DERIVED FUNCTORS FOR MAPS OF SIMPLICIAL SPACES

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Abstract. In this paper we discuss in detail a site for simplicial spaces which is particularly suitable for defining derived functors for maps between simplicial spaces. It is shown that the derived category of sheaves on this site is closely related to the derived category of sheaves on another well known site. Applications to equivariant derived categories associated to algebraic group actions in positive characteristics are also discussed.

0. Introduction. The derived category of a simplicial space is often defined using a Grothendieck topology originally defined by Deligne. (See [De].) This topology plays a fundamental role in the cohomological study of simplicial spaces (see [De], [Fr] and [J-1] for some applications). However, this topology is not often convenient for defining derived functors for maps between simplicial spaces. To see the difficulty, consider a map $f_\bullet : X_\bullet \to Y_\bullet$ of simplicial spaces. Then one may define points of $Y_\bullet$ in such a way so that if $p_\bullet$ is a point of $Y_\bullet$, the fiber of $f_\bullet$ over $p_\bullet$ (denoted $X_{\bullet p_\bullet}$) is a sub-simplicial space of $X_\bullet$. If f is proper and F is a sheaf on $X_\bullet$ it would often be desirable to identify the stalks $Rf_*(F)_p$ with $H^*(X_{\bullet p_\bullet}; F_{|X_{\bullet p_\bullet}})$. However, this is never possible with the site in [De]. This problem arises in the setting of equivariant derived categories and one reason for often adopting a non-simplicial setting to define these is to circumvent this problem. However such approaches do not apply in general to group-scheme actions in positive characteristics (see Theorem (5.1) for example) nor to general simplicial algebraic spaces or schemes which, for example, arise in the study of algebraic stacks. For this purpose we had introduced a topology called the simplicial topology in [J-T] and discussed briefly in [J-1]. One of the goals of this short paper is to give a better and more detailed formulation of the basic ideas introduced there in a more general setting. This is clearly justified in view of the fact that this simplicial topology plays a key role in establishing derived functors associated to maps of simplicial spaces in *positive characteristics*: apart from a much more technically complicated approach using algebraic stacks (see [Be]) this seems to be the only technique available for defining such derived functors. In fact one of the original motivations for the present paper is a *new* application to actions of non-connected groups which plays a crucial role in extending the results of the paper [B-J] on vanishing of odd dimensional intersection cohomology to positive characteristics. This is discussed in the last section. It should be pointed out that in order to establish decent properties for sheaves on the simplicial site, one has to still invoke the topology due to Deligne. In fact, as we show below, much is gained by relating these two topologies and the two topologies seem to complement each other rather nicely.

Here is an outline of the paper. Sections and 1 and 2 are devoted to basic definitions. Section 3 is devoted to a detailed comparison of the simplicial site with a more familiar site and in section 4 we discuss Theorem (4.2) which is one of our key techniques. The discussion here extends the brief discussion of some of these techniques in [J-T] and [J-1], making them strong enough for the applications considered in the last section. Our discussion in the first 3 sections, which form the foundations for the rest of the paper, is kept sufficiently general (and somewhat detailed) so as to be applicable to other contexts. We thank the referee for several helpful comments.

1. The basic definitions.

(1.0) Throughout the paper we will adopt the following conventions. We will only consider schemes and algebraic spaces defined over a fixed Noetherian base scheme S. These will be provided with a Grothendieck topology; typically this will be either the Zariski, étale or Nisnevich topologies. The category of all algebraic spaces (schemes) over S will be denoted $(alg.spaces/S)$ $((schemes/S)$, respectively). Our results also hold for locally compact Hausdorff topological spaces with reasonable properties (as in [Verd] (1.2)), though we do not consider them explicitly and are left to the reader.

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A simplicial space will always mean a simplicial object in the category of algebraic spaces over S. The category of simplicial algebraic spaces over S (simplicial schemes over S) will be denoted $(simpl.spaces/S)$ ($(simpl.schemes/S)$, respectively). We put *three basic hypotheses* on the sites we consider. These will be marked **A.1, A.2 and A.3** and will be spread out in the first section.

Our first goal is to extend Grothendieck topologies defined on algebraic spaces to simplicial spaces. For this we will begin with a *class* $\mathcal{P}$ of maps of algebraic spaces that satisfy the following conditions:

(i) If $\sigma : X' \to X$ is an isomorphism, σ belongs to $\mathcal{P}$

(ii) If $\sigma : X'' \to X'$ belongs to $\mathcal{P}$ and $\tau : X' \to X$ belongs to $\mathcal{P}$, then the composition $\tau \circ \sigma$ belongs to $\mathcal{P}$ and

(iii) If $\sigma : X' \to X$ belongs to $\mathcal{P}$ and $Y \to X$ is an arbitrary map, the pull-back $X' \underset{X}{\times} Y \to Y$ belongs to $\mathcal{P}$.

The language of fibered categories (see [SGA] 1, Expose VI) seems to provide a convenient technique for handling both big and small sites simultaneously for simplicial spaces. (From the description below, one may see that such sites are put together from the corresponding sites on the algebraic spaces in each degree. Since we need to relate the sites of the algebraic spaces in each degree by means of the structure maps of the simplicial space, the language of fibered categories is unavoidable.) We define a *Grothendieck topology* Top, on $(alg.spaces/S)$ as a *fibered category*, fibered over $(alg.spaces/S)$, provided with a class $\mathcal{P}$ as above and satisfying the following hypotheses:

A.1(i) for each algebraic space X, $Top(X)$ is a *full* sub-category of the category of all algebraic spaces Y over X,

(ii) closed under finite inverse limits and such that for any $Y \to X$ in $Top(X)$ and any map $U \to Y$ in $\mathcal{P}$, the composition $U \to Y \to X$ belongs to $Top(X)$. We also require that $X \xrightarrow{id} X$ belong to $Top(X)$. (Observe as a consequence that any $\alpha : Y \to X$ in $\mathcal{P}$ also belongs to $Top(X)$.)

We define the coverings of any object Y in $Top(X)$ to be given by families $\{f_\alpha : U_\alpha \to Y \quad in \quad \mathcal{P} | \alpha\}$ so that $\bigcup_\alpha f_\alpha(U_\alpha) = Y$.

Remarks 1. Observe that the property of being a fibered category implies that for each map $f : Y \to X$ of algebraic spaces, there is an induced functor $f^{-1} : Top(X) \to Top(Y)$, sending $U \in Top(X)$ to $f^{-1}(U) = U \underset{X}{\times} Y$. Moreover, if $g : Z \to Y$ is another map of algebraic spaces, there is given a natural isomorphism between the two functors $g^{-1} \circ f^{-1} \simeq (f \circ g)^{-1}$ and such natural isomorphisms are required to satisfy certain compatibility conditions.

2. This definition is broad enough to include both big and small topologies: if one lets $Top(X)$ denote the category of all algebraic spaces over X, one obtains a *big* topology. On the other hand, if one lets the objects in $Top(X)$ to be the maps $f : Y \to X$ in $\mathcal{P}$, then one obtains a *small* topology, provided for each X, the category $\{f : Y \to X | f \quad in \quad \mathcal{P}\}$ is a small (or skeletally small) category. Given a big topology, $Top(X)$, we may consider the associated small topology $Top(X, \mathcal{P})$, where the objects are maps $Y \to X$ that are in $\mathcal{P}$.

3. When considering small topologies, it will be convenient to add the following hypothesis on the class $\mathcal{P}$ which will ensure that the category $Top(X)$ is closed under finite inverse limits for any X:

(iv) if $\sigma : X' \to X$ is in $\mathcal{P}$, the diagonal map $X' \to X' \underset{X}{\times} X'$ is also in $\mathcal{P}$.

4. It is clear that the (small) étale topology where the objects of $Top(X)$ are étale maps $Y \to X$ is a small topology in the above sense satisfying all the four conditions on $\mathcal{P}$, whereas the flat (h-, qfh -) topology on an algebraic space X which is the category of all algebraic spaces $Y \to X$ and where the coverings are all *flat* maps (topological epimorphisms, topological epimorphisms that are also quasi-finite, respectively) forms a big topology.

(1.1.1) (**The basic topologies on simplicial algebraic spaces**). Let Top denote a Grothendieck topology on $(alg.spaces/S)$ satisfying **A.1**. We will extend this to the following two Grothendieck topologies on $(simpl.spaces/S)$.

(i) Let $X_\bullet$ denote a simplicial algebraic space. We let $Top(X_\bullet)$ denote the following category. The objects are $U \to X_n$ in $Top(X_n)$ for some n. Given two objects $U \to X_n$ and $V \to X_m$ in $Top(X_\bullet)$, a map $(U \to X_n) \to (V \to X_m)$ is a commutative square

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X_n \arrow[r] & X_m \end{tikzcd}$$

where the bottom row is a structure map of the given simplicial space $X_\bullet$. Given any object $U \to X_n$ in $Top(X_\bullet)$, the *coverings* of $U \to X_n$ are maps $V_\alpha \xrightarrow{\epsilon_\alpha} U$ in $Top(X_n)$ belonging to the class $\mathcal{P}$ so that $\underset{\alpha}{\cup}\epsilon_\alpha(V_\alpha) = U$. (This topology is originally due to Deligne - see [De](5.1.6).)

(ii) We will next define another Grothendieck topology, $STop$, on simplicial algebraic spaces (again as a fibered category, fibered over $(simpl.spaces/S)$). We define the objects of $STop(X_\bullet)$ to consist of all maps $f_\bullet : Y_\bullet \to X_\bullet$ of simplicial algebraic spaces so that each $f_n : Y_n \to X_n$ belongs to $Top(X_n)$. Morphisms between two such objects will be defined to be commutative triangles in the obvious manner. One defines a family of maps $\{f_{\bullet\alpha} : V_{\bullet\alpha} \to U_\bullet | \alpha\}$ to be a covering if each $\{f_{n_\alpha} : V_{n_\alpha} \to U_n | \alpha\}$ is a *covering* in $Top(X_n)$ i.e. f_{n_α} belongs to $\mathcal{P}$ and $\bigcup_\alpha f_{n_\alpha}(V_{n_\alpha}) = U_n$ for all n. We will often call this *the simplicial topology associated to the given topology Top*. We may define a class of maps $\mathcal{P}_\bullet$ between simplicial spaces as follows: a map $f_\bullet : V_\bullet \to U_\bullet$ belongs to $\mathcal{P}_\bullet$ if each $f_n : V_n \to U_n$ belongs to the class $\mathcal{P}$.

(1.1.2) One may observe readily that, for each fixed $X_\bullet$, the category $STop(X_\bullet)$ is closed under finite inverse limits. (This follows from the hypotheses that each $Top(X_n)$ is closed under finite inverse limits. Observe that the inverse limit of a diagram of simplicial schemes may be computed in each simplicial degree.)

(1.1.3) The small étale (smooth) topology on any algebraic space X will be denoted $Et(X)$ $(Smt(X))$. In case we are using the small étale topology on algebraic spaces, the induced topologies on simplicial algebraic spaces $X_\bullet$ will be denoted $Et(X_\bullet)$ and $SEt(X_\bullet)$ respectively. The corresponding smooth topologies will be denoted $Smt(X_\bullet)$ and $SSmt(X_\bullet)$.

(1.2) **Hypercoverings**. We may define hypercoverings in $STop(X_\bullet)$ to be simplicial objects $V_\bullet$ in $STop(X_\bullet)$ (i.e. bisimplicial algebraic spaces over $X_\bullet$) so that for each $t \geq 0$, the map $V_t \to (\mathrm{cosk}_{t-1}^{X_\bullet} V_\bullet)_t$ is a covering. (Here $(\mathrm{cosk}_{-1}^{X_\bullet} V_\bullet)_0 = X_\bullet$.) The hypercoverings in $Top(X_\bullet)$ are defined in [Fr] p. 23 to be bisimplicial algebraic spaces $V_{\bullet,\bullet}$ in $Top(X_\bullet)$ satisfying the same conditions. It follows that the hypercoverings in the above two categories are the same. The category of hypercoverings in $STop(X_\bullet)$ $(Top(X_\bullet))$ will be denoted $HR(STop(X_\bullet))$ $(HR(Top(X_\bullet))$, respectively).

One may define the homotopy category of hypercoverings as in [SGA4] Expose V, (7.3.2). These will be denoted $HHR(STop(X_\bullet))$ and $HHR(Top(X_\bullet))$: as shown in [SGA4] Expose V, (7.3.2) the opposite of these categories are *filtered* categories.

(1.3.1) **Presheaves and sheaves**. Let **C** denote a category that is closed under all small limits and colimits. A presheaf on any of the above topologies with values in the category **C** is a contravariant functor taking values in **C**. A presheaf is a sheaf if it satisfies the usual sheaf axiom. For the topology $Top(X_\bullet)$, such a presheaf (sheaf)F corresponds to a collection of presheaves (sheaves, respectively) $\{F_n | n\}$ on each $Top(X_n)$ so that for each structure map $\alpha : X_n \to X_m$ of the simplicial space $X_\bullet$, there is given a map $\Phi_\alpha : \alpha^*(F_m) \to F_n$ satisfying certain obvious compatibility conditions. Given a presheaf F on $Top(X_\bullet)$, F_n always will denote the restriction of F to $Top(X_n)$ for each $n \geq 0$. Given two presheaves (sheaves) $F = \{F_n | n\}$ and $F' = \{F'_n | n\}$, a morphism $\alpha : F \to F'$ is given by a compatible collection of maps $\{\alpha_n : F_n \to F'_n | n\}$.

(1.3.2) We say such a presheaf (sheaf) $F = \{F_n | n\}$ has *descent* if each of the above structure maps Φ_α is an isomorphism. If $\mathbf{C} = \mathbf{A}$ is an abelian category, the sub-category of presheaves (sheaves) with descent defines a *full abelian sub-category* of the category of all presheaves (sheaves, respectively) on $Top(X_\bullet)$ with values in **A**. *The key observation* is that the sub-category of presheaves (sheaves) with descent *is also closed under extensions* in the abelian category of all presheaves (sheaves, respectively). For the most part we will restrict our discussion to the case where **A** is the category of modules over a given ring R, but most of our results readily extend to other situations as well - this will be left to the reader. If R is a commutative ring with a unit, the category of all presheaves (sheaves) of R-modules on $Top(X_\bullet)$ will be denoted $Presh(Top(X_\bullet); R)$ ($Sh(Top(X_\bullet); R)$, respectively). The corresponding categories on $STop(X_\bullet)$ are denoted $Presh(STop(X_\bullet); R)$ and $Sh(STop(X_\bullet); R)$, respectively.

(1.3.3) One can see readily that the categories of presheaves and sheaves are closed under all small limits and colimits and that filtered colimits are exact.

Remark. Assume that $X_\bullet$ is the classifying simplicial space associated to an atlas $x : X \to \mathcal{S}$ for an algebraic stack. (i.e. x is a smooth surjective map from an algebraic space to the stack $\mathcal{S}$ and $X_\bullet = cosk_0^{\mathcal{S}}(x)$.) Then a sheaf F on $Smt(X_\bullet)$ has descent if and only if it descends to a sheaf on the smooth site of the stack. We have adopted this terminology to all simplicial spaces, even if they are not the classifying simplicial spaces associated to algebraic stacks.

Points.

Since points of different types are used extensively in the paper, we begin with a general discussion on points: see [SGA] 4, Expose IV, section 6 for further details.

(1.4.0) Given any site $\mathcal{C}$, a *point* of $\mathcal{C}$ is a map of sites $p : (sets) \to \mathcal{C}$ where $(sets)$ denotes the obvious site whose objects are all small sets, morphisms being maps of sets and coverings being surjective maps of sets. Let $p^{-1} : \mathcal{C} \to (sets)$ denote the associated functor. The point p now defines maps of topoi: $p^* : Sh(\mathcal{C}) \to Sh((sets))$ and $p_* : Sh((sets)) \to Sh(\mathcal{C})$ where Sh denotes the category of abelian sheaves on the corresponding sites. Associated to each point p, one may define a fiber (or stalk) functor sending a sheaf F on the site $\mathcal{C}$ to its stalk F_p as in [SGA] 4, Exposé IV (6.8.2). Under the assumption that the category $\mathcal{C}$ is closed under finite inverse limits, one may show that this functor is exact on abelian sheaves. We say the site $\mathcal{C}$ has a ***conservative family of points*** (or equivalently there are *enough* points) if it is provided with a small family of points $\{p\}$ so that a sequence of abelian sheaves $0 \to F' \to F \to F'' \to 0$ is exact if and only if $0 \to F'_p \to F_p \to F''_p \to 0$ is exact for all points p.

(1.4.1) Next we make the following observation. A given family of points is conservative if and only if the following holds: a family $\{U_\alpha \to U|\alpha\}$ in $\mathcal{C}$ is a covering family if and only if for each of the given points p, $\{p^{-1}(U_\alpha) \to p^{-1}(U)|\alpha\}$ is a covering. This is proved in [SGA] 4, Exposé IV, Proposition (6.5).

A.2: We assume that there exists (a small set of) algebraic spaces $*$ which are *acyclic* in cohomology with respect to any abelian sheaf on $Top(*)$ and so that the only objects of $Top(*)$ are finite disjoint unions of copies of $*$. We assume that for each simplicial space $X_\bullet$ and each $n \geq 0$, one is provided with a set $\bar{X}_n$ so that for each $\bar{p}_n \,\epsilon\, \bar{X}_n$, one is given a map $\bar{p}_n : * \to X_n$ of algebraic spaces (where $*$ is as above). *We require that each such* $\bar{p}_n$ *defines a point of the site* $Top(X_n)$ *sending a* $U \overset{\alpha}{\to} X_n$ *to all possible liftings of* $\bar{p}_n$ *to* U *and that the set of all such points forms a conservative family of points* for the site $Top(X_n)$. We will identify the set $\bar{X}_n$ with the corresponding set of points of the site $Top(X_n)$.

(1.4.2) A *simplicial point (or simply point)* of $X_\bullet$ is defined as a map of simplicial spaces $\bar{p}_\bullet : \Delta[n] \otimes * \to X_\bullet$ so that in each degree, k, $(\bar{p}_\bullet)_k$ is a disjoint union of the given points of X_k. (Recall $(\Delta[n] \otimes *)_k = \underset{\alpha \,\epsilon\, \Delta[n]_k}{\sqcup} *$ with the structure maps of $\Delta[n] \otimes *$ induced from the structure maps of the simplicial set $\Delta[n]$). Let $\bar{p}_n$ denote a point of X_n as above. Then such a point defines a map $(\bar{p}_n)_\bullet : \Delta[n] \otimes * \to X_\bullet$ of simplicial spaces so that $(\bar{p}_n)_\bullet$ restricted to $i_n \otimes *$ is $\bar{p}_n$ where $i_n \,\epsilon\, \Delta[n]_n$ is the generator of $\Delta[n]$. It also defines a map of sites $(sets) \to STop(X_\bullet)$ by sending a $U_\bullet$ to the set of all liftings of $(\bar{p}_n)_\bullet$ to $U_\bullet$. (In fact we may identify liftings of the simplicial point $(\bar{p}_n)_\bullet$ to $U_\bullet$ with the set of all liftings of the point $\bar{p}_n$ to U_n.) This observation shows that simplicial points define points of the site $STop(X_\bullet)$ in the usual sense. *Notation: given a point* $\bar{p}_n$ *of* X_n*, the associated simplicial point of* $X_\bullet$ *will be denoted* $(\bar{p}_n)_\bullet$.

(1.4.3) In the case of the étale site, one may obtain a more explicit definition of a simplicial (geometric) point of $X_\bullet$ as a map $\bar{x}_\bullet : (\text{Spec } \Omega) \otimes \Delta[n] \to X_\bullet$ of simplicial algebraic spaces, where Ω is a separably closed field.

Observe also that the cohomology of $\Delta[n] \otimes *$ with respect to any abelian sheaf on $Top(\Delta[n] \otimes *)$ that has descent is trivial in all positive degrees. (See (3.7.7) below for this computation.)

(1.5.1) **(Simplicial) neighborhoods of a point**. Let $p_\bullet : \Delta[n] \otimes * \to X_\bullet$ denote a point of $X_\bullet$. A (simplicial) neighborhood of $p_\bullet$ is a commutative triangle

$$\begin{tikzcd}
 & U_\bullet \arrow[d] \\
\Delta[n] \otimes * \arrow[ur] \arrow[r, "p_\bullet"] & X_\bullet
\end{tikzcd}$$

where the map $U_\bullet \to X_\bullet$ is in $STop(X_\bullet)$. Observe that for small topologies, this implies the last map is in the class $\mathcal{P}_\bullet$ as well. (In the setting of étale topologies, this definition is originally due to Friedlander.)

(1.5.2) Let $U_\bullet \,\epsilon\, STop(X_\bullet)$ and let $\{U_\bullet^\alpha|\alpha\}$ denote a family of maps to $U_\bullet$ in $STop(X_\bullet)$. Let $\{p_\bullet\}$ denote the family of simplicial points of $STop(X_\bullet)$ defined as in (1.4.2) associated to a family of points of X_n for all $n \geq 0$: recall these define points of the site $STop(X_\bullet)$. In view of the hypothesis in **A.2**, (1.4.1) shows the family $\{U_n^\alpha \to U_n|\alpha\}$ is a covering if and only if for each given point p_n of X_n, $\{p_n^{-1}(U_n^\alpha) \to p_n^{-1}(U_n)|\alpha\}$ is a covering. Therefore the family $\{U_\bullet^\alpha|\alpha\}$ is a covering family of $U_\bullet$ if and only if the corresponding family of sets $\{p_\bullet^{-1}(U_\bullet^\alpha)|\alpha\}$ is a covering of $p_\bullet^{-1}(U_\bullet)$ for each given point $p_\bullet$. Therefore, it follows from the discussion in (1.4.1), that the given family of simplicial points is a *conservative* family of points for the site $STop(X_\bullet)$.

A.3: We will assume that the system of neighborhoods in both $Top(X_\bullet)$ and $STop(X_\bullet)$ of any point has a *small* cofinal family.

(1.5.3) Let $j_{U_\bullet} : U_\bullet \to X_\bullet$ denote the obvious map corresponding to a simplicial neighborhood of a point $p_\bullet$. Even though the sites we consider are not necessarily small, one may see from [SGA]4, I, (5.10) (or [Mi] p.78) that the restriction functor $j_{U_\bullet}^* : Presh(STop(X_\bullet); R) \to Presh(STop(U_\bullet); R)$ has a left adjoint denoted $j_{U_\bullet!}$. The same holds for the functor $j_{U_\bullet}^* : Presh(Top(X_\bullet); R) \to Presh(Top(U_\bullet); R)$ as well as the corresponding restriction functors on the categories of sheaves. Using this functor and the hypothesis on the existence of a small cofinal system of neighborhoods of any point, one may show readily that the categories $Sh(Top(X_\bullet); R)$ and $Sh(STop(X_\bullet); R)$ have a generator and are Grothendieck categories. In particular they have enough injectives and in fact an injective co-generator. (The hypothesis **A**.3 is necessary since we are in general considering big sites.)

(1.6) **Morphisms**. Let $f : X_\bullet \to Y_\bullet$ denote a map of simplicial algebraic spaces. Then f induces a map of sites: $f^* : Top(Y_\bullet) \to Top(X_\bullet)$ sending $(V_n \to Y_n)$ to $(X_n \underset{Y_n}{\times} V_n) \to X_n$ and also ${}_sf^* : STop(Y_\bullet) \to STop(X_\bullet)$ sending $(V_\bullet \to Y_\bullet)$ to $(X_\bullet \underset{Y_\bullet}{\times} V_\bullet \to X_\bullet)$

2. Cohomology and Derived functors.

Throughout this section $f : X_\bullet \to Y_\bullet$ will denote a map of simplicial algebraic spaces. Let R denote a commutative Noetherian ring with unit.

(2.1) Then f defines a direct image functor $f_* : Presh(Top(X_\bullet); R) \to Presh(Top(Y_\bullet); R)$. One may readily verify that $f_* = \{f_{n*} : Presh(Top(X_n); R) \to Presh(Top(Y_n); R)|n\}$. Similarly f induces an inverse image functor $f^* : Presh(Top(Y_\bullet); R) \to Presh(Top(X_\bullet); R)$ and $f^* = \{f_n^* : Presh(Top(Y_n); R) \to Presh(Top(X_n); R)|n\}$. The obvious functor induced by f^* at the level of sheaves will also be denoted f^*. One may readily verify that f_* sends sheaves to sheaves and both f_* and f^* are exact functors at the level of presheaves. At the level of sheaves f_* is left-exact while f^* is exact. The exactness of the functor f^* depends on our hypothesis that each of the categories $Top(Y_n)$ is closed under finite limits.

(2.2) f also defines a direct image functor $Presh(STop(X_\bullet); R) \to Presh(STop(Y_\bullet); R)$. We will denote this by ${}_sf_*$. The inverse image functor $Presh(STop(Y_\bullet); R) \to Presh(STop(X_\bullet); R)$ is denoted ${}_sf^*$. Both are exact functors.

(The exactness of the inverse image functors ${}_sf^*$ once again depends on our hypothesis that the topologies are closed under finite limits.) Observe that ${}_sf_*$ sends sheaves to sheaves: at the level of sheaves this functor is only left-exact. The functor induced by ${}_sf^*$ at the level of sheaves will still be denoted ${}_sf^*$.

(2.3) Since the categories $Sh(Top(X_\bullet); R)$ and $Sh(STop(X_\bullet); R)$ have enough injectives, we may define the right derived functors of f_* and ${}_sf_*$ in the usual manner.

(2.4) **Proposition**. Let $X_\bullet$ be a simplicial algebraic space as before and let F be an abelian sheaf on $STop(X_\bullet)$. (i)Then $H^*_{STop(X_\bullet)}(X_\bullet;\ F) \cong \lim_{\substack{\rightarrow \\ U_{\bullet,\bullet}}} H^*(\Gamma(U_{\bullet\bullet},\ F))$ where the direct limit is taken over the (filtered) homotopy category of hypercoverings and the left-hand-side denotes the cohomology of $X_\bullet$ computed on the site $STop(X_\bullet)$.

(ii) Similarly, if $f : X_\bullet \to Y_\bullet$ is a map of simplicial algebraic spaces,

$$\Gamma(U_\bullet, R_sf_*(F)) = \lim_{\substack{\rightarrow \\ V_{\bullet,\bullet}}} \Gamma(V_{\bullet\bullet};\ F)$$

where $U_\bullet \,\epsilon\, STop(Y_\bullet)$ and $V_{\bullet\bullet}$ varies in the the homotopy category of all hypercoverings of $U_\bullet \underset{Y_\bullet}{\times} X_\bullet$.

Proof. Since the proof of (ii) is entirely similar, we will only consider (i). To prove the proposition, it suffices to show that the functor sending F to the right hand side is an effaceable δ-functor. The proof is quite standard and follows as in [SGA4] Expose V. □

(2.5) **Derived categories**. If $\mathbf{A}$ is an abelian category, $C(\mathbf{A})$ ($C_+(\mathbf{A})$, $C_b(\mathbf{A})$, $C_0(\mathbf{A})$) will denote the category of all unbounded complexes (complexes that are bounded below, complexes that are bounded, complexes that are trivial in negative degrees, respectively). One may then define the homotopy categories and derived categories associated to the first three in the usual manner: these are denoted $D(\mathbf{A})$, $D_+(\mathbf{A})$ and $D_b(\mathbf{A})$. For the most part we will only consider the derived category of bounded below complexes, i.e. $D_+(\mathbf{A})$.

If $f : X_\bullet \to Y_\bullet$ is a map of simplicial algebraic spaces, it is clear the derived functors Rf_* and R_sf_* are in fact functors at the level of the associated derived categories of bounded below complexes.

If $X_\bullet$ is a simplicial algebraic space $D^{des}_+(Absh(Top(X_\bullet)))$ will denote the full subcategory of $D_+(Absh(Top(X_\bullet)))$ consisting of complexes K whose cohomology sheaves have *descent*. The discussion in (1.3.2) shows that the full (abelian) subcategory $Absh^{des}(Top(X_\bullet))$ of sheaves with descent is closed under extensions in the category $Absh(Top(X_\bullet))$. Therefore, $D^{des}_+(Absh(Top(X_\bullet)))$ is indeed a triangulated category.

(2.6) **Remarks**. (i) Observe that (2.4) extends readily to the case where F is replaced by a complex $K \,\epsilon\, C_0(Absh(STop(X_\bullet)))$ and $\lim_{\substack{\rightarrow \\ U_{\bullet,\bullet}}} H^*(\Gamma(U_{\bullet\bullet},\ F))$ is replaced by $\lim_{\substack{\rightarrow \\ U_{\bullet,\bullet}}} H^*(\Delta\Gamma(U_{\bullet\bullet},\ DN(K)))$. Here $DN(K)$ denotes the obvious cosimplicial object associated to the co-chain complex K and Δ is the diagonal. (See the appendix.) In the next section (see (3.7.3)) we show that there is a functor $\eta_* : Absh(Top(X_\bullet)) \to C_0(Absh(STop(X_\bullet)))$. One may show from the definition of the functor η_* that $\Delta\Gamma(U_{\bullet,\bullet}, DN(\eta_*(F))) = \Gamma(\Delta(U_{\bullet,\bullet}), F)$, as $U_{\bullet,\bullet}$ varies in the category $HHR(STop(X_\bullet))$ and for each $F \,\epsilon\, Absh(Top(X_\bullet))$. But $\lim_{\substack{\rightarrow \\ U_{\bullet,\bullet}}} H^*(\Delta\Gamma(U_{\bullet,\bullet}, DN(\eta_*(F)))) \cong \mathbb{H}^*_{STop(X_\bullet)}(X_\bullet, \eta_*(F))$ as shown above. ((3.7.7) below shows that one may replace the complex $\eta_*(F)$ by the sheaf $\bar{\eta}_*(F)$ if $F \,\epsilon\, Absh^{des}(Top(X_\bullet))$.)

On the other hand, [Fr] Corollary (3.10) shows $H^*_{Top(X_\bullet)}(X_\bullet, F) \cong \lim_{\substack{\rightarrow \\ U_{\bullet,\bullet}}} H^*(\Gamma(\Delta(U_{\bullet\bullet}),\ F))$. Recall from (1.2) that the hypercoverings in the two sites $Top(X_\bullet)$ and $STop(X_\bullet)$ are the same. Therefore, the above observations readily provide the isomorphism $H^*_{STop(X_\bullet)}(X_\bullet, \eta_*(F)) \cong H^*_{Top(X_\bullet)}(X_\bullet, F)$ for any sheaf $F \,\epsilon\, Absh(Top(X_\bullet))$ (established by different techniques in (3.11).) A similar argument using (2.4)(ii) will provide a different proof of (3.11)(ii).

(ii) Let $K \,\epsilon\, D_+(Absh(Top(X_\bullet)))$. Then one obtains a spectral sequence

$$E_1^{s,t} = \mathbb{H}^t(X_s; K_s) \Rightarrow \mathbb{H}^{s+t}(X_\bullet; K)$$

exactly as in [Fr] p. 19. This is strongly convergent since the complex K is bounded below by assumption.

3. Comparison of sites.

In this section we compare the derived category on the two sites $STop(X_\bullet)$ and $Top(X_\bullet)$ associated to a simplicial algebraic space.

(3.1) Let $X_\bullet$ denote a simplicial space and let $k \geq 0$ be a fixed integer. Observe that the functor sending $U_\bullet \epsilon STop(X_\bullet)$ to U_k defines a map of sites $\eta_k : Top(X_k) \rightarrow STop(X_\bullet)$: the underlying functor sends $U_\bullet$ to U_k.

(3.2) The functor η_k sending a $V_\bullet \epsilon STop(X_\bullet)$ to V_k has a right adjoint L^k defined as follows. Let V be in $Top(X_k)$. Then $L^k(V)_\bullet$ is the simplicial space with each $L^k(V)_n$ defined by the cartesian square

$$\begin{tikzcd}
L^k(V)_n \arrow[r] \arrow[d] & \underset{\alpha \,\epsilon\, Hom_\Delta([k],[n])}{\times} V \arrow[d] \\
X_n \arrow[r] & \underset{\alpha \,\epsilon\, Hom_\Delta([k],[n])}{\times} X_k
\end{tikzcd} .$$

Here the bottom row sends X_n to the factor X_k indexed by α using the map $X_\bullet(\alpha) : X_n \rightarrow X_k$. Therefore, we may define a map of sites $\psi_k : STop(X_\bullet) \rightarrow Top(X_k)$ by $V \mapsto L^k(V)_\bullet$, $V \epsilon Top(X_k)$.

(3.3) **Proposition.**

(i) Let $\bar{p}_\bullet : \Delta[n] \otimes * \rightarrow X_\bullet$ denote a given point of $X_\bullet$. Let $k \geq 0$ be an integer and let W denote a neighborhood of $(\bar{p}_\bullet)_k$. Then there exists a neighborhood $V_\bullet$ of $\bar{p}_\bullet$ so that the map $V_k \rightarrow X_k$ factors through the given map $W \rightarrow X_k$. Moreover, the system of neighborhoods W of $(\bar{p}_\bullet)_k$ for which the obvious map $\pi_0((\bar{p}_\bullet)_k) \rightarrow \pi_0(W)$ is bijective is cofinal in the set of all neighborhoods of $(\bar{p}_\bullet)_k$.

(ii) Let $\bar{x}_k$ denote a given point of X_k. Given any map $W \rightarrow X_k$ which is a neighborhood of $\bar{x}_k$ as well as $X_\bullet(\alpha)(\bar{x}_k)$, for all structure maps $\alpha : [k] \rightarrow [k]$, there exists a neighborhood $W_\bullet$ of the corresponding point $(\bar{x}_k)_\bullet : \Delta[k] \otimes * \rightarrow X_\bullet$ in the topology $STop(X_\bullet)$ so that the map $(W_\bullet)_k \rightarrow X_k$ factors through the given map $W \rightarrow X_k$.

(iii) Given any covering $W \rightarrow X_k$ in $Top(X_k)$ there exists a simplicial object $V_\bullet \epsilon STop(X_\bullet)$ so that for each n, $V_n \rightarrow X_n$ is a covering in $Top(X_n)$ and the map $V_k \rightarrow X_k$ factors through the given map $W \rightarrow X_k$.

Proof. (i) Take $V_\bullet = L^k(W)_\bullet$. In order to show $V_\bullet$ is a neighborhood of $\bar{p}_\bullet$, it suffices to show there is a lift of the point $\bar{p}_n : i_n \otimes * \rightarrow X_n$ to $L^k(W)_n$ (here i_n is the generator of $\Delta[n]$): this follows from the definition of $L^k(W)_\bullet$ and the hypothesis that W is a neighborhood of $(\bar{p}_\bullet)_k = \{X_\bullet(\alpha)(\bar{p}_n) | \alpha \epsilon Hom_\Delta([k],[n])$. The assertion that the map $V_k \rightarrow X_k$ factors through $W \rightarrow X_k$ follows from the definition of $L^k(W)_\bullet$. This proves the first assertion in (i). (iii) is proved similarly.

Now we consider the second assertion in (i). First observe that, given any neighborhood W of $(\bar{p}_\bullet)_k$, one may find a neighborhood W' satisfying the given condition and dominating W. Next, given any two neighborhoods V and W of $\bar{p}_\bullet$ with two maps $f, g : V \rightarrow W$ of neighborhoods, one may let $U = V \underset{W}{\times} V =$ the the equalizer of f and g; U is also a neighborhood of $(\bar{p}_\bullet)_k$. At this point, one may find a neighborhood satisfying the given condition and dominating U. This completes the proof of (i).

Next we consider (ii). Let k denote a fixed integer ≥ 0 and let $\bar{x}_k$ denote a point of X_k. Let $W \rightarrow X_k$ denote an object in $Top(X_k)$ so that it is a neighborhood of $\bar{x}_k$ as well as all $X_\bullet(\alpha)(\bar{x}_k)$, for all α. Then the hypothesis in (i) is satisfied for $\bar{p}_\bullet = (\bar{x}_k)_\bullet$ by W. Therefore, $(L^k(W))_k \rightarrow X_k$ is a neighborhood of $\bar{x}_k$ and the above map factors through the given map $W \rightarrow X_k$. Therefore, let $W_\bullet = L^k(W)_\bullet$. This proves (ii). $\square$

(3.4.1) **Proposition**. Let $\bar{x}_\bullet : \Delta[n] \otimes * \rightarrow X_\bullet$ denote a fixed simplicial point of $X_\bullet$. Let $k \geq 0$ and let W denote a fixed neighborhood of $\bar{x}_k = (\bar{x}_\bullet)_k$. Then there exists a simplicial neighborhood $V_\bullet$ of $\bar{x}_\bullet$ so that:

i) for every $m \geq 0$, the map $\pi_0(\bar{x}_m) \rightarrow \pi_0(V_m)$ is bijective and

ii) the map $V_k \rightarrow X_k$ factors through the given map $W \rightarrow X_k$

Proof. We will define $V'_\bullet = L^k(W)_\bullet$. Therefore, the last assertion is clear for $V'_\bullet$ and it is a neighborhood of $\bar{x}_\bullet$ by (3.3)(i). Let $\bar{v}_n \epsilon V'_n$ be the image of $i_n \otimes *$, where i_n is the generator of $\Delta[n]$.

Let $V'_{\bar{v}_n}$ be the connected component of V'_n containing $\bar{v}_n$. For each map $\alpha : [m] \to [n]$ in Δ, let $V'_{\bar{\alpha}(\bar{v}_n)}$ denote the connected component of V'_m containing $\bar{\alpha}(\bar{v}_n)$. (Here $\bar{\alpha} : V'_n \to V'_m$ is the map induced by α.) Then the collection $\{V'_{\bar{\alpha}(\bar{v}_n)}|\alpha\}$ defines a sub-simplicial object of $V'_\bullet$. We define the simplicial neighborhood $V_\bullet$ by letting $V_m = \underset{\alpha:[m]\to[n]}{\sqcup} V'_{\bar{\alpha}(\bar{v}_n)}$ where $\sqcup$ denotes the *disjoint union*. The simplicial structure is defined by letting $V'_{\bar{v}_n}$ map to the summand $V'_{\bar{\alpha}(\bar{v}_n)}$ by the map $\bar{\alpha}$ restricted to $V'_{\bar{v}_n}$. (i.e. $V_\bullet = \Delta[n] \otimes V'_{\bar{v}_n}$.) Now it is clear that for every $m \geq 0$, the map $\pi_0(\bar{x}_m) \to \pi_0(V_m)$ is bijective. Moreover, since $V_\bullet$ dominates $V'_\bullet$, it is clear that the map $V_k \to X_k$ factors through the given map W. □

We *define* a category $\mathcal{C}$ to be *left-filtered* if the opposite category $\mathcal{C}^{op}$ is *filtered* in the sense of [Mac] Chapter IX, section 1.

(3.4.2) **Corollary**. Assume the hypotheses of the proposition. Then the category of all simplicial neighborhoods of a simplicial point $\bar{x}_\bullet$ is left-filtered and the sub-category of all simplicial neighborhoods satisfying the hypotheses in (3.4.1) is cofinal in the category of all such simplicial neighborhoods. Moreover, when the site considered is the small étale site, there is at most one map between two neighborhoods satisfying the hypothesis in (3.4.1).

Proof. Clearly, given any two simplicial neighborhoods $V_\bullet$ and $W_\bullet$, of the point $\bar{x}_\bullet$, one may form the fibered product $V_\bullet \underset{X_\bullet}{\times} W_\bullet$ which dominates both $V_\bullet$ and $W_\bullet$. The proof of (3.4.1) shows that one may find a simplicial neighborhood satisfying the conditions there and dominating this fibered product. Given any two maps $f, g : V_\bullet \to W_\bullet$ of two simplicial neighborhoods of $\bar{x}_\bullet$, one may form their inverse limit $U_\bullet$ which is also a simplicial neighborhood of $\bar{x}_\bullet$. (In the case of small topologies (satisfying the hypothesis (iv) of **A.1** on the class of maps in $\mathcal{P}$) this inverse limit $U_\bullet$ is defined by the cartesian square:

(3.4.2.*)

$$\begin{tikzcd}
U_\bullet \arrow[r] \arrow[d] & V_\bullet \arrow[d, "\Gamma_f"'] \\
V_\bullet \arrow[r, "\Gamma_g"] & V_\bullet \underset{X_\bullet}{\times} W_\bullet
\end{tikzcd}$$

The hypothesis (iv) shows the maps Γ_f and Γ_g belong to $\mathcal{P}_\bullet$ and the hypotheses (i) and (iii) (of **A.1**) show $U_\bullet \to X_\bullet$ belongs to $\mathcal{P}_\bullet$.) Now the map $U_\bullet \to V_\bullet$ is the equalizer of f and g. This shows the category of all simplicial neighborhoods of a point $\bar{x}_\bullet$ is left-filtered.

Let $\bar{x}_\bullet : \Delta[n] \otimes * \to X_\bullet$ denote the given point. If $U_\bullet$ denotes the simplicial neighborhood given in the above paragraph, one may replace it with $\Delta[n] \otimes U_{n,0}$ where $U_{n,0}$ denotes the connected component of U_n containing the point $\bar{x}_n = \bar{x}(i_n \otimes *)$. This shows that the neighborhoods satisfying the conditions in (3.4.1) are cofinal in the category of all neighborhoods of $\bar{x}_\bullet$. Let $V_\bullet$ ($V'_\bullet$) denote a neighborhood of $\bar{x}_\bullet$ with $\bar{v}_\bullet : \Delta[n] \otimes * \to V_\bullet$ ($\bar{v}'_\bullet : \Delta[n] \otimes * \to V'_\bullet$, respectively) denoting the given lifting of $\bar{x}_\bullet$. Recall that a morphism $\phi : V_\bullet \to V'_\bullet$ is a map of simplicial objects over $X_\bullet$ sending the lifting $\bar{v}_\bullet$ to $\bar{v}'_\bullet$. Now it is clear that there is at most one map between simplicial neighborhoods satisfying the hypothesis in (3.4.1) in case the site is the small étale site. (See [Mi] Chapter I, Corollary (3.13).) □

(3.5) **Definition** (The stalk or the fiber functor associated to a simplicial point). Let $\bar{x}_\bullet : \Delta[n] \otimes * \to X_\bullet$ denote a simplicial point as before and let F' denote a sheaf on $STop(X_\bullet)$. We let $F'_{\bar{x}_\bullet} = \underset{\overrightarrow{U_\bullet}}{\lim} \Gamma(U_\bullet, F')$ where the colimit is over all simplicial neighborhoods of the point $\bar{x}_\bullet$. The above colimit is a filtered colimit and therefore exact. In particular, it defines a fiber functor in the sense of [SGA]4, Exposé IV, section 7.

Remark. (3.4.2) shows that the sub-category, $\{V_\bullet\}$, of all simplicial neighborhoods of $\bar{x}_\bullet$ so that the map $\pi_0(\bar{x}_m) \to \pi_0(V_m)$ is bijective for every $m \geq 0$ is cofinal in the category of all simplicial neighborhoods. Therefore, without loss of generality, one may take the colimit over only such simplicial neighborhoods to define the stalk. While (3.4.2) is not essential for us, it shows that, in the étale case, the filtered colimit involved in the definition of the stalk may replaced by the colimit over a directed set.

(3.6) Next observe that the functors associated to η_k and ψ_k are adjoint. Using the observation in (3.3)(ii), one may readily show that the functors $\eta_{k*} : Sh(Top(X_k); R) \to Sh(STop(X_\bullet); R)$, $k \geq 0$ are all *exact*. If

$F \epsilon Sh(STop(X_\bullet); R))$, the stalk of $\eta_k^*(F)$ at a point $\bar{x}_k$ of X_k may be computed to be isomorphic to the stalk of F at the corresponding simplicial point $(\bar{x}_k)_\bullet$. Therefore, it follows that the functors η_k^* are also exact. Since η_k^* is left adjoint to η_{k*} it follows that each of the functors η_{k*} sends injectives to injectives.

(3.7.1) We may now define a functor $\bar{\eta}_* : Sh(Top(X_\bullet); R) \rightarrow Sh(STop(X_\bullet); R)$ as follows. Let $F = \{F_k|k\} \quad \epsilon\ Sh(Top(X_\bullet); R)$. Then $\{\eta_{k*}(F_k)|k\}$ forms a cosimplicial object in $Sh(STop(X_\bullet); R)$. Therefore, its inverse limit $\lim\limits_{\Delta}\{\eta_{k*}(F_k)|k\}$ defines an object in $Sh(STop(X_\bullet); R)$. We let $\bar{\eta}_*(F) = \lim\limits_{\Delta}\{\eta_{k*}(F_k)|k\}$.

(3.7.2) Next observe that the derived functor of the inverse limit functor $\lim\limits_{\Delta}$ from the category of cosimplicial R-modules to the category of R-modules is given as follows. Let $\{K^k|k\}$ be a cosimplicial R-module. Then $R^s\lim\limits_{\Delta}(\{K^k|k\}) = H^s(N(\{K^k|k\}))$ = the s-th cohomology of the associated normalized chain complex $N(\{K^k|k\})$. (See for example [B-K] p. 310.) Since each η_{k*} was observed to be an exact functor, it follows that $R^s\bar{\eta}_*(F) = R^s\lim\limits_{\Delta}(\{\eta_{k*}F_k|k\}) = H^s(N(\{\eta_{k*}F_k|k\}))$, $F = \{F_k|k\} \epsilon Sh(Top(X_\bullet); R)$. (See (3.7.6) below.)

Next let $C_0(Sh(STop(X_\bullet)); R)$ denote the category of all co-chain complexes in $Sh(STop(X_\bullet); R)$ that are trivial in negative degrees. Then one may define a functor

(3.7.3) $\eta_* : Sh(Top(X_\bullet); R) \rightarrow C_0(Sh(STop(X_\bullet); R))$

as follows. Given a sheaf $F = \{F_k|k\} \epsilon Sh(Top(X_\bullet); R)$, we let $\eta_*(F)$ be the co-chain complex obtained by normalizing the cosimplicial object $\{\eta_{k*}(F_k)|k\}$. One may extend this functor to a functor

$\eta_*^0 : C_0(Sh(Rop(X_\bullet); R)) \rightarrow C_0(Sh(STop(X_\bullet); R))$

as follows. For this, first consider a co-chain complex K in $Sh(Top(X_\bullet); R)$ trivial in negative degrees. By denormalizing it, it defines a cosimplicial object $DN(K)$ in $Sh(Top(X_\bullet); R)$. Then $\{\eta_{k*}(DN(K_k)|k\}$ defines a double cosimplicial object in $Sh(STop(X_\bullet); R)$. One may now define $\eta_*^0(K) = N(\Delta\{\eta_{k*}(DN(K_k))|k\})$= the co-chain complex obtained by normalizing the cosimplicial object $\Delta\{\eta_{k*}(DN(K_k))|k\}$. One may further extend the above functor to a functor

(3.7.4) $\eta_*^+ : C_+(Sh(Top(X_\bullet); R)) \rightarrow C_+(Sh(STop(X_\bullet); R))$

as follows. Let $K \epsilon C_+(Sh(Top(X_\bullet); R))$ so that $K^i = 0$ for all $i < n$, for some $n < 0$. Then $K[n]$ is a complex that is trivial in negative degrees. One defines $\eta_*^+(K) = (\eta_*^0(K[n]))[-n]$. This functor is defined on $C_n(Sh(Top(X_\bullet); R))$ which is the full sub-category of $C_+(Sh(Top(X_\bullet); R))$ consisting of complexes K that are trivial in degrees below n. One may readily verify that this defines a functor η_* as in (3.7.4). Since each of the functors η_{k*} is exact, the functor η_*^+ preserves distinguished triangles and provides a spectral sequence

$$E_2^{s,t} = H^s(\eta_*(\mathcal{H}^t(K)))(= R^s\bar{\eta}_*(\mathcal{H}^t(K))) \Rightarrow H^{s+t}(\eta_*^+(K))$$

This spectral sequence converges strongly since K is a bounded below complex. It follows that the functor η_*^+ preserves quasi-isomorphisms and therefore induces a derived functor $R\bar{\eta}_* : D_+(Sh(Top(X_\bullet); R)) \rightarrow D_+(Sh(STop(X_\bullet); R))$.

(3.7.5) Observe that the above functor $\bar{\eta}_*$ preserves all small limits. Next observe that the category $Sh(Top(X_\bullet); R)$ is *well-powered* i.e. the sub-objects of each object can be indexed by a small set. (This follows readily since the sub-objects of any fixed stalk forms a small set and there is a small conservative family of points.) Moreover, the above category is closed under all small limits, has small hom-sets and a co-generator. The category $Sh(STop(X_\bullet); R)$ has small hom-sets. Therefore, one may apply the special adjoint functor theorem (see [Mac] Chapter 5, section 8) to conclude that $\bar{\eta}_*$ has a *left adjoint.* We denote this by $\bar{\eta}^*$. We proceed to show that this functor is *exact*: clearly it suffices to show its right adjoint $\bar{\eta}_*$ sends injectives to injectives.

First we begin by recalling the construction of *standard injectives* from [Ill-2] Chapter VI, section 6 and also [Fr] chapter 2. Let $\Delta^{op,dis}$ denote the discrete category associated to Δ^{op}, i.e. it has the same objects as Δ^{op}, but the morphisms are only the identity maps. Given a simplicial space $X_\bullet$, let $X_\bullet^{dis}$ denote the space $\sqcup_{n \epsilon \Delta} X_n$. (i.e. $X_\bullet$ is the diagram of spaces associated to the discrete category $\Delta^{op,dis}$.) A sheaf F on the space $X_\bullet^{dis}$ is simply a collection of sheaves $\{F_n|n\}$ with F_n a sheaf on $Top(X_n)$: this category of sheaves of R-modules on $X_\bullet^{dis}$ will be denoted $Sh(Top(X_\bullet^{dis}); R)$. It is shown in [Ill-2] p.57 that there is a direct image functor $e_* : Sh(Top(X_\bullet^{dis}); R) \rightarrow Sh(Top(X_\bullet); R)$ that sends injectives to injectives. Let $\bar{X}^{dis} = \underset{n}{\sqcup}\bar{X}_n$ be a set of points for $\underset{n}{\sqcup}X_n$ and let $\bar{p} : \bar{X}^{dis} \rightarrow X_\bullet^{dis}$ denote the obvious map. Any injective sheaf I on $Top(X_\bullet)$ of the form $e_*(\bar{p}_*(J))$ for some injective sheaf J on $Top(\bar{X}^{dis})$ will be called a *standard injective.* A more explicit description of such injectives is given in [Fr] chapter 2; these are the injectives denoted $\underset{n}{\Pi}R_n(\bar{p}_{n*}J_n)$, where $R_n : Sh(Top(X_n); R) \rightarrow Sh(Top(X_\bullet); R)$ is a right adjoint to the obvious restriction functor $(\quad)_n : Sh(Top(X_\bullet); R) \rightarrow Sh(Top(X_n); R)$, J_n is an injective in $Sh(Top(\bar{X}_n); R)$ and $\bar{p}_n : \bar{X}_n \rightarrow X_n$ is the obvious map. It is shown there that any sheaf F in $Sh(Top(X_\bullet); R)$ can be imbedded in a standard injective.

(3.7.6) **Proposition** Let $I = e_*(\bar{p}_*(J))$, for some injective sheaf $J = \{J_n|n\}$ in $Sh(Top(\bar{X}^{dis}); R)$. Then the following hold:

(i) $\bar{\eta}_*(I)$ is an injective in $Sh(STop(X_\bullet); R)$

(ii) $\eta_*(I)$ is an (injective) resolution of the sheaf $\bar{\eta}_*(I)$

(iii) If I' is any injective in $Sh(Top(X_\bullet); R)$, $\bar{\eta}_*(I')$ is an injective in $Sh(STop(X_\bullet); R)$

Proof. Recall $I = e_*(\bar{p}_*(J)) = \underset{n}{\Pi}R_n(\bar{p}_{n*}J_n)$, where $R_n(\bar{p}_{n*}(J_n))_m = \underset{\alpha:[m]\rightarrow[n] \quad in \quad \Delta}{\Pi} X_\bullet(\alpha)_*(\bar{p}_{n*}(J_n))$. Moreover, recall that $\bar{\eta}_*(I) = ker(\delta^0 - \delta^1 : \eta_{0*}(I_0) \rightarrow \eta_{1*}(I_1))$. In view of the description of the functor R_n as above, one may observe that $ker(\delta^0 - \delta^1 : \eta_{0*}(I_0) \rightarrow \eta_{1*}(I_1))$ is a product of sheaves of the form $ker(\delta^0 - \delta^1 : \eta_{0*}X(\beta \circ \delta^0)_*(\bar{p}_{n*}(J_n)) \times \eta_{0*}X_\bullet(\beta \circ \delta^1)_*(\bar{p}_{n*}(J_n)) \rightarrow \eta_{1*}X_\bullet(\beta)_*(\bar{p}_{n*}(J_n)))$ where $\beta : [1] \rightarrow [n]$ is a map in Δ. Therefore, an explicit computation shows that $\bar{\eta}_*(I)$ is a product of sheaves of the form $\eta_{n*}(\bar{u}_*(L))$, where $u \epsilon \bar{X}_n$ is a point of X_n, L is an injective sheaf on $*$ and $\bar{u} : Top(*) \rightarrow Top(X_n)$ is the associated map. Since these are injectives (see (3.6)), it follows that $\bar{\eta}_*(I)$ is an injective in $Sh(STop(X_\bullet); R)$ thereby proving (i). In order to prove (ii), it suffices to show that the stalk of $\eta_*(I)$ at a simplicial point $\bar{x}_\bullet$ of $X_\bullet$ is acyclic. Therefore, let $\bar{x}_\bullet : \Delta[n] \otimes * \rightarrow X_\bullet$ denote a fixed simplicial point of $X_\bullet$ and let $U_\bullet$ denote a neighborhood of $\bar{x}_\bullet$. Now one may use the isomorphism in (3.11)(i) (or (2.6)(i)) to show that $H^i(\Gamma(U_\bullet, (\eta_*(I))) \cong H^i_{STop(U_\bullet)}(U_\bullet, \eta_*(I)) \cong H^i_{Top(U_\bullet)}(U_\bullet, I) \cong 0$ for all $i > 0$ since I is an injective sheaf on $Top(X_\bullet)$. Therefore taking the direct limit over all neighborhoods of the simplicial point $\bar{x}_\bullet$, we see that $\bar{x}_\bullet^*(\eta_*(I))$ is acyclic. This proves (ii).

Now (i) shows that $Hom(\quad, \bar{\eta}_*(I))$ is an exact functor on $Sh(STop(X_\bullet); R)$. In particular $0 = Ext^1(M, \bar{\eta}_*(I)) \cong H^1(Hom(M, \eta_*(I)))$, for any $M \epsilon Sh(STop(X_\bullet); R)$: the last isomorphism makes use of (ii) which shows that $\eta_*(I)$ is an injective resolution of $\bar{\eta}_*(I)$. Finally, in order to prove (iii), observe that any injective I' imbeds into a standard injective I: since I' is also an injective, it in fact splits as a summand of I. Therefore $\eta_*(I')$ $(Hom(M, \eta_*(I')))$ is a split summand of $\eta_*(I)$ $(Hom(M, \eta_*(I))$, respectively). It follows that $H^1(Hom(M, \eta_*(I'))) = 0$ for any injective $I' \epsilon Sh(Top(X_\bullet); R)$ and any sheaf $M \epsilon Sh(STop(X_\bullet); R)$. Recall from (3.6)) that $\eta_*(I')$ is a complex of injectives in $Sh(STop(X_\bullet); R)$; therefore, $Hom(\quad, \eta_*(I'))$ sends a short-exact-sequence of sheaves in $Sh(STop(X_\bullet); R)$ to a short exact sequence of complexes. Since $Hom(M, \bar{\eta}_*(I')) = H^0(Hom(M, \eta_*(I'))$ and $H^1(Hom(M, \eta_*(I')) = 0$ for any sheaf $M \epsilon Sh(STop(X_\bullet); R)$, it follows that $Hom(\quad, \bar{\eta}_*(I'))$ is an exact functor in the first argument. This proves $\bar{\eta}_*(I')$ is an injective in $Sh(STop(X_\bullet); R)$ for any injective $I' \epsilon Sh(Top(X_\bullet); R)$. $\square$

It follows that $\bar{\eta}^*$ is an exact functor and hence preserves quasi-isomorphisms between objects in $C_+(Absh(STop(X_\bullet)))$. Since it has a right adjoint it also commutes with colimits, hence with sums and therefore with mapping cones. Therefore, one may show readily that $\bar{\eta}^*$ preserves distinguished triangles as well as quasi-isomorphisms and induces a derived functor $\bar{\eta}^* : D_+(Sh(STop(X_\bullet); R)) \rightarrow D_+(Sh(Top(X_\bullet); R))$.

Remark. It may be worth pointing out the need to define the functor $\bar{\eta}^*$ as we have done above. The main difficulty is that the collection of functors $\{\eta_k^*|k\}$ do not, in general, send a sheaf on $STop(X_\bullet)$ to a sheaf on $Top(X_\bullet)$: this difficulty may be seen from (3.9)(i) and (iii) below. (The maps in (3.9)(i) go in the wrong direction for $\{\eta_m^*(F'_m)|m\}$ to define a sheaf on $Top(X_\bullet)$.) Therefore, we need to invoke the special adjoint functor theorem to be able to define the functor $\bar{\eta}^*$ as a left adjoint to $\bar{\eta}_*$ in general. However, as shown in (3.9)(iii), if one restricts to sheaves F' on $STop(X_\bullet)$ with descent, $\bar{\eta}^*(F') \cong \{\eta_k^*(F')|k\}$.

We proceed to show that $R^s\bar{\eta}_*(F) = 0$ if $s > 0$ for any sheaf $F \epsilon Sh(Top(X_\bullet); R)$ with *descent.*

(3.7.7) Let $F = \{F_k|k\} \epsilon\ Sh(Top(X_\bullet); R)$ have *descent*. Since F has descent, we will show that $\mathcal{H}^t(\eta_*(F)) = 0$ if $t \neq 0$. To see this we may argue as follows. Let $\bar{x}_\bullet : (\Delta[n] \otimes *) \rightarrow X_\bullet$ be a point of $X_\bullet$. It follows from (3.3) and (3.5) that $\mathcal{H}^t(\eta_*(F)_{\bar{x}_\bullet})$ is the $t-th$ cohomology of the cosimplicial abelian group

$$\Gamma((\Delta[n]_0 \otimes *); (\bar{x}_\bullet)_0^* F_0) \rightarrow \Gamma((\Delta[n]_1 \otimes *); (\bar{x}_\bullet)_1^* F_1) \rightarrow ...$$

As F has descent, we may identify this cosimplicial abelian group with the cosimplicial abelian group

$$\Delta[n]_0 \otimes F_{\bar{x}_0} \overset{\rightarrow}{\longrightarrow} \Delta[n]_1 \otimes F_{\bar{x}_0} .....$$

where $F_{\bar{x}_0}$ is the stalk of F_0 at the point $\bar{x}_0 = d_\alpha(\bar{x}_n)$ of X_0. (Here $d_\alpha : X_n \rightarrow X_0$ is any structure map of the simplicial scheme $X_\bullet$ and $\bar{x}_n = i_n \otimes * \rightarrow X_n$ is the point of X_n. i_n is the generator of $\Delta[n]$.) Clearly $\mathcal{H}^t(\eta_*(F))_{x_\bullet} \cong 0$ for $t \neq 0$ and $\mathcal{H}^0(\eta_*(F))_{\bar{x}_\bullet} \cong F_{\bar{x}_0}$. It follows that we obtain the identification:

$$R\bar{\eta}_*(F) = H^0(\eta_*(F)) = \bar{\eta}_*(F),\ F \ \epsilon\ Sh^{des}(Top(X_\bullet); R)$$

(3.8.1) Let $\bar{x}_m$ be a given point of X_m, let $\alpha : X_m \rightarrow X_0$ be a structure map of the simplicial scheme $X_\bullet$ and let $\bar{x}_0 = X(\alpha)(\bar{x}_m)$ denote the corresponding point of X_0. Let $(\bar{x}_m)_\bullet$ and $(\bar{x}_0)_\bullet$ denote the associated simplicial points. (Observe that there exists a natural map $(\bar{x}_0)_\bullet \rightarrow (\bar{x}_m)_\bullet$ and therefore every neighborhood of $(\bar{x}_m)_\bullet$ is also a neighborhood of $(\bar{x}_0)_\bullet$.)

(3.8.2) **Definition**. Let $F' \ \epsilon\ Sh(STop(X_\bullet), R)$. We say that F' has *descent* if the induced map $F'_{(\bar{x}_0)_\bullet} \leftarrow F'_{(\bar{x}_m)_\bullet}$ is an isomorphism for all points $\bar{x}_m$ as in (3.8.1), all $\alpha : X_m \rightarrow X_0$ and all $m \geq 0$. If $K' \ \epsilon\ D_+(Sh(STop(X_\bullet), R)$, we say that K' has *descent* if the corresponding maps are quasi-isomorphisms. The full subcategory of complexes K' having descent will be denoted $D_+^{des}(Sh(STop(X_\bullet), R))$.

(3.9) **Proposition**. Let $F' \ \epsilon\ Sh(STop(X_\bullet); R)$.

(i) For each map $\alpha : [n] \rightarrow [m]$ in Δ, there exists a map $\eta_m^*(F') \rightarrow X_\bullet(\alpha)^*(\eta_n^*(F'))$, natural in F', satisfying certain obvious compatibility conditions.

(ii) If $F = \{F_k|k\} \ \epsilon\ Sh(Top(X_\bullet); R)$ is a sheaf with descent and $F' = \bar{\eta}_*(F)$, the above maps are all isomorphisms so that $\{\eta_m^*(\bar{\eta}_*(F)|m\}$ defines a sheaf with descent on $Top(X_\bullet)$.

(iii) If $F' \ \epsilon\ Sh(STop(X_\bullet), R)$ has *descent*, the collection $\{\eta_m^*(F')|m\}$ defines a sheaf with descent in $Sh(Top(X_\bullet); R)$.

(iv) For each $F \ \epsilon\ Sh^{des}(Top(X_\bullet); R)$, there exists an isomorphism $\bar{\eta}^*(\bar{\eta}_*(F)) \cong \{\eta_m^*(\bar{\eta}_*(F))|m\}$. Similarly if $F' \ \epsilon\ Sh(STop(X_\bullet); R)$ has descent, $\bar{\eta}^*(F') \cong \{\eta_m^*(F')|m\}$.

(v) For each $F \ \epsilon\ Sh^{des}(Top(X_\bullet); R)$, the natural map $\bar{\eta}^*(\bar{\eta}_*(F)) \rightarrow F$ is an isomorphism.

(vi) The functor $\bar{\eta}_*$ sends a sheaf with descent on $Top(X_\bullet)$ to sheaf with descent on $STop(X_\bullet)$.

Proof. (i) is equivalent to showing that there exist natural transformations $\eta_{n*} \circ X(\alpha)_* \rightarrow \eta_{m*}$. This follows readily from the definition of η_{n*} and η_{m*}.

(ii) Clearly it suffices to consider the case $n = 0$. Let $\bar{x}_m$ be a fixed point of X_m and let $\bar{x}_0 = X_\bullet(\alpha)(\bar{x}_m)$. One may observe that

$$X_\bullet(\alpha)^*(\eta_0^*(\bar{\eta}_*(F)))_{\bar{x}_m} \cong \eta_0^*(\bar{\eta}_*(F))_{\bar{x}_0} \cong (\bar{\eta}_*(F))_{(\bar{x}_0)_\bullet}$$

By (3.7.7) the latter is isomorphic to $F_{0,\bar{x}_0}$. On the other hand $\eta_m^*(\bar{\eta}_*(F))_{\bar{x}_m} \cong \bar{\eta}_*(F)_{(\bar{x}_m)_\bullet} \cong F_{0,\bar{x}_0}$ as well by (3.7.7).

Moreover, it again follows from (3.7.7) (see also (3.3)) that the map of the right-hand-sides above is the identity. Since the map $\eta_m^*(\bar{\eta}_*(F)) \xrightarrow{\Psi_\alpha} X_\bullet(\alpha)^*(\eta_n^*(\bar{\eta}_*(F)))$ is an isomorphism, so is its inverse $\Phi_\alpha = (\Psi_\alpha)^{-1}$. The sheaves $\{\eta_n^*(\bar{\eta}_*(F))|n\}$ along with the maps $\{\Phi_\alpha|\alpha\}$ now define a sheaf with descent on the simplicial space $X_\bullet$. This proves (ii).

(iii) Let $\bar{x}_m$ and $\bar{x}_0$ be as in (ii). Then $(\eta_m^* F')_{\bar{x}_m} \cong F'_{(\bar{x}_m)_\bullet}$ while $(\eta_0^* F')_{\bar{x}_0} \cong F_{(\bar{x}_0)_\bullet}$. The hypothesis on F' implies that one obtains an isomorphism $X_\bullet(\alpha)^*(\eta_0^*(F'))_{\bar{x}_m} \cong \eta_m^*(F')_{\bar{x}_m}$. It follows that $\{\eta_m^*(F')|m\}$ defines a sheaf with descent on $Top(X_\bullet)$.

(iv) The first assertion follows readily in view of the natural isomorphism:

$$Hom_{Sh(Top(X_\bullet);R)}(\{\eta_k^*(\bar\eta_*(F))|k\}, L) \cong Hom_{Sh(STop(X_\bullet);R)}(\bar\eta_*(F), \bar\eta_*(L))$$

$$\cong Hom_{Sh(Top(X_\bullet);R)}(\bar\eta^*(\bar\eta_*(F)), L)$$

for all $L \in Sh(Top(X_\bullet);R)$. The first isomorphism follows from the definition of $\bar\eta_*$ as in (3.7.1) and the observation that each η_{k*} is right adjoint to η_k^*. Observe that giving an element in the left-hand-side corresponds to giving a map from the constant cosimplicial object $\bar\eta_*(F)$ (in $Sh(STop(X_\bullet);R)$) to the cosimplicial object $\{\eta_{k*}(L_k)|k\}$; this in turn corresponds to giving a map from the object $\bar\eta^*(F)$ to the inverse limit $\underset{\Delta}{\lim}\{\eta_{k*}(L_k)|k\}$. The last isomorphism follows from the fact $\bar\eta^*$ is left adjoint to $\bar\eta_*$ by definition.) The last assertion in (iv) also follows by a similar argument in view of (iii).

(v) Since both sheaves have descent, it suffices to show that there exists an isomorphism on restriction to $Top(X_0)$. This is clear in view of the above discussion.

(vi) follows readily in view of the computation in (3.7.7). □

(3.10)**Corollary**. The natural transformation of functors

$\bar\eta^* \circ R\bar\eta_* \to id : D_+^{des}(Sh(Top(X_\bullet);R)) \to D_+^{des}(Sh(Top(X_\bullet);R))$ is an isomorphism. The functor $R\bar\eta_* : D_+^{des}(Sh(Top(X_\bullet);R)) \to D_+^{des}(Sh(STop(X_\bullet);R))$ is *fully-faithful.*

Proof. Since both $\bar\eta^*$ and $R\bar\eta_*$ preserve distinguished triangles one obtains a spectral sequence:

$$E_2^{s,t} = \mathcal{H}^s(\bar\eta^*(R\bar\eta_*\mathcal{H}^t(K))) \Rightarrow \mathcal{H}^{s+t}(\bar\eta^*(R\bar\eta_*(K)))$$

In view of the hypothesis that the cohomology sheaves of K have descent, this spectral sequence degenerates and $E_2^{s,t} = 0$ if $s > 0$ and $\cong \bar\eta^*\bar\eta_*(\mathcal{H}^t(K))$ if $s = 0$ by the arguments earlier and by (3.9)(iv). By (3.9)(v) this is isomorphic to $\mathcal{H}^t(K)$. This proves the first assertion. The second is now clear since $\bar\eta^*$ is left-adjoint to $R\bar\eta_*$. □

(3.11) **Proposition**.

(i) If $K \in D_+(Sh(Top(X_\bullet);R))$, $H^*_{Top(X_\bullet)}(X_\bullet; K) \cong H^*_{STop(X_\bullet)}(X_\bullet; R\bar\eta_*(K))$. If, in addition, $K \in Sh^{des}(Top(X_\bullet);R)$, the last term is isomorphic to $H^*_{STop(X_\bullet)}(X_\bullet; \bar\eta_*(K))$.

(ii). Let $f : X_\bullet \to Y_\bullet$ be a map of simplicial algebraic spaces and let $K \in D_+(Sh(Top(X_\bullet);R))$. If ${}_sf_{\bullet *}$ denotes the induced functor (as in (2.2)),

$$R\bar\eta_*(Rf_{\bullet *}K) = R_sf_{\bullet *}R\bar\eta_*(K).$$

Proof. The first isomorphism in (i) follows from the definition of the functors $R\bar\eta_*$ and η_*^+ along with the observations that $R\bar\eta_*(K) = \eta_*^+(K)$ and that each η_{k*} sends injectives to injectives and is also exact. The second isomorphism in (i) follows from the computation in (3.7.7). Now we consider (ii). Observe that for each $k \geq 0$, there is a natural isomorphism: $\eta_{k*} \circ f_{k*} \simeq {}_sf_* \circ \eta_{k*}$. Next observe that both η_{k*} and f_{k*} send injectives to injectives while the functor ${}_sf_*$ commutes with inverse limits. Therefore, the identification in (ii) is clear from the definition of $R\bar\eta_*$. □

4. Hyper-cohomology of fibers for a cohomologically proper map

(4.0) **Definition**. Let $f : X \to Y$ be a map of algebraic spaces and $K^\bullet$ denote a bounded below complex in $Absh(Top(X))$. Then $(K^\bullet, f)$ is *cohomologically proper* if for every map $g : Y' \to Y$, the canonical base-change $g^*Rf_*K^\bullet \to Rf'_*g'^*K^\bullet$ is a quasi-isomorphism, where f', g' are defined by the cartesian square

$$\begin{tikzcd}
X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\
Y' \arrow[r, "g"] & Y
\end{tikzcd}$$

(4.1)*Examples.*(i). Let the topology be the étale topology, f be proper and F a torsion sheaf; then the proper-base-change theorem (see [Mi] chapter 4 or [SGA]4 Expose XVI) shows (F, f) is cohomologically proper.

(ii). Let $p_2 : X \times Z \to Z$ denote the projection to the second factor between schemes of finite type over an algebraically closed field k. If F' is an l-adic sheaf on $Et(Z)$, $(F = p_2^*(F'), p_2)$ is cohomologically proper. To see this, one may identify Z with $Spec \quad k \times Z$ so that the projection p_2 identifies with the map $p \times id$. Here $p : X \to Spec \quad k$ is the structure map. Then $F \cong \underline{\mathbb{Q}}_l \boxtimes F'$ so that $Rp_{2*}(F) \simeq Rp_*(\underline{\mathbb{Q}}_l)\overset{L}{\boxtimes}F'$. Next consider a point $\bar{z} : Spec \quad k \to Z$; let $i_{\bar{z}} : Spec \quad k \xrightarrow{\Delta} Spec \quad k \times Spec \quad k \xrightarrow{id \times \bar{z}} Spec \quad k \times Z$ and let $\tilde{i}_{\bar{z}} : X \times Spec \quad k \xrightarrow{id \times i_{\bar{z}}} X \times Z$. Let $p_{\bar{z}} : X \times Spec \quad k \to Spec \quad k \times Spec \quad k \cong Spec \quad k$ be the obvious projection. Clearly $i_{\bar{z}}^*(Rp_{2*}(F)) = Rp_*(\underline{\mathbb{Q}}_l) \boxtimes F'_{\bar{z}}$ while $Rp_{\bar{z},*}(\tilde{i}_{\bar{z}}^*(F)) = Rp_{\bar{z},*}(\underline{\mathbb{Q}}_l \boxtimes F'_{\bar{z}}) \simeq Rp_*(\mathbb{Q}_l) \boxtimes F'_{\bar{z}}$.

(iii). Let k denote an algebraically closed field, X a scheme of finite type over k and G an algebraic group acting on X. Let H denote a closed subgroup of G and let $G\underset{H}{\times}X$ denote the object defined in (4.3.1) below. If F is a G-equivariant l-adic sheaf on $Et(G \times X)$ and $p : G \times X \to G\underset{H}{\times}X$ is the projection, (F, p) is cohomologically proper.

To see this we may proceed as follows. First one defines maps $pr_1 : G\underset{H}{\times}X \to G/H$ and $m : G\underset{H}{\times}X \to X$ as follows: pr_1 is induced by the projection sending all of X to $Spec \quad k$, while the map m sends (g, x) to $g.x$. (One may observe that $(gh^{-1}, h.x) \mapsto gh^{-1}.h.x = g.x$, for any $h \,\epsilon\, H$, so that the map m is well defined as stated.) Then pr_1 and m define a map $pm : G\underset{H}{\times}X \to G/H \times X$ which one may verify readily to be *bijective* (on points) and hence purely inseparable. Therefore, it suffices to show that $(F, \pi = pm \circ p)$ is cohomologically proper. Next consider the automorphism $\sigma : G \times X \to G \times X$ defined by $\sigma(g, x) = (g, gx)$, $g \,\epsilon\, G$, $x \,\epsilon\, X$. One may now observe that $\pi = (\bar{p} \times id) \circ \sigma$ where $\bar{p} : G \to G/H$ is the obvious projection. Therefore, it suffices to show that $(F, \bar{p} \times id)$ is cohomologically proper.

Next the G-equivariance of the sheaf F shows, as in (4.4.5) below, that $F \cong \underline{\mathbb{Q}}_l \boxtimes F'$, for some l-adic sheaf F' on $Et(X)$. Therefore, $R(\bar{p} \times id)_*(F) \simeq R\bar{p}_*(\mathbb{Q}_l) \boxtimes F'$.

Let $i_{\bar{z}} : \bar{z} \to G/H$, $i_{\bar{x}} : \bar{x} \to X$ denote two points and let $\tilde{i}_{\bar{z}} : H \to G$ be the closed immersion of the geometric fiber over $\bar{z}$ into G. Then $(i_{\bar{z}} \times i_{\bar{x}})^*(R(\bar{p} \times id)_*(F)) \simeq Ri_{\bar{z}}^* R\bar{p}_*(\underline{\mathbb{Q}}_l) \boxtimes F'_{\bar{x}}$. Using the observation that the cohomology sheaves of $R\bar{p}_*(\underline{\mathbb{Q}}_l)$ are G-equivariant and hence *lisse* (i.e. each term in the inverse system defining the corresponding l-adic sheaf is locally constant) on $Et(G/H)$, one may identify the last term with $R\bar{p}_{\bar{z},*}(\tilde{i}_{\bar{z}}^*(\underline{\mathbb{Q}}_l)) \boxtimes F'_{\bar{x}}$ where $\bar{p}_{\bar{z}}$ is the projection from the geometric fiber above $\bar{z}$ ($\cong H$) to $\bar{z}$. Then the last term identifies with $R\pi_{\bar{z},\bar{x},*}(\tilde{i}_{\bar{z}} \times i_{\bar{x}})^*(F)$ where $\pi_{\bar{z},\bar{x}} : H \times \bar{x} \to \bar{z} \times \bar{x}$ is the obvious projection.

(4.2) **Theorem** (Joshua: see [J-T] appendix C.) Let $f_\bullet : X_\bullet \to Y_\bullet$ be a map of simplicial algebraic spaces. If $\bar{K} \epsilon\ D_+(Absh(STop(X_\bullet)))$, we obtain a Leray spectral sequence:

$$E_2^{p,q} = H^p(Y_\bullet;\ R^q{}_s f_* \bar{K}) \Rightarrow \mathbb{H}^{p+q}(X_\bullet;\ \bar{K}).$$

If, in addition, $\bar{K} = R\bar{\eta}_*(K)$, $K \epsilon\ D_+(Absh(Top(X_\bullet)))$ and each $(K_n,\ f_n)$ is *cohomologically proper*, then we obtain the identification of the stalks:

$$(R^q{}_s f_{\bullet *}\bar{K})_{\bar{y}_\bullet} \simeq \mathbb{H}^q(\bar{y}_\bullet \underset{Y_\bullet}{\times} X_\bullet;\quad K|\bar{y}_\bullet \underset{Y_\bullet}{\times} X_\bullet).$$

for any point $\bar{y}_\bullet$ of $Y_\bullet$

Proof. We begin with the hypercohomology spectral sequence

$$E_2^{p,q} = H^p(Y_\bullet;\ \mathcal{H}^q(R_s f_* \bar{K})) \Rightarrow \mathbb{H}^{p+q}(Y_\bullet;\ R_s f_* \bar{K}) \simeq \mathbb{H}^{p+q}(X_\bullet;\ \bar{K}).$$

This clearly provides the required spectral sequence. We proceed to identify the stalks of $R^q{}_s f_* \bar{K}$. Let $\bar{y}_\bullet$ denote a simplicial point of $Y_\bullet$. Observe that

$$(R^q{}_s f_* \bar{K})_{\bar{y}_\bullet} = \lim_{\overrightarrow{U_\bullet}} \mathbb{H}^q(U_\bullet \underset{Y_\bullet}{\times} X_\bullet;\ \bar{K})$$

where the colimit is over all (simplicial) neighborhoods of $\bar{y}_\bullet$ as in (3.5) and the cohomology is computed on the site $STop(U_\bullet \underset{Y_\bullet}{\times} X_\bullet)$. Next recall the isomorphism

$$\mathbb{H}^q_{STop(X_\bullet)}(U_\bullet \underset{Y_\bullet}{\times} X_\bullet;\, \bar{K}) \simeq \mathbb{H}^q_{Top(X_\bullet)}(U_\bullet \underset{Y_\bullet}{\times} X_\bullet;\, K)$$

since $\bar{K} = R\bar{\eta}_*(K)$. Therefore, we are able to make use of the spectral sequence

$$E_1^{r,s} = \mathbb{H}^s(U_r \underset{Y_r}{\times} X_r;\, K_r) \Rightarrow \mathbb{H}^{r+s}(U_\bullet \underset{Y_\bullet}{\times} X_\bullet;\, K)$$

as in (2.6)(ii). Taking the direct limit over all simplicial neighborhoods $U_\bullet$ of $\bar{y}_\bullet$, we obtain the spectral sequence

$$(4.2.1)\ \underset{\overrightarrow{U_\bullet}}{\lim} E_1^{r,s} = \underset{\overrightarrow{U_\bullet}}{\lim} \mathbb{H}^s(U_r \underset{Y_r}{\times} X_r;\, K_r) \Rightarrow \underset{\overrightarrow{U_\bullet}}{\lim} \mathbb{H}^{r+s}(U_\bullet \underset{Y_\bullet}{\times} X_\bullet;\, K)$$

Now we use the assumption that each $(K_n,\, f_n)$ is cohomologically proper along with (3.3) to identify the left-side as $\mathbb{H}^s(\bar{y}_r \underset{Y_r}{\times} X_r;\, K_r|\bar{y}_r \underset{Y_r}{\times} X_r)$. Comparison with the spectral sequence in (2.6)(ii) for the simplicial scheme $\bar{y}_\bullet \underset{Y_\bullet}{\times} X_\bullet$ shows we may identify the right side of (4.2.1) as $\mathbb{H}^{r+s}(\bar{y}_\bullet \underset{Y_\bullet}{\times} X_\bullet; K|\bar{y}_\bullet \underset{Y_\bullet}{\times} X_\bullet)$. $\square$

We will conclude this section by recalling an *important* application of the above theorem to equivariant derived categories in positive characteristics. (Another application is the main theorem of the next section.)

(4.3.0) We assume throughout the rest of the paper that all schemes and algebraic spaces are defined over an algebraically closed field k of characteristic $p \geq 0$ and that these are provided with the étale topology. Now sheaves are l-adic sheaves. (In case $k = \mathbb{C}$, one may consider complex analytic spaces and varieties using the transcendental topology; now sheaves will be sheaves of $\mathbb{Q}$ or $\mathbb{C}$-vector spaces.) Let X denote an algebraic space provided with the action of an algebraic group G, which we will assume is *not* in general connected. The action $G \times X \to X$ will be denoted σ. In this situation, one may define the simplicial algebraic spaces $EG \underset{G}{\times} X$ and BG in the usual manner. One defines the equivariant derived category $D^G_+(X; \mathbb{Q}_l)$ as the full sub-category of $D_+(Et(EG \underset{G}{\times} X); \mathbb{Q}_l)$ consisting of complexes that have G-equivariant and constructible cohomology sheaves or equivalently have cohomology sheaves which are constructible and have descent. (See [B-J] (1.3.1) for more details.) We let $D^G_+(BG; \mathbb{Q}_l) = D^G_+(EG \underset{G}{\times}(Spec\quad k); \mathbb{Q}_l)$. We let $D^G(X) = D^G_b(X; \mathbb{Q}_l)$ which is the full sub-category of $D^G_+(X; \mathbb{Q}_l)$ consisting of bounded complexes.

(4.3.1) Let $i : H \to G$ denote the closed immersion of a *closed algebraic subgroup* of G, let $\sigma_H : H \times X \to X$ denote the induced action and let $\bar{i} : EH \underset{H}{\times} X \to EG \underset{G}{\times} X$ denote the induced map where X is a scheme with G-action.

Let H act on $G \times X$ by $h.(g,\, x) = (g.h^{-1},\quad hx)$, $h \in H$, $g \in G$ and $x \in\ X$. Then a geometric quotient $G \underset{H}{\times} X$ exists for this action and the obvious map $s : G \times X \to G \underset{H}{\times} X$ is smooth with fibers isomorphic to H.

(4.3.2) Then G has an action on $G \times X$ induced from its action by translation on the first factor of $G \times X$; this induces a G-action on $G \underset{H}{\times} X$ which will be denoted σ_1. One verifies that the map s is equivariant for these actions of G.

(4.3.3). Let $m : G \underset{H}{\times} X \to X$ denote the map induced by the action $\sigma : G \times X \to X$. One verifies that m is G-equivariant for the G-action on $G \underset{H}{\times} X$ as in (4.3.2) and the G-action on X. It follows that m defines a map $\bar{m} : EG \underset{G}{\times}(G \underset{H}{\times} X) \to EG \underset{G}{\times} X$.

(4.3.4) Let $r : G \times X \to X$ denote the projection to the second factor.

(4.3.5) Next let $G \times H$ act on $G \times X$ by $(g_1,\, h_1).(g,\, x) = (g_1 g h_1^{-1},\, h_1 x)$, $g_1,\, g \in G$, $h_1 \in H$ and $x \in X$. This action will be denoted σ_2. We observe that the maps r and s are such that we obtain the commutative squares:

$$\begin{tikzcd}
(G \times H) \times (G \times X) \arrow[r, "\sigma_2"] \arrow[d, "pr_1 \times s"'] & G \times X \arrow[d, "s"] \\
G \times (G \underset{H}{\times} X) \arrow[r, "\sigma_1"] & G \underset{H}{\times} X
\end{tikzcd}$$

and

$$\begin{tikzcd}
(G \times H) \times (G \times X) \arrow[r, "\sigma_2"] \arrow[d, "pr_2 \times r"'] & G \times X \arrow[d, "r"'] \\
H \times X \arrow[r, "\sigma_H"] & X
\end{tikzcd}$$

It follows that r and s induce maps $\bar{r} : E(G \times H) \underset{G\times H}{\times} (G \times X) \to EH \underset{H}{\times} X$ and $\bar{s} : E(G \times H) \underset{G\times H}{\times} (G \times X) \to EG \underset{G}{\times} (G \underset{H}{\times} X)$.

(4.4) **Theorem**. (See [J-1] Theorem (6.4).) Assume the above hypotheses. Then we obtain the equivalences of categories:

$$D^H(X) \xrightarrow{\bar{r}^*} D^{G\times H}(G \times X) \text{ and } D^G(G \underset{H}{\times} X) \xrightarrow{\bar{s}^*} D^{G\times H}(G \times X)$$

Outline of Proof. Observe first that each r_n has as geometric fibers G^{n+1} while each s_n has geometric fibers $\cong H^{n+1}$. Therefore, observe that the geometric fiber of $\bar{r}$ over a point $\bar{x}_\bullet : \Delta[n] \otimes Spec \quad k \to EH \underset{H}{\times} X$ (of $\bar{s}$ over a point $\bar{x}_\bullet : \Delta[n] \otimes Spec \quad k \to EG \underset{G}{\times} (G \underset{H}{\times} X)$) is isomorphic to the simplicial space $\Delta[n] \otimes EG$ ($\Delta[n] \otimes EH$, respectively); hence they have trivial cohomology with respect to any locally constant abelian sheaf. Since $(EH)_0 = H$ ($(EG)_0 = G$) any constructible H-equivariant (G-equivariant) abelian sheaf on $Et(EH)$ ($Et(BG)$, respectively) is locally constant; if F (K) is a constructible H-equivariant (G-equivariant, respectively) abelian sheaf on $Et(EH \underset{H}{\times} X)(Et(EG \underset{G}{\times} (G \underset{H}{\times} X)))$, $\bar{r}^*(F)(\bar{s}^*(K)$, respectively) is a $G \times H$-equivariant constructible sheaf on $Et(E(G \times H) \underset{G\times H}{\times} (G \times X))$. Recall the geometric fibers of $\bar{r}$ ($\bar{s}$) were observed to be $\Delta[n] \otimes EG$ ($\Delta[n] \otimes EH$, respectively). Therefore, one may make the following observations :

(i) the cohomology sheaves of $\bar{r}^*(F)$ ($\bar{s}^*(K)$) are lisse (and therefore, in fact, constant) on the geometric fibers of $\bar{r}(\bar{s}$, respectively) and hence

(ii) the geometric fibers of $\bar{r}$ ($\bar{s}$) are acyclic with respect to $\bar{r}^*(F)(\bar{s}^*(K)$, respectively). Moreover $H^*(\Delta[n] \otimes EG; \bar{r}^*(F)) \cong H^*(\Delta[n] \otimes Spec \quad k; \bar{x}_\bullet^*(F))$ ($H^*(\Delta[n] \otimes EH; \bar{s}^*(K)) \cong H^*(\Delta[n] \otimes Spec \quad k; \bar{x}_\bullet^*(K))$, respectively) if $\Delta[n] \otimes EG$ ($\Delta[n] \otimes EH$) is the geometric fiber over the point $\bar{x}_\bullet$.

Next (4.1)(ii) ((4.1)(ii) and (4.1)(iii)) applies to show that $(\bar{r}_n^*(F_n), \bar{r}_n)$ $((\bar{s}_n^*(K_n), \bar{s}_n)$, respectively) is cohomologically proper for each $n \geq 0$. In view of the above observations, Theorem (4.2) shows that the natural map

$F = \{F_n|n\} \to \{Rr_{n*}r_n^*F_n|n\}, \ \ (K = \{K_n|n\} \to \{Rs_{n*}s_n^*K_n|n\})$
induces an isomorphism

(4.4.1) $H^t(EH \underset{H}{\times} X;\, F) \simeq H^t(E(G \times H) \underset{G\times H}{\times} (G \times X);\, \bar{r}^*F)$

((4.4.2) $(H^t(EG \underset{G}{\times} (G \underset{H}{\times} X);\, K) \simeq H^t(E(G \times H) \underset{G\times H}{\times} (G \times X);\, \bar{s}^*K))$, respectively.)

As these isomorphisms are natural in K and F they induce a map of the hypercohomology spectral sequences proving thereby that such an isomorphism holds for any $F \epsilon\, D^H(X)$ ($K \epsilon\, D^G(G \underset{H}{\times} X)$, respectively). (Recall that the above derived categories consist of bounded complexes.)

Now we show that $\bar{r}^*$ and $\bar{s}^*$ are fully faithful. For this it is necessary to define an internal hom functor, $\mathcal{H}om$, for the category $D_+^{des}(Sh(Et(Z_\bullet)); \mathbb{Q}_l)$ on any simplicial space $Z_\bullet$. The main defining property of this functor is that we obtain the isomorphism

(4.4.3) $Hom(M \overset{L}{\otimes} N, P) \cong Hom(M, \mathcal{H}om(N, P))$

where M, N and P belong to $D_+(Sh(Et(Z_\bullet)); \mathbb{Q}_l)$ and the two Homs denote the external Homs in the derived category $D_+^{des}(Sh(Et(Z_\bullet)); \mathbb{Q}_l)$. Taking $M =$ the constant sheaf $\mathbb{Z}_{Z_\bullet}$, we obtain:

$Hom(N, P) \cong H^0(Z_\bullet, \mathcal{H}om(N, P))$

where the right-hand-side is cohomology computed on $Et(Z_\bullet)$. Next assume that all the face maps $d_i : Z_n \rightarrow Z_{n-1}$, for all i and all $n \geq 1$ are *smooth*. One may then show that if N and P belong to $D^{des}_+(Sh(Et(Z_\bullet)); \mathbb{Q}_l)$, one obtains the quasi-isomorphism

(4.4.4) $\mathcal{H}om(N, P)_n = \mathcal{H}om(N_n, P_n)$ for each $n \geq 0$

Using this and the observation that each of the maps r_n is smooth, one may also show that if N, $P \,\epsilon\, D^H(X)$, the natural map $r_n^*(\mathcal{H}om(N_n, P_n)) \rightarrow \mathcal{H}om(r_n^*N_n, r_n^*P_n)$ is a quasi-isomorphism. It follows $\bar{r}^*\mathcal{H}om(N, P) \simeq \mathcal{H}om(\bar{r}^*N, \bar{r}^*P)$.

Now we may show $\bar{r}^*$ is fully-faithful as follows. Let M, $N \epsilon\, D^H(X)$ and let $F = \mathcal{H}om(M,\ N)$; observe that $F \epsilon\, D^H(X)$. The left-side of (4.4.1) is $Ext^t(M,\ N)$ while the right-side is $Ext^t(\bar{r}^*M,\ \bar{r}^*N)$. This proves that the functor $\bar{r}^*$ is fully faithful; the proof for $\bar{s}^*$ is similar.

Once $\bar{r}^*$ and $\bar{s}^*$ are known to be fully-faithful, in order to show they are equivalences of categories, it suffices to show they are equivalences on the hearts of the appropriate derived categories. (See for example : [Beil] Lemma (1.4) .) i.e. It suffices to establish that $\bar{r}^*$ $(\bar{s}^*)$ provides the equivalence:

(4.4.5) (H-equivariant sheaves of $\mathbb{Q}_l$-modules on $Et(EH \underset{H}{\times} X)$
$\simeq$ ($G \times H$-equivariant sheaves of $\mathbb{Q}_l$-modules on $Et(E(G \times H) \underset{(G\times H)}{\times} (G \times X))$

((4.4.6) (G-equivariant sheaves of $\mathbb{Q}_l$-modules on $Et(EG \underset{G}{\times} (G \underset{H}{\times} X))) \simeq$
(($G \times H$)-equivariant sheaves of $\mathbb{Q}_l$-modules on $Et(E(G \times H) \underset{(G\times H)}{\times} (G \times X)))$, respectively.)

The key observation here is that $G \times H$-equivariance corresponds to suitable descent conditions. □

5. An application to representations of finite groups and to the equivariant derived categories of non-connected groups in positive characteristics

(5.0) Throughout this section we will assume the hypotheses of (4.3.0). In addition, we will assume that the group G is not necessarily connected and that H is a *connected closed normal subgroup* so that the quotient $\bar{G} = G/H$ is finite. If $K \,\epsilon\, D^G_+(X; \mathbb{Q}_l)$, we define $\mathbb{H}^*_H(X; K) = \mathbb{H}^*(EG \underset{G}{\times} (G \underset{H}{\times} X); \phi^*(K))$ where $\phi : EG \underset{G}{\times} (G \underset{H}{\times} X) \rightarrow EG \underset{G}{\times} (G \underset{G}{\times} X) = EG \underset{G}{\times} X$ is the obvious map and $\mathbb{H}^*_G(X; K) = \mathbb{H}^*(EG \underset{G}{\times} X; K)$.

In order to motivate the following theorem, first consider the special case $X = Spec \quad \mathbb{C}$, $H = G^o$= the connected component of G containing the identity and $K = \mathcal{L}$ a G-equivariant sheaf on BG. Since the fundamental group of BG is now $\bar{G}$, the G-equivariant sheaf $\mathcal{L}$, which is automatically a local system on BG, has an action by $\bar{G}$. Therefore, there is an induced action of $\bar{G}$ on the equivariant cohomology $H^*(BH; \mathcal{L})$. The following theorem is a generalization of this fact and it is already established in [B-J] Theorem 1 in characteristic 0. (See [B-J] (1.3.3) for a discussion of local systems in positive characteristics.)

The only non-simplicial approach to the classifying space of an algebraic group in positive characteristics is in terms of infinite Grassmanians. However, this does not provide a *fibration* $BG \rightarrow B\bar{G}$ *with fibers* BH *in positive characteristics*. The existence of such a fibration is crucial in characteristic 0 for the proof of the following theorem. The simplicial techniques introduced in the earlier sections are adequate substitutes for this (and seem to be essential) for the proof of the following theorem in positive characteristics.

(5.1) **Theorem**. Let $K \,\epsilon\, D^G_+(X; \mathbb{Q}_l)$. (i) Then there exists an action of $\bar{G}$ on $\mathbb{H}^*_H(X; K)$ that is natural in K. (ii) Moreover, one obtains an identification $\mathbb{H}^*_G(X; K) \cong (\mathbb{H}^*_H(X; K))^{\bar{G}}$.

Proof. Let $K \,\epsilon\, D^G(X)$ and let $\pi^G : EG \underset{G}{\times} X \rightarrow BG$ denote the obvious map. Now one considers $R^n\pi^G_*(K)$, where π^G_* denotes the direct image functor for sheaves on the site considered in (1.1.1)(i). One may identify the stalks of this sheaf at any point $\bar{x}_n$ of BG_n with $\mathbb{H}^*(X; K)$. Therefore, one may readily see that $R^n\pi^G_*K$ is sheaf with descent or equivalently a G-equivariant sheaf on BG. Since any G-equivariant sheaf on BG is lisse, the fundamental group of $BG \cong G/G^o$ has an action on the stalks which are identified with $\mathbb{H}^*(X; K)$. Moreover, it is clear that this action is natural in K. (Recall that if $\alpha : K' \to K$ is a map in $D^G_+(X, \mathbb{Q}_l)$, each $R^n\alpha_*$ is a map of G-equivariant sheaves on BG.)

We will next consider the first statement of the Theorem. Let $\bar{\pi} : BG \to B\bar{G}$ denote the obvious maps. Let the composition $\bar{\pi} \circ \pi^G = \pi$. One of the key ideas of the proof is to identify $\mathbb{H}^*_H(X; K)$ with the stalks of the sheaf $\underset{n}{\oplus} R^n({}_s\pi)_*K$ at any simplicial geometric point on $B\bar{G}$. Since $\bar{G}$ is finite one may observe that $(L_k, \bar{\pi}_k)$ is cohomologically proper for any $L \,\epsilon\, D_+(Et(BG); \mathbb{Q}_l)$ and any $k \geq 0$. One may also readily observe that (K_k, π^G_k) is cohomologically proper for any $K \,\epsilon\, D^G_+(X; \mathbb{Q}_l)$. Therefore, it follows that (K_k, π_k) is cohomologically proper for any $k \geq 0$. Let $\bar{x}_0$ denote a fixed geometric point of $(B\bar{G})_0$ and let $\bar{x}_\bullet$ denote the associated simplicial geometric point of $B\bar{G}$. One may clearly identify the geometric fiber of π over $\bar{x}_\bullet$ with $EH \underset{H}{\times} X$. Therefore, (4.2) provides the identification of the stalk at $\bar{x}_\bullet$:

(5.1.1) $(\mathcal{H}^n(R_s\pi_*R\bar{\eta}_*(K)))_{\bar{x}_\bullet} \cong \mathbb{H}^n_H(X; K)$, $K \,\epsilon\, D^G_+(X; \mathbb{Q}_l)$, $n \geq 0$.

Observe that $\mathcal{H}^n(R_s\pi_*(R\bar{\eta}_*K))$ is a sheaf on $SEt(B\bar{G})$. In order to show there exists an action of $\bar{G}$ on the above sheaf that is natural in K, we will first show that for each n, $\mathcal{H}^n(R_s\pi_*(R\bar{\eta}_*(K)))$ is a sheaf with descent in the sense of (3.8.2).

First we reduce to the case $X = Spec \quad k$ as follows. By (3.11)(ii), $R_s\pi_*R\bar{\eta}_*(K) = R_s\bar{\pi}_*R_s\pi^G_*R\bar{\eta}_*(K) = R_s\bar{\pi}_*R\bar{\eta}_*R\pi^G_*(K)$. Therefore, we may replace $EG \underset{G}{\times} X$ (π, the complex K) by BG ($\bar{\pi}$, $K' = R\pi^G_*(K)$, respectively).

Assume that $K' \,\epsilon\, D^G_+(Spec \quad k; \mathbb{Q}_l)$. Let $\bar{x}_m$ be a fixed (geometric) point of $B\bar{G}_m = \bar{G}^{\overset{m}{\times}}$ and let $\bar{x}_\bullet = (\bar{x}_m)_\bullet : \Delta[m] \otimes * \to B\bar{G}$ denote the associated simplicial point. Let $\bar{\alpha} : (B\bar{G})_m \to (B\bar{G})_0$ denote a structure map of the simplicial scheme $B\bar{G}$, let $\bar{x}_0 = \bar{\alpha}(\bar{x}_m)$ be the corresponding geometric point of BG_0 and let $(\bar{x}_0)_\bullet$ denote the associated simplicial geometric point of $B\bar{G}$. There exists a map $(\bar{x}_0)_\bullet \to \bar{x}_\bullet$ of simplicial points and therefore an induced map of the corresponding geometric fibers of the map $\bar{\pi}$. These are isomorphic to the simplicial scheme $\Delta[0] \otimes BH$ (= the diagonal of the bi-simplicial scheme $\{\Delta[0]_l \otimes H^r | r, l\}$) and $\Delta[m] \otimes BH$ (= the diagonal of the bisimplicial scheme $\{\Delta[m]_l \otimes H^r | r, l\}$). One also obtains an induced map of the stalks:

(5.1.2) $(R_s\bar{\pi}_*(R\bar{\eta}_*(K')))_{(\bar{x}_0)_\bullet} \leftarrow (R_s\bar{\pi}_*(R\bar{\eta}_*(K')))_{\bar{x}_\bullet}$

Assume for the time being that the map in (5.1.2) is a quasi-isomorphism for all structure maps $\bar{\alpha}$. We immediately observe that if $\bar{\alpha} : (B\bar{G})_m \to (B\bar{G})_0$ is a structure map of BG, the natural map $\bar{\alpha}^*(\eta^*_0(R_s\bar{\pi}_*(R\bar{\eta}_*(K')))) \leftarrow \eta^*_m(R_s\bar{\pi}_*(R\bar{\eta}_*(K')))$ (see (3.9)(i)) is a quasi-isomorphism. (To see this observe that the stalk on left-hand-side at $\bar{x}_m$ identifies with $(R_s\bar{\pi}_*(R\bar{\eta}_*(K')))_{(\bar{x}_0)_\bullet}$ while the stalk on the right-hand-side at $\bar{x}_m$ identifies with $(R_s\bar{\pi}_*(R\bar{\eta}_*(K')))_{\bar{x}_\bullet}$.) It will follow therefore as in (3.9)(iii), that for each fixed n, $\{\eta^*_k(\mathcal{H}^n(R_s\bar{\pi}_*(R\bar{\eta}_*(K'))))|k\}$ is a sheaf with descent on $Et(B\bar{G})$ and that therefore there exist an action of $\bar{G}$ on the stalks of the above sheaf thereby proving the first statement of the theorem. (Observe the action of $\bar{G}$ is natural since there is an equivalence of categories between $\bar{G}$-equivariant sheaves and sheaves with descent on $B\bar{G}$.)

We proceed to show that the map in (5.1.2) is a *quasi-isomorphism.* In view of the identification of the stalks as in (4.2), the map in (5.1.2) corresponds to a map

$\mathbb{H}^*(\Delta[0]_\bullet \otimes BH; K') \leftarrow \mathbb{H}^*(\Delta[m]_\bullet \otimes BH; K')$

Observe that the above map is induced by the map of bi-simplicial schemes: $\{\Delta[0]_l \otimes H^r | r, l\} \xrightarrow{\alpha_l \otimes id} \{\Delta[m]_l \otimes H^r | r, l\}$, where $\alpha : [0] \to [m]$ in Δ induces the map $\bar{\alpha}$. Using the spectral sequence for the hypercohomology of a bisimplicial scheme (see [Fr] p. 19), one reduces to showing that the corresponding maps

(5.1.3) $\mathbb{H}^*(\Delta[0]_\bullet \otimes H^r; K'_r) \leftarrow \mathbb{H}^*(\Delta[m]_\bullet \otimes H^r; K'_r)$

are isomorphisms for each fixed r. Recall that we have already replaced X by $Spec \quad k$ and the map π by $\bar{\pi}$. Therefore, the geometric fiber of the map $\bar{\pi}_r$, over each geometric point of $B\bar{G}_r$, is isomorphic to H^r; moreover all the geometric fibers of π_r over $\Delta[m]_r$ are contained in $G^r = (BG)_r$. Since K' has G-equivariant cohomology sheaves, one may now compute $\mathbb{H}(\Delta[m]_\bullet \otimes H^r; K'_r) \simeq \Delta[m]_\bullet \otimes \mathbb{H}(H^r; K'_r)$. Therefore, the map in (5.1.3) is indeed an isomorphism. We have therefore completed the proof that the map in (5.1.2) is a quasi-isomorphism and therefore also the proof of the first statement in the Theorem.

Next consider the Leray spectral sequence for the map $\bar{\pi} : BG \rightarrow B\bar{G}$ and for the complex $R\bar{\eta}_* R\pi^G_*(K) \,\epsilon\, D_+(SEt(BG); \mathbb{Q}_l)$. We obtain:

$$E_2^{u,v} = H^u(B\bar{G}; R^v({}_s\bar{\pi})_*(R\bar{\eta}_* R\pi^G_*(K))) \Rightarrow H^{u+v}(BG; R\pi_*(K)) \cong \mathbb{H}^{u+v}_G(X; K).$$

Since $\bar{G}$ is a finite group, $E_2^{u,v} = 0$ unless $u = 0$ in which case it is given by $(R^v({}_s\bar{\pi})_*(R\bar{\eta}_*(R\pi^G_*(K))))^{\bar{G}} \cong (\mathbb{H}^v_H(X;K))^{\bar{G}}$ by the above results. This completes the proof of the theorem. □

The remainder of this section will be devoted to a proof of (5.6) below: this is needed in [B-J]. We will make a slight change of notation: the map $BG \rightarrow B\bar{G}$ denoted $\bar{\pi}$ above will now be denoted p. The composite functor $R_s p_* \circ R\bar{\eta}_*$ will be henceforth denoted $R\bar{p}_*$.

(5.2.1) Assume the situation of (5.1). If $\underline{\mathbb{Q}}_l$ is the obvious constant sheaf on $Et(BG)$, $R\bar{p}_*(\underline{\mathbb{Q}}_l) = R({}_s p)_* \bar{\eta}_*(\underline{\mathbb{Q}}_l)$ is a complex of l-adic sheaves on $SEt(B\bar{G})$. The obvious pairing $\underline{\mathbb{Q}}_l \otimes \underline{\mathbb{Q}}_l \rightarrow \underline{\mathbb{Q}}_l$ induces an associative pairing: $R\bar{p}_*(\underline{Q}_l) \otimes R\bar{p}_*(\underline{\mathbb{Q}}_l) \rightarrow R\bar{p}_*(\underline{\mathbb{Q}}_l)$. This shows $R\bar{p}_*(\underline{\mathbb{Q}}_l)$ is an l-adic sheaf of differential graded algebras on $SEt(B\bar{G})$.

(5.2.2) Let $BiMod_+(SEt(B\bar{G}); R\bar{p}_*(\underline{\mathbb{Q}}_l))$ denote the category of bounded below complexes of l-adic sheaves of bi-modules over the l-adic sheaf of differential graded algebras $R\bar{p}_*(\underline{\mathbb{Q}}_l)$ on $SEt(B\bar{G})$. An object in this category is a bounded below complex complex of l-adic sheaves M on $SEt(B\bar{G})$ provided with (coherently) associative pairings $R\bar{p}_*(\underline{\mathbb{Q}}_l) \otimes M \rightarrow M$ and $M \otimes R\bar{p}_*(\underline{\mathbb{Q}}_l) \rightarrow M$ that make the obvious diagrams commute. Morphisms $M' \rightarrow M$ in this category are morphisms of complexes that are compatible with the extra structure. (See [K-M] for details on differential graded algebras and modules over them.)

(5.2.3) Given $K \,\epsilon\, D^G_+(BG; \mathbb{Q}_l)$, the associative pairings $\underline{\mathbb{Q}}_l \otimes K \rightarrow K$ and $K \otimes \underline{\mathbb{Q}}_l \rightarrow K$ imply that $R\bar{p}_*(K)$ belongs to $BiMod_+(SEt(B\bar{G}); R\bar{p}_*(\underline{\mathbb{Q}}_l))$.

(5.2.4) Let $\Delta[1] \otimes \underline{\mathbb{Q}}_l$ denote the normalization of the simplicial abelian group given by $n \rightarrow \Delta[1]_n \otimes \underline{\mathbb{Q}}_l$. We will define two maps $f, g : M \rightarrow N$ in $BiMod_+(SEt(B\bar{G}); R\bar{p}_*(\underline{\mathbb{Q}}_l))$ to be homotopic if there exists a map $H : M \otimes \Delta[1] \otimes \underline{\mathbb{Q}}_l \rightarrow N$ so that $f = H \circ d_0$ and $g = H \circ d_1$, with $d_i : M \cong M \otimes \Delta[0] \otimes \underline{\mathbb{Q}}_l \rightarrow M \otimes \Delta[1] \otimes \underline{\mathbb{Q}}_l$ being the obvious maps. The associated homotopy category (i.e. where a morphism is an equivalence class of maps in the equivalence relation generated by the above definition of homotopy) is denoted $HBiMod_+(SEt(B\bar{G}); R\bar{p}_*(\underline{\mathbb{Q}}_l))$. The corresponding derived category is obtained from $HBiMod_+(SEt(B\bar{G}); R\bar{p}_*(\underline{\mathbb{Q}}_l))$ by inverting maps that induce isomorphisms on cohomology. This will be denoted $D(BiMod_+(SEt(B\bar{G}); R\bar{p}_*(\underline{\mathbb{Q}}_l)))$.

(5.3.1) Let $\bar{x}_n$ be a fixed geometric point of $B\bar{G}_n$ and let $\alpha : (B\bar{G})_n \rightarrow (B\bar{G})_0$ be a structure map of the simplicial scheme $B\bar{G}$. Let $\bar{x}_0 = \alpha(\bar{x}_n)$ be the associated geometric point of $(B\bar{G})_0$. If $(\bar{x}_n)_\bullet$ and $(\bar{x}_0)_\bullet$ are the corresponding simplicial geometric points of $B\bar{G}$, as observed above, there exists a natural map $(\bar{x}_0)_\bullet \rightarrow (\bar{x}_n)_\bullet$. If F' is a sheaf of differential graded modules on $SEt(B\bar{G})$, we will say F' has *descent* (or is $\bar{G}$-equivariant) if the induced map of stalks $(\bar{x}_0)^*_\bullet F' \leftarrow (\bar{x}_n)^*_\bullet F'$ is a quasi-isomorphism for all geometric points $\bar{x}_n$ and for all $n \geq 0$. Next we let $D^{\bar{G}}(BiMod_+(SEt(B\bar{G});\quad R\bar{p}_*(\underline{\mathbb{Q}}_l)))$ denote the full subcategory of $D(BiMod_+(SEt(B\bar{G}); R\bar{p}_*(\underline{\mathbb{Q}}_l)))$ consisting of l-adic sheaves of differential graded bi-modules over $R\bar{p}_*(\underline{\mathbb{Q}}_l)$ whose cohomology sheaves are $\bar{G}$-equivariant.

(5.3.2) **Lemma**. Let $M \,\epsilon\, D^{\bar{G}}(BiMod_+(SEt(B\bar{G}); R\bar{p}_*(\underline{\mathbb{Q}}_l)))$. Then $\mathbb{H}^n_{SEt(B\bar{G})}(B\bar{G}, M)) = H^n(\bar{\eta}^*(M))^{\bar{G}}$ for all n.

Proof. Let F' denote a sheaf on $SEt(B\bar{G})$ with *descent*. It follows from (3.9)(iii) and (3.7.6) that the natural map $F' \rightarrow R\bar{\eta}_* \bar{\eta}^*(F')$ is a quasi-isomorphism. Moreover $H^n_{SEt(B\bar{G})}(B\bar{G}, F') \cong H^n_{Et(B\bar{G})}(B\bar{G}, \bar{\eta}^*(F')) \cong \bar{\eta}^*(F')^{\bar{G}}$ if $n = 0$ and $\cong 0$ otherwise. Therefore, one observes that the spectral sequence

$$E_2^{s,t} = H^s_{SEt(B\bar{G})}(B\bar{G}; \mathcal{H}^t(M)) \Rightarrow \mathbb{H}^{s+t}_{SEt(B\bar{G})}(B\bar{G}; M)$$

degenerates with $E_2^{s,t} = 0$ for $s > 0$ providing the isomorphism $\mathbb{H}^t_{SEt(B\bar{G})}(B\bar{G}; M) \cong \Gamma(B\bar{G}; \bar{\eta}^* H^t(M)) \cong H^t(\bar{\eta}^*(M))^{\bar{G}}$. □

Next let $M,\ N \in D^{\bar{G}}(BiMod_+(Et(B\bar{G});\quad R\bar{p}_*(\underline{\mathbb{Q}}_l)))$. Then there exists a spectral sequence:

(5.3.3) $E_2^{s,t} = \mathcal{E}xt^{s,t}_{\mathcal{H}^*(R\bar{p}_*(\underline{\mathbb{Q}}_l))}(\mathcal{H}^*(M),\ \mathcal{H}^*(N)) \Rightarrow \mathcal{H}^{s+t}(\mathcal{RH}om_{R\bar{p}_*(\underline{\mathbb{Q}}_l)}(M,\ N)$

Moreover, if at each geometric point $\bar{x}$ of $B\bar{G}$, $\mathcal{H}^*(M)_{\bar{x}}$ is a projective module over $\mathcal{H}^*(R\bar{p}_*(\underline{\mathbb{Q}}_l))_{\bar{x}} \cong H^*(BH;\mathbb{Q}_l)$, then the above spectral sequence degenerates and $E_2^{s,t} = 0$ for all $s > 0$. (The spectral sequence is established in [K-M] Theorem (7.3), Part V. Recall that the cohomology sheaves $\mathcal{H}^*(M)$ are $\bar{G}$-equivariant and hence lisse. Therefore the stalk of the term in (5.3.3) at a geometric point $\bar{x}$ identifies with $Ext^{s,t}_{H^*(R\bar{p}_*(\underline{\mathbb{Q}}_l)_{\bar{x}})}(\mathcal{H}^*(M)_{\bar{x}}, \mathcal{H}^*(N)_{\bar{x}})$.)

(5.3.4) This holds for example if $M = R\bar{p}_*(F)$, $F \in Sh^G(Et(BG), \mathbb{Q}_l)$. To see this, take the stalks of the E_2-terms at a geometric point $\bar{x}$ of $B\bar{G}$. Since the cohomology sheaves $\mathcal{H}^*(R\bar{p}_*(F))$ are *lisse*, $E_{2,\bar{x}}^{s,t} \cong Ext^{s,t}_{\mathcal{H}^*(R\bar{p}_*(\underline{\mathbb{Q}}_l))_{\bar{x}}}(\mathcal{H}^*(R\bar{p}_*(F))_{\bar{x}}, \mathcal{H}^*(R\bar{p}_*(\underline{\mathbb{Q}}_l))_{\bar{x}})$. (5.1) shows that$\mathcal{H}^*(R\bar{p}_*(\underline{\mathbb{Q}}_l))_{\bar{x}} \cong H^*(BH;\mathbb{Q}_l)$ and $\mathcal{H}^*(R\bar{p}_*F)_{\bar{x}} \cong H^*(BH; F_{|BH})$. Observe that $(BH_{et})\hat{}$ is simply connected, i.e. $\pi_1(BH, *)\hat{} = 0$ where π_1 denotes the étale fundamental group where $\hat{}$ denotes completion away from the characteristic p. Therefore $F_{|BH}$ is the constant l-adic sheaf associated to a (free) $\mathbb{Z}_l$-module. Therefore $H^*(BH; F_{|BH})$ is a free module over $H^*(BH;\mathbb{Q}_l)$.

Next we define the functor

(5.4.1) $L_sp^* : D(BiMod_+(SEt(B\bar{G}); R\bar{p}_*(\underline{\mathbb{Q}}_l))) \rightarrow D_+(SEt(BG);\mathbb{Q}_l)$ by $\underline{\mathbb{Q}}_l \overset{L}{\underset{{}_sp^{-1}(R\bar{p}_*(\underline{\mathbb{Q}}_l))}{\otimes}} {}_sp^{-1}(N)$.

(See [K-M] part III for a definition of the derived tensor product functors considered above in a somewhat more general setting.)

(5.4.2) **Proposition**. (i) If $K \in D_b^G(BG;\mathbb{Q}_l)$, there exists a map $\bar{\eta}^* L_sp^*(R\bar{p})_*(K) \rightarrow K$ which is a quasi-isomorphism and is natural in K.

(ii) The functor $R\bar{p}_* : D_b^G(BG;\mathbb{Q}_l) \rightarrow D\bar{G}(Bi - Mod_+(SEt(B\bar{G}); R\bar{p}_*(\underline{\mathbb{Q}}_l)))$

is *fully-faithful.*

(iii) If $K,\ L \in D_b^G(BG;\mathbb{Q}_l)$ the map in (i) induces a quasi-isomorphism:

$R\bar{p}_*(\mathcal{RH}om(K,L)) \rightarrow \mathcal{RH}om_{R\bar{p}_*(\underline{\mathbb{Q}}_l)}(R\bar{p}_*(K), R\bar{p}_*(L))$. (The two $\mathcal{RH}om$ denote derived functors of the appropriate internal Homs.)

Proof. Observe that L_sp^* is the left-adjoint to the functor

$R_sp_* : D_+(SEt(BG);\mathbb{Q}_l) \rightarrow D(BiMod_+(SEt(B\bar{G}); R\bar{p}_*(\underline{\mathbb{Q}}_l)))$;

similarly $\bar{\eta}^*$ is left adjoint to $\bar{\eta}_*$. Therefore, the naturality of the map in (i) is clear. To show it is a quasi-isomorphism, it suffices to show the restriction of the map in (i) to each geometric fiber (of the map p over each simplicial geometric point of $B\bar{G}$) is a quasi-isomorphism. Therefore, we readily reduce to the situation where the map p is replaced by the obvious map from the geometric fiber of p over a fixed simplicial geometric point $\bar{x}_\bullet$ of $B\bar{G}$. If $F_{\bar{x}_\bullet}$ denotes the geometric fiber of p over $\bar{x}_\bullet$, one may readily compute $\mathbb{H}^*(F_{\bar{x}_\bullet}; K) \cong \mathbb{H}^*(BH; K)$. One may also compute the stalk of $R\bar{p}_*(K) = R({}_sp)_* R\bar{\eta}_*(K)$ at $\bar{x}_\bullet$ using (4.2) to be isomorphic to $\mathbb{H}^*(BH; K)$. Therefore, (i) reduces to the case where the group G is replaced by H and G/H by H/H. In this case (i) may be established as follows. Since the map in (i) is natural in K, it suffices to prove (i) with K replaced by an H-equivariant sheaf $\mathcal{L}$. Next observe that $\pi_1(BH)\hat{} = 0$, where π_1 denotes the étale fundamental group and the completion is away from the characteristic. Therefore the H-equivariant sheaf $\mathcal{L}$ is the constant sheaf associated to a free $\mathbb{Z}_l$-module. In this case, (i) follows readily. It follows from (i) that the functor $R\bar{p}_*$ is fully-faithful thereby proving (ii).

Observe that the map in (iii) is adjoint to a map $R\bar{p}_*(\mathcal{RH}om(K,L) \overset{L}{\underset{R\bar{p}_*(\underline{\mathbb{Q}}_l)}{\otimes}})R\bar{p}_*(K) \rightarrow R\bar{p}_*(\mathcal{RH}om(K,L) \otimes K) \rightarrow R\bar{p}_*(L)$. This exists in view of the adjunction between $R\bar{p}_*$ and $\bar{\eta}^* L_sp^*$. In order to prove this is a quasi-isomorphism, once again we reduce to the case where $\bar{G}$ is trivial as follows. First observe that the cohomology sheaves of $R\bar{p}_*(K)$ are *lisse* on $B\bar{G}$. Therefore, if $\bar{x}$ is a fixed (simplicial) geometric point of $B\bar{G}$,

$(\mathcal{RH}om_{R\bar{p}_*(\underline{\mathbb{Q}}_l)}(R\bar{p}_*(K), R\bar{p}_*(L))_{\bar{x}} \simeq \mathcal{RH}om_{(R\bar{p}_*(\underline{\mathbb{Q}}_l))_{\bar{x}}}((R\bar{p}_*(K))_{\bar{x}}, (R\bar{p}_*(L))_{\bar{x}})$. Then the computation of the stalks as in the proof of Theorem (5.1) reduces to the case $\bar{G}$ is trivial or where G itself is connected. Since the map in (iii) is natural in K and L, we may reduce to the case these are themselves G-equivariant sheaves. Since $(BG_{et})\hat{}$ is now simply connected, we may assume that K is the constant sheaf associated to a free $\mathbb{Z}_l$-module. The quasi-isomorphism in (iii) is now clear. □

(5.5.1) Given $M \epsilon D(BiMod_+(SEt(B\bar{G}); R\bar{p}_*(\underline{\mathbb{Q}}_l))))$, we define the dual of M to be $\mathcal{RH}om_{R\bar{p}_*(\underline{\mathbb{Q}}_l))}(M, \quad R\bar{p}_*(\underline{\mathbb{Q}}_l)))$. This will be denoted $D_{R\bar{p}_*(\underline{\mathbb{Q}}_l)}(M)$.

One may re-interpret (5.4.2)(iii) with $L = \mathbb{D}_{\mathbb{Q}_l} \simeq \underline{\mathbb{Q}}_l$ as

(5.5.2) $D_{R\bar{p}_*(\underline{\mathbb{Q}}_l)}(R\bar{p}_*(K)) \simeq R\bar{p}_*(D(K))$

where $D(K)$ denotes the Verdier-dual of K.

(5.6) **Proposition**. Let $K \epsilon D^G_+(BG; \mathbb{Q}_l)$ so that for each simplicial geometric point $\bar{x}_\bullet$ of $B\bar{G}$, the stalk $\mathcal{H}^*(R\bar{p}_*(K))_{\bar{x}_\bullet}$ is a projective module over $\mathcal{H}^*(R\bar{p}_*(\mathbb{Q}_l))_{\bar{x}_\bullet}$. Then

$$\mathbb{H}^*(BG; D(K)) \cong \mathbb{H}^*(BH; D(K))^{\bar{G}} \cong (Hom_{H^*(BH;\mathbb{Q}_l)}(\mathbb{H}^*(BH; K), H^*(BH; \mathbb{Q}_l)))^{\bar{G}}.$$

Proof. The first isomorphism follows from (5.1)(ii) with $X = Spec \quad k$. Observe that the stalks of $\mathcal{H}^*(R\bar{p}_*D(K))$ on $B\bar{G}$ are $\mathbb{H}^*(BH; D(K))$ and that the cohomology sheaves $\mathcal{H}^*(R\bar{p}_*D(K))$ are $\bar{G}$-equivariant. Therefore, one may identify $\mathbb{H}^*(BH; D(K))^{\bar{G}}$ with $H^*(\bar{\eta}^*(R\bar{p}_*D(K)))^{\bar{G}}$. By (5.3.2) Lemma, one obtains the isomorphisms: $H^*(\bar{\eta}^*(R\bar{p}_*D(K)))^{\bar{G}} \cong H^*_{SEt(B\bar{G})}(B\bar{G}; R\bar{p}_*D(K))) \cong H^*_{SEt(B\bar{G})}(B\bar{G}; D_{R\bar{p}_*(\underline{\mathbb{Q}}_l)}(R\bar{p}_*(K))))$. By definition the last term

(5.6.*) $= H^*_{SEt(B\bar{G})}(B\bar{G}; \mathcal{RH}om_{R\bar{p}_*(\underline{\mathbb{Q}}_l)}(R\bar{p}_*(K), R\bar{p}_*(\mathbb{Q}_l)))$.

One may observe $\mathcal{RH}om_{R(\bar{p})_*(\underline{\mathbb{Q}}_l))}(R\bar{p}_*(K), R\bar{p}_*(\mathbb{Q}_l))$ is a complex whose cohomology sheaves are $\bar{G}$-equivariant. (See the proof of (5.1)(ii)). By an argument as in (5.3.2), the term in (5.6.*) may now be identified with $H^*(\bar{\eta}^*\mathcal{RH}om_{R\bar{p}_*(\mathbb{Q}_l)}(R\bar{p}_*(K), \ R\bar{p}_*(\mathbb{Q}_l)))^{\bar{G}}$. Now the arguments in (5.3.3), (5.3.4) along with the observation that $\bar{\eta}^*$ preserves distinguished triangles applies to identify this with $(Hom_{H^*(BH;\mathbb{Q}_l)}(\mathbb{H}^*(BH; K), \ H^*(BH; \mathbb{Q}_l)))^{\bar{G}}$. □

Appendix: The Dold-Puppe correspondence. Let $\mathbf{A}$ denote an abelian category; a co-chain complex K in $\mathbf{A}$ will denote a sequence $K^i \ \epsilon \ \mathbf{A}$ provided with maps $d : K^i \to K^{i+1}$ so that $d^2 = 0$. Let $C_0(\mathbf{A})$ denote the category of co-chain complexes in $\mathbf{A}$ that are trivial in negative degrees. One defines the denormalizing functor:$DN : C_0(\mathbf{A}) \to$ (*Cosimplicial objects in* $\mathbf{A}$) as in [Ill-1] pp. 8-9. (In [Ill-1] pp.8-9, the corresponding functors between simplicial objects and chain complexes are considered. Making use of the observation that a simplicial object (a chain complex) in an abelian category corresponds to a cosimplicial object (co-chain complex, respectively) in the opposite abelian category, one may adapt these to the present situation.) DN will be inverse to the functor N : (*Cosimplicial objects in* $\mathbf{A}$) $\to C_0(\mathbf{A})$ defined by $(NK)^n = \underset{i \neq 0}{+} coker(d^i : K^n \to K^{n+1})$ with $\delta : (NK)^n \to (NK)^{n+1}$ induced by d^0.

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