))$ for all n. Since $\mathcal{H}^*(F)$ has bounded cohomology sheaves and $\mathcal{S}$ is quasi-compact, there exists a $k >> 0$ so that the last term is isomorphic to $\mathcal{H}om_{\mathcal{O}_{\mathcal{S}}}(\mathcal{O}_{\mathcal{S}}/\mathcal{I}^k, \mathcal{H}^n(F)) \cong \mathcal{H}^n(F)$ for all n. This proves the lemma. □

Proposition 6.5. *(Devissage) The induced map* $G(\mathcal{S}', \mathcal{A}') \to G_{\mathcal{S}'}(\mathcal{S}, \mathcal{A})$ *is a weak-equivalence.*

Proof. Let $Mod(\mathcal{S}, \mathcal{A}/\mathcal{I}^k)$ denote the full sub-category of $\mathcal{A}$-modules that are killed by $\mathcal{I}^k$. Clearly this inherits the structure of a Waldhausen category and one obtains natural maps $G(\mathcal{S}', \mathcal{A}') \to K(Mod(\mathcal{S}, \mathcal{A}/\mathcal{I}^k)) \to G_{\mathcal{S}'}(\mathcal{S}, \mathcal{A})$. Moreover, the obvious induced map $\lim_{\infty \to k} K(Mod(\mathcal{S}, \mathcal{A}/\mathcal{I}^k)) \to G_{\mathcal{S}'}(\mathcal{S}, \mathcal{A})$ is a weak-equivalence. Now we will fix an integer $k_0 > 0$ and consider the functor $F_{k_0} = \mathcal{RH}om_{\mathcal{O}_{\mathcal{S}}}(\mathcal{O}_{\mathcal{S}}/\mathcal{I}^{k_0}, \quad) : Mod(\mathcal{S}, \mathcal{A}) \to Mod(\mathcal{S}, \mathcal{A})$. (This may be defined using the canonical Godement resolution.) Clearly, the above functor, restricted to $Mod(\mathcal{S}, \mathcal{A}/\mathcal{I}^{k_0})$ induces the identity on the associated derived catgeories. One may now observe that, the functors $\mathcal{RH}om_{\mathcal{O}_\mathcal{S}}(\mathcal{O}_\mathcal{S}/I^j, \quad)$, $1 \leq j \leq k_0$ define functors $Mod(\mathcal{S}, \mathcal{A}/\mathcal{I}^{k_0}) \rightarrow Mod(\mathcal{S}, \mathcal{A}/\mathcal{I}^{k_0})$; since they preserve weak-equivalences they induce maps of the corresponding K-theory spectra. Moreover one has a distinguished triangle $F_{i-1}(M) \rightarrow F_i(M) \rightarrow F_i/F_{i-1}(M) = \mathcal{RH}om_{\mathcal{O}_\mathcal{S}}(\mathcal{I}^{i-1}/\mathcal{I}^i, M)$, $M \varepsilon Mod(\mathcal{S}, \mathcal{A})$.

However, one needs to show that each $F_i(M)$ has bounded cohomology sheaves. To prove this one may proceed as follows. Let N denote an $\mathcal{O}_\mathcal{S}/I$-module. Then $\mathcal{RH}om_{\mathcal{O}_\mathcal{S}}(\mathcal{O}_\mathcal{S}/I^n, N) \simeq N$ for all n and therefore, by the distinguished triangle

$$\mathcal{RH}om_{\mathcal{O}_\mathcal{S}}(I^{n-1}/I^n, N) \rightarrow \mathcal{RH}om_{\mathcal{O}_\mathcal{S}}(\mathcal{O}_\mathcal{S}/I^n, N) \rightarrow \mathcal{RH}om_{\mathcal{O}_\mathcal{S}}(\mathcal{O}_\mathcal{S}/I^{n-1}, N),$$

one may conclude that $\mathcal{RH}om_{\mathcal{O}_\mathcal{S}}(I^{n-1}/I^n, N)$ has bounded cohomology sheaves for all $n \geq 1$. If N is an $\mathcal{O}_\mathcal{S}/I^k$-module for some $k \geq 1$, one may use ascending induction on k along with the obvious filtration of N by submodules F_iN so that $F_iN/F_{i+1}N$ is an $\mathcal{O}_\mathcal{S}/I$-module, to conclude that the same conclusion now holds for N. Next let $F \varepsilon Mod(\mathcal{S}, \mathcal{A})$ so that all the cohomology sheaves of F are $\mathcal{O}_\mathcal{S}/I^k$-modules. Then in the spectral sequence $E_2^{s,t} = \mathcal{E}xt^s_{\mathcal{O}_\mathcal{S}}(\mathcal{O}_\mathcal{S}/\mathcal{I}^k, \mathcal{H}^t(F)) \Rightarrow \mathcal{E}xt^{s+t}_{\mathcal{O}_\mathcal{S}}(\mathcal{O}_\mathcal{S}/\mathcal{I}^k, F)$, there exist integers $N >> 0$ and $M >> 0$ so that $E_2^{s,t} = 0$ for all $s > N$ and all $t > M$, all $s < 0$ and $t < -M$. Therefore, the abutment has cohomology only in finitely many degrees. This proves the required assertion.

By additivity, one now obtains:

(6.2.1) $$M \simeq F_{k_0}(M) = \Sigma_i(-1)^i \mathcal{RH}om_{\mathcal{O}_\mathcal{S}}(\mathcal{I}^{i-1}/\mathcal{I}^i, M), \quad M \varepsilon Mod(\mathcal{S}, \mathcal{A}/\mathcal{I}^{k_0})$$

It follows that the identity map of $K(Mod(\mathcal{S}, \mathcal{A}/\mathcal{I}^{k_0}))$ factors as $\Sigma_i(-)^i F_i/F_{i-1} : K(Mod(\mathcal{S}, \mathcal{A}/\mathcal{I}^{k_0})) \rightarrow G(\mathcal{S}', \mathcal{A}')$ followed by the obvious map of the latter into $K(Mod(\mathcal{S}, \mathcal{A}/\mathcal{I}^{k_0}))$. (Observe that the composition $G(\mathcal{S}', \mathcal{A}') \rightarrow K(Mod(\mathcal{S}, \mathcal{A}/\mathcal{I}^{k_0})) \rightarrow G(\mathcal{S}', \mathcal{A}')$ is clearly the identity.) It follows, therefore, that the obvious map $G(\mathcal{S}', \mathcal{A}') \rightarrow K(Mod(\mathcal{S}, \mathcal{A}/\mathcal{I}^{k_0}))$ is a weak-equivalence. Taking the direct limit as $k_0 \rightarrow \infty$, one obtains the weak-equivalence: $G(\mathcal{S}', \mathcal{A}') \rightarrow G_{\mathcal{S}'}(\mathcal{S}, \mathcal{A})$. □

Proposition 6.6. *(Localization for G-theory) Let $i : (\mathcal{S}', \mathcal{A}') \rightarrow (\mathcal{S}, \mathcal{A})$ denote a closed immersion of DG-stacks with open complement $j : (\mathcal{S}'', \mathcal{A}'') \rightarrow (\mathcal{S}, \mathcal{A})$. Now one obtains the fibration sequence $G(\mathcal{S}', \mathcal{A}) \rightarrow G(\mathcal{S}, \mathcal{A}) \rightarrow G(\mathcal{S}'', \mathcal{A}'') \rightarrow \Sigma G(\mathcal{S}', \mathcal{A}')$ of spectra*

Proof. This follows from Waldhausen's fibration theorem making use of the last proposition to identify $G_{\mathcal{S}'}(\mathcal{S}, \mathcal{A})$ with $G(\mathcal{S}', \mathcal{A}')$. In more detail, one lets w denote the category of weak-equivalences on $Mod(\mathcal{S}, \mathcal{A})$ defined by quasi-isomorphisms, while one lets v denote the coarser category of weak-equivalences on $Mod(\mathcal{S}, \mathcal{A})$ given by morphisms that are quasi-isomorphisms after restriction to $\mathcal{S}''$. One may show readily that any map $\alpha : F'' \rightarrow j^*(F)$, $F \varepsilon Mod(\mathcal{S}, \mathcal{A})$, $F'' \varepsilon Mod(\mathcal{S}'', \mathcal{A}'')$, may be factored as the composition of a quasi-isomorphism $F'' \rightarrow j^*(\tilde{F})$ and a map $j^*(c) : j^*(\tilde{F}) \rightarrow j^*(F)$. (Recall that we have let $Coh(\mathcal{S}, \mathcal{A})$ denote all complexes of $\mathcal{O}_\mathcal{S}$-modules M having the structure of an $\mathcal{A}$-module and whose cohomology sheaves are all bounded and finitely generated $\mathcal{H}^*(\mathcal{A})$-modules. Therefore, one may simply let $\tilde{F} = j_!(F'')$.) Therefore, the approximation theorem of Waldhausen (see [Wald] (1.6.7)) applies to provide a weak-equivalence $K(Mod(\mathcal{S}, \mathcal{A}), v) \simeq G(\mathcal{S}'', \mathcal{A}'')$; the fibration theorem of Waldhausen (see [Wald] (1.6.4)) then provides the fibration sequence $G_{\mathcal{S}'}(\mathcal{S}, \mathcal{A}) \rightarrow G(\mathcal{S}, \mathcal{A}) \rightarrow G(\mathcal{S}'', \mathcal{A}'')$. Finally the proposition above provides the weak-equivalence $G(\mathcal{S}', \mathcal{A}') \simeq G_{\mathcal{S}'}(\mathcal{S}, \mathcal{A})$ to complete the proof. □

Next we consider the homotopy property and projective space bundle formulae for G-theory. This will follow by suitable modifications of Quillen's arguments and is proved in detail in [J-7]. Therefore we will merely quote this result.

Proposition 6.7. *(Homotopy property of G-theory) Let $(\mathcal{S}, \mathcal{A})$ denote a DG-stack and let $\pi : \mathcal{S} \times \mathbb{A}^1 \rightarrow \mathcal{S}$ denote the obvious projection. Now $\pi^* : G(\mathcal{S}, \mathcal{A}) \rightarrow G(\mathcal{S} \times \mathbb{A}^1, \pi^*(\mathcal{A}))$ is a weak-equivalence.*

Proposition 6.8. *(Projective space bundle formula) Let $(\mathcal{S}, \mathcal{A})$ denote a DG-stack and let $\mathcal{E}$ denote a vector bundle of rank r on $\mathcal{S}$. If $\pi : Proj(\mathcal{E}) \rightarrow \mathcal{S}$ is the obvious map, $G(Proj(\mathcal{E}), \pi^*(\mathcal{A})) \simeq \underset{i=0,\cdots,r-1}{\sqcup} G(\mathcal{S}, \mathcal{A}).[\mathcal{O}_{Proj(\mathcal{E})}(-i)]$.*

.

Remark 6.9. In view of the localization sequence, we reduce to proving both statements for the case where the stack $\mathcal{S}$ is smooth. In this case, the projective space bundle formula is proved in detail in [J-7] adapting the arguments in [T-T]. The homotopy property may be established again using the projective space bundle formula and the localization sequence.

Equivariant cohomology for smooth group-scheme actions on algebraic stacks

If G is a smooth affine group scheme acting on an algebraic stack $\mathcal{S}$, the quotient stack $[\mathcal{S}/G]$ is also an algebraic stack as shown in [J-4] section 7. Therefore one may define equivariant cohomology and homology of $\mathcal{S}$ with respect to the action of G as the cohomology of the stack $[\mathcal{S}/G]$ as in [J-2] and [J-3]. If the complexes $\Gamma(\bullet)$ and $\Gamma^h(\bullet)$ extend to the smooth site of all algebraic stacks, one may define these as hypercohomology on the smooth site of the stack $[\mathcal{S}/G]$. When this is not the case, as in the case of the higher cycles complexes, one may define these theories as in [J-2] and [J-3]. Then we may make the following observations readily.

6.2.2.

- Let $\pi : [\mathcal{S}/G] \to [B/G] = BG$ denote the obvious map, where B is the base scheme. Then there exist spectral sequences : $E_2^{s,t} = H^s(BG, R^t\pi_*(\Gamma^h(\bullet))) \otimes \mathbb{Q} \Rightarrow \mathbb{H}_G^{s+t}([\mathcal{S}/G], \Gamma^h(\bullet)) \otimes \mathbb{Q}$ and similarly for the complex $\Gamma(\bullet)$. These converge for each fixed complex $\Gamma^h(r)$ and $\Gamma(r)$ in view of our hypotheses. The stalks of $R^t\pi_*(\Gamma(r))$ identify with $\mathbb{H}^t(\mathcal{S}, \Gamma(r))$ and similarly for the complex $\Gamma^h(r)$. In particular the sheaves $R^t\pi_*(\Gamma(r))$ and $R^t\pi_*(\Gamma^h(r))$ are locally constant on BG. (The last assertion follows readily from the definition of a locally constant sheaf on the étale site of a simplicial stack or simplicial algebraic space- see [Fr] p. 14, for example.)
- A key observation now is the following. If $\mathcal{F}$ is a perfect complex of $\mathcal{O}$-modules on $[\mathcal{S}/G]$ (i.e. a G-equivariant perfect complex of $\mathcal{O}$-modules on $\mathcal{S}$), then the Chern classes $c_i(\mathcal{F})$ define classes in $E_2^{0,2i} = H^0(BG, R^{2i}\pi_*(\Gamma(i))) \otimes \mathbb{Q}$ which are in fact *infinite cycles* and produce the equivariant Chern Class $c_i^G(\mathcal{F}) \varepsilon \mathbb{H}^{2i}([\mathcal{S}/G], \Gamma(i))$. (The fact that such Chern classes are defined for G-equivariant perfect complexes on any of the truncated simplicial stacks obtained from $EG \underset{G}{\times} \mathcal{S}$ by truncating at the n-th stage shows the classes above are in fact infinite cycles: see the description of the differentials of spectral sequences as in [C-E], Chapter XV.)
- In case the stack $\mathcal{S}$ is an algebraic space $\mathfrak{M}$ and $\mathcal{G}\varepsilon\pi_0(G[\mathcal{S}/G])$, then each term of $\tau_{\mathfrak{M}}(\mathcal{G})$ of degree $2i$ and weight i also belongs to $E_2^{0,2i} = H^0(BG, R^{2i}\pi_*(\Gamma^h(i)))$. This also is an infinite cycle and produces a class in $\mathbb{H}^{-2i}([\mathcal{S}/G], \Gamma^h(i))$.

7. Appendix B: Operational Chern classes for vector bundles on Deligne-Mumford stacks

Here we will outline how to extend the operational Chern classes defined, for example, in [Ful] to Deligne-Mumford stacks. The Chow groups of algebraic stacks may be defined as in [J-2] section 4: we will denote this Chow group of dimension n cycles (= integral linear combination of closed integral sub-stacks of dimension n modulo rational equivalence) by $CH_n(\mathcal{S})$. It is shown in [J-2] that this naive Chow group is isomorphic modulo torsion to the intrinsic Chow group $CH_n(\mathcal{S}, 0) \otimes \mathbb{Q}$ which is defined as the hypercohomology on $\mathcal{S}_{et}$ (in degree 0) with respect to the higher cycle complex $Z_n(\quad, \bullet) \otimes \mathbb{Q}$.

Lemma 7.1. *Assume the above situation. If $x : X \to \mathcal{S}$ is an atlas and $B_x\mathcal{S}$ is the associated simplicial classifying space of $\mathcal{S}$, then*

$$CH_n(\mathcal{S}, 0) \otimes \mathbb{Q} = \pi_0(Kernel(Z_n(X, \bullet) \otimes \mathbb{Q} \overset{\delta_0^* - \delta_1^*}{\to} Z_n(X \underset{\mathcal{S}}{\times} X, \bullet) \otimes \mathbb{Q})) \tag{7.0.3}$$

Moreover, if $\mathcal{S} = [X/G]$ is a finite quotient stack for the action of a constantétale group scheme on a scheme X, one obtains the isomorphism

$$CH_n(\mathcal{S}, 0) \otimes \mathbb{Q} = Kernel(CH_n(X, 0) \otimes Q \overset{\delta_0^* - \delta_1^*}{\to} CH_n(G \times X, 0) \otimes \mathbb{Q}) \tag{7.0.4}$$

Proof. (Outline). By the results in [J-2], $CH_n(\mathcal{S}, 0) \otimes \mathbb{Q} = \pi_0(\underset{\Delta}{\text{holim}}\{\mathbb{H}_{et}(B_x\mathcal{S}_m, Z_n(\quad, \bullet) \otimes \mathbb{Q}) | m\})$

$\cong \pi_0(\underset{\Delta}{\text{holim}}\{Z_n(B_x\mathcal{S}_m, \bullet) \otimes \mathbb{Q} | m\})$. Since generically the stack $\mathcal{S}$ is a finite quotient stack, a localization sequence argument as in [J-2] section 4, shows that one of the spectral sequences for this homotopy inverse limit degenerates providing the isomorphism as in the first statement of the lemma. The second statement is clear since one uses $\mathbb{Q}$-coefficients. □

Let $\mathcal{E}$ denote a rank r vector bundle on a Deligne-Mumford stack $\mathcal{S}$. Let $\mathcal{P}(\mathcal{E}) = Proj(\mathcal{E})$ denote the associated projective space bundle. In view of the projective space bundle formula in [J-2] Theorem 1(ii), $c_1(\mathcal{O}_{\mathcal{P}})$ may be defined as the class of a divisor on $\mathcal{P}(\mathcal{E})$. Now one may define the *first Chern class* of the tautological bundle $\mathcal{O}_{\mathcal{P}}(1)$ as the operation $c_1(\mathcal{O}_{\mathcal{P}}) \cap - : CH_n(\mathcal{P}(\mathcal{E})) \to CH_{n-1}(\mathcal{P}(\mathcal{E}))$ sending the class $[T]$ of a closed integral sub-stack T

to the class of the corresponding divisor on T pushed forward to $\mathcal{P}(\mathcal{E})$. (This image is denoted $c_1(\mathcal{O}_{\mathcal{P}}(1)) \cap [T]$.) Now one may define the i-th *Segre class* $s_i(\mathcal{E})$ as the operation $CH_n(\mathcal{S}) \to CH_{n-i}(\mathcal{S})$, $\alpha \mapsto p_*(c_1(\mathcal{O}_{\mathcal{P}})^{r+i} \cap p^*(\alpha))$ where $p : \mathcal{P}(\mathcal{E}) \to \mathcal{S}$ is the obvious projection.

In view of (7.0.3 and 7.0.4) one may extend all the properties of the Segre class as in [Ful] Proposition 3.1 to this setting, by doing local computations on the étale site of the stacks and by replacing all maps by representable maps of stacks.

Given a rank r vector bundle on the stack $\mathcal{S}$, one may now define *Chern classes* $c_i(\mathcal{E})$ by the same formula as in [Ful] 3.2: i.e. one first defines the Segre series $s_t(\mathcal{E}) = \Sigma_{i=0}^{\infty} s_i(\mathcal{E}) t^i$ and then the Chern polynomial $c_t(\mathcal{E}) = \Sigma_{i=0}^{\infty} c_i(\mathcal{E}) t^i$ as its formal inverse. A local calculation on the étale site of the stacks making use of (7.0.3 and 7.0.4) shows that obtains the following results as in [Ful] Theorem 3.2.

Theorem 7.2. *(i) If $\mathcal{E}$ is a rank r vector bundle on $\mathcal{S}$, then $c_i(\mathcal{E}) = 0$ for all $i > r$.*

(ii) If $\mathcal{E}$ is a vector bundle on a stack $\mathcal{S}$, $\alpha \varepsilon CH_n(\mathcal{S}', 0)$ and $f : \mathcal{S}' \to \mathcal{S}$ is a proper representable map, then $f_*(c_i(f^*(\mathcal{E})) \cap \alpha) = c_i(\mathcal{E}) \cap f_*(\alpha)$ for all i.

(iii) If $\mathcal{E}$ is a vector bundle on a stack $\mathcal{S}$, $\alpha \varepsilon CH_n(\mathcal{S}, 0)$ and $f : \mathcal{S}' \to \mathcal{S}$ is a representable flat map, then $c_i(f^*(\mathcal{E})) \cap f^*(\alpha) = f^*(c_i(\mathcal{E}) \cap \alpha)$.

(iv) Whitney sum formula If $0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$ is a short exact sequence of vector bundles on a stack $\mathcal{S}$, then $c_k(\mathcal{E}) = \Sigma_{i+j=k} c_i(\mathcal{E}') \circ c_j(\mathcal{E}'')$ where $\circ$ denotes the composition of the operations corresponding to $c_i(\mathcal{E}')$ and $c_j(\mathcal{E}'')$.

Definition 7.3. For each vector bundle $\mathcal{E}$ on the stack $\mathcal{S}$, one may now define the Chern character $Ch(\mathcal{E}) : CH_*(\mathcal{S}, 0) \otimes \mathbb{Q} \to CH_*(\mathcal{S}, 0) \otimes \mathbb{Q}$ by the usual universal polynomial in the Chern classes $c_i(\mathcal{E})$.

One establishes readily using (7.0.3) that the Chern character satisfies the usual properties: if $\mathcal{E}'$ and $\mathcal{E}$ are two vector bundles on the stack $\mathcal{S}$, then

(i) $Ch(\mathcal{E} \oplus \mathcal{E}') = Ch(\mathcal{E}) + Ch(\mathcal{E}')$ and (ii) $Ch(\mathcal{E} \otimes \mathcal{E}') = Ch(\mathcal{E}) \circ Ch(\mathcal{E}')$ where $\circ$ once again denotes the composition of the operations corresponding to $Ch(\mathcal{E})$ and $Ch(\mathcal{E}')$.

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