A WEAK GROWTH DICHOTOMY FOR D-MINIMAL EXPANSIONS OF THE REAL FIELD*

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An expansion $\mathfrak{R}$ of the real line is **d-minimal** if for every $n \in \mathbb{N}$ and definable $A \subseteq \mathbb{R}^{n+1}$, there exists $N \in \mathbb{N}$ such that for every $x \in \mathbb{R}^m$, the fiber $\{\, t \in \mathbb{R} : (x,t) \in A \,\}$ either has interior or is a union of N (not necessarily distinct) discrete sets. For more information on d-minimality, see [1, 2, 3, 5, 6, 8].

Until further notice, $\mathfrak{R}$ denotes an expansion of the real field; definability is with respect to $\mathfrak{R}$. We obtain a generalization of a result that holds for o-minimal expansions of $\overline{\mathbb{R}}$.

1. **Theorem.** *If $\mathfrak{R}$ is d-minimal, then either $\mathfrak{R}$ is exponential or every definable power function is $\emptyset$-definable.*

The above is not quite a dichotomy—every power function is $\emptyset$-definable in the o-minimal (hence d-minimal) structure $\big(\mathbb{R}, +, e^x, (r)_{r\in\mathbb{R}}\big)$—but having $\operatorname{dcl}(\emptyset) = \mathbb{R}$ is the only obstruction to a dichotomy.

The property of all definable power functions being $\emptyset$-definable has been quite useful in o-minimality; perhaps the same will be true for d-minimality (but it's probably too early to tell).

By the usual tricks, the theorem is an easy consequence of the next four results.

2. **Lemma.** *Let $F\colon \mathbb{R}^{m+1} \to \mathbb{R}$ be $\emptyset$-definable. Then $\{\, (a,r) \in \mathbb{R}^{m+1} : F(a,\cdot) = x^r \,\}$ is $\emptyset$-definable.*

Proof. For all $(a,r) \in \mathbb{R}^{m+1}$, we have $F(a,\cdot) = x^r$ if and only if $F(a,1) = 1$, $F(a,\cdot){\restriction}(0,\infty)$ is differentiable, and $t(\partial F/\partial t)(a,t) = rF(a,t)$ for all $t > 0$. □

3. **Lemma.** *If $\mathfrak{R}$ has definable Skolem functions and there exist $m \in \mathbb{N}$ and definable $F\colon \mathbb{R}^{m+1} \to \mathbb{R}$ such that $\{\, r \in \mathbb{R} : \exists a \in \mathbb{R}^m,\ F(a,\cdot) = x^r \,\}$ has interior, then $\mathfrak{R}$ is exponential.*

Proof. Argue as in [4, 4.1], using the previous lemma. □

4. **Proposition** ([3]). *If $\mathfrak{R}$ is d-minimal, then $\mathfrak{R}$ has definable Skolem functions.*

5. **Lemma.** *Suppose that every $\emptyset$-definable subset of $\mathbb{R}$ either has interior or is a finite union of discrete sets.*

(1) *If $A \subseteq \mathbb{R}$ is $\emptyset$-definable and has no interior, then $A \subseteq \operatorname{dcl}(\emptyset)$.*
(2) *Every nonempty $\emptyset$-definable set (of any arity) contains a $\emptyset$-definable point.*

Proof. (1). Since $\operatorname{dcl}(\emptyset)$ is dense in $\mathbb{R}$, every nonempty open interval intersects $\operatorname{dcl}(\emptyset)$, and every isolated point of a $\emptyset$-definable subset of $\mathbb{R}$ is $\emptyset$-definable. The set of isolated points of a $\emptyset$-definable set is $\emptyset$-definable.

(2) follows from (1) by an easy induction. □

*This is **not** a preprint; please do not refer to it as such.

By similar arguments:

6. **Theorem.** *If* $\mathfrak{R}$ *is a d-minimal expansion of* $(\mathbb{R}, <, +, 1)$*, then either* $\mathfrak{R}$ *defines multiplication or every definable scalar function is* $\emptyset$*-definable.*

Details are left to the interested reader; see *e.g.* [7] for some relevant tricks.

References

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