Integrability, Nonintegrability and the Poly-Painlevé Test

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Nonlinearity 2003, 1997, *preprint*, *Meth.Appl.An.* 1997, *Inv.Math.* 2001

Talk dedicated to my professor, Martin Kruskal.

Happy birthday!

Historical perspective:

Motivated by the idea that

new functions can be defined as solutions of *linear* differential equations

(Airy, Bessel, Mathieu, Lamé, Legendre...),

a natural question arose:

Can nonlinear equations define new, "transcendental" functions?

Fuchs' intuition: A given equation does define functions if movable singularities in $\mathbb{C}$ of general solution are no worse than poles (no branch points, no essential singularities).

Such an equation has solutions meromorphic on a (common) Riemann surface.

$$u' = \frac{1}{2u} \qquad \text{gen sol} \quad u = (x - C)^{1/2} \quad \text{movable branch point}$$

$$u' = \frac{1}{2x^{1/2}} \qquad \text{gen sol} \quad u = x^{1/2} + C \quad \text{fixed branch point}$$

The property of a diff.eq. to have general solution without movable singularities (except, perhaps, poles) is now called "the Painlevé Property" (PP).

Directions: discovery of Painlevé equations (1893-1910), development of Kowalevski integrability test (1888).

- Q: Fuchs 1884: find the nonlinear equations with the Painlevé property.

  Among first and second order equations

class: $F(w', w, z) = 0$, algebraic; $w'' = F(z, w, w')$ rational in w, w', analytic in z

equations with (PP) were classified.

Among them 6 define new functions:

the Painlevé transcendents.

$$P_I \quad w'' = 6w^2 + z \qquad\qquad P_{II} \quad w'' = 2w^3 + zw + a$$

$$P_{III} \quad w'' = \frac{1}{w}(w')^2 - \frac{1}{z}w' + \frac{1}{z}\left(aw^2 + b\right) + cw^3 + \frac{d}{w}$$

$$P_{IV} \quad w'' = \frac{1}{2w}(w')^2 + \frac{3}{2}w^3 + 4zw^2 + 2(z^2 - a)w + \frac{b}{w}$$

$$P_V \quad w'' = \left(\frac{1}{2w} + \frac{1}{w-1}\right)(w')^2 - \frac{1}{z}w' + \frac{(w-1)^2}{z^2}\left(aw + \frac{b}{w}\right) + c\frac{w}{z} + d\frac{w(w+1)}{w-1}$$

$$P_{VI} \quad w'' = \frac{1}{2}\left(\frac{1}{w} + \frac{1}{w-1} + \frac{1}{w-z}\right)(w')^2 - \left(\frac{1}{z} + \frac{1}{z-1} + \frac{1}{w-z}\right)w'$$
$$+ \frac{w(w-1)(w-z)}{z^2(z-1)^2}\left[a + \frac{bz}{w^2} + c\frac{z-1}{(w-1)^2} + d\frac{z(z-1)}{(w-z)^2}\right]$$

Painlevé transcendents: in many physical applications (critical behavior in stat mech, in random matrices, reductions of integrable PDEs,...).

In another direction:

Sophie Kowalevski linked the idea of absence of movable singularities (other than poles) to integrability. She developed a method of identifying integrable cases, and applied it to the equations describing the motion of a rigid body (a top) rotating about a fixed point. Kowalevski determines parameters for which there are no movable branch points (in $\mathbb{C}$).

She found 3 cases:
1) Euler, 2) Lagrange, 3) Kowalevski.

In the new case Kowalevski expressed solutions using theta-functions of two variables. 1888

The method: of finding parameters for absence of movable branch points

has been refined (Painlevé-Kowalevski test),

generalized to PDEs,

used extensively, and successfully to identify integrable systems.

Kruskal, Ablowitz, Clarkson, Deift, Flaschka, Grammaticos, Newell, Ramani, Segur, Tabor, Weiss...

Many aspects of the Painlevé-Kowalevski test remain, however, not completely understood.

Kruskal:

- Is not clear if all the movable singularities of a given eq. have been determined.
- Movable *essential* sing are usually not looked for. In fact, this is a very difficult question.
- Painlevé property is not invariant under coordinate transformations. E.g. $u' = \frac{1}{2u}$ has gen sol $u = (x - C)^{1/2}$. Set $v = u^2$.
- A general connection to existence of first integrals is not rigorously known.

Classical problem: For a given DE establish or disprove existence of first integrals (smooth functions constant on the integral curves).

Important: exact number of first integrals determines the dimension of the set filled by an integral curve.

Extensive research (theoretical and applications).

Poincaré series: identify (or introduce) a small parameter in equation, assume first integral analytic, and expand in power series.

Ziglin, extended and generalized by Ramis, Morales: expand eq. around special solutions. Study first approximation using differential Galois theory: if it does not have a meromorphic integral, then original DE does not have a meromorphic integral.

However: this is a restrictive assumption, generic integrals are not meromorphic.

Kruskal - The Poly-Painlevé Test

Q: When does a diff. eq. have single-valued first integrals?

Will illustrate the concepts on two equations which can be explicitly solved.

$Ex\ 1:$ $u'(x) = \frac{a}{x-b}$ has gen sol $u(x) = a\ln(x-b) + C$

A single-valued first integral $F(x,u) = \Phi(C)$ satisfies

$$\Phi(C) = \Phi(u - a\ln(x-b)) = \Phi(u - a\ln(x-b) - 2n\pi i a) = \Phi(C - 2n\pi i a) \quad \text{for all } n \in \mathbb{Z}$$

We found the monodromy: how constants change upon analytic continuation along closed loops.

Hence we may take $\Phi(C) = \exp(C/a)$, so $F(x,u) = \frac{\exp(u/a)}{x-b}$

$Ex\ 2:\quad u'(x) = \frac{a_1}{x-b_1} + \frac{a_2}{x-b_2} + \frac{a_3}{x-b_3}$ gen. sol.

$$u = a_1 \ln(x-b_1) + a_2 \ln(x-b_2) + a_3 \ln(x-b_3) + C$$

A single-valued first integral $F(x,u) = \Phi(C)$ should satisfy

$$\Phi(C) = \Phi(C - 2n_1\pi i a_1 - 2n_2\pi i a_2 - 2n_3\pi i a_3)\ , n_{1,2,3} \in \mathbf{Z}$$

so $\Phi(C) = \Phi(C+s)$ for all $s \in \mathcal{S} = \{2n_1\pi i a_1 + 2n_2\pi i a_2 + 2n_3\pi i a_3\, ;\, n_{1,2,3} \in \mathbf{Z}\}$

For generic (a_1, a_2, a_3) the set $\mathcal{S}$ is dense in $\mathbf{C}$. Hence $\Phi\ \equiv$const: no single-valued first integrals.

Obstructions to integrability are encoded in the monodromy — branching properties of solutions . Dense branching $\rightsquigarrow$ no single-valued first integrals.

How can one find branching properties?

Study equations in singular regions.

Results: non-integrability analysis, and criteria for

- systems with one regular singular point,
- with several reg sing points, also: obstructions to linearization, classification;
- for polynomial systems; new monodromy groups.

•Irregular singular point: general structure of movable singularities solutions develop.

Region with one regular singular point

$$\boxed{x\mathbf{u}' = M\mathbf{u} + \mathbf{h}(x, \mathbf{u})}$$

$\mathbf{u}, \mathbf{h} \in \mathbb{C}^n$, $x \in \mathbb{C}$, $M \in \mathcal{M}_{n,n}(\mathbb{C})$, $\mathbf{h}$ analytic on $D = \{(x, \mathbf{u}); |\mathbf{u}| < r', r'' < |x| < r'''\}$, having a zero of order 2 at $\mathbf{u} = 0$.

Theorem (RDC, *Nonlin.97*)

For generic matrices M a complete description of the number of single-valued independent integrals on D is given. Generically integrals exist on D, and are not meromorphic.

Illustrate approach on 1-d: $xu' - \mu u = h(x, u)$.

Set $u = \epsilon U$. Eq. becomes $xU' - \mu U = \epsilon U^2 \tilde{h}(x, \epsilon U)$

Solutions $U(x) = U_0(x) + \epsilon U_1(x) + \epsilon^2 U_2(x) + \ldots$

Then $xU_0' - \mu U_0 = 0$ so gen. sol. $U_0(x) = Cx^\mu$

It is relatively easy to see if this first approximation is integrable.

(i) $\mu \in \mathbf{C} \setminus \mathbf{R}$ is the generic case.

Then $F_0(x, U_0) = \mathcal{P}(\ln U_0 - \mu \ln x)$ is single-valued if $\mathcal{P}$ is doubly periodic, with periods $2\pi i$, $2\pi i\mu$.

Note: First integral is not meromorphic
(singularities accumulate at $U_0 = 0$). There is no meromorphic integral.

(ii) If $\mu \in \mathbf{Q}$ then $F_0(x, U_0) = U_0^q x^{-p}$ for $\mu = \frac{p}{q}$.

(iii) No single-valued integrals if $\mu \in \mathbf{R} \setminus \mathbf{Q}$:

$$\Phi(C) = F_0(x, Cx^\mu) = F_0(x, Cx^\mu e^{2n\pi i\mu}) = \Phi\left(Ce^{2n\pi i\mu}\right)$$

$\forall n \in \mathbf{Z}$. The set $\{e^{2n\pi i\mu}\ ;\ n \in \mathbf{Z}\}$ is dense in S^1; hence $F(x, u) \equiv$const. No single-valued integrals.

If $\mu \in \mathbf{R} \setminus \mathbf{Q}$ no single-valued first integrals.

Conjecture: It is natural to expect that the original, nonlinear equation $xu' - \mu u = h(x, u)$ is nonintegrable as well.

Q: How to prove?

$\mathbf{A_1}$: If one addresses the question of existence of regular integrals, they can be easily ruled out. Indeed, if there was one, then

$$F(x,u) = F(x,\epsilon U) = F(x,\epsilon U_0 + \epsilon^2 U_1 + ...)$$

$$= F_0(x) + \epsilon F_1(x) U_0 + ...$$

Then F_0=const and the first j with $F_j \neq 0$ would give a first integral $F_j(x)U_0^j$ for the reduced equation.

A similar argument works for F meromorphic.

However: in our case we can not expect meromorphic first integrals (we saw that first approximation had, generically, nonmeromorphic integrals).

$\mathbf{A_2}$: Show that the original, nonlinear equation is analytically equivalent to its linear part.

Normal Form of diff.eq. in an annulus around a regular singular point (*Nonlinearity*, 1997)

Theorem Let $\boxed{x\mathbf{u}' = M\mathbf{u} + \mathbf{h}(x, \mathbf{u})}$

$\mathbf{u}, \mathbf{h} \in \mathbb{C}^n$, $x \in \mathbb{C}$, $M \in \mathcal{M}_{n,n}(\mathbb{C})$, $\mathbf{h}$ holomorphic for $|\mathbf{u}| < r'$, $r'' < |x| < r'''$, having a zero of order 2 at $\mathbf{u} = 0$.

Assume eigenvalues $\mu_1, ..., \mu_n$ of M satisfy the Diophantine condition:

$$|\sum_{j=1}^{n} k_j\mu_j + l - \mu_s| > C(\sum_{j=1}^{n} k_j + |l|)^{-\nu} \text{for all } s \in \{1, ..., n\},\ l \in \mathbf{Z},\ \mathbf{k} \in \mathbf{N}^n,\ |\mathbf{k}| \geq 2$$

Then the system is biholomorphically equivalent to its linear part in $D' \subset D$.

The proof has some similarity with the case of periodic coefficients. Steps: 1) Find equivalence map as a formal power series. 2) Showing convergence. There is a small denominator problem, overcome by a generalization of Newton's method.

Note: The set of all Diophantine M has full measure.

Consequences:

For such matrices, study of nonlinear case reduces to the study of the linear equation (done in the paper).

Generically: equations do have first integrals in D, and they are not meromorphic.

Q: What about wider regions, containing several regular singular points?

Several regular singular points

$$\boxed{u' = \mathcal{A}u + f(x,u)} \quad \mathcal{A} = \tfrac{1}{x}\mu_1 + \tfrac{1}{x-1}\mu_2, \quad f = \frac{u^2 \tilde{f}(x,u)}{x(x-1)}\ ,\ x \in D \supset \{0;1\}\ ,\ |u| < r$$

Valid in more dimensions (at least for special classes of matrices $\mu_{1,2}$).

Theorem Obstructions to analytic linearization: (RDC, *preprint*)

Assume $\Re\mu_{1,2} > 0$. For any $f(x,u)$ there exists a unique $\phi(u)$, analytic at $u = 0$, so that eq. with $f - \phi$ is analytically equivalent to the linear part.

Note Obstructions to integrability of Hamiltonian systems: conjectured by Gallavotti, proved by Ehrenpreis.

Theorem Analytic classification (RDC, *preprint*)

Assume $\Re\mu_{1,2} > 0$. For any $f(x,u)$ there exists a unique $\psi(u)$ so that $u' = \mathcal{A}u + f(x,u)$ is analytically equivalent to $w' = \mathcal{A}w + \psi(w)$

$$\boxed{u' = \mathcal{A}u + f(x,u)} \quad , \quad with \quad \mathcal{A} = \tfrac{1}{x}\mu_1 + \tfrac{1}{x-1}\mu_2$$

Study the integrability properties:

If the system is not analytically linearizable then:

Theorem (RDC, Kruskal, *Nonlinearity*, 2003)

For generic μ_1, μ_2 (precise conditions given) solutions have dense branching:

no single-valued integrals exist.

Among integrable cases, first integrals are not meromorphic (generically).

For the proof we introduce a nonlinear monodromy group generated by germs of analytic functions $$G = < \gamma_0, \gamma_1 >$$

(γ_j is the monodromy map at the singular point $x = j$)

Note Nonlinear Galois groups were recently proposed by Malgrange. (Monodromy and Galois groups are classically defined for linear equations, and are sometimes connected.)

Need $\{\gamma(C); \gamma \in G\}$. It turns out that noncommutativity of G is source of generic density.

One surprising ingredient: uniform asymptotic behavior of iterations of analytic maps. Denote repeated composition by $\gamma^{\circ n} \equiv \gamma \circ \gamma \circ \ldots \circ \gamma$ (n times).

Lemma Let γ be a germ of analytic function $\gamma(z) = z + \omega q^{-1} z^{q+1} + O(z^{q+2})$, $\omega \neq 0$, $q \geq 1$. We have uniform convergence near $z = 0$:

$$n^{1/q}\gamma^{\circ n}\left(zn^{-1/q}\right) \to \frac{z}{(1-\omega z^q)^{1/q}}$$

Note Generically $q = 1$; limit group is a Möbius group.

Conclusions:

Obstructions to integrability (by single-valued functions, not necessarily meromorphic) are encoded in special limiting groups, generically Möbius.

When single-valued first integrals exist, they are not meromorphic (generically).

Polynomial systems

Ex. Henon-Heiles system (RDC, *Meth.Appl.An.)*

$$\boxed{H = \frac{1}{2}\left(p_1^2 + p_2^2\right) + aq_1^2 q_2 + \frac{b}{3}q_2^3}$$

Chaotic sol found numerically. Extensive research. Meromorphic integrals found for $a/b = 0, 1, \frac{1}{2}, \frac{1}{6}$; conjectured more integrable cases exist. Ramis found conditions for absence of meromorphic integrals.

Equations: $\ddot{q}_1 = -2aq_1q_2 \quad , \quad \ddot{q}_2 = -aq_1^2 - bq_2^2$

Eliminate t and use $H =$const. to reduce order. Look near special solutions. First approximation:

$$\boxed{(x^3 - 1)u'' + \tfrac{3}{2}x^2u' - \lambda xu = 0}$$

linear, second order, Fuchsian equation:

four singular points in $\mathbb{C} \cup \infty$, all regular.

Branching properties encoded in monodromy group.

Monodromy groups were previously known for Fuchsian equations with ≤ 3 singularities. For 3 singularities:

- order 2: hypergeometric equations (celebrated classical result, Riemann)

- order≥ 3: for hypergeometric function ${}_iF_{n-1}$

(recent: Beukers, Heckman, *Inv.Math.* 1989)

Not known for eq. with ≥ 4 singularities.

$$\boxed{(x^3-1)u'' + \frac{3}{2}x^2u' - \lambda xu = 0}$$

Monodromy group found (RDC, *Meth.Appl.An.*, 1997).

Consequences:

There exists one real-analytic invariant function, providing a non-meromorphic integral, real-valued (for generic parameters).

Differential Galois methods results show no meromorphic integral exists. (Theorem: for Fuchsian equations: differential Galois group=closure of monodromy group in Zariski topology.)

Generalization (limits of polyn. systems higher degree yield):

$$(x^n - 1)\frac{d^2U}{dx^2} + \frac{n}{2}x^{n-1}\frac{dU}{dx} + \mu x^{n-2}U = 0 \quad (n \geq 3)$$

Theorem **a)** Monodromy group is generated by

$X_j = M^{-j}AM^j$, $(j = 0, 1.., n-1)$, $A = \mathrm{diag}(1, -1)$

$$M = \begin{bmatrix} e^{il}(\cos l - i\cos p) & 1 \\ e^{2il}(\cos^2 l + \cos^2 p - 1) & e^{il}(\cos l + i\cos p) \end{bmatrix}$$

$l = \pi(1 - \frac{1}{n})$, $p = \frac{\pi}{n}\left[((n-2)^2 - \mu)^{1/2} + (n_+ - n_-)\right]$, $n_+ + n_- = n - 2$

b) If $\sqrt{\frac{n-2}{4} - \mu} \notin \mathbb{Q}$ the invariant function is $|c_1(x, u, u')|^2 + \tau_\mu |c_2(x, u, u')|^2$

where $\tau_\mu \in \mathbb{R}$, $c_{1,2} = \frac{\phi'_{1,2}}{W}u - \frac{\phi_{1,2}}{W}u'$,where $\phi_{1,2}$ fundamental system, $W = [\phi_1, \phi_2]$.

Conclusions and further directions

• Existence or, nonexistence, of first integrals for DE can be established in a general and rigorous setting.

• First integrals, when they exist, are generically not meromorphic.

• We will apply these new ideas in other systems of importance and look forward to some (pleasant) surprises.

Irregular singular points:

Structure of movable singularities

(R.D.C, O. Costin, *Invent. Math.* 2001)

Fundamental **Q**: Find position, distribution and type of singularities of solutions.

We show general results for nonlinear analytic differential systems (genericity) in a region of $\mathbb{C}$ near an irregular singular point of the system.

We show: the regular distribution of movable (spontaneous) singularities observed in many examples is in fact generic.

We introduce: practical methods to predict the position and type of the singularities.

Systems $\mathbf{y}' = \mathbf{f}(x^{-1}, \mathbf{y})$, regularity and genericity assumptions to make $x = \infty$

a rank 1 irregular singular point.

Consider the solutions with $\mathbf{y}(x) \to 0$ for $x \to \infty$ in a sector.

After a standard normalization of the system:

$$\mathbf{y}' = \left(\Lambda + \tfrac{1}{x}A\right)\mathbf{y} + \mathbf{g}(x) + \mathbf{f}(x^{-1}, \mathbf{y})$$

Theorem (i) Exists unique asymptotic expansion

$$\boxed{\mathbf{y}(x) \sim \sum_{m=0}^{\infty} x^{-m}\mathbf{F}_m(C\xi(x)) \quad (|x| \to \infty \text{ in } \mathcal{D})}$$

where $\boxed{\xi = e^{-\lambda_1 x}x^{\alpha_1}}$

Functions $\mathbf{F}_m$ are analytic at $\xi = 0$ and $\mathbf{F}_0'(0) = \mathbf{e}_1$.

(ii) Singularities of $\mathbf{y}(x)$: assume $\mathbf{F_0}$ has an isolated singularity at ξ_s. Then $\mathbf{y}(x)$ is singular at the points in the nearly periodic array

$$x_n = 2n\pi i + \alpha_1 \ln(2n\pi i) - \ln \xi_s + \ln C + o(1) \quad (n \in \mathbb{N})$$

Singularities x_n of y :: same type as singularities ξ_s of F_0.

Theorem If the differential equation satisfies some estimates (is weakly nonlinear) then generically the array of singularities are square root branch points.

Remark: $F_0(\xi)$ satisfies much simpler ODE, which can often be explicitly calculated.

The *proofs* rely on methods of exponential asymptotics; are delicate.

An example: Painlevé equation P_I $\quad \frac{d^2y}{dz^2} = 6y^2 + z$

Near $\arg z = 4\pi/5$: normalization, expansion yields

$$\xi^2 F_0'' + \xi F_0 = F_0 + \frac{1}{2}F_0^2 \quad , \text{ elliptic equation}$$

Since $F_0(\xi) = \xi + O(\xi^2)$, then F_0 is a degenerate elliptic function: $F_0(\xi) = \frac{\xi}{(\xi/12-1)^2}$

Then solutions of P_I have arrays of poles, whose positions can be found using the formula, then work back through normalization substitutions.

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