Integrability Properties of a Generalized Lamé Equation; Applications to the Hénon-Heiles System

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Abstract

In the first part of the paper we investigate the integrability properties of a second order linear ODE, which is a reduction of the Hénon-Heiles systems and a generalization of the Lamé equation. The equation has four regular singular points in the extended complex plane. The investigation of the poly-Painlevé property (the Kruskal test) is done by calculating explicitly the monodromy group of the equation. The result is that there is only one single-valued and necessarily non-analytic first integral, which is explicitly found using the branching properties of the solutions.

In the second part we prove non-integrability of the Hénon-Heiles systems in a comprehensive setting. The method used is the Kruskal test and is general.

Short title: Integrability properties of some ODEs

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1 Introduction

Relatively recently, Martin Kruskal proposed a new test for integrability [1],[4]. It is a generalization of the Painlevé test, in the sense that movable branch points for the solutions are allowed, as long as the branching of the solutions is not "too rich".

More precisely, consider a first order ODE in the complex plane

$$\frac{dx}{dt} = f(x, t)$$

where f is analytic on a domain $D_x \times D_t \subset \mathbf{C}^2$. The solutions $\mathbf{x}(t)$ might be multi-valued, and yet there might exist an analytic first integral in $D_x \times D_t$: a non-constant function $F(x, t)$ such that $F(x(t), t)$ is constant for all the branches x of $\mathbf{x}$ in D_t.

Let us fix some $t \in D_t$ and consider the set

$$S_t = \{x(t) \; ; \; x \in \mathbf{x}\}$$

If, on the other hand, the set S_t is dense in some open set $\tilde{D} \subset D_x$ (in which case we say that the solution has bad branching) then any continuous function F which is constant on the trajectories must be constant on $\tilde{D}$, and the equation has no first integrals on $\tilde{D} \times D_t$.

This idea is illustrated in [4], [6] on Abel's equation $\frac{dx}{dt} = x^3 + t$, which is shown not to be integrable. In order to study the multi-valuedness of the solutions, the authors study their behavior near the singular point $(t = \infty)$, where the branch points may accumulate, giving rise to bad multi-valuedness. The study is performed using a generalization of the Painlevé α-method. A small parameter is introduced in the equation and the series solutions are calculated using a perturbative approach. The terms of the series involve repeated integration on a Riemann surface; by studying the values of these integrals on different paths around the branch points it is concluded that the solutions are badly branched. Thus, the equation has no first integrals.

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The object of the present research is the investigation of the way this integrability test can be carried out for equations of order larger than 1.

The first part is devoted to the study of the second order linear family of ODEs

$$(x^3 - 1)\frac{d^2u}{dx^2} + \frac{3}{2}x^2\frac{du}{dx} + \mu x u = 0 \tag{1}$$

(μ is a real parameter).

This is a generalization of the Lamé equation, which, in its algebraic form, is [2]

$$(x^3 - 1)\frac{d^2u}{dx^2} + \frac{3}{2}x^2\frac{du}{dx} + \left(h + \frac{n(n+1)}{4}x\right)u = 0 \qquad (2)$$

The equation is known to have a uniform solution if n is a positive integer (h is a constant, taken to be 0 in (1)).

The equation (1) is also a reduction of the Hénon-Heiles system (which is studied in the second section of the paper).

The equation being linear, all the branch points of the solutions are fixed, so a perturbative method does not seem to be productive. Instead, the multi-valuedness of the solutions is described by the monodromy group and the analysis suggests that the essential obstruction to the existence of integrals is encoded in this group.

In the case of linear equations, while winding around the branch point singularities of the equation, the solutions transform into each other. If the orbit of some solution covers densely an open set in the space of solutions, this prevents a single-valued continuous (non-locally-constant) function from being constant along the trajectories. In the contrary case, the monodromy group generates level surfaces of functions which turn out to be integrals of the motion.

In the case of the equation (1), the monodromy group $\mathcal{G}$ is found almost explicitly and we show that the equation has no global first integrals. (There are, of course, local integrals, since the equation is linear; here "global" means that the first integral should be defined on a domain whose projection on the t-coordinate contains closed curves surrounding all the singular points.)

More generally, the information about the branching of the solutions in the complex plane is relevant for investigating the existence of real-analytic first integrals and we use this information to actually calculate the integrals.

The method we use is the study of the action of $\mathcal{G}$ on complex vectors. In abstract terms, we do this by calculating the Lie algebra of (the closure) of the group and by looking at its action on $\mathcal{G}$-invariant functions.

The method illustrated on the equation (1) is based on the following features of the example treated:

-we are able to find the monodromy group (because all the singular points are regular and because the equation has a discrete symmetry);

-the monodromy group $\mathcal{G}$ acts (on vectors in $\mathbf{C}^2$) "almost" densely: for generic vectors c the set $\{cG \; ; \; G \in \mathcal{G}\}$ is dense on a manifold of real dimension 3 in $\mathbf{C}^2$, which implies that there are no holomorphic integrals, but there is a real-analytic one.

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The second section is devoted to the study of the Hénon-Heiles system, which is defined by the Hamiltonian

$$H = \frac{1}{2}\left(\dot{q}_1^2 + \dot{q}_2^2 + aq_1^2 + bq_2^2\right) + dq_1^2 q_2 - \frac{1}{3}eq_2^2 \qquad (3)$$

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Different aspects and results on the non-integrability of this system can be found in many papers (see, for example [3],[5],[6],[7]).

The investigation of the Painlevé property shows [5] the existence of solutions with behavior

$$q_1(t) \sim A(t - t_0)^{r_1} \quad , \ t \longrightarrow t_0 \quad \text{where } r_1 = \frac{1}{2} \pm \frac{1}{2}\sqrt{1 - 48d/e} \qquad (4)$$

(A is arbitrary) as well as resonances

$$q_1(t) \sim A_0(t - t_0)^{-2} + ... + C(t - t_0)^{r_2} \quad , \ t \longrightarrow t_0 \quad \text{where } r_2 = \frac{1}{2} \pm \frac{1}{2}\sqrt{1 - 24(1 + e/d)} \qquad (5)$$

(A_0 has a definite value, C is arbitrary).

Therefore, the system has the Painlevé property only if the numbers r_1, r_2 are integers, which is the case for values of d/e equal to -1, $-1/2$, $-1/6$. Integrability has also been proved in the case $d/e = -1/16$, $b = 16a$ [5], for which the numbers r_1, r_2 are rational.

Ziglin [7] applies his result on the existence of meromorphic integrals in Hamiltonian systems to the Hénon-Heiles system in the case when $a = b = 1, d = -1$; he shows that the system has no meromorphic first integral on the surface of constant energy, for sufficiently low values of the energy.

Starting from Ziglin's theorem, Ito [3] proves a criterion for analytic non-integrability of Hamiltonian systems and applies it to the Hénon-Heiles case, for $a = b \neq 0$ (the inhomogeneity of the potential being important in his treatment) and concludes that the system has a second analytic first integral in $\mathbf{C}^4$ only if $d/e = 0, -1/6, -1/2, -1$.

In the case $a = b = 0$ we prove that if the numbers r_1, r_2 are irrational, then there are no first integrals (independent of the Hamiltonian) under fairly mild conditions on the regularity of the first integrals under consideration. The method used is the Kruskal test and the approach does not use the Hamiltonian structure; it is therefore suitable for the study of more general systems of ODEs. The approach is rigorous and it allows for first integrals more singular than the meromorphic functions.

The actual calculation seems to rely on similar characteristics of the system as the ones used in the proof of Ziglin.

2   The Kruskal Integrability Test on a Second Order ODE

The equation (1) being linear, we can immediately write down two local constants of the motion, as follows.

The points $x = 1$, $x = \omega$, $x = \omega^2$ $(\omega^3 = 1)$ and $x = \infty$ are regular singular points for the equation. In a neighborhood of $x = 1$ there are two independent solutions

$$\phi(x), \ \gamma(x) = \sqrt{x-1}\psi(x) \tag{6}$$

with ϕ, ψ analytic. To fix them we choose $\phi(1) = 1, \psi(1) = 1$ and the usual branch of the radical.

All the solutions in the complex plane can be written as $u = c_1\phi + c_2\psi$ for some constants c_1, c_2. Two independent local integrals for the equation can be immediately obtained:

$$c_1 = \frac{u\gamma'(x) - u'\gamma(x)}{\phi(x)\gamma'(x) - \phi'(x)\gamma(x)} \tag{7}$$

$$c_2 = \frac{u'\phi(x) - u\phi'(x)}{\phi(x)\gamma'(x) - \phi'(x)\gamma(x)}$$

The fundamental solutions ϕ, γ are multi-valued in the complex plane, having branch points at the three singular points of the equation (and so are the local integrals (7)).

In what follows we will use the expression "the domain $D \subset \mathbf{C}$ permits winding around the points x_1, x_2, x_3" in the sense that there is a point $\hat{x} \in D$ different from x_j and there are three closed paths γ_j in D, starting and ending at $\hat{x}$, such that γ_j surrounds only x_j. (Note that the points x_j need not be included in D.)

We address the question of existence of global holomorphic and real-analytic first integrals for the equation.

Proposition
Consider the equation (1) .
Let Ω be a domain in $\mathbf{C}^3$ whose projection on the x-coordinate permits winding around the three singular points $x = 1$, $x = \omega$, $x = \omega^2$.
If μ is such that $\sqrt{\frac{1}{16} - \mu}$ is irrational, then:
a) There are no holomorphic first integrals $F(u, u', x)$ on Ω.
b) More generally, any holomorphic (in 6 variables) first integral

$F(u, u', x, \bar{u}, \bar{u}', \bar{x})$ *(i.e. real-analytic) defined on* $\Omega \times \bar{\Omega}$ *is a function of*

$$|c_1|^2 + \tau_\mu |c_2|^2$$

where c_1, c_2 *are given by (7) and* τ_μ *is a real number.*

Proof

Step 1 : The Monodromy Group

Consider a fundamental set of solutions near $x = 1$,

$$V_0(x) = \begin{pmatrix} \phi(x) \\ \gamma(x) \end{pmatrix}$$

as in (6). Let ω be the cubic root of 1 with $\Im \omega > 0$.

Then $V_1(x) = V_0(x/\omega)$ and $V_2(x) = V_0(x/\omega^2)$ are fundamental sets of solutions near $x = \omega$ and $x = \omega^2$, respectively and V_0, V_1, V_2 are analytic in a neighborhood of the origin. For x in that neighborhood, define the transition matrices M, N, P by $V_1(x) = M V_0(x)$, $V_2(x) = N V_1(x)$, $V_0(x) = P V_2(x)$. These relations also hold if x is replaced by x/ω or by x/ω^2. It follows that $M = N = P$ and that $M^3 = I$.

The point at infinity is also a regular singular point and a fundamental set of solutions near infinity has the form

$$V_\infty(x) = \begin{pmatrix} \phi_\infty(x) \\ \gamma_\infty(x) \end{pmatrix}$$

where

$$\phi_\infty(x) = x^{r_-} \tilde{\phi}(x) \ , \ \gamma_\infty(x) = x^{r_+} \tilde{\gamma}(x)$$

with $\tilde{\phi}$, $\tilde{\gamma}$ analytic functions at infinity ; $r_\pm$ are the characteristic exponents at infinity :

$$r_\pm = -\frac{1}{4} \pm \nu \ \ , \text{where} \ \ \nu = (\frac{1}{16} - \mu)^{1/2} \tag{8}$$

Define the transition matrix Q by $V_\infty = Q^{-1} V_0$.

Let

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

denote the monodromy matrix around a finite singular point and

$$B = \begin{pmatrix} e^{2\pi i r_-} & 0 \\ 0 & e^{2\pi i r_+} \end{pmatrix}$$

denote the monodromy matrix at infinity. Then the change of values of the fundamental solutions V_0 upon closed circuits around the singular points is given by the action of the group $\mathcal{G}$ of matrices generated by the monodromy matrices corresponding to loops around each of the three finite singular points: A — corresponding to a loop around $x = 1$, $T = MAM^2$ — corresponding to a loop enclosing $x = 1$ and $x = \omega^2$ and $S = M^2AM$ — corresponding to a loop enclosing $x = 1$ and $x = \omega$.

The relations among these generators are: $M^3 = I, (AM)^3 = QBQ^{-1}$. It turns out that there is almost enough information in these relations to calculate the transition matrices. Simple algebra yields:

$$M = \begin{pmatrix} \frac{1}{2}(z - \Delta^2) & 1 \\ -\frac{1}{4}(z^2 + 3\Delta) & -\frac{1}{2}(z + \Delta^2) \end{pmatrix}$$

where

$$z = \omega\alpha_+ + \delta\alpha_- \quad , \quad \alpha_\pm = e^{2\pi i r_\pm/3} \quad , \quad \det(M) = \Delta = \omega\delta e^{2\pi i/3} \quad , \quad \omega^3 = \delta^3 = 1$$

Then the generators S, T of $\mathcal{G}$ have the expressions

$$S = \Delta^2 \begin{pmatrix} -\frac{1}{2}(z^2 + \Delta^2) & -(z + \Delta^2) \\ \frac{1}{4}(z^2 + 3\Delta)(z - \Delta^2) & \frac{1}{2}(z^2 + \Delta) \end{pmatrix}$$

and

$$T = \Delta^2 \begin{pmatrix} -\frac{1}{2}(z^2 + \Delta^2) & -(z - \Delta^2) \\ \frac{1}{4}(z^2 + 3\Delta)(z + \Delta^2) & \frac{1}{2}(z^2 + \Delta) \end{pmatrix}$$

Step 2

Let

$$u(x) = \begin{pmatrix} c_1 & c_2 \end{pmatrix} \begin{pmatrix} \phi(x) \\ \gamma(x) \end{pmatrix} \tag{9}$$

be a solution of (1) near $x = 1$. After a closed circuit around the singular points, the value of solution becomes

$$\begin{pmatrix} c_1 & c_2 \end{pmatrix} G \begin{pmatrix} \phi(x) \\ \gamma(x) \end{pmatrix} \tag{10}$$

where G is the corresponding matrix in $\mathcal{G}$.

Note that, if $u(x)$ satisfies (9) then

$$\overline{u(x)} = \begin{pmatrix} \overline{c_1} & \overline{c_2} \end{pmatrix} \begin{pmatrix} \overline{\phi(x)} \\ \overline{\gamma(x)} \end{pmatrix}$$

After a closed circuit in the x-plane, if the new value of the solution $u(x)$ is given by (10), then the new value of $\overline{u(x)}$ is

$$\begin{pmatrix} \overline{c_1} & \overline{c_2} \end{pmatrix} \overline{G} \begin{pmatrix} \overline{\phi(x)} \\ \overline{\gamma(x)} \end{pmatrix}$$

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Assume that there is a holomorphic first integral: $F(u, u', x, \overline{u}, \overline{u'}, \overline{x})$, defined on the domain $\Omega \times \overline{\Omega}$.

Let (u_0, u_0', x_0) be a point in Ω, where x_0 is not one of the singular points. Depending on the position of x_0 in the complex plane, we can write any solution with initial conditions at x_0 as $u(x) = (c_1, c_2) V_j(x)$ where $j = 0, 1, 2$ or ∞. We assume here $j = 0$ and the same arguments hold for all other cases.

Let $u(x)$ be the solution with $u(x_0) = u_0$, $u'(x_0) = u_0'$. Then (9) holds, for some constants c_1, c_2. Denote, for short, $c = (c_1, c_2)$.

We will consider closed paths around the three (finite) singular points of the equation, starting and ending at x_0.

If $G \in \mathcal{G}$ is any matrix in the monodromy group, after going on a corresponding path in Ω, we should have

$$F(cGV_0(x_0), \overline{c}\overline{G}\overline{V_0(x_0)}) = F(cV_0(x_0), \overline{c}\overline{V_0(x_0)}) \tag{11}$$

if the point $(cGV_0(x_0), \overline{c}\overline{G}\overline{V_0(x)})$ is in the domain of F.

After a linear change of coordinates, we may omit the fundamental solutions in (11) and simply write

$$F(cG, \overline{c}\overline{G}) = F(c, \overline{c}) \tag{12}$$

where F is holomorphic in a domain $U \times \overline{U}$.

Consider the following element of $\mathcal{G}$: $X = (AM)^3 = QBQ^{-1}$. Since ν is irrational, the set $\{12nr_+ (\text{mod} 1) \; ; \; n \in \mathbf{Z}\}$ is dense in the interval $[0, 1]$ hence $\{X^{12n} \; ; \; n \in \mathbf{Z}\}$ is dense in the set

$$\{QD_\alpha Q^{-1} \; ; \; D_\alpha = \text{diag}(e^{i\alpha}, e^{-i\alpha}) \; , \; \alpha \in \mathbf{R}\}$$

It follows that

$$F(c, \overline{c}) = F(cQD_\alpha Q^{-1}, \overline{c}\overline{Q}\overline{D_\alpha Q^{-1}})$$

must also be true, for all α for which $cQD_\alpha Q^{-1} \in U$, therefore for α small enough.

Denote $d = cQ$ and let $\tilde{F}$ be defined by $F(c, \overline{c}) = F(dQ^{-1}, \overline{dQ^{-1}}) = \tilde{F}(d, \overline{d})$. Then $\tilde{F}$ satisfies $\tilde{F}(d, \overline{d}) = \tilde{F}(dD_\alpha, \overline{d}D_{-\alpha})$ for α small, or,

$$\tilde{F}(d_1, d_2, \overline{d_1}, \overline{d_2}) = \tilde{F}(e^{i\alpha}d_1, e^{-i\alpha}d_2, e^{-i\alpha}\overline{d_1}, e^{i\alpha}\overline{d_2})$$

Taking the derivative at $\alpha = 0$ we get the equation

$$d_1 \frac{\partial \tilde{F}}{\partial d_1} - \overline{d_1}\frac{\partial \tilde{F}}{\partial \overline{d_1}} - d_2 \frac{\partial \tilde{F}}{\partial d_2} + \overline{d_2}\frac{\partial \tilde{F}}{\partial \overline{d_2}} = 0$$

which has as solutions arbitrary functions of $|d_1|$, $|d_2|$, $\Re(d_1 d_2)$.

Thus any integral has the form

$$F(c) = \Phi(|(cQ)_1|, |(cQ)_2|, \Re((cQ)_1 (cQ)_2))$$

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Pick now another element in $\mathcal{G}$ having the spectrum on the unit circle, namely $Y = (AM^2)^3$. Let R be the matrix which diagonalizes Y: $Y = R(-B)R^{-1}$.

By the above argument, there is a function Ψ such that

$$F(c) = \Psi(|(cR)_1|, |(cR)_2|, \Re((cR)_1\overline{(cR)_2})) = \Phi(|(cQ)_1|, |(cQ)_2|, \Re((cQ)_1\overline{(cQ)_2}))$$

Denote $cQ = z$, $P = Q^{-1}R$. Then

$$\Phi(|z_1|, |z_2|, \Re(z_1\overline{z_2})) = \Psi(|(zP)_1|, |(zP)_2|, \Re((zP)_1\overline{(zP)_2})) \tag{13}$$

for all pairs of complex numbers z.

Simple algebra yields for the matrix P:

$$P = \begin{pmatrix} 1+m & m \\ -m & 1-m \end{pmatrix}$$

where

$$m = (-1)^{k+l}(2\sin(\frac{\pi}{3}(k-l+2\nu)))^{-1}$$

so m is a real number. The integers k and l are defined by $\omega = \exp(2k\pi i/3)$, $\delta = \exp(2l\pi i/3)$.

It is convenient to write (13) in polar coordinates: if $z = (z_1, z_2)$ with $z_k = r_k \exp(i\theta_k)$ and if we denote $\theta = \theta_1 + \theta_2$, $x = \theta_1 - \theta_2$, then the condition (13) becomes

$$\tilde{\Phi}(r_1, r_2, \theta) = \tilde{\Psi}(A, B, C)$$

where

$$A = (1+m)^2 r_1^2 + m^2 r_2^2 - 2m(1+m)r_1 r_2 \cos(x)$$
$$B = m^2 r_1^2 + (1-m)^2 r_2^2 + 2m(1-m)r_1 r_2 \cos(x)$$
$$C = m(1+m)r_1^2 \cos(\theta+x) - m(1-m)r_2^2 \cos(\theta-x) + (1-2m^2)r_1 r_2 \cos(\theta)$$

Since the relation must hold for all x, by differentiating with respect to x at $x = 0$ and at $x = \pi/2$ we get $\tilde{\Psi}_C = 0$ and, then that $\tilde{\Psi}$ depends only on the combination $(1-m)A + (1+m)B$. Substituting back the variable $c = zQ^{-1}$ we get, after a few calculations, that both Φ and Ψ must be functions of the expression

$$|c_1|^2 + \frac{1-m^2}{4m^2}|c_2|^2 \tag{14}$$

It can now be directly checked that the above expression is invariant under the generators of the monodromy group. The contention b) of the proposition is thus proved, and so is a).

3   The Hénon-Heiles System

Consider the Hénon-Heiles system (3) for $a = b = 0$:

$$H = \frac{1}{2}\left(\dot{q}_1^2 + \dot{q}_2^2\right) - \lambda q_1^2 q_2 - \frac{1}{3}q_2^3 \tag{15}$$

(we consider $e = 1$ since the variable q_2 can be rescaled).

The corresponding equation of motion are

$$\begin{aligned}\ddot{q}_1 &= 2\lambda q_1 q_2 \\ \ddot{q}_2 &= \lambda q_1^2 + q_2^2\end{aligned} \tag{16}$$

Relying on the intuitive picture that the absence of additional first integrals of motion can be due to the presence of almost periodic motions with "badly" related "periods", we choose to study the system in a neighborhood of particular periodic solutions.

Note that there are three two-parameter families of doubly periodic solutions, namely

$$\begin{aligned}q_1 &= \alpha q_2 \\ \ddot{q}_2 &= \left(\lambda\alpha^2 + 1\right) q_2^2\end{aligned}$$

where $\alpha(\lambda\alpha^2 + 1 - 2\lambda) = 0$.

After the linear change of coordinates (non-canonical)

$$q_1 = \alpha q + U \quad , \quad q_2 = q \tag{17}$$

the equations of motion are

$$\begin{aligned}\ddot{U} &= 2(1 - \lambda)Uq - \lambda\alpha U^2 \\ \ddot{q} &= 2\lambda q^2 + 2\lambda\alpha qU + \lambda U^2\end{aligned} \tag{18}$$

Consider first α to be one of the roots of $\lambda\alpha^2 + 1 - 2\lambda = 0$.

Assume that (18) has a first integral $F(U, \dot{U}, q, \dot{q})$, functionally independent of $H(U, \dot{U}, q, \dot{q})$. We assume that F is meromorphic on a domain $D_U \times D_q$ and analytic on $(D_U \setminus (0,0)) \times D_q$ where $D_U \subset \mathbf{C}^2$ is a neighborhood of the origin and $D_q \subset \mathbf{C}^2$ is a domain whose projection on the q-coordinate permits winding around the three cubic roots of some real number. We show that, under these assumptions, the number r_2 in (5) must be rational. Note that the space of first integrals F that we allow for is quite large. It contains, for example, functions having essential singularities along analytic varieties. The essential restriction here is the condition that F be meromorphic when $U = \dot{U} = 0$ for values of q of all arguments.

In order to perform the Kruskal test, we introduce a small parameter ϵ by setting $U = \epsilon u$.

The system (18) becomes

$$\begin{aligned} \ddot{u} &= 2(1 - \lambda)uq - \epsilon\lambda\alpha u^2 \\ \ddot{q} &= 2\lambda q^2 + 2\epsilon\lambda\alpha qu + \epsilon^2\lambda u^2 \end{aligned} \tag{19}$$

We study the integrability of the reduced system (i.e at $\epsilon = 0$)

$$\begin{aligned} \ddot{u} &= 2(1 - \lambda)uq \\ \ddot{q} &= 2\lambda q^2 \end{aligned} \tag{20}$$

The second equation can be integrated once and we get

$$\begin{aligned} \ddot{u} &= 2(1 - \lambda)uq \\ \dot{q}^2 &= \tfrac{4\lambda}{3}q^3 - K \end{aligned} \tag{21}$$

Note that the constant of integration K corresponds to the Hamiltonian at $\epsilon = 0$.

We first choose K: let K be such that the projection of D_q on the q-coordinate permits winding around the three roots of the polynomial $\frac{4\lambda}{3}q^3 - K$.

Next, we show that the integral F produces a first integral for (21). Since F is meromorphic, there is an integer p such that

$$F(\epsilon u, \epsilon\dot{u}, q, \dot{q}) = \epsilon^p G(\epsilon, u, \dot{u}, q, \dot{q}) = \epsilon^p \sum_{n=0}^{\infty} \epsilon^n G_n(u, \dot{u}, q, \dot{q})$$

For ϵ small the functions G_n have the same regularity as F and the series converges for $(u, \dot{u}, q, \dot{q})$ in the domain of F.

Then G_0 is a first integral for the reduced system (20) (or is constant). If G_0 depends on u or $\dot{u}$, then it is independent of $H|_{\epsilon=0}$, hence $G(0, u, \dot{u}, q, \sqrt{\frac{4\lambda}{3}q^3 - K})$ is a first integral for (21).

If G_0 depends only of $(q, \dot{q})$ or is constant, we can reduce to the preceding case in the following way. The function G_0 will be an integral for the second equation in (20) (or a constant); therefore $\epsilon^{-p-1}(F - G_0)$ is either a first integral for (21) or a constant. We repeat the procedure, until, for some n_0, G_{n_0} depends on u or $\dot{u}$. (There is such a number n_0 because (19) obviously does not admit integrals depending only on $(q, \dot{q})$.) The upshot is that there exists a first integral $G(u, \dot{u}, q, \dot{q})$ for (21), having the same regularity as F.

The next step is eliminating the time in (21). A formal calculation gives:

$$\frac{d^2u}{dq^2}\left(\frac{4\lambda}{3}q^3 - K\right) + 2\lambda q^2\frac{du}{dq} + 2(\lambda - 1)qu = 0 \tag{22}$$

It is clear that $G(u, \frac{du}{dq}\sqrt{\frac{4\lambda}{3}\lambda q^3 - K}, q, \sqrt{\frac{4\lambda}{3}\lambda q^3 - K})$ is a first integral for (22). Denote it by $\tilde{G}(u, u', q, \sqrt{\frac{4\lambda}{3}\lambda q^3 - K})$ (here u' stands for the q-derivative

of u). By restricting $\tilde{G}$ to a sub-domain $\tilde{D}_U \times D_q$, we may assume that $\tilde{G}$ is analytic in $(u, u', q, \dot{q})$.

Rescaling the variable q by

$$q = \left(\frac{3K}{4\lambda} \right)^{1/3} x \tag{23}$$

the equation (22) becomes (1) with $\mu = 3/2(1 - 1/\lambda)$ and

$$\tilde{G}(u, u', q, \sqrt{\frac{4\lambda}{3}\lambda q^3 - K}) = \hat{G}(u, u', x, \sqrt{x^3 - 1})$$

where $\hat{G}$ is analytic in four variables on $\tilde{D}_U \times D_x$ (where, clearly, D_x is a dilation of D_q: $D_x = (3K/4/\lambda)^{-1/3} D_q$).

In the first part of the paper it was shown that if $\sqrt{1/16 - \mu}$ is irrational then there are no first integrals for (1). In that argument, only paths surrounding the singular points an even number of times are used. Therefore, the argument also applies to first integrals of the type $\hat{G}(u, u', x, \sqrt{x^3 - 1})$. Hence $r_\pm$ given by (8) must be rational.

Note the connection between the numbers $r_\pm$ and the resonances r_2 from (5): $r_\pm = -r_2/2$. Hence r_2 must be rational.

Consider now the value $\alpha = 0$ (the change of coordinates (17) is trivial in this case). Assume that there is a second first integral F with the same properties as before.

Substituting $U = \epsilon u$ the system (18) becomes

$$\ddot{u} = 2\lambda u q$$
$$\ddot{q} = q^2 + \epsilon^2 \lambda u^2$$

As before, we study the reduced system which is, after one integration,

$$\ddot{u} = 2\lambda u q$$
$$\dot{q}^2 = \tfrac{2}{3}q^3 - K$$

Eliminating the time and rescaling, we get the equation (1) for $\mu = -3\lambda$, which has a first integral.

Since in this case the connection between the numbers $r_\pm$ and the resonances r_1 given by (4) is $r_\pm = -r_2/2$, it follows that r_1 must be rational.

Conclusions

We have thus proved that if one of the resonances r_1, r_2 is irrational then there are no additional first integrals $F(q_1, \dot{q}_1, q_2, \dot{q}_2)$ such that F is meromorphic on a domain $D_1 \times D_2$, analytic on $(D_1 \times D_2) \setminus V_\alpha \setminus V_0$ where V_α is the variety $\{q_1 = \alpha q_2, \dot{q}_1 = \alpha \dot{q}_2\}$ and V_0 is the variety $\{q_1 = 0, \dot{q}_1 = 0\}$ and the projection

of D_2 on the q_2-coordinate permits winding around the three roots of a real number.

In the notations (15), the condition that r_1 or r_2 be irrational is: $\sqrt{1+48\lambda} \in \mathbf{R} \setminus \mathbf{Q}$, or $\sqrt{-23+24/\lambda} \in \mathbf{R} \setminus \mathbf{Q}$.

The analysis above is not exhaustive: there is still more information to be deduced from the monodromy group of the equation (1). That could answer the question whether there also exist rational or non-real values of ν for which the equation is not integrable in the complex domain. Also, the Kruskal test was performed only in the 0 order, and to that order there exists a real-analytic first integral (14) (besides the reduced Hamiltonian), which is in fact an integral in the real domain. The multi-valuedness of the solutions needs to (and can) be studied in the next orders to possibly rule out the existence of an integral in the real domain.

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