Differential systems with Fuchsian linear part: correction and linearization, normal forms and matrix valued orthogonal polynomials

Rodica D. Costin (The Ohio State University)

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Study a class of differential systems in $\mathbb{C}^d$ (first order, nonlinear), studied in domains containing two, or more singularities (*semi-local* study).

Motivation: the study integrability - *existence of single-valued first integrals*.

* Near a regular point all equations are equivalent $\rightsquigarrow$ integrable (locally).
* Non-integrability can be detected in regions which contain singularities.

Integrability has deep connections to other important mathematical objects, such as orthogonal polynomials. In some cases this is understood, via the Riemann-Hilbert or inverse scattering reformulation of integrable systems, in some others it is less well understood.

The study of higher dimensional systems turns out to connect to interesting, and new, higher dimensional, usually non-commutative generalizations of the classical orthogonal polynomials.

I will discuss theses problems separately, as well as their interconnections, as far as we understand them now.

Linear systems

A first order **linear** system $\quad \dfrac{d\mathbf{w}}{dx} = A(x)\mathbf{w} \qquad \mathbf{w} \in \mathbb{C}^d,\ A(x) \in \mathcal{M}_d(\mathbb{C})$

is **Fuchsian** if *all its singularities in* $\mathbb{C} \cup \{\infty\}$ *are regular*.

Singularity - point $x = x_0$ where $A(x)$ is not analytic.

If x_0 is an *isolated* singularity $\Rightarrow$ fundamental system $Y(x) = \Phi(x)(x - x_0)^P$ with $\Phi(x), P$ matrices, and $\Phi(x)$ has an *isolated* singularity at x_0.

If x_0 is at most pole of Φ, then $x_0 =$ regular singularity (Fuchsian point) $\Rightarrow$ solutions $=$ *convergent series* in powers of $x - x_0$ [& possibly $\ln(x - x_0)$].

For a Fuchsian system, with singularities at $p_0, p_1, \ldots, p_{S+1}, \infty$ (all Fuchsian)

$$\Longrightarrow \quad A(x) = \sum_{j=0}^{S+1} \frac{1}{x - p_j} A_j, \qquad \text{with} \quad A_j = \text{constant matices.}$$

Fuchsian systems appear in a wide range of problems of mathematics and physics, and have been the topic of extensive studies.

Nonlinear systems with Fuchsian linear part

$$\boxed{\frac{d\mathbf{u}}{dx} = A(x)\,\mathbf{u} + \frac{1}{Q(x)}\mathbf{f}(x,\mathbf{u})} \qquad \text{with } A(x) = \sum_{j=0}^{S+1} \frac{1}{x-p_j} A_j$$

∘ for $x \in D \ni \{p_0, p_1, \ldots, p_{S+1}\}$, $D \subset \mathbb{C}$ and $\mathbf{u} \in \mathbb{C}^d$, $|\mathbf{u}| < r$

∘ $\mathbf{f} = O(|\mathbf{u}|^2)$, $\mathbf{f}$ analytic for $x \in D$, $|\mathbf{u}| < r$

∘ the denominator $Q(x) = (x-p_0)(x-p_1)\ldots(x-p_{S+1})$ simply shows that the nonlinear term can have at most first order poles at p_j.

Q: which systems are analytically equivalent to their linear part for $x \in D$?

Q: More generally, classify!

Motivation

$\diamondsuit$ The question of linearization and, more generally, of classification, is *a fundamental problem* in the theory of differential equations.

$\diamondsuit$ Vector fields with an eigenvalue 1: $\dfrac{d\mathbf{u}}{dx} = A(x)\,\mathbf{u} + \dfrac{1}{Q(x)}\mathbf{f}(x,\mathbf{u}) \rightleftharpoons$

$$\begin{cases} \dot{x} = Q(x) & \text{... } polynomial\ in\ x,\ deg.\ S+1 \\ \dot{\mathbf{u}} = Q(x)A(x)\mathbf{u} + \mathbf{f}(x,\mathbf{u}) & \text{... } \mathbf{u} \times polynom.\ deg.\ S + O(\mathbf{u}^2) \end{cases}$$

studied in a domain containing the $S+1$ singular points $(p_j, \mathbf{0})$ where $Q(p_j) = 0$.

$\diamondsuit$ *Irregular* singularities can be obtained, and studied, as limits when two, or more, regular singular points tend to coincide: *coalescence*.

$\diamondsuit$ The study of integrability: reductions of Hamiltonian systems, with polynomial potentials, near doubly periodic solutions (RDC - '96, '97).

$\diamondsuit$ Classification of Schrödinger-type equations w.r.t. global behavior (work in progress).

$\diamondsuit$ The study reveals:

◦ inter-connections between analysis and algebra (as in the linear case).

In particular, it gives rise to:

-Matrix-valued generalizations of the Jacobi polynomials and of multiple-orthogonal polynomials;

-Generalization of the notion of orthogonality for Jacobi polynomials in the case of general weights.

◦ A close connection between three concepts: linearizability, integrability and multiple orthogonal polynomials.

Prior results - local study

$\diamondsuit$ In *dimension one, near one singular point* equations - Martinet and Ramis.

$\diamondsuit$ Near one singularity: vector fields have been also studied, have generated deep results, and are relatively well understood.

The present talk considers regions containing two or more singularities - semi-local study.

Other types of results concerning correction and linearization/integrability:
* Écalle and Vallet showed that resonant systems are linearizable after appropriate correction (1998);
* Gallavotti showed that there exists appropriate corrections of Hamiltonian systems so that the new system is integrable (1982), convergence proved by Eliasson (1988).

Systems near one singularity $\frac{d\mathbf{u}}{dx} = \frac{1}{x}L(x)\,\mathbf{u} + \frac{1}{x}\mathbf{f}(x,\mathbf{u})$, $L(x)$=analytic at 0.

If $\sigma(L(0))$ is nonresonant, then: analytic transf. $\rightsquigarrow L(x) \equiv L(0) = L$.

Theorem Analytic linearization near one singularity holds for generic systems: if $\sigma(L)$ is 'not too close' to resonance $\exists\ \mathbf{u} = \mathbf{h}(x,\mathbf{w})$ analytic for $|x| < \epsilon,\ |\mathbf{u}| < \epsilon_1$

$$\frac{d\mathbf{u}}{dx} = \frac{1}{x}L\,\mathbf{u} + \frac{1}{x}\mathbf{f}(x,\mathbf{u}) \iff \frac{d\mathbf{w}}{dx} = \frac{1}{x}L\,\mathbf{w}$$

Consequence The study of the local analytic properties of nonlinear systems reduces to the study of the linear ones. Eq. $x\mathbf{w}' = L\mathbf{w}$ is easy to study!

In particular $\rightsquigarrow$ local integrability: generically equations do have first integrals in D, and not meromorphic (accumulation of poles at $x = 0$ and/or $\mathbf{w} = \mathbf{0}$).

Q: What about wider regions, containing several regular singular points?

$$(*) \qquad \boxed{\mathbf{u}' = (\sum_{j=0}^{S+1} \frac{1}{x - p_j} A_j)\mathbf{u} + \frac{\mathbf{f}(x, \mathbf{u})}{\prod(x - p_j)}}, \qquad Q(x) = \prod(x - p_j)$$

Region: $x \in D \subset \mathbb{C}$ simply connected domain $D \ni p_0, \ldots, p_{S+1}$ & $\mathbf{u} \in \mathbb{C}^d$, $|\mathbf{u}| < r$

$\mathbf{f} = O(|\mathbf{w}|^2)$, holomorphic on $D \times \{|\mathbf{u}| < r\}$.

Is the system linearizable?

Q: What about wider regions, containing several regular singular points?

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Region: $x \in D \subset \mathbb{C}$ simply connected domain $D \ni p_0, \ldots, p_{S+1}$ & $\mathbf{u} \in \mathbb{C}^d$, $|\mathbf{u}| < r$

$\mathbf{f} = O(|\mathbf{w}|^2)$, holomorphic on $D \times \{|\mathbf{u}| < r\}$.

A: Systems are not necessarily linearizable - the analytic map which provide analytic linearization near one singularity is (usually) ramified at the other singularities.

This has important consequences!

Illustrate in 1-d, for two singularities ($S = 0$):

Theorem (RDC, M.D. Kruskal, *Nonlin.*'03)

If the nonlinear eq. $\frac{du}{dx} = \left(\frac{a_0}{x - p_0} + \frac{a_1}{x - p_1}\right) u + \frac{f(x, u)}{(x - p_0)(x - p_1)}$
is *not* analytically linearizable
then no single-valued integrals exists (it is *not* integrable) for generic a_0, a_1.

Among integrable cases, first integrals are not meromorphic (generically).

Q: Which equations are linearizable, and which are not**?**

Q: Classify: find the equivalence classes w.r.t. analytic equivalence.

Theorem **Correction and linearization**

Let $\frac{d\mathbf{u}}{dx} = A(x)\,\mathbf{u} + \frac{\mathbf{f}(x,\mathbf{u})}{Q(x)}$ $\quad (*)$ with $A(x) = \sum_{j=0}^{S+1} \frac{1}{x-p_j} A_j,\ Q(x) = \prod (x-p_j)$

assuming $A_0, \ldots, A_{S+1}, A_\infty = \sum A_j$ nonresonant
$(k + \boldsymbol{\lambda}\cdot\mathbf{m} - \lambda_i \neq 0,$ for all $k \in \mathbb{N},\ \mathbf{m} \in \mathbb{N}^d,\ |\mathbf{m}| \geq 2,\ i = 1, \ldots, d)$

Then $\exists$ **unique** *correction* $\boldsymbol{\phi}(x,\mathbf{u}) = \sum_{\mathbf{m}\in\mathbb{N}^d,\ |\mathbf{m}|\geq 2} \boldsymbol{\phi}_{\mathbf{m}}(x)\mathbf{u}^{\mathbf{m}}$ (formal series)

where $\boldsymbol{\phi}_{\mathbf{m}}(x)$ are **polynomials in x of deg.** $\leq S$, such that the corrected system

$\frac{d\mathbf{u}}{dx} = A(x)\,\mathbf{u} + \frac{\mathbf{f}(x,\mathbf{u}) - \boldsymbol{\phi}(x,\mathbf{u})}{Q(x)}$ is (formally) linearizable.

Note. Equation (*) is linearizable iff $\boldsymbol{\phi}(x,\mathbf{u}) \equiv 0$, so
the unique correction $\boldsymbol{\phi}$ is the obstruction to linearizability.

Convergence of the correction $\boldsymbol{\phi}(x, \mathbf{u})$?

Theorem (RDC, Nonlin, 2008)
Convergence holds in the commutative case, for two singularities, eigenvalues with positive real parts.

Proof:
steepest descent $\rightsquigarrow$ small denominators $\rightsquigarrow$ improvement of a rapidly convergent algorithm.

* * *

If a system is not formally linearizable, then it is not analytically linearizable either.

Since equations are not necessarily linearizable, then they are not all equivalent either. Classification of these equations by specifying formal normal forms:

Theorem **Normal form**

(Assume non-resonance.) For any $\mathbf{f}(x,\mathbf{w})$ analytic on $D \times \{|\mathbf{w}| < r\}$

there exists a unique formal series $\boldsymbol{p}(x,\mathbf{w}) = \sum_{\mathbf{m}\in\mathbb{N}^d,\ |\mathbf{m}|\geq 2} \boldsymbol{p}_{\mathbf{m}}(x)\mathbf{w}^{\mathbf{m}}$

where $\boldsymbol{p}_{\mathbf{m}}(x)$ are **polynomials in x of degree at most S**, such that

$$\frac{d\mathbf{u}}{dx} = A(x)\,\mathbf{u} + \frac{\mathbf{f}(x,\mathbf{u})}{Q(x)} \Longleftrightarrow$$

$$\frac{d\mathbf{w}}{dx} = A(x)\,\mathbf{w} + \frac{\boldsymbol{p}(x,\mathbf{w})}{Q(x)}$$

through $\mathbf{u} = \mathbf{h}(x,\mathbf{w}) = \mathbf{w} + \sum \mathbf{h}_{\mathbf{m}}(x)\mathbf{w}^{\mathbf{m}}$ with $\mathbf{h}_{\mathbf{m}}(x)$ analytic on D.

Normal forms:

Near a regular point: $\dfrac{d\mathbf{u}}{dx} = M(x)\mathbf{u} + \mathbf{f}(x,\mathbf{u}) \quad \Leftrightarrow \quad \dfrac{d\mathbf{w}}{dx} = 0$
(keep no terms)

Near one reg. sing. point: $x\dfrac{d\mathbf{u}}{dx} = L(x)\mathbf{u} + \mathbf{f}(x,\mathbf{u}) \;\Leftrightarrow\; x\dfrac{d\mathbf{w}}{dx} = L(0)\mathbf{w}$ (generic)
(keep the linear part)

Near two reg. sing. point: $x(x-p_1)\dfrac{d\mathbf{u}}{dx} = (L_0 + xL_1)\mathbf{u} + \mathbf{f}(x,\mathbf{u}) \;\Longleftrightarrow$

$$x(x-p_1)\frac{d\mathbf{w}}{dx} = (L_0 + xL_1)\mathbf{w} + \boldsymbol{\psi}_0(\mathbf{w}) \text{ (generic)}$$

(keep some nonlinear terms)

Near three reg. sing. point: $x(x-p_1)(x-p_2)\dfrac{d\mathbf{u}}{dx} = (L_0 + xL_1 + x^2L_2)\mathbf{u} + \mathbf{f}(x,\mathbf{u}) \;\Leftrightarrow$

$$x(x-p_1)\frac{d\mathbf{w}}{dx} = (L_0 + xL_1 + x^2L_2)\mathbf{w} + \boldsymbol{\psi}_0(\mathbf{w}) + x\boldsymbol{\psi}_1(\mathbf{w}) \text{ (generic)}$$

(keep more nonlinear terms) Etc.

Proofs. A change of variables $\mathbf{u} = \mathbf{h}(x, \mathbf{w})$ provides a linearization iff

$$(**) \qquad \partial_x \mathbf{h} + d_{\mathbf{w}}\mathbf{h}\, A\mathbf{w} = A\mathbf{h} + \frac{1}{Q(x)}\left[\mathbf{f}(x, \mathbf{w}+\mathbf{h}) - \boldsymbol{\phi}(x, \mathbf{w}+\mathbf{h})\right]$$

Power series in $\mathbf{w}$: denote by $\mathbf{h}_n$ the homogeneous part degree n of $\mathbf{h}(x, \mathbf{w})$:

$$\mathbf{h}_n(x, \mathbf{w}) = \sum_{|\mathbf{m}|=n} \mathbf{h}_{\mathbf{m}}(x)\mathbf{w}^{\mathbf{m}}, \quad (n \geq 2), \quad \text{similarly } \mathbf{f}_n,\ \boldsymbol{\phi}_n$$

(**) splits into blocks of systems of ordinary differential equations for $\{\mathbf{h}_{\mathbf{m}}\}_{|\mathbf{m}|=n}$:

$$\partial_x \mathbf{h}_n + \mathrm{d}_{\mathbf{w}}\mathbf{h}_n\, A\mathbf{w} - A\mathbf{h}_n = \frac{1}{Q(x)}\mathbf{R}_n(x, \mathbf{w}), \qquad n \geq 2$$

where $\mathbf{R}_n = \mathbf{f}_n - \boldsymbol{\phi}_n + \tilde{\mathbf{R}}_n$ with $\tilde{\mathbf{R}}_n$ a polynomial in $\boldsymbol{\phi}_{\mathbf{m}}$, $\mathbf{h}_{\mathbf{m}}$, $\mathbf{f}_{\mathbf{m}}$ with $|\mathbf{m}| < n$, and $\tilde{\mathbf{R}}_2 = 0$. Each $\mathbf{h}_n$ and $\boldsymbol{\phi}_n$ are to be determined from inductively on n.
The system is complicated due to non-commutativity (unlike near 1 sing. or 1d)...

$$\partial_x \mathbf{h}_n + \mathrm{d}_{\mathbf{w}} \mathbf{h}_n \, A\mathbf{w} - A\mathbf{h}_n = \frac{1}{Q(x)} \mathbf{R}_n(x, \mathbf{w}), \qquad n \geq 2$$

Remarkably, the system for $\mathbf{h}_n$ is a Fuchsian non-homogeneous system! (If properly organized...)

Denote $\mathcal{P}_n$ the space of $\mathbb{C}^d$-valued polynomials in $\mathbf{w} \in \mathbb{C}^d$, homog. degree n:

$$\mathcal{P}_n = \left\{ \mathbf{q} \; ; \; \mathbf{q}(\mathbf{w}) = \sum_{\mathbf{m} \in \mathbb{N}^d, |\mathbf{m}|=n} \mathbf{q}_{\mathbf{m}} \mathbf{w}^{\mathbf{m}}, \; \mathbf{q}_{\mathbf{m}} \in \mathbb{C}^d \right\},$$

canonical basis $\mathbf{r}_{\mathbf{m},i} = \mathbf{w}^{\mathbf{m}} \mathrm{e}_i, \quad |\mathbf{m}| = n, \; i = 1, \ldots, d$

Denote $N =$dim $\mathcal{P}_n = d(n+d-1)!/n!/(d-1)!$.

Denote by $B(x)$ the linear operator on $\mathcal{P}_n$: $B(x)\mathbf{h}_n = \mathrm{d}_{\mathbf{w}} \mathbf{h}_n \, A\mathbf{w} - A\mathbf{h}_n$

Note that $B(x)$=a Fuchsian matrix (in the canonical basis) since

$$B(x) = \sum_{j=0}^{S+1} \frac{1}{x - p_j} B_j \qquad \text{where} \quad B_j \mathbf{q} = \mathrm{d}_{\mathbf{w}} \mathbf{q} \, A_j \mathbf{w} - A_j \mathbf{q}.$$

The recursive system $\partial_x \mathbf{h}_n + \mathrm{d}_{\mathbf{w}}\mathbf{h}_n\, A\mathbf{w} - A\mathbf{h}_n = \dfrac{1}{Q(x)}\mathbf{R}_n(x,\mathbf{w}), \qquad n \geq 2$

has the structure (*) $\dfrac{d}{dx}\mathbf{h}_n + B(x)\mathbf{h}_n = \dfrac{1}{Q(x)}\mathbf{R}_n(x,\mathbf{w})$, where

$$B(x) = \sum_{j=0}^{S+1} \frac{1}{x - p_j} B_j \qquad \text{where} \quad B_j\mathbf{q} = \mathrm{d}_{\mathbf{w}}\mathbf{q}\, A_j\mathbf{w} - A_j\mathbf{q}$$

therefore (*) is a Fuchsian non-homogeneous system.

The Theorems now follow using recursively the following results concerning non-homogeneous Fuchsian equation:

Fundamental Lemma

Consider $\mathbf{y}'(x) + B(x)\,\mathbf{y}(x) = \dfrac{\mathbf{g}(x)}{Q(x)}$

a Fuchsian equation with a non-homogeneous term ($\mathbf{y} \in \mathbb{C}^N$) where $B(x) = \sum_{j=0}^{S+1} \dfrac{1}{x - p_j} B_j$, and $Q(x) = \prod_{j=0}^{S+1}(x - p_j)$. Let $D \ni p_0, \ldots p_{S+1}$.

Non-resonance: $k + B_j$ are invertible for all j and $j = \infty$ (where $B_\infty = \sum B_j$).

Then for any function $\mathbf{g}(x)$ analytic on D there exists a unique $\boldsymbol{\phi}(x) \in \mathbb{C}^N[x]$, deg$\boldsymbol{\phi} \leq S$ so that the corrected equation

$$\mathbf{y}'(x) + B(x)\,\mathbf{y}(x) = \frac{\mathbf{g}(x) - \boldsymbol{\phi}(x)}{Q(x)}$$

has a solution $\mathbf{y}(x)$ analytic on D.

The proof consists of several steps:

(1) Show the uniqueness of the correction.

(2) Prove the Lemma when $\mathbf{g}(x)$ is a polynomial.
Solutions are found as expansions in terms of matrix-valued generalizations of orthogonal, or multiple orthogonal, polynomials which are introduced here.

(3) Proof of the Lemma for the case when the eigenvalues of all matrices B_j have positive real parts. An analytic approach is used (int. factor, integrals).

(4) Show that the general case (when integrals diverge) reduces to (3), thus completing the proof of the Lemma.
The algebraic results of (2) are used to construct this reduction.

(Bonus) A generalized integral is defined, so that the general case (4) can be proved using analytic methods just like in (3).

Matrix-valued polynomials generalizing the classical orthogonal polynomials

Given an interval J and a weight function $w(x)$, a **sequence of orthogonal polynomials** $p_0, p_1, \ldots, p_n, \ldots$ is a sequence such that $\deg p_n = n$ and $< p_k, p_n > := \int_J p_k \, p_n \, w = 0$ whenever $k \neq l$ (are $\perp$ w.r.t. a bilinear form).

Classical orthogonal polynomials:

- **Legendre** for $J = [-1, 1]$ and $w(x) = 1$;
- **Jacobi** (and sub-classes) for $J = [-1, 1]$ and $w(x) = (1-x)^\alpha (1+x)^\beta$;
- Associated **Laguerre** for $J = [0, \infty)$ and $w(x) = x^\alpha \mathrm{e}^{-x}$;
- **Hermite** for $J = \mathbb{R}$ and $w(x) = \mathrm{e}^{-x^2}$.

They were introduced in connection to continued fractions, and now have an impressive number of applications in mathematics and mathematical physics (approximation theory, differential systems, integrable systems, random matrices...)

They all have an impressive number of beautiful common properties: two-step recurrence relations, are eigenfunctions to Sturm-Liouville operators (...), they satisfy a Rodrigues' formula: $p_n(x) = w(x)^{-1}\frac{d^n}{dx^n}\left[Q(x)^n w(x)\right]$ (with Q=pol.)

Generalizations of orthogonal polynomials to matrix-valued ones - early work of Krein (on Jacobi matrices), more recently - Aptekarev, Nikishin, very active in recent years - B. Simon, A. Grünbaum [...]

The matrix-valued generalizations which appear in the study of Fuchsian systems are polynomials which satisfy a Rodrigues' formula - because

$$\mathbf{y}' + B(x)\mathbf{y} = \mathbf{g}/Q \iff \frac{d}{dx}\left[W(x)\mathbf{y}\right] = \tilde{W}(x)\mathbf{g}, \text{ with } \tilde{W} = WQ^{-1},\ W' = WB$$

Plan: we define them using a Rodrigues' formula, then show that they satisfy many of the properties usually associated to the classical orthogonal polynomials. (R.D.C. JAT '09, JAT '09)

V=complex vector space; $\mathcal{M} = \mathcal{L}(V, V)$, $\mathcal{M}[x]$=$\mathcal{M}$-valued polynomials.
Let $Q(x)$=polynomial (deg≤ 2): $Q(x) = \sigma x^2 + \tau x + \delta$.
Let $L_1, L_2 \in \mathcal{M}$ with L_1 (nonresonant): $L_1 + k\sigma$ is invertible $\forall k = 1, 2, \ldots$
$W(x)$ is $\mathcal{M}$-valued, s.t. $Q(x)\,W'(x) \;=W(x)\,(x\,L_1 \;+\; L_2)$ (the Pearson eq.)
(a Fuchsian system if deg $Q = 2$, or higher)

Definition Let $P_n(x)$ be the $\mathcal{M}$-valued function defined by the Rodrigues formula
$P_n(x) \;=\; W(x)^{-1}\,\frac{d^n}{dx^n}\,[\,Q(x)^n\,W(x)\,]$.

Notes:

If dim$V = 1$ then $P_n(x)$ are the classical orthogonal polynomials.
If $L_1 \wedge L_2$ are diagonal, then $P_n(x)$ is diagonal (entries classical orth. pol.).
If $L_1 \vee L_2$ are not diagonal, it is not clear that P_n are polynomials...
A direct calculation gives:
$P_0 = I,\; P_1(x) = (2\sigma + L_1)x + \tau + L_2,\; \ldots$

Non-commutativity made earlier studies very hard. A new representation:

Proposition Denote by $\mathcal{A}_k$ the following linear operators on $\mathcal{M}(x)$:

$\mathcal{A}_k = k\,Q'(x) + x\,L_1 + L_2 + Q(x)\partial_x$ Then $P_n = \mathcal{A}_1\mathcal{A}_2\ldots\mathcal{A}_n\,I$. (Not ladder!)

- Therefore P_n are polynomials.
- The leading coeff. is $C_n = (L_1 + 2n\sigma)\ [L_1 + (2n-1)\sigma]\ \ldots\ [L_1 + (n+1)\sigma]$

which is invertible, therefore degP_n=n.

- Expan.: $\forall p \in \mathcal{M}[x]$, deg $p = n \quad \exists! q_0, \ldots, q_n \in \mathcal{M}$ with $p(x) = \sum_{k=0}^{n} P_k(x) q_k$

The following (orthogonality) relations hold:

$$\int_J P_j(x)^*\,W(x)\,P_k(x)\,dx = 0 \text{ for } j < k \text{ and}$$

$$\int_J P_j(x)^*\,W(x)^*\,P_k(x)\,dx = 0 \text{ for } j > k$$

Moreover, other properties characteristic to orthogonal polynomials hold:

Proposition Two-step recurrence relation:

$$x\,P_n(x) = P_{n+1}(x)\alpha_n + P_n(x)\,\beta_n + P_{n-1}(x)\gamma_n$$

Proposition If L_1 and L_2 commute then P_n are eigenfunctions for the operator $\mathcal{A}_1\partial_x$ and $\mathcal{A}_1\partial_x P_n = n\,[(n+1)\sigma + L_1]\,P_n$

Proofs rely on commutation relations of $\mathcal{A}_k = k\,Q'(x) + x\,L_1 + L_2 + Q(x)\partial_x$ - which are key to the beautiful properties of the classical orthogonal polynomials (and make non-commutative cases manageable):

Proposition For $r \in \mathcal{M}(x)$ the operators $\mathcal{A}_k$ satisfy the identities

(a) $\mathcal{A}_k(x\,r) = x\,\mathcal{A}_k\,r + Q\,r$
(b) $Q\,\mathcal{A}_k\,r = \mathcal{A}_{k-1}\,(Q\,r)$
(c) $\partial_x\mathcal{A}_k\,r = \mathcal{A}_{k+1}\partial_x r + (2\sigma k + L_1)r$

We have $P_n = \mathcal{A}_1\mathcal{A}_2 \ldots \mathcal{A}_n\, I$ with $\mathcal{A}_k \;=\; k\, Q'(x) \;+\; x\, L_1 \;+\; L_2 \;+\; Q(x)\partial_x$.
(a) $\mathcal{A}_k(x\, r) \;=\; x\, \mathcal{A}_k\, r \;+\; Q\, r$
(b) $Q\, \mathcal{A}_k\, r \;=\; \mathcal{A}_{k-1}\, (Q\, r)$
(c) $\partial_x \mathcal{A}_k\, r \;=\; \mathcal{A}_{k+1}\partial_x r \;+\; (2\sigma k + L_1) r$

Example: show that in the commutative case (this includes the classical polynomials!) P_n are eigenfunctions for the operator $\mathcal{A}_1\partial_x$ and find the eigenvalues.

An iterative calculation $\mathcal{A}_1\partial_x P_n \;=\; \mathcal{A}_1\partial_x\mathcal{A}_1\mathcal{A}_2 \ldots \mathcal{A}_n I \;=\; (2\sigma + L_1)P_n + \mathcal{A}_1\mathcal{A}_2\partial_x\mathcal{A}_3 \ldots \mathcal{A}_n I = \ldots = \left[(2\sigma + L_1) + \ldots + (2n\sigma + L_1)\right] P_n$. The End

A non-commutative example: find the two-step recurrence relation
$x\, P_n(x) \;= P_{n+1}(x)\alpha_n + P_n(x)\, \beta_n + P_{n-1}(x)\gamma_n$

Note: $\mathcal{A}_1\mathcal{A}_2 \ldots \mathcal{A}_n(x\, q) = \mathcal{A}_1\mathcal{A}_2 \ldots \mathcal{A}_{n-1}\, (x\, \mathcal{A}_n q) \;+\; \mathcal{A}_1\mathcal{A}_2 \ldots \mathcal{A}_{n-1}\, (Q\, q) = \ldots \;= x\, \mathcal{A}_1\mathcal{A}_2 \ldots \mathcal{A}_n q \;+\; n\, \mathcal{A}_1\mathcal{A}_2 \ldots \mathcal{A}_{n-1}\, (Q\, q)$

We found $\mathcal{A}_1\mathcal{A}_2\ldots\mathcal{A}_n(x\,q) = x\,\mathcal{A}_1\mathcal{A}_2\ldots\mathcal{A}_n q \,+\, n\,\mathcal{A}_1\mathcal{A}_2\ldots\mathcal{A}_{n-1}\,(Q\,q)$.

So $\mathcal{A}_n x - nQ = \mathcal{A}_n\mathcal{A}_{n+1}\alpha_n + \mathcal{A}_n\beta_n + \gamma_n$ which expanded yields an identity of quadratic polynomials in x, and by identifying the coefficients we obtain three equations for $\alpha_n, \beta_n, \gamma_n$ and solve them. Q.E.D!

Matrix-valued Jacobi-Angelesco polynomials

$$\mathbf{y}'(x) + B(x)\,\mathbf{y}(x) = \frac{\mathbf{g}(x)}{Q(x)} \text{ with } B(x) = \sum_{j=0}^{S+1} \frac{1}{x-p_j} B_j,\ Q(x) = \prod_{j=0}^{S+1}(x-p_j).$$

For two singularities $S=0$ ($\longrightarrow$deg $Q=2$) $\rightsquigarrow$ the above matrix-valued polynomials.

For more singularities ($\longrightarrow$ higher deg Q) the Rodrigues's formula only yields polynomials of degree multiple of $S+1$ and these polynomials do not form a basis.

A basis of matrix-valued polynomials is $P_n(x) = W(x)^{-1} \dfrac{d^m}{dx^m} \left[x^i\, Q(x)^m\, W(x) \right]$

where $n = (S+1)m + i$ with $m = \lfloor n/(S+1) \rfloor$, and $i = 0, 1, \ldots, S$.

In the scalar case and for $S=1$ these are the Jacobi-Angelesco polynomials.

Orthogonality of Jacobi and Laguerre polynomials for general weights

Jacobi polynomials $P_n^{(\alpha,\beta)}$: on $J=[-1,1]$ w.r.t. $w(x)=(1-x)^\alpha(1+x)^\beta$:

$$< f,g >= \int_{-1}^{1} f(x)\,g(x)\,(1-x)^\alpha(1+x)^\beta\,dx$$

Laguerre polynomials $L_n^{(\alpha)}$: on $J=[0,\infty)$ w.r.t. $w(x)=x^\alpha \mathrm{e}^{-x}$:

$$< f,g >= \int_{0}^{\infty} f(x)\,g(x)\,x^\alpha \mathrm{e}^{-x}\,dx$$

The integrals are defined only for $\Re\alpha,\beta>-1$, but:

1) other properties (e.g. 2-step recurrence) hold for most $\alpha,\beta\in\mathbb{C}$ and

2) it is known that they are orthogonal w.r.t. a bilinear form (Favard's Theorem).

Question: find this bilinear form!

The natural approach: by analytic continuation in the parameters of the Borel measure of the classical case.

◦ Carlson - uses an integral kernel - proves Jacobi series expansions for functions analytic on ellipses.

◦ Recently Kuijlaars, Martínez-Finkelshtein and Orive establish orthogonality - in some cases - by integration on special paths in the complex plane.

Another approach (RDC, J.Approx.Theory, '09): using **the Hadamard finite part** of the (divergent) integrals.

Advantage Since these can be manipulated much like integrals, the classical formulas which are analytic in the parameters are formally similar. (They do not behave well with respect to inequalities.)

Definition Let $\alpha \in \mathbb{C} \setminus (-\mathbb{N})$. For $f(x)$ analytic at $x = 0$ and x small define

$$\int_0^x t^{\alpha-1} f(t)\, dt = \int_0^x t^{\alpha-1} \sum_{n=0}^{\infty} f_n t^n \, dt := \sum_{n=0}^{\infty} \frac{f_n}{n+\alpha} x^n$$

Example The general solution of $x\, y' + \alpha y = f(x)$

is $y = Cx^{-\alpha} + x^{-\alpha} \int_0^x t^{\alpha-1} f(t)\, dt$

Theorem The operator $f \mapsto x^{-\alpha} \int_0^x t^{\alpha-1} f(t)\, dt$ is compact and analytic in α (between suitable Banach spaces of analytic functions).

Once analyticiy in parameters of the Hadamard finite part is established the usual properties of orthogonal polynomials should follow by analytic continuation.