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POWER AND EXPONENTIAL-POWER SERIES SOLUTIONS OF EVOLUTION EQUATIONS

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1. Introduction

1.1. **Brief overview.** The theory of partial differential equations when one, or more variables, is in the complex domain, and approaches a characteristic variety has only recently started to develop.

In their paper [11], generalized in [12], O. Costin and S. Tanveer proved existence and uniqueness of solutions with given initial conditions, for quasilinear systems of evolution equations in a large enough sector of $\mathbb{C}$.

Borel summability of divergent solutions of the heat equation was proved by Lutz, Miyake, and Schäfke [13], and more generally, Borel summability of series solutions of linear equations with constant coefficients was proved, in a general setting, by Balser (see [1], and the references therein).

A natural question is to find what formal objects lie beyond formal power series solutions, and what is their connection to power series. The present paper contains initial results in this direction.

For ordinary differential equations a comprehensive and general theory of formal solutions (transseries), in a one-to-one correspondence with true solutions, is presented in the fundamental work of Écalle [3]-[5]. The correspondence between transseries and solutions was later proved under nonresonance assumptions by O. Costin, who constructed a generalized Borel transform [6], [7]. O. Costin and Kruskal showed how formal solutions can be used to produce the Stokes constants [9], [8]. Transseries solutions can be used to find the type and location of movable arrays of singularities toward the irregular singular point [8], [10].

Braaksma has recently extended the theory of transseries representations to nonlinear difference equations [2]. The structure of singularities of solutions of difference equations has been obtained by Kuik [15].

1.2. **General Remarks.** The present paper considers a few simple evolution equations, and examines the formal solutions that can be

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built with exponentials and powers. Some conclusions are mentioned below.

A main distinction between formal solutions of ordinary, versus partial differential equations is that there is an tremendous freedom in the formal solutions of a partial differential equation: after each monomial there is a functional freedom.

Perhaps surprisingly however, there is a maximal order possible for exponential solutions (for evolution equations of order at least two); these maximal exponentials are also distinguished in another way: they generate the terms beyond all orders of divergent power solutions.

For the linear equations examined in this paper, the heat equation and Airy equation, solutions can be obtained by superposition of simpler solutions (which solve similarity reductions to ordinary equations). A remarkable fact is that the PDE transseries are also obtained by superpositions of the transseries of the similarity solutions. This provides robustness to a theory of transseries for solutions of partial differential equations.

Formal series and exponential series solutions of partial differential equations are deduced in this paper using standard tools in asymptotic analysis; their basic principles were exposed by Kruskal in [14]. Such calculations apply to solutions that do have a (trans)asymptotic representation, and use the assumption that monomials in this representation do preserve their ordering after operations with functions. In the case of algebraic, ordinary differential or difference equations, with analyzable coefficients (as is the case of equations arising in applications), the general solutions seem to be in a one-to-one isomorphic correspondence with such algebraic representations (transseries) (see [3] for ordinary differential equations). For partial differential equations however, due to a rich functional freedom in the set of solutions, it is clear that only subclasses of solutions can be represented by algebraic objects. The steps of specific calculations will be shown in some detail to uncover that they can be justified for algebraic representations that have the properties of Écalle's transseries: they are based on monomials, that can be well ordered with respect to the *much larger* relation ($\gg$), and for which all operations preserve the ordering.

1.3. **Setting.** *The formal solutions will be derived under the assumption that x is real positive $x \to +\infty$ and that t varies in a compact subinterval of $(0, +\infty)$.*

1.4. **Main results.** Formal solutions of the heat equation, and their association to solutions obtained by inverse Laplace transform are studied in §3.

It is shown that power series are linear combinations (finite, or infinite, in the latter case possibly transfinite) of pure series solutions, in which all freedoms besides the first term are taken to be 0 (3.8); these series are generically divergent. It is shown that exponential terms cannot have arbitrary order. In fact, there is a maximal order of decay. These distinguished exponential series (3.12), together with the pure power series satisfy the same ordinary differential equations (3.14), hence it may be inferred that these exponentials are the possible terms beyond all orders of the pure power series.

Complex plane solutions u of the heat equation are then studied in §3.4 using inverse Laplace transform, after proper normalization in the sense of O. Costin. The power series asymptotic to Laplace integrals are precisely the corresponding superpositions of pure power series. Moreover, the corresponding superposition of the distinguished exponentially small terms coincides with the loop integral that encircles all the singularities of the inverse Laplace transform of u, then generating all the terms beyond all orders of u (Proposition 1).

In §3.6 it is shown that initial data determines the power series and boundary data fixes the exponentially small terms as well.

Similar results hold for the Airy equation and are briefly presented in §4.

A simple first order, nonlinear example is examined in §5. Power series solutions cannot have arbitrary order, and there is a maximal power.

2. The simplest partial differential equation

Consider the simplest first order evolution equation

$$u_t = u_x \tag{2.1}$$

Equation (2.1) has the general solution $u = \Phi(x + t)$ where Φ is *any* differentiable function. Then its formal solutions are any expressions in $x + t$ and nothing more specific seems to emerge.

3. The Heat Equation

Consider the simplest second order evolution equation: the heat equation

$$u_t = \frac{1}{4} u_{xx} \tag{3.2}$$

The question is to find the formal series solutions that can be written in terms of powers of x or exponentials of powers of x, in the asymptotic limit of §1.3. It will shortly appear that the same formal solutions satisfy another limit, namely for $t \to +0$ and $x = O(1)$.

3.1. **Power series solutions.** Formal calculation of power series solutions of (3.2) is standard. The main steps are as follows.

Looking for solutions $u(x,t)$ behaving like a power of x, substitute

$$u(x,t) = f(t)x^n + \Delta(x,t) \qquad \text{with } \Delta \ll x^n \ , \ n \in \mathbb{C} \tag{3.3}$$

Then (3.2) becomes

$$f'(t)\, x^n \ + \ \Delta_t \ = \ \frac{1}{4}\, n(n-1)\, f(t)\, x^{n-2} \ + \ \frac{1}{4}\, \Delta_{xx} \tag{3.4}$$

The usual assumption at this point is that since $\Delta \ll x^n$ then also $\Delta_t, \Delta_{xx} \ll x^n$. This certainly holds if Δ is in a "good" class of functions (but is clearly not true in full generality, and an easy counterexample is $\Delta = x^{n-1}\sin(1/x)$).

Then in (3.4) the term $f'(t)\, x^n$ is much larger than all others, and it must therefore vanish: $f(t) = \text{const} = c_n$. Then $u = c_n x^n + \Delta$ where

$$\Delta_t \ = \ \frac{1}{4}\, n(n-1)\, c_n\, x^{n-2} \ + \ \frac{1}{4}\, \Delta_{xx} \qquad (\Delta \ll x^n) \tag{3.5}$$

The main behavior of Δ contains a functional freedom, since then $\Delta = \Phi(x) + \delta(x,t)$, where Φ is *any* function satisfying $\Phi \ll x^n$, $\Phi_{xx} \ll x^{n-2}$ and δ satisfies $\delta \ll \Phi$.

For simplicity (and definiteness) only powers of x are considered here, so $\Phi(x) = c_k x^k$ (where $\Re k < \Re n$).

Then $u = c_n x^n + c_k x^k + \delta(x,t)$ (where $\delta \ll x^k$) and the steps above are repeated to determine the leading behavior of δ (which is the largest monomial in its asymptotic representation).

It turns out that there is a functional freedom after *every* monomial of the series solution u; a simple calculation yields the general form of a power series solution of (3.2):

$$u(x,t) \sim c_n x^n + c_k x^k + ... + \left[c_n \frac{n(n-1)}{4}\, t + c_{n-2} \right] x^{n-2} + ... \tag{3.6}$$

Equation (3.2) is linear, hence (3.6) is a superposition (finite, or infinite, in the latter case it can be transfinite)

$$u(x,t) = \sum_{n \in S} c_n \hat{u}_n \qquad (c_n \in \mathbb{C} \ , \ S \subset \mathbb{C}) \tag{3.7}$$

where $\hat{u}_n$ are "pure series": series solutions with leading behavior x^n and where all arbitrary freedoms were chosen zero; a simple calculation yields

$$\hat{u}_n = x^n \left[1 + \sum_{j=1}^{\infty} \frac{n(n-1)(n-2)...(n-2j+1)}{j!} \left(\frac{t}{4x^2} \right)^j \right] \tag{3.8}$$

Note that the series (3.8) is finite if $n \in \mathbb{N}$ and diverges otherwise.

The series (3.7) is an asymptotic object only if its terms can be arranged in a decreasing way. For this to be possible the set S must be well ordered with respect to the relation $\succcurlyeq$: $n \succcurlyeq k$ iff $x^n \gg x^k$ $(x \to \infty)$.

The decomposition (3.7), (3.8) allows to distinguish between the divergence intrinsic to the equation (seen in (3.8)) and possible divergence due to given data (seen in the behavior of c_n, for n going towards points of accumulation in $S \cup \{+\infty\}$).

It is interesting to note that t appears only in the form of powers (even if no assumptions on t were made). Also, the series (3.7) is asymptotic in the another limit as well: $t \to +0$, and x varying in a compact subinterval of $\mathbb{R}_+$.

3.2. **Exponential series. Distinguished exponentials.** Unlike the case of the first order equation (2.1), where any exponential growth of solutions was possible, for the heat equation (and seemingly, for most other second or higher order equations) it turns out that there is a maximal order of increase (for representations in a "good class").

Formal calculation of exponential series solutions is the WKB method, whose main steps are outlined below together with necessary assumptions.

With the substitution $u = \exp(W)$ (where $|W| \gg 1$) equation (3.2) becomes

$$W_t = \frac{1}{4} W_x^2 + \frac{1}{4} W_{xx} \tag{3.9}$$

The term W_{xx} can be neglected in a first approximation, since $W_{xx} \ll W_x^2$. (Indeed, otherwise W_{xx}/W_x^2 is much larger, or of order 1 which by integration gives that W has at most logarithmic order, so u does not have exponential growth.)

For the dominant order of W one needs then to solve $W_t \approx \frac{1}{4} W_x^2$, which gives

$$tW_x = -2x + \Phi(W_x) \tag{3.10}$$

where Φ is an arbitrary function.

It turns out that solutions W of (3.10) belonging to a "good" class have the maximal order of growth x^2. Indeed, if W_x is much larger than, or of the same order as $\Phi(W_x)$ then (3.10) implies that W_x is of order x. Otherwise $\Phi(W_x)$ must have order x, hence $W_x \ll x$. Thus W_x has at most order x, so W has at most order x^2.

Note that these considerations hold for functions W in a class for which the relation $\gg$ is conserved under operations in x ($W(\cdot, t)$ is analyzable).

Looking for the specific form of a maximal W, substitute $W = f(t)x^2 + o(x^2)$ in (3.9), which gives $W = -\frac{x^2}{t-\tau} + o(x^2)$.

Since the heat equation is invariant under translations in t, take $\tau = 0$, and find the distinguished exponential

$$W_{max} = -\frac{x^2}{t} + o(x^2) \tag{3.11}$$

Other solutions of (3.10) have the form $W = cx^n + o(x^n)$ with $\Re n < 2$ and $c \in \mathbb{C}$.

There is a functional freedom after each monomial in the expansion of W, and this freedom is an arbitrary function of $\frac{x}{t}$:

$$W = -\frac{x^2}{t} + \Phi\left(\frac{x}{t}\right) + ...$$

An interesting case is $\Phi_1(z) = 2\xi z$ (with $\xi \in \mathbb{C}$). Taking all other freedoms to be 0 we get the exact solution

$$W = -\frac{x^2}{t} + 2\xi\frac{x}{t} - \xi^2\frac{1}{t} - \frac{1}{2}\ln t \qquad \text{, so} \qquad u = t^{-\frac{1}{2}}\mathrm{e}^{-\frac{(x-\xi)^2}{t}}$$

and the freedom Φ_1 corresponds to invariance of (3.2) under translations in x.

Another special freedom is $\Phi_0(z) = (-n-1)\ln z$ (with $n \in \mathbb{C}$). Again taking all other freedoms 0 one gets the formal solutions

$$\hat{\hat{u}}_n = e^{-\frac{x^2}{t}} t^{-\frac{1}{2}} \left(\frac{x}{t}\right)^{-n-1} \left[1 + \sum_{j=1}^{\infty} \frac{(n+1)(n+2)(n+3)...(n+2j)}{j!} \left(-\frac{t}{4x^2}\right)^j\right] \tag{3.12}$$

Note that the series (3.12) is finite if $n \in \mathbb{Z}_-$ and diverges otherwise.

3.3. **Similarity solutions and transseries.** In this section it is shown that the exponentially small terms following a series solution (3.8) are precisely (3.12). Indeed, it will be shown that the series (3.8) and (3.12) solve the same differential equation (for fixed n); based on existing results in the theory of transseries representations for solutions

of ordinary differential equations it follows then that if (3.8) diverges, then (3.12) must constitute the terms beyond all orders of that series.

3.3.1. *Transseries.* Noting that the formal solutions (3.8), (3.12) are powers of t multiplying series in $\frac{x^2}{t}$, denote

$$u_n(x,t) \;=\; t^{n/2}\, g_n\left(\frac{x^2}{t}\right) \tag{3.13}$$

which transforms (3.2) into the ordinary differential equation

$$zg''(z) + \left(z+\frac{1}{2}\right) g'(z) - \frac{n}{2} g(z) = 0 \quad , \qquad z = \frac{x^2}{t} \tag{3.14}$$

Equations (3.14) are usually called similarity reductions of (3.2) and their solutions (3.13) are similarity solutions.

Transseries solutions of (3.14) are of course, linear combinations of two independent solutions, which have the form

$$g_n(z) = C_1 z^{\frac{n}{2}}(1+o(1)) + C_2 \mathrm{e}^{-z} z^{-\frac{n+1}{2}}(1+o(1)) \quad (z \to +\infty) \tag{3.15}$$

Then formal, as well as actual, solutions of the heat equation are obtained from (3.13) and (3.14), (3.15): they are

$$A_n x^n \left(1 + O\left(\frac{t}{x^2}\right)\right) + B_n \mathrm{e}^{-\frac{x^2}{t}} t^{-\frac{1}{2}} \left(\frac{x}{t}\right)^{-n-1} \left(1 + O\left(\frac{t}{x^2}\right)\right) \tag{3.16}$$

The series multiplying the constant A_n is (3.8), and B_n is followed by (3.12).

3.3.2. *Representations using inverse Laplace transform.*

Assumption: *For simplicity only negative integer powers will be considered in the following:* $n \in \mathbb{Z}_-$.

Positive values of n correspond to solutions of the heat equation that do not decrease to 0; they can also be studied using inverse Laplace transform after subtracting the increasing terms in the expansions (see [7], [10], [11]). Noninteger values of n make the inverse Laplace transform branched at the origin, but no major differences exist otherwise in using the Borel space techniques [7].

Transseries (3.15) are obtained using generalized inverse Laplace transform [4], [7] in the following way. The substitution

$$g(z;\ell) \;=\; \int_\ell \mathrm{e}^{-pz}\, G(p)\, dp \tag{3.17}$$

(where ℓ is a path starting at $p=0$, going to ∞ in the right-half plane) transforms (3.14) to[1]

$$p(p-1)G' + \left(\frac{3}{2}p - \frac{n}{2} - 1\right) G = 0$$

whose solution is

$$G(p) = Cp^{-\frac{n}{2}-1}(1-p)^{\frac{n}{2}-\frac{1}{2}} \quad (C \in \mathbb{C}) \tag{3.18}$$

Since n was assumed negative, $G(p)$ is integrable at $p=0$, and is singular at $p=1$.

The integral (3.17) depends on the path of integration ℓ only relative to its homotopy class in the right half-plane minus the point $p=1$; therefore ℓ can be assumed to be either $d^+ = \mathrm{e}^{i\theta}\mathbb{R}_+$ or $d^- = \mathrm{e}^{-i\theta}\mathbb{R}_+$ (where θ is any number in $(0, \frac{\pi}{2})$).

A solution of (3.14) whose transseries is a power series with no exponentially small terms is the balanced average [4], [7], [6]

$$g_n^{[p]}(z) \;=\; \frac{1}{2}\, g(z; d^+) \;+\; \frac{1}{2}\, g(z; d^-)$$

and a solution whose transseries is exponentially small is

$$g_n^{[e]}(z) \;=\; \frac{1}{2}\, g(z; d^+) \;-\; \frac{1}{2}\, g(z; d^-)$$

(which amounts to integration in (3.17) on a loop around $p=1$).

Stating these facts in terms of solutions of the heat equation, using (3.13), (3.14), (3.15) consider the solutions of (3.2)

$$u_n(x,t; d^\pm) \;=\; t^{n/2} \int_{d^\pm} \mathrm{e}^{-px^2/t}\, p^{-\frac{n}{2}-1}\, (1-p)^{\frac{n-1}{2}}\, dp \tag{3.19}$$

and define

$$u_n^{[p]}(x,t) \;=\; \frac{1}{2}\, u(x,t; d^+) \;+\; \frac{1}{2}\, u(x,t; d^-) \tag{3.20}$$

$$u_n^{[e]}(x,t) \;=\; \frac{1}{2}\, u(x,t; d^+) \;-\; \frac{1}{2}\, u(x,t; d^-) \tag{3.21}$$

Then

$$u_n^{[p]}(x,t) \;\sim\; \Gamma\left(-\frac{n}{2}\right)\, \hat{u}_n \tag{3.22}$$

(see §6.1 for details); the transseries of $u_n^{[p]}$ has no exponentially small terms.

[1] It must be assumed that G is such that the integral (3.17) exists and, when integrating by parts, the boundary values vanish.

Also

$$u_n^{[e]}(x,t) \sim \frac{\pi i}{\Gamma\left(\frac{1-n}{2}\right)}\hat{\hat{u}}_n \tag{3.23}$$

(see §6.2 for details).

3.4. **Solutions of the heat equation by inverse Laplace transform.** Solutions of linear, or nonlinear partial differential equations have been studied using inverse Laplace transform methods by O. Costin and S. Tanveer [11], [12].

Looking for solutions of (3.2) which go to 0 for $x \to +\infty$ using inverse Laplace transform, first normalize the equation by substituting $y = x^2$

$$u(x,t) = v(y,t) \quad , \quad y = x^2 \tag{3.24}$$

and (3.2) becomes

$$v_t = y\, v_{yy} + \frac{1}{2}\, v_y \tag{3.25}$$

which after inverse Laplace transform gives

$$q^2\, V_q + \frac{3}{2}\, q\, V = V_t$$

with the general solution

$$V(q,t) = (1-tq)^{-3/2}\, F\left(\frac{q}{1-tq}\right) \tag{3.26}$$

where F is arbitrary.

This gives solutions of the heat equation in the form

$$u_F(x,t;\ell) = \int_\ell \mathrm{e}^{-q\,x^2}\,(1-tq)^{-3/2}\, F\left(\frac{q}{1-tq}\right)\, dq \tag{3.27}$$

where ℓ is a path starting at $q = 0$, which for t in a specified interval lies in the right-half plane and avoids the singularities of the integrand.

3.4.1. *Assumptions on F.* At this point a discussion is required on the assumptions on the arbitrary function F that are needed, or useful.

First of all, formula (3.27) defines a solution of (3.2) if the integral exists and can be differentiated with respect to t and x.

Also, formula (3.27) defines uniquely, by Borel summation, a function u_F if $F(\zeta)$ is analytic at $\zeta = 0$ (or, at least has a convergent Frobenius series).

It should be noted that the path of integration ℓ in (3.27) will have to vary with t, since the singularities of the integrand do vary. But the value of the integral (3.27) should not depend on small variations of ℓ (otherwise u_F may not solve (3.2)). Then F must be assumed

mostly analytic, in an appropriate sense which assures that when t is varied, the path of integration can be accordingly varied in a domain of analyticity of the integrand, so that the value of the integral is locally constant in t.

In addition, considering functions F in a more regular class, more information on u_F can be obtained using analytic methods. For example, restricting the considerations to solutions u_F which are (1) analytic for x in a sector containing $\mathbb{R}_+$ (possibly excepting a discrete set of points), and (2) are defined for all $t > t_0$ then $F(\zeta)$ would be assumed (a) analytic in the complex plane less two half-lines $\mathcal{S}_{\zeta_0} = \{\zeta \in \mathbb{R}\,;\, |\zeta| > \zeta_0\}$, and (b) the increase of F at points on $\mathcal{S}_{\zeta_0}$ should allow integrals (3.27) to converge (see §6.3 for details).

3.4.2. *Asymptotic and transasymptotic expansions.*

The asymptotic power series of the solution u_F in (3.27) is found in a straightforward way, using Watson's Lemma (which amounts to a formal integration, term by term, of the Taylor series of $(1-tq)^{-3/2}\, F\left(\frac{q}{1-tq}\right)$ at $q = 0$).

As a natural generalization of the techniques used for ordinary differential equations, it is to be expected that exponentially small terms of u_F are generated by taking linear combinations of $u_F(x, t; \ell)$ on different paths ℓ. For linear equations the study of exponentially small series is easier, and easier to put to test.

3.5. **Superpositions of similarity solutions and transseries.**

3.5.1. *Similarity solutions.* For $F(\zeta) = \zeta^m$ (with $\Re m > -1$) formula (3.27) has the form (3.19) for $n = -2m - 2$, hence

$$u_{\zeta^m} = u_{-2m-2} \tag{3.28}$$

3.5.2. *Finite superpositions of similarity solutions.* For $F(\zeta)$ a polynomial:

$$F(\zeta) = \sum_{m=0}^{M} F_m \zeta^m$$

(3.27) is a finite superposition of (3.19):

$$u_F = \sum_{m=0}^{M} F_m u_{-2m-2} \tag{3.29}$$

and the transseries of u_F is obtained by a direct summation and rearrangement of the transseries of u_{-2m-2}, $m = 0..M$.

Similar results hold if $F(\zeta)$ a polynomial in $\zeta^{1/2}$.

3.5.3. *Infinite superpositions of similarity solutions.*

Since the function F of (3.27) is assumed analytic at $\zeta = 0$, it has a convergent Taylor series expansion

$$F(\zeta) = \sum_{m\geq 0} F_m \zeta^m \tag{3.30}$$

A natural question is to investigate what formal objects are obtained by the corresponding superposition of the transseries of u_{-2m-2}, $m \geq 0$.

Consider then the formal objects obtained by replacing $u^{[p]}_{-2m-2}$ and $u^{[e]}_{-2m-2}$ with their formal series (3.22), (3.8), respectively (3.23), (3.12):

$$\hat{u}_F = \sum_{m=0}^{\infty} F_m \Gamma(m+1)\, \hat{u}_{-2m-2} \tag{3.31}$$

$$\hat{\hat{u}}_F = \sum_{m=0}^{\infty} F_m \frac{\pi i}{\Gamma\left(m+\frac{3}{2}\right)}\, \hat{\hat{u}}_{-2m-2} \tag{3.32}$$

The power series $\hat{u}_F$ can be rearranged to become asymptotic (meaning that the terms are decreasing), and this is clearly the power series asymptotics of u_F (by Watson's Lemma).

The exponential series $\hat{\hat{u}}_F$ cannot be immediately rearranged in an asymptotic way (since the powers of x multiply the same exponential term and have no upper bound).

However, $\hat{\hat{u}}_F$ is an analytic function which sums all the exponentially small terms:

Proposition 1. *Let F have a convergent Taylor series (3.30) at the origin. Let $\hat{\hat{u}}_F$ be defined by (3.32).*

Then there exists $t_0 \geq 0$ and $c > 0$ such that

$$\hat{\hat{u}}_F(x,t) = \frac{1}{2}\int_{c+i\mathbb{R}} \mathrm{e}^{-q\,x^2}\,(1-tq)^{-3/2}\,F\left(\frac{q}{1-tq}\right)\,dq \tag{3.33}$$

for all $t > t_0$.

The integral (3.27) is on the loop which encircles all the singularities of the integrand.

To be more specific, t_0 and c are any positive constants for which $F(\zeta)$ is analytic on the disk $|\zeta + \frac{1}{t_0}| < \frac{1}{c}$. Such constants always exist since F was assumed analytic at $\zeta = 0$; for example if F is analytic on $|\zeta| < r$ then one can take $t_0 > 1/r$. In particular, t_0 can be chosen 0 if F is analytic on the half-plane $\Re\zeta \leq 0$.

The proof of Proposition 1 is given in §6.4.

3.5.4. *Example 1.* If F is entire then the only singularity of the integrand in (3.27) is $q = 1/t$, which is an essential singularity. The paths of integration ℓ can be taken to be rays $d^{\pm}$ as in (3.19).

To examine the transseries of solutions on a simple example in this class consider

$$F(\zeta) = \mathrm{e}^{a^2\zeta} \qquad (a > 0)$$

For this F the exponential series $\hat{\hat{u}}_F$ can be calculated explicitly (either from (3.33), or, directly from (6.52)), yielding

(3.34)
$$\hat{\hat{u}}_F \;=\; \frac{1}{2}u(\cdot; d^+) \;-\; \frac{1}{2}u(\cdot; d^-) \;=\; \frac{\sqrt{\pi}i}{2a}\, t^{-1/2} \left(\mathrm{e}^{-\frac{(x-a)^2}{t}} \;-\; \mathrm{e}^{-\frac{(x+a)^2}{t}} \right)$$

Note that the loop integral (3.34) contains more than one type of exponential terms, in spite of the fact that there is only one singularity.

3.5.5. *Example 2.* If F is a rational function, then it can be written as a polynomial plus a sum of poles. Polynomials were considered already in §3.5.2. Consider next the case of a simple pole

$$F(\zeta) = \frac{1}{1 - a\zeta} \qquad (a > 0) \tag{3.35}$$

so

$$F(\zeta) = \sum_{n\geq 0} a^n \zeta^n \qquad \text{for } \; |\zeta| < \frac{1}{a}$$

Loop integrals

For F given by (3.35) formula (3.27) is

$$u_F(x,t;\ell) \;=\; \int_\ell \mathrm{e}^{-q\,x^2}\, \frac{(1-tq)^{-1/2}}{1-(a+t)q}\, dq \tag{3.36}$$

and the integrand is singular at $q_1 = \frac{1}{t+a}$ and $q_2 = \frac{1}{t}$. There are four paths of integration ℓ avoiding q_1 and q_2.

Denote by d^{σ_1,σ_2} ($\sigma_j = \pm$) the paths in the right half-plane starting at $q = 0$, avoiding q_j from above (respectively, below) if $\sigma_j = +$ (respectively, $-$). Elementary calculations give

(3.37)
$$u_F(x,t;d^{++}) - u_F(x,t;d^{-+}) = 2\pi i a^{-\frac{1}{2}}(t+a)^{-\frac{1}{2}}\mathrm{e}^{-\frac{x^2}{t+a}} \equiv 2u_F^{[e,1]}(x,t)$$

and

(3.38)
$$u_F(x,t;d^{++}) - u_F(x,t;d^{+-}) = u_F(x,t;d^{-+}) - u_F(x,t;d^{--})$$
$$= -2ia^{-\frac{1}{2}}(a+t)^{-\frac{1}{2}}\mathrm{e}^{-\frac{x^2}{t}} \int_0^\infty \mathrm{e}^{-s\frac{ax^2}{t(a+t)}} s^{-\frac{1}{2}}(1+s)^{-1}\, ds \equiv 2u_F^{[e,2]}(x,t)$$

$$\sim -\frac{2i\sqrt{\pi}}{a} t^{\frac{1}{2}} x^{-1} \mathrm{e}^{-\frac{x^2}{t}}$$

Main loop integral

On the other hand, direct superposition of the exponentially small terms (see (3.33)) give, using (3.37), (3.38),

$$\hat{\hat{u}}_F = \frac{1}{2} u_F(x,t;d^{++}) - \frac{1}{2} u_F(x,t;d^{--}) = u_F^{[e,1]} + u_F^{[e,2]}(x,t)$$

3.6. **Relations between initial data and transseries representations.** The structure of transseries solutions makes visible the type of initial, or boundary data that specifies uniquely a solution with a given class. Indeed, consider similarity solutions with $n < 0$; their transseries have the form (3.16), and it is intuitive that looking at the limit $t \to +0$ the small exponential term must vanish, thus fixing A_n, while in the limit $x \to +0$ the constant B_n becomes visible, since the exponential is no longer beyond all orders. This means that conditions on solutions given at $x = 0+$ and at $t = 0+$ specifies uniquely a solution. Remarks 2 and 3 state these properties.

Remark 2. *The initial condition determines the dominant power series. Indeed:*

(i) The similarity solutions satisfy

$$\lim_{t\to 0+} u_n(x,t;d^{\pm}) \;=\; \Gamma\left(-\frac{n}{2}\right) x^n \tag{3.39}$$

for $x > 0$ and $n < 0$; therefore

$$\lim_{t\to 0+} u_n^{[p]}(x,t) \;=\; \Gamma\left(-\frac{n}{2}\right) x^n \quad , \quad \lim_{t\to 0+} u_n^{[e]}(x,t) \;=\; 0$$

(ii) More generally, if $F(\zeta)$ is entire, then

$$\lim_{t\to 0+} u_F(x,t;\ell) \;=\; \int_0^\infty \mathrm{e}^{-qx^2}\, F\,(q)\, dq$$

and the function F is determined by the initial condition at $t = 0$.

The proof follows immediately from (3.19) and (3.27).

Further Remarks.

The general case, when F is not entire, is very interesting and rich in consequences, for both finding an initial time when specifying an initial condition determines F uniquely, and for the study of backwards evolution (in the sense of analytic continuation for t less than the initial time). These issues however will not be pursued here.

Remark 3. *If n is not an even integer, boundary conditions determine the exponentially small terms. Indeed:*

$$\lim_{x\to 0+} u_n(x,t;d^{\pm}) \;=\; \mathrm{e}^{\mp i\pi n/2}\,\frac{\sqrt{\pi}\Gamma\left(-\frac{n}{2}\right)}{\Gamma\left(-\frac{n}{2}+\frac{1}{2}\right)}\,t^{n/2} \tag{3.40}$$

for $t>0$ and $n<0$.

Formula (3.40) follows by substituting $p/(1-p)=r$ in (3.19) for $x=0$ and using a formula for an Eulerian integral of the first kind [16].

Remark 4. The existence of a maximal order of increase of exponential terms implies that a solution asymptotic to a given power series in a sector large enough is unique. This type of property insures uniqueness also in the general results of O. Costin and S. Tanveer in [11], [12].

For the heat equation, since the largest possible exponent is of order x^2 (see (3.11)) then the requirement that a solution be asymptotic to a power series on a sector larger than $\arg x \in [0,\pi/4]$ implies uniqueness of the solution. (Intuitively, exponential terms are not small after continuation in x beyond $|\arg x| \geq \pi/4$.)

4. The Airy Equation

Consider the simplest third order evolution equation: the Airy Equation

$$u_t = \frac{4}{27}\,u_{xxx} \tag{4.41}$$

4.1. **Formal solutions.** Consider power series and exponential-power series solutions of the Airy equation (4.41) in the limit of §1.3.

The considerations on the structure of formal solutions are very similar to the case of the heat equation, and are not repeated here.

Power solutions are superpositions of

$$\hat{u}_n = x^n\left[1+\sum_{j\geq 0}\frac{n(n-1)(n-2)...(n-3j+1)}{j!}\left(\frac{4t}{27x^3}\right)^j\right] \tag{4.42}$$

The series (4.42) terminates if $n\in\mathbb{N}$ and diverges otherwise.

The largest order of increase of solutions are the exponentials of $x^{3/2}$; the distinguished exponentials are

$$\mathrm{e}^{\pm i\frac{x^{3/2}}{t^{1/2}}}$$

and there are exponential-power series solutions of the form

$$\hat{\hat{u}}_{n;\pm} = \mathrm{e}^{\pm i\frac{x^{3/2}}{t^{1/2}}} \left(\frac{x^{3/2}}{t^{1/2}}\right)^{-\frac{n}{3}-\frac{1}{2}} \times \tag{4.43}$$

$$\left[1 \mp i\frac{12n^2+48n+41}{72}\frac{t^{1/2}}{x^{3/2}} - \left(\frac{1}{72}n^4+\frac{5}{27}n^3+\frac{365}{432}n^2+\frac{55}{36}n+\frac{9241}{10368}\right)\frac{t}{x^3}+\cdots\right]$$

The series (4.42) and (4.43) are formal solutions of similarity reduction equations: substituting

$$u = t^{\frac{n}{3}} g\left(\frac{x^{3/2}}{t^{1/2}}\right) \quad , \quad z = \frac{x^{3/2}}{t^{1/2}}$$

equation (4.41) becomes the ordinary differential equation

$$z^2 g''' + z g'' + \left(z^2 - \frac{1}{9}\right) g' - \frac{2n}{3} z g = 0 \tag{4.44}$$

which, for each n, links the series $\hat{u}_n$ to the exponential series $\hat{\hat{u}}_{n;\pm}$. Inverse Laplace transform of (4.44) gives

$$\left(\frac{1}{2}+\frac{n}{3}+p^2\right) G(p) + \frac{1}{2}p(p^2+1)G'(p) = \frac{1}{18}\int_0^p rG(r)\,dr$$

with solution a combination of hypergeometric functions of the form

$$A\ {}_2F_1\left(\frac{5}{6},\frac{7}{6},\frac{3}{2}+\frac{n}{3},-p^2\right) + B\,p^{-1-\frac{2n}{3}}\ {}_2F_1\left(\frac{1-n}{3},\frac{2-n}{3},\frac{1}{2}-\frac{n}{3},-p^2\right) \tag{4.45}$$

for appropriate constants A, B (see §6.5.3 for the definition of ${}_2F_1$).

Next, considering solutions of the Airy equation which go to 0 as $x \to +\infty$, they can be expressed by inverse Laplace transform after the proper normalization $x^{3/2} = y$. Substituting $u(x,t) = v(y,t)$ equation (4.41) becomes

$$v_t - \frac{y}{2}v_{yyy} - \frac{1}{2}v_{yy} + \frac{1}{18y}v_y = 0 \tag{4.46}$$

which after inverse Laplace transform $v(y,t) = \int_\ell \mathrm{e}^{-qy} V(q,t)dq$ gives

$$V_t + q^2 V + \frac{1}{2}q^3 V_q = \frac{1}{18}\int_0^q rV(r,t)dr \tag{4.47}$$

As in the case of the heat equation, solutions of (4.47) can be expressed as superpositions of similarity solutions; in fact, for n a negative integer (not a multiple of 3) formulas (4.45) can be written in terms of elementary functions.

5. A nonlinear example

Consider the simple first order, nonlinear evolution equation

$$u_t + u_x\, u = 0 \tag{5.48}$$

whose general solution is given implicitly by

$$t\, u = x + \Phi(u) \tag{5.49}$$

where Φ is an arbitrary function.

Consider the formal solutions of (5.48) in the limit of §1.3.

A simple analysis shows that the maximal order of a power series solution is x: $u = \frac{x}{t} + o(x)$; other series solutions may start with any lower power: $u = cx^n + ...$ (with $\Re n < 1$).

Let us focus on the distinguished power series. Then substitute $u = \dfrac{x}{t} + v$ (where v is assumed much smaller than x), which yields

$$v_t + \frac{x}{t} v_x + \frac{1}{t} v = -v_x v \tag{5.50}$$

Since $v \ll x$, then $v_x \ll 1$ (if v is in a good class), therefore the linear part of (5.50) contains the largest terms, which by solving gives $v \sim \frac{1}{t}\Phi\left(\frac{x}{t}\right)$ (with $\Phi \ll 1$). Successive perturbations give, iteratively, the series solution

$$\begin{aligned} v \sim \frac{1}{t}\Phi + \frac{1}{t^2}\Phi'\Phi + \frac{1}{t^3}\left(\Phi'^2\Phi + \frac{1}{2}\Phi''\Phi^2\right) \\ + \frac{1}{t^4}\left(\Phi'^3\Phi + \frac{3}{2}\Phi''\Phi'\Phi^2 + \frac{1}{6}\Phi'''\Phi^3\right) + ... \end{aligned} \tag{5.51}$$

which can be viewed as a "nonlinearization" of the formal and actual solutions $\frac{1}{t}\Phi$ of the linear part, generalizing the same phenomenon seen for ordinary differential equations.

Exponential terms can be included in the function Φ (which is the same as in (5.49)); so there are clearly no distinguished exponentials.

6. Appendix

6.1. **Appendix 1.** From (3.17), (3.18) and Watson's Lemma we have

$$g(z; d^+) = \int_{d^+} \mathrm{e}^{-pz}\, p^{-\frac{n}{2}-1}\, (1-p)^{\frac{n}{2}-\frac{1}{2}}\, dp \sim \int_0^{+\infty} \mathrm{e}^{-pz}\, p^{-\frac{n}{2}-1}\, dp$$

which yields (3.22).

6.2. **Appendix 2.** From (3.17), (3.18) we have

$$g^{[e]}(z) \,=\, \frac{1}{2}\int_{\ell} \mathrm{e}^{-pz}\, p^{-\frac{n}{2}-1}\,(1-p)^{\frac{n}{2}-\frac{1}{2}}\,dp$$

$$=\, \frac{1}{2}\,\mathrm{e}^{-z}\, z^{-\frac{n}{2}-\frac{1}{2}} \int_{\ell_0} \mathrm{e}^{-r}\,\left(1+\frac{r}{z}\right)^{-\frac{n}{2}-1}\,(-r)^{\frac{n}{2}-\frac{1}{2}}\,dr$$

$$=\, \mathrm{e}^{-z}\, z^{-\frac{n}{2}-\frac{1}{2}}\,\left(C_n \,+\, O(z^{-1})\right)$$

where

$$C_n \,=\, \frac{1}{2}\int_{\ell_0} \mathrm{e}^{-r}\,(-r)^{\frac{n}{2}-\frac{1}{2}}\,dr \,=\, \frac{\pi i}{\Gamma\left(\frac{1-n}{2}\right)}$$

where the last equality follows by Hankel's formula for Γ-functions [16].

6.3. **Appendix 3.** (i)Assuming the integrand of (3.27) is analytic along $\ell = \mathrm{e}^{i\theta}\mathbb{R}_+$, it follows that $F(\zeta)$ must be analytic on the path $\zeta = q/(1-tq)$, $(q \in \mathrm{e}^{i\theta}\mathbb{R}_+)$ which is the arc $\mathcal{A}_{t,\theta}$ of the circle centered at $-1/(2t) \,+\, i\cos\theta/(2t\sin\theta)$ passing through the origin and the point $-1/t$ which lies in the upper half-plane for $\theta > 0$, respectively in the lower half-plane for $\theta < 0$.

(ii) Assuming in addition that the solution u_F is defined for all $t \geq t_0$ (for some $t_0 > 0$) it follows that F is analytic in the region bounded by $\mathcal{A}_{t_0,\theta}$ and the x-axis (possible excluding a finite number of singular points).

(iii) Assuming (i) and (ii) for all $\theta \in (0, \theta_0]$ (possibly excepting a discrete set of points), this entails that F is analytic in the outer region bounded by $\mathcal{A}_{t,\theta_0}$ and the x-axis (possible excluding a finite number of singular points).

6.4. **Appendix 4.** Each term $\hat{\hat{u}}_{-2m-2}$ is a finite sum and it will be shown that the series converges if for all $t > t_0 > 0$ if $F(\zeta)$ is analytic on the disk $|\zeta + \frac{1}{t_0}| < \frac{1}{c}$

In view of §3.5.1 we need to consider $\sum_{m\geq 0} F_m u^{[e]}_{-2m-2}$. By (3.23), (3.12) the superposition of the corresponding formal series is

$$\sum_{m\geq 0} F_m \frac{\pi i}{\Gamma(m+\frac{3}{2})}\hat{\hat{u}}_{-2m-2}$$

which is a series of finite sums (by (3.12))

$$= \pi i \mathrm{e}^{-\frac{x^2}{t}} t^{-\frac{1}{2}} \sum_{m\geq 0} F_m \frac{\Gamma(2m+2)}{\Gamma(m+\frac{3}{2})}\left(\frac{x}{t}\right)^{2m+1} \sum_{j=0}^{m} \frac{1}{j!(2m-2j+1)!}\left(-\frac{t}{4x^2}\right)^j$$

and using the duplication formula for Γ-function (see (6.53))

$$= \sqrt{\pi} i \mathrm{e}^{-\frac{x^2}{t}} t^{-\frac{1}{2}} \sum_{m\geq 0} F_m \left(\frac{2x}{t}\right)^{2m+1} \sum_{r=0}^{m} \frac{m!}{(m-r)!(2r+1)!} \left(-\frac{t}{4x^2}\right)^{m-r}$$

This series cannot be reordered in an asymptotic way. It will however be shown that it converges absolutely for t large enough.

Doing for the moment a formal calculation, the change of the order of summation gives

$$= 2\sqrt{\pi} i \mathrm{e}^{-\frac{x^2}{t}} t^{-\frac{3}{2}} x \sum_{r\geq 0} \frac{1}{(2r+1)!} \left(-\frac{4x^2}{t}\right)^r \sum_{m\geq r} F_m \frac{m!}{(m-r)!} (-t)^{-m}$$

and with the notation: $\Phi_r(\tau) = \frac{1}{r!} F^{(r)}(\tau)$ it follows

$$= 2\sqrt{\pi} i \mathrm{e}^{-\frac{x^2}{t}} t^{-\frac{3}{2}} x \sum_{r\geq 0} \frac{r!}{(2r+1)!} \left(\frac{4x^2}{t^2}\right)^r \Phi_r(-\frac{1}{t})$$

$$= \pi i \mathrm{e}^{-\frac{x^2}{t}} t^{-\frac{1}{2}} \sum_{r\geq 0} \frac{1}{\Gamma(r+\frac{3}{2})} \left(\frac{x^2}{t^2}\right)^{r+\frac{1}{2}} \Phi_r(-\frac{1}{t})$$

and using (6.55)

$$= \frac{1}{2} \mathrm{e}^{-\frac{x^2}{t}} t^{-\frac{1}{2}} \sum_{r\geq 0} \int_{c-i\infty}^{c+i\infty} \mathrm{e}^{\xi \frac{x^2}{t^2}} \frac{1}{\xi^{r+\frac{3}{2}}} \, d\xi \, \Phi_r(-\frac{1}{t}) \tag{6.52}$$

$$= \frac{1}{2} \mathrm{e}^{-\frac{x^2}{t}} t^{-\frac{1}{2}} \int_{c-i\infty}^{c+i\infty} \mathrm{e}^{\xi \frac{x^2}{t^2}} \xi^{-\frac{3}{2}} \sum_{r\geq 0} \frac{1}{\xi^r} \, d\xi \, \Phi_r(-\frac{1}{t})$$

$$= \frac{1}{2} \mathrm{e}^{-\frac{x^2}{t}} t^{-\frac{1}{2}} \int_{c-i\infty}^{c+i\infty} \mathrm{e}^{\xi \frac{x^2}{t^2}} \xi^{-\frac{3}{2}} F\left(\frac{1}{\xi} - \frac{1}{t}\right) \, d\xi$$

where substituting $\xi = t(1-tq)$ gives

$$= \frac{1}{2} \int_{c'-i\infty}^{c'+i\infty} \mathrm{e}^{-x^2 q} (1-tq)^{-\frac{3}{2}} F\left(\frac{q}{1-tq}\right) \, dq \qquad (c' = \frac{t-c}{t^2})$$

$$= \frac{1}{2} u^{++} - \frac{1}{2} u^{--}$$

Clearly the series converge absolutely, justifying thus the calculation above, if the Taylor series of F at the point $-\frac{1}{t}$, $F\left(\frac{1}{\xi} - \frac{1}{t}\right) = \sum_{r\geq 0} \frac{1}{\xi^r} \Phi_r(-\frac{1}{t})$ converges absolutely, for all ξ on the line of integration $c + i\mathbb{R}$, which follows if F is analytic on the disk centered at $-\frac{1}{t}$ and radius exceeding $\frac{1}{c}$. This holds for all $t > t_0$ if the positive numbers c and t_0 are such that $F(\zeta)$ is analytic on the disk $|\zeta + \frac{1}{t_0}| \leq \frac{1}{c}$ (such a t_0 and c always exist since F was assumed analytic at $\zeta = 0$). In

particular, the series converges absolutely for all $t > 0$ if F is analytic on the half-plane $\Re\zeta \leq 0$.

6.5. **Appendix 5: Some formulae.**

6.5.1. *Duplication formula for the* Γ*-function:* [16]

$$\sqrt{\pi}\Gamma(2z) = 2^{2z-1}\Gamma(z)\Gamma(z+\frac{1}{2}) \tag{6.53}$$

6.5.2. *Laplace and Inverse Laplace transformations.*

$$\int_0^\infty \mathrm{e}^{-p\xi} p^{n-1}\, dp = \frac{\Gamma(n)}{\xi^n} \quad (n \geq 1) \tag{6.54}$$

and conversely,

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \mathrm{e}^{py}\frac{1}{y^n}\, dy = \frac{p^{n-1}}{\Gamma(n)} \qquad (c > 0) \tag{6.55}$$

6.5.3. *The extended hypergeometric function* ${}_2F_1$.

$${}_2F_1(n_1, n_2, d) = \sum_{k=0}^{\infty}\prod_{i=1}^{2}\frac{\Gamma(n_i+k)}{\Gamma(n_i)}\,\frac{\Gamma(d)}{\Gamma(d+k)}\,\frac{z^k}{k!}$$

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