APPLICATIONS OF THE POLY-PAINLEVÉ TEST

BY RODICA DANIELA COSTIN

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ABSTRACT OF THE DISSERTATION

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by Rodica Daniela Costin

Dissertation Director: Professor Martin Kruskal

The present thesis contains results on integrability of ordinary differential equations, in the sense of finding whether a given equation, or class of equations, has single-valued first integrals in the complex domain. The method used is the poly-Painlevé test.

The first chapter contains an overview of different approaches to integrability, with special emphasis on the Painlevé property and the Painlevé test. The ideas of the poly-Painlevé test are presented.

The second chapter contains the study of the integrability properties of nonlinearly perturbed Euler equations near the singular point. (An Euler equation is an ordinary linear differential equation which is invariant under scaling transformations of the independent variable.) We allow first integrals to have essential singularities and give sufficient conditions for nonintegrability of the equations in the complex domain. We extend normal form theorems for singular equations and provide rigorous proofs for the results.

The first part of the third chapter contains the calculation of the monodromy group of generalized Lamé equations—a class of second order linear equations characterized by the presence of $k+1$ singular points in the extended complex plane, all of them regular, and a discrete symmetry. Based on the properties of this group, it is found that the only single-valued first integrals are functions of a hermitian form. The second part of

Chapter 3 uses the poly-Painlevé test to give sufficient conditions of nonintegrability for a comprehensive class of differential equations, which includes Hamiltonian systems with polynomial potentials. The analysis shows that nonintegrability of such equations follows from the same property of certain generalized Lamé equations.

The last chapter contains further applications of the poly-Painlevé test. The examples illustrate different techniques and approaches to the test.

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Dedication

To Denise Miriam

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List of Abbreviations

N	the set of nonnegative integers
Z	the set of integer numbers
R	the set of real numbers
C	the set of complex numbers

Chapter 1

Introduction

One of the fundamental questions in the theory of differential equations is to determine whether an equation is integrable or not. "When, however, one attempts to formulate a precise definition of integrability, many possibilities appear, each with a certain theoretical interest." (D. Birkhoff) [1]

1.1 An Historical Perspective

From the 17th throughout the mid 19th century, a lot of effort in the study of differential equations was directed towards finding methods of explicitly solving equations (in the sense of constructing solutions out of already known functions). Leibnitz, three of the Bernoullis (James, John, and Daniel), Riccati, Euler, Lagrange, Laplace are some of the most preeminent names in this field, and many classes of explicitly solvable equations bear their names. It was noticed very early that the primitive of a given function could only in very special cases be expressed in terms of previously known functions; this is then a fortiori the case for solutions of differential equations.

The study of linear equations brought to light some of their special features. The fact that a linear combination of solutions is again a solution made possible a comprehensive theory of the family of all solutions of a given linear equation. The idea that new functions can be defined as solutions of equations was brought to light and "transcendental" functions defined (Bessel functions, Airy functions, etc.).

The question (which is closely linked to the problem of expressing solutions in terms of already known functions), "which equations can be considered to define new functions?", is equally applicable to nonlinear equations. As the superposition principle is no longer valid, how could a generalization proceed? For linear equations, it is a consequence of the linearity of the space of solutions that the singular points of any solution can occur only at predetermined locations (the singular points of the equation). They are therefore *fixed*.[1] As a consequence, all the solutions of a given equation are defined on a common Riemann surface, which covers the complex plane with the singular points of the equation removed (see also [2]).

In his overview of the mathematical work of Paul Painlevé [3] Hadamard explains the historic context in which this work appears:

"Un beau problème sur les équations différentielles du premier ordre avait été inspiré, en 1884, à L. Fuchs par ses résultats relatifs aux équations linéaires. Ne pouvait-il, en dehors du cas linéaire, exister des équations différentielles ayant sinon toutes leurs singularités, du moins leurs points critiques fixes, les seules singularités variables avec la constante d'intégration étant celles autour desquelles l'intégrale générale reste uniforme?" [2]

Fuchs, and Briot and Bouquet, identified the first order equations (of a certain form) whose only critical points are fixed. This analysis was rendered rigorous by Painlevé, who proved a missing step in the earlier work, namely that first order equations (of a type regular enough) cannot have movable essential singularities (1888). He also extended the earlier results; the upshot was that first order equations with fixed critical points can be solved in terms of already known functions and therefore did not define new transcendents. (An overview can be found in [17].) Painlevé went on to consider second order equations.

Fuchs believed that the same methods used for first order equations could be applied to second order. However, second order equations may have movable essential singularities. Painlevé notes that this may happen only for equations of very special form:

"Cependant, la discordance avec le cas du premier ordre, sur ce point, n'est pas

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[1] It is also said that the locations of the singularities "do not vary [continuously] with the constants of integration".

[2] In the terminology used by Painlevé and his contemporaries the notion of "integral" refers to "solution", and the notion of "critical point" refers to a point at which branching takes place.

aussi profonde qu'elle le parait a première vue. Pour l'équation (3)

$$y'' = R(y', y, x) \ , \quad R \text{ rationnel en } y, y' \qquad (3)$$

[les coefficients de l'équation étant toujours supposés analytiques en x], elle ne peut se présenter que moyennant certaines restrictions très particulières apportées a la forme de cette équation." (Hadamard, [3])

It was thus possible to generalize the methods of analysis of first order equations to second order and Painlevé identifies the "equations which have no movable branch points".[3] His analysis, completed by Gambier and Fuchs, yields six equations, now known as P-I,...,P-VI, whose solutions cannot be expressed in terms of already known functions. Their solutions are now known as the *Painlevé transcendents*.

It is interesting to note that, although these new functions were discovered from strictly mathematical considerations, they have recently appeared in many physical applications (see [17] for an overview).

The question formulated by Fuchs led to other paths of research as well. The first important application of the property of an equation of having no movable critical points—which is now called the Painlevé property—as a method of identifying integrable cases of a system of differential equations (dependent on parameters) is due to Sophie Kowalewski. In her outstanding work on the theory of the motion of a rigid body (a top) rotating about a fixed point (1888) she determines the parameters of the problem for which the system of equations has no movable critical points. She then finds first integrals for these cases, and integrates the equations (using hyperelliptic integrals), thus solving a problem which had remained open from the eighteenth century.

In fact, Liouville integrability of Hamiltonian systems and integration by quadratures are closely connected. Indeed, if a Hamiltonian system with n degrees of freedom

[3]This often used terminology actually means that the solutions of the given equations have no movable branch points, in the sense that the location of such singular points do not vary continuously with the solution.

has n independent integrals in involution then it can be integrated by quadratures. The result was proven by E. Bour and generalized by Liouville [1].

We must also mention that many of the early rigorous results on nonintegrability of Hamiltonian systems are due to Poincaré.

A recent rigorous result on nonintegrability of analytic Hamiltonian systems is due to Ziglin [18]. Using "variational equations for known single-valued solutions" (perturbations around special orbits)—method proposed by Lyapunov—and investigating their monodromy groups (consisting of symplectic transformations) he finds necessary conditions for a system to possess meromorphic integrals, and he applies these conditions to various examples. The proof relies on the symplectic structure, and on the fact that only meromorphic integrals are considered. Under these assumptions, the results of Ziglin were generalized using differential Galois theory ([8], [9], etc).

A common feature of many methods of investigating the existence of invariants of Hamiltonian systems is the assumption of analyticity, or meromorphicity, of the first integrals. These assumptions often worked well. For example, many of the problems of Hamiltonian mechanics that have been integrated have constants of motion which can be continued to meromorphic functions in the complex domain. But this is not always the case. Recently, new examples of integrable rational Hamiltonians with nonmeromorphic invariants were found [10].

Also, in the case of systems presented as regular perturbations of simpler systems (with a small parameter, say ϵ) the common technique is to look for first integrals expressed as power series in ϵ (Poincaré series). However, there conceivably are cases when first integrals have essential singularities when $\epsilon = 0$.

1.2 The Painlevé Test

Very often, especially for applications, the notion of "integrable" is used with the same meaning as "having the Painlevé property": absence of movable branch points. Ablowitz, Ramani, and Segur [4] (see also [17]) use an algorithm, essentially the method used by Kowalevski, for determining whether solutions of a given differential equation have movable branch points. This method, now called the Painlevé test, has become one of the main tools for investigating integrability ([19] contains an overview). Kruskal pointed out [2] several further aspects that should be taken into account.

Kruskal proposes the following method of performing the Painlevé test. Since the test can be carried out only for "mostly analytic" equations, and more precisely, those which are presented in polynomial (or rational) form, some substitutions may be first performed to bring a given equation to a suitable form. Then one looks for generalized asymptotic series solutions for $t \to t_0$ (with terms containing powers and possibly $\log$, $\log\log$, ...). The point t_0 is generic (special values may be excluded, since one is not interested in fixed singularities). If one of the terms of the series solution contains noninteger powers or logs, then the solutions are multivalued, the point t_0 is a movable branch point (it is movable since t_0 depends on the solution), and the equation does not have the Painlevé property. A systematic way of searching for all the possible local behaviors of solutions as $t \to t_0$ is the method of dominant balance. In the limit, at least two terms in the equation (1.1) should be of the same order and dominant, and should balance in a first approximation.

We illustrate this procedure on the equation [4]

$$x\frac{d^2x}{dt^2} - \frac{dx}{dt} + 1 = 0 \tag{1.1}$$

Denote for simplicity $\tau = t - t_0$.

Case I: all three terms are of the same order. In particular, $\frac{dx}{dt}$ is of order 1. We then have $x \sim a + b\tau$ (with $b \neq 0$). Write $x = a + b\tau + u$, with $u << \tau$, $(\tau \to 0)$ and substitute in equation (1.1) to obtain the next term in the expansion of $x(t)$. We get

$$(a + b\tau + u)\frac{d^2u}{d\tau^2} - \frac{du}{d\tau} + 1 - b = 0$$

Case IA: If $a \neq 0$, then the terms $b\tau + u$ are much smaller than a and can be

[4] The equation (1.1) can be explicitly integrated using a substitution $v(x) = dx/dt$. We will comment later on this fact.

neglected in a first approximation.

Case IAi): If $1-b \neq 0$, then $\frac{du}{d\tau}$ is much smaller than $1-b$ and can be omitted. We get the equation[5]

$$a\frac{d^2u}{d\tau^2} + 1 - b \approx 0$$

with the solution $u \sim (b-1)/(2a)\tau^2$. Hence $x \sim a + b\tau + (b-1)/(2a)\tau^2$. We have a two parameter family of solutions. (The apparent three parameters: a, b, t_0 can be combined into only two arbitrary constants, as one expects.) One should make sure that the expansion can go on indefinitely as an integer power series in τ.

Case IAii) If $1-b=0$, then we get the equation

$$a\frac{d^2u}{d\tau^2} - \frac{du}{d\tau} \approx 0$$

with the solution $u \sim k_1 + k_2\exp(\tau/a) \not\ll \tau$, which contradicts the assumption of approximation.

Case IB: If $a=0$, then the term $b\tau$ cannot be neglected. The term u is much smaller than $b\tau$ and can be omitted in a first approximation.

Case IBi) If $b \neq 1$, then by neglecting $\frac{du}{d\tau}$ we get

$$b\tau\frac{d^2u}{d\tau^2} + 1 - b \approx 0$$

with solutions $u \sim O(\tau \ln \tau) \not\ll \tau$, hence this case does not yield an asymptotic series.

Case IBii) If $b=1$ then

$$\tau\frac{d^2u}{d\tau^2} - \frac{du}{d\tau} \approx 0$$

so $u \sim c\tau^2$. We get

$$x \sim \tau + c\tau^2 \tag{1.2}$$

[5]The sign " $\approx$ " is used in the equation in the usual sense that the quantity on the left side of the equality is much smaller than the individual terms of this quantity.

a two parameter series solution. (Unlike the expansion obtained in case IAi, here t_0 and c count as two parameters: a re-expansion of the series (1.2) around a different point t_0' yields a series which has a constant coefficient.)

Case II: The first two terms of equation (1.1) are of the same order, and the third one is much smaller.

To find the leading behavior of $x(t)$, as $t \to t_0$, in this case we solve the equation

$$x\frac{d^2x}{dt^2} - \frac{dx}{dt} \approx 0 \tag{1.3}$$

to leading order, in a region of the phase space where $\tau \ll 1$, $|\frac{dx}{dt}| \gg 1$.

Dividing by x and integrating (1.3) we get

$$\int_{x_0}^{x} \frac{d\xi}{\ln(c\xi)} \approx \tau$$

with c, x_0 arbitrary. The solutions are singular for $x = 0$ and $x = \infty$. Integrating by parts we get

$$\int_{x_0}^{x} \frac{d\xi}{\ln(c\xi)} \sim \frac{x}{\ln(cx)} \quad , \quad (x \to 0)$$

hence one possible leading behavior for x is obtained for

$$\frac{x}{\ln(cx)} = \tau$$

To solve this implicit equation for x small we denote $y = cx$ and $\tau' = c\tau$. Taking the logarithm, we get the equation

$$\ln y = \ln \tau' + \ln \ln y$$

For small y we have $\ln y \gg \ln \ln y$, hence to a first approximation y is $y_0 = \tau'$. The asymptotic expansion for y is obtained by iteration [6]:

$$\ln y_{n+1} = \ln \tau' + \ln \ln y_n \quad , \quad y_0 = \tau'$$

Therefore

$$\ln y_1 = \ln \tau' + \ln\ln \tau'$$

$$\begin{aligned}\ln y_2 &= \ln \tau' + \ln\left(\ln \tau' + \ln\ln \tau'\right)\\ &\sim \ln \tau' + \ln\ln \tau' + \frac{\ln\ln \tau'}{\ln \tau'} \qquad (\tau' \to 0)\end{aligned}$$

Thus $y \sim \tau' \ln \tau'$ which implies $x \sim c\tau \ln \tau$.

After a few calculations [to find the next term of the asymptotic series of $x(t)$] we get $x \sim \tau \ln \tau + k\tau$ (in fact $\tau \ln \tau + k\tau$ is an exact solution).

As a consequence, equation (1.1) has movable branch points (of logarithmic type), hence does not have the Painlevé property. (Further investigation of possible behaviors of solutions at $t = t_0$ is not necessary.)

We chose the simple example (1.1) to illustrate the Painlevé test because it shows some aspects which require care.

This example fails the Painlevé test due to the presence of solutions with behavior $x \sim (t - t_0)\log(t - t_0)$. Hence looking for power-type leading behaviors of solutions is not enough; logarithms must also be taken into account.

Also, even if equation (1.1) does not have the Painlevé property, it can be integrated once yielding the first integral $(\dot{x} - 1)\exp(\dot{x})/x = k$. So, in fact, the negative result of the test does not rule out the existence of single-valued first integrals.

Kruskal points out [2] other aspects which require care in the Painlevé test.

Firstly, it is not always clear when all the movable branch points have been determined (e.g. there are equations with no singular solutions).

Secondly, the so-called "negative resonances" should be taken into account, and nonintegral such resonances should be interpreted as a failure to pass the Painlevé test (see also [7]).

Thirdly, movable essential singularities can occur and they should be taken into account. Kruskal gives as example the equation

$$3u'''u' = 5u''^2 - u''u'^2/u - u'^4/u^2$$

with the general solution $u = a\exp(b(z-z_0)^{-1/2})$, where a, b, z_0 are arbitrary constants.

Commenting on this last issue, Hadamard notes [3]:

"Mais un fait inattendu qui apparaît, lorsque l'ordre dépasse deux, montre combien ces études difficiles peuvent être aussi—et les deux choses vont ensembles —fécondes en surprises. Il peut arriver, comme on le constate sur un exemple simple, qu'une famille particulière de solutions, définie par certaines relations entre les constantes d'intégration, présente des singularités essentielles mobiles, alors que l'intégrale générale en est dépourvue."

Moreover, another phenomenon which complicates a clear-cut definition of the Painlevé property (or a Painlevé test) for higher order equations is the possibility of movable boundaries of analyticity. An example is the Chazy equation (see e.g. [7])

$$u''' - 2u''u + 3u'^2 = 0$$

The maximal connected domain of definition of a generic solution of the Chazy equation is the interior or the exterior of a(n arbitrary) circle. However, the general solution can be obtained by simple transformations of elementary functions and quadratures. The question whether such an equation should be considered to have the Painlevé property is still debated.

A further unsatisfactory aspect of the definition of integrability by the Painlevé property is that absence of movable branch points is a coordinate dependent property (a system which is not "Painlevé" in one set of coordinates may become Painlevé in another). Equation (1.1) is such an example; we also cite the equation [17]

$$\frac{d^2y}{dt^2} = -\frac{1}{y}\left(\frac{dy}{dt}\right)^2 + y^5 + ty + \frac{\alpha}{y}$$

where α is a constant. The equation has movable branch points, since it has series solutions of the form

$$y(t) = \sum_{n\geq 0} y_n (t-t_0)^{n-\frac{1}{2}}$$

with y_n constants. However, the transformation $x = y^2$ yields the P-II equation.

Sometimes it is not clear whether substitutions can give a system having the Painlevé property. (To deal with some such cases, the "weak Painlevé property" was introduced, but this is not fully satisfactory either. See [19], [17].)

In spite of the many aspects not fully clarified yet, the Painlevé test was used extensively in applications as a way of investigating integrability (Grammaticos, Ramani, and colleagues considered many Hamiltonian and non-Hamiltonian systems), and successfully: practically all equations which were determined to have the Painlevé property were subsequently integrated (in some reasonable sense).

To conclude this short presentation of the Painlevé property as an indicator of integrability, we note that it does not constitute a rigorous theory (at present), and its dependence on the system of coordinates makes its application more difficult. Not all the specialists agree on all its aspects.

1.3 The Poly-Painlevé Test

Recently, Kruskal suggested and applied the idea that the important distinction, from the point of view of integrability of a differential equation, is not between single-valued and multivalued solutions, but rather between nondensely and densely multivalued solutions [5], [17].

We illustrate how the branching of solutions is linked to existence of single-valued first integrals on some very simple examples.

Consider the equation

$$\frac{dx}{dt} = \frac{a_1}{t - t_1}$$

in the complex plane. It has the general solution

$$x(t) = a_1 \ln(t - t_1) + C$$

hence

$$C = x - a_1 \ln(t - t_1)$$

is a (multivalued) first integral. Any other integral $F(x,t)$ of the equation is a function of the constant of integration:

$$F(x,t) = \Phi(C) = \Phi(x - a_1 \ln(t - t_1))$$

If F is single-valued, then Φ must be constant when $x(t)$ is analytically continued along closed paths, i.e. Φ must satisfy

$$\Phi(x - a_1 \ln(t - t_1)) = \Phi(x - a_1 \ln(t - t_1) + 2\pi i a_1 n_1)$$

for all $n_1 \in \mathbf{Z}$, and all complex $x, t,\ (t \neq t_1)$ in the domain of F. Therefore

$$\Phi(C) = \Phi(C + 2\pi i a_1 n_1)$$

for all C in the domain of Φ, and for all $n_1 \in \mathbf{Z}$. So Φ is a function of the exponential; for $\Phi(C) = \exp(C/a_1)$ we obtain $F(x,t) = (t - t_1)^{-1} \exp(x/a_1)$, a single-valued first integral.

Next, consider the equation

$$\frac{dx}{dt} = \frac{a_1}{t - t_1} + \frac{a_2}{t - t_2} \tag{1.4}$$

which has the general solution

$$x(t) = a_1 \ln(t - t_1) + a_2 \ln(t - t_2) + C$$

Hence the function

$$C = x - a_1 \ln(t - t_1) - a_2 \ln(t - t_2)$$

is a (multivalued) first integral. A single-valued integral has the form $F(x,t) = \Phi(C)$ with

$$\Phi(C) = \Phi(C + 2\pi i a_1 n_1 + 2\pi i a_2 n_2)$$

for all $n_1, n_2 \in \mathbf{Z}$, and for all C in the domain of Φ.

There are three possible cases.

Case I: if a_2/a_1 is real, and in particular rational, $a_2/a_1 = p/q$ with p, q relatively prime integers, then Φ has the same value at points which differ by an element of the set

$$\mathcal{S} = \{2\pi i a_1 n_1 + 2\pi i a_2 n_2 \ ; \ n_1, n_2 \in \mathbf{Z} \ \}$$

which equals $\mathbf{Z}$; therefore, just as in the previous example, the exponential is a uniformizing function for C and $F(x,t) = (t-t_1)^{-q}(t-t_2)^{-p}\exp(qx/a_1)$ is a single-valued first integral.

Case II: $a_2/a_1 = r$ is real, and in particular irrational. Then the set $\mathcal{S}$ is dense on the line of direction ia_1. We may write a constant of integration as

$$C' = \frac{x}{a_1} - \ln(t-t_1) - r\ln(t-t_2)$$

Any continuous function of C' must be constant on the vertical lines in the complex plane. Therefore, it is not holomorphic. However, there is a real-valued analytic function (defined in $\mathbf{C} \equiv \mathbf{R}^2$) with this property, namely the real part. We obtain the integral

$$F(x,t) = \Re\left(\frac{x}{a_1}\right) - \ln|t-t_1| - r\ln|t-t_2| \tag{1.5}$$

There is something peculiar about the integral (1.5): it has real values, hence the complex trajectories $(x(t), t)$ lie in (and fill densely) a set of real dimension 3 in $\mathbf{C}^2$. For a given equation, the existence of a holomorphic first integral implies that the trajectories lie on a surface of complex dimension 1 (hence real dimension 2), and the absence of any continuous first integral implies that the trajectories fill densely a set of complex dimension 2 (hence real dimension 4). Because of the dimension of the trajectories in cases such as (1.5), Kruskal proposed the notion of "half an integral".

Finally, Case III: a_2/a_1 is not real. Then the set $\mathcal{S}$ is a 2-dimensional lattice in the complex plane, and a function taking the same values at the points that are equivalent modulo the lattice $\mathcal{S}$ is a doubly periodic function. A single-valued first integral is found for Φ a doubly periodic function, of periods $2\pi i a_1$ and $2\pi i a_2$.

To illustrate dense branching, the following prototypical example is considered in [17]. The equation

$$\frac{dx}{dt} = \frac{a_1}{t-t_1} + \frac{a_2}{t-t_2} + \frac{a_3}{t-t_3} \tag{1.6}$$

in the complex domain, has the general (multivalued) solution

$$x(t) = a_1 \ln(t-t_1) + a_2 \ln(t-t_2) + a_3 \ln(t-t_3) + C \tag{1.7}$$

The question is, again, whether equation (1.6) admits a single-valued first integral.

Let $F(x,t)$ be a function which is constant on the solutions (1.7): $F(x(t),t) \equiv k$. Then F remains constant when $x(t)$ is analytically continued on paths in the complex plane. Consider a closed path in the t-plane, which encircles the points t_j a number n_j of times, $j = 1,2,3$. After analytic continuation on such a path, the initial value $x(t)$ of a solution becomes $x(t) + 2\pi i a_1 n_1 + 2\pi i a_2 n_2 + 2\pi i a_3 n_3$; hence we must have

$$k = F(x(t),t) = F(x(t) + 2\pi i a_1 n_1 + 2\pi i a_2 n_2 + 2\pi i a_3 n_3, t) \tag{1.8}$$

for all $n_j \in \mathbf{Z},\ j = 1,2,3$. For generic $(a_1, a_2, a_3) \in \mathbf{C}^3$ the set

$$\mathcal{S} = \{2\pi i a_1 n_1 + 2\pi i a_2 n_2 + 2\pi i a_3 n_3 \ ;\ n_j \in \mathbf{Z},\ j = 1,2,3\}$$

is dense in the complex plane (cf. Section 4.1), hence any continuous function F satisfying (1.8) must be a constant. Equation (1.6) has no continuous (single-valued) first integrals in the complex domain.

We have illustrated on examples a rigorous way of finding and proving whether a differential equation has first integrals, and of determining their regularity. Namely, we first found the general solution. We then solved with respect to the constants of integration, thus obtaining (multivalued, in general) first integrals. The question of existence of single-valued first integrals is equivalent to the question of existence of uniformizing functions for the constants of integration (whether there are functions which yield a single-valued function when applied to the constants).

In the case of an arbitrary equation one needs to find branching properties for the solutions without solving explicitly the equation. To this end, Kruskal proposed the study of asymptotic expansions of solutions, performed in appropriate regions of the phase space. The method used for obtaining such expansions is close to the Painlevé α-method.

This idea is applied in [17] to the equation

$$\frac{dx}{dt} = x^3 + t \tag{1.9}$$

The poly-Painlevé test applied to (1.9) goes as follows. Based on the intuition that nonintegrability of (1.9) can be seen in the behavior of solutions for large t, where the distance between different branch points of a solution is much smaller than the magnitude of t , the solutions are expanded for t in a patch near infinity, for large x, using the change of variables

$$t = \epsilon^{-1} + \epsilon^p T \quad , \quad x = \epsilon^q X \tag{1.10}$$

where $\epsilon \ll 1,\ p > -1$, and X, T vary in a finite domain. Equation (1.9) becomes

$$\epsilon^{q-p}\frac{dX}{dT} = \epsilon^{3q}X^3 + \epsilon^{-1} + \epsilon^p T$$

The maximal dominant balance (which enables the equation in the dominant order to have several branch points) is obtained for

$$q - p = 3q = -1, \quad \text{i.e. } p = \frac{2}{3},\ q = -\frac{1}{3}$$

hence

$$\frac{dX}{dT} = X^3 + 1 + \tilde{\epsilon}T, \quad \text{where } \tilde{\epsilon} = \epsilon^{5/3} \tag{1.11}$$

Equation (1.11) is analyzed by perturbation theory. Since it is a regular perturbation, the solutions are given, locally, by convergent power series in $\tilde{\epsilon}$.

It is more convenient to represent T as a function of X:

$$T(X) = T_0(X) + \tilde{\epsilon} T_1(X) + \tilde{\epsilon}^2 T_2(X) + ... \tag{1.12}$$

The terms T_n are calculated order by order. For T_0 one obtains

$$T_0(X) = k + \int_{\infty}^{X} \frac{1}{z^3+1} dz$$

Since the solutions of (1.9) form a one parameter family, and a parameter, $\tilde{\epsilon}$, has already been introduced, we may choose a particular value for k (generic, $O(1)$ when $\epsilon \to 0$). The function $T_0(X)$ is branched, with the uncertainty forming a parallelogram lattice in the T-space. Hence, an appropriate elliptic function applied to $T_0(X)$ yields a uniform function (i.e. $T_0(X)$ can be regularized). Further analysis shows that the same is true for $T_0(X)+\tilde{\epsilon}T_1(X)$. However, the uncertainty for the function $T_0(X)+\tilde{\epsilon}T_1(X)+\tilde{\epsilon}^2T_2(X)$ is dense in the T-space. Hence, there is no regularization for $T_0(X)+\tilde{\epsilon}T_1(X)+\tilde{\epsilon}^2T_2(X)$ and it is argued that the same will be true for the whole series, thus establishing the nonintegrability of (1.9).

One natural question that appears in connection with this method is: what is the significance of the small parameter ϵ which is introduced in the equation? A substitution like (1.10) determines the form of a region of the space (where an expansion will be obtained)—by giving absolute and relative orders of magnitude for t and x. In the series solution (1.12) the parameter ϵ can be absorbed in a constant of integration.

Another natural question is: can we consider a region where, after introducing the small parameter ϵ, the equation becomes singularly perturbed? It is known that for singularly perturbed equations the form of an asymptotic series solution may depend on sectors (Stokes' phenomena); multivalued expansions may correspond to single-valued solutions and the analysis may become very difficult.

The last step in establishing the nonintegrability of (1.9), namely deducing dense branching for the solutions of (1.9) from the dense branching of a truncation [in this case $T_0(X)+\tilde{\epsilon}T_1(X)+\tilde{\epsilon}^2T_2(X)$] of the series solution, is believed to be generally true, and can be formulated as follows:

Conjecture: Consider an ordinary differential equation. A small parameter is introduced using a substitution, such that in the new variables the equation is presented as a regular perturbation. If truncations to a certain order of the perturbation series solutions have dense branching, then the true solutions also have dense branching (at least in generic cases); therefore, the given equation has no continuous first integrals.

It is not difficult to show that such a result holds under supplementary assumptions — for example, if one assumes that first integrals have power series developments in the small parameter. However, in the general case such regularity assumptions may be too restrictive.

Finally, Kruskal pointed out a further aspect. If one is concerned with the dimensionality of the set formed by trajectories, then the notion of single-valued first integral is not necessarily the most appropriate one, and some branched integrals should be accepted (for example, integrals with an algebraic branch point). So again, the distinction is not necessarily between single-valuedness and multivaluedness but rather between different types of multivaluedness.

1.4 Description of the Results and Conclusions

In recent years integrability analysis, in the sense of finding whether an equation has first integrals (and how many independent ones) has become an important issue. The study of complex phenomena often has to be described in terms of global properties of the time evolution of a system, such as ergodicity, or the onset of chaos. The number of independent first integrals of a system of ordinary differential equations determines the dimensionality of the space that the trajectories fill.

Nonintegrability, ergodicity, and being chaotic are distinct qualitative features. If a differential system has solutions which exhibit chaotic behavior (in a region of the phase space), then, of course, the system will not be integrable (in that region). On the other hand, nonintegrability does not imply chaos [6], since a nonintegrable system can have a very "orderly" evolution (e.g. harmonic oscillators with irrationally related

[6] This is sometimes tacitly assumed.

frequencies, discussed in Example 1 of Section 3.2.1).

The present thesis contains applications and extensions of the poly-Painlevé test to higher order equations. The test is used for determining whether a given differential equation has first integrals which are holomorphic on certain domains of the (complex) phase space and how many independent integrals there are. Criteria for nonintegrability of classes of equations are found and rigorous proofs are given.

In applying the poly-Painlevé test to an equation believed to be nonintegrable, one of the first issues to be resolved is to find an appropriate domain for expanding the solutions. One must find an expansion such that a truncation of the series solution exhibits dense branching. However, the terms of the series should have a form simple enough that the branching can be studied. For first order equations intuitive considerations on this issue are found in [17]; as a rule, one studies the equation near singular points, where the branch points of solutions may accumulate. For higher order equations one must consider singular manifolds (as used in Chapter 2, and in Section 4.2).

In the case of higher order equations, expansions near singular manifolds may not yield bad branching even in nonintegrable cases. Chapter 3 investigates a comprehensive class of such equations (which includes Hamiltonian systems with polynomial potentials). It is argued that if solutions have oscillatory behavior nonintegrability may be due to the presence of oscillations with "badly" related frequencies. The poly-Painlevé test is therefore performed in a region of the phase space encompassing periodic trajectories.

In these regions, the first order approximation of an equation in the class considered in Chapter 3 is a system of generalized Lamé equations. The first part of the chapter is devoted to the analysis of these linear equations, characterized by the presence of $k+1$ singular points in the extended complex plane, all of them regular.

The branching properties of the solutions of a linear equation are encoded in the monodromy group. A discrete symmetry of the equations allows explicit calculation of this group (whose determination is a question interesting in itself). Sufficient criteria for nonexistence of single-valued first integrals are found (in a class of functions which may have essential singularities).

The next step in the poly-Painlevé test is to show that nonintegrability of the reduced system of Lamé equations implies nonintegrability of the original equation. Then the sufficient criteria for nonintegrability that were found in Section 3.1 also apply to the original (nonlinear) systems studied in Section 3.2. A rigorous proof of this fact is given in the case when only first integrals which are locally meromorphic are considered (in a neighborhood of the particular trajectory around which the expansion is made).

We believe that a rigorous proof can be obtained even if one allows first integrals with essential singularities, and that these more singular integrals should also be considered. One possible way of reasoning may be analogous to the one the Chapter 2: using reduction to a normal form. The investigation of this issue remains open for further research.

It is interesting to note that some of the equations studied in Chapter 3, using the poly-Painlevé test, are also studied by Ziglin [18] and other researchers following his path (see for example [12]).

A specific equation which is analyzed in both Chapter 3 and [18] is the Hénon-Heiles system. A lot of effort, by many people, went into deducing nonintegrability of this equation, since it is the first example on which chaotic behavior was found using numerical simulations.

Our approach suggests that Ziglin's method is less general than the poly-Painlevé test. Indeed, Ziglin's theorem applies only to Hamiltonian systems. Also, it can be viewed as a lowest order poly-Painlevé test (see [14] for an example). The reduced system is studied in both approaches: it may or may not turn out to be integrable. In the latter case, both tests yield nonintegrability. In the former case, Ziglin's theorem can imply nothing about the integrability properties of the original system, while in the poly-Painlevé test one could proceed to the study of the higher order terms of the solutions, which might display bad branching (hence, nonintegrability may be found).

The numerical studies on the Hénon-Heiles system were performed in the real domain; the main interest for this Hamiltonian system is also in the real phase space. Then the question raised by Kruskal [17] of finding the relevance of the analysis in the complex domain for the real one is of high interest (at least in this example). The results in Chapter 3 give more information than Ziglin's theorem: we show that in the first approximation the Hénon-Heiles system (and of all others in the class that we study in Chapter 3) has an integral in the real domain: a real analytic constant (half an integral in the complex domain) is found (situation similar to that of equation (1.4), case II). The poly-Painlevé test enables (at least in principle) the study of the next order approximation, where the existence of this integral may be contradicted. (It is fairly typical when applying the poly-Painlevé test that one finds more and more branching values when the order of the approximation analyzed increases. See for example [17] or the equation studied in Section 4.3.)

Another important issue to the mathematician applying the poly-Painlevé test is to find a rigorous proof for the conjecture presented in Section 1.3 (in general, or in specific cases). The test uses dense branching for an asymptotic approximation for the solution to deduce dense branching for the exact solution, and the question is to prove this deduction without additional assumptions on the first integrals (such as meromorphicity, or algebraic character).

The main difficulty in a rigorous proof is the following. Consider the case of equation (1.9). General techniques in the theory of regularly perturbed equations can be used to show that the series solution (1.12) converges for X confined on a compact set on the Riemann surface of $T_0(X)$. However, to establish density of the values on all branches, one way would be to show that the series converges on an infinitely long path (which encircles each singular point of the solution an arbitrary number of times). For a generic regularly perturbed equation the radius of convergence of the perturbation series solution decreases to 0 as the length of the path goes to ∞. [There is no contradiction between this general fact and the need, for a proof of the poly-Painlevé test, that the series converges on an infinitely long path, since equations in which a small parameter is introduced by a substitution (as in the α-method) form a particular class among the regularly perturbed equations: the small parameter is internal, in the sense that all the equations with $\epsilon \neq 0$ are equivalent. We note that the *reduced* equation (i.e. the equation for $\epsilon = 0$) is generally not equivalent to the original equation; it is normally chosen to be "simpler" than the original one, at least with respect to finding the multivaluedness of the solutions.]

We deal with the issue of finding rigorous proofs without assumptions of meromorphicity in Chapter 2. We study the integrability properties of a nonlinear differential equation whose linearization has one regular singular point in the complex plane.

We argue that, if equations in this class have single-valued integrals, then, generically, these integrals have essential singularities. We consider the problem of finding sufficient criteria for the existence of single-valued first integrals (in a class of functions which may have essential singularities). Sufficient conditions for nonintegrability are found. To transform these criteria into rigorous results, we extend normal form theorems and argue that equivalence to normal forms captures the spirit of the poly-Painlevé test and is a powerful tool for a rigorous approach to nonintegrability.

The last chapter of the thesis contains further applications of the poly-Painlevé test.

Section 4.1 gives the correct formulation, and the proof, of a density result which was incorrectly stated in [17]. The lemma gives necessary and sufficient conditions for an integer lattice with three generators to be dense in the complex plane. This result is needed in many applications, such as the equation (1.9) analyzed in [17], the one studied in Sections 4.2 and 4.3, and for (1.6).

Section 4.2 contains the study of a model in statistical mechanics; nonintegrability is established.

Section 4.3 contains an application of the poly-Painlevé test to a first order equation with two singular points in the complex plane; the problem was left open in [17].

Finally, Section 4.4 illustrates, on two examples, how the poly-Painlevé test can be used, in cases when the result is nondense branching, to obtain asymptotic approximations for conserved quantities.

We add some final remarks. The definition of integrability, as given by Kruskal [17], gives necessary and sufficient conditions for a differential equation to have single-valued first integrals in the complex plane. The method of investigation that he proposes constitutes a powerful and productive tool.

We applied the test and extensions of it to classes of higher order equations and found criteria and rigorous proofs of nonintegrability. In the course of the investigation, new questions were brought to light. We believe that the ideas and techniques of the poly-Painlevé test constitute a basis for further rigorous investigation of integrability of differential equations, and that our research is just a beginning in this field.

Chapter 2

Integrability Properties of Nonlinearly Perturbed Euler Equations

2.1 Introduction

We study the integrability properties of nonlinearly perturbed Euler equations near the singular point. (An Euler equation is an ordinary linear differential equation which is invariant under scaling transformations of the independent variable.) We allow first integrals to have essential singularities and give sufficient conditions for the nonintegrability of the equations in the complex domain. We extend normal form theorems for singular equations and provide rigorous proofs for the results.

The present chapter is based on results presented in [15].

2.1.1 Motivation, and Some Known Results

Perhaps the simplest example of using the poly-Painlevé test to deduce nonintegrability (in the sense of nonexistence of holomorphic first integrals on given domains) is provided by the equation

$$xu' = \mu u + h(x, u) \tag{2.1}$$

where $h(x, u)$ is holomorphic at $x = u = 0$, has a zero of order 2 at $u = 0$ (so that $h(x, u) = u^2 h_1(x, u)$ with $h_1(x, \cdot)$ holomorphic at $u = 0$) and μ is a complex parameter.

In order to study the multivaluedness of the solutions, the poly-Painlevé test devised by Kruskal uses a technique analogous to the α-method: introduce a small parameter into the equation, calculate the series solutions, and study the multivaluedness of its terms. This amounts to study of the equation in a certain region of the phase space.

In our example (2.1), we choose to study the equation for small values of u. So we set up the equation in a regularly perturbed form by introducing a small parameter ϵ via the substitution $u = \epsilon U$. The equation becomes

$$xU' = \mu U + \epsilon^{-1} h_1(x, \epsilon U) \quad , \quad \text{where} \quad \epsilon^{-1} h_1(x, \epsilon U) = O(\epsilon) \tag{2.2}$$

and the solutions U can be found as perturbation series in ϵ:

$$U = U_0 + \epsilon U_1 + \epsilon^2 U_2 + ... \tag{2.3}$$

The nonintegrability test is very easy to perform in this example because of the following features: the reduced equation (i.e. the equation with $\epsilon = 0$) has only one singular point ($x = 0$); the multivaluedness of U_0 can be readily found and the integrability, or nonintegrability, depending on the value of μ, can be easily established.

Indeed, the test proceeds as follows.

First, the perturbation series for the solutions U is calculated. We solve (2.2) with any initial condition $U = k$ at $x = x_0$. The point x_0 is arbitrary ($x_0 \neq 0, \infty$). Say $x_0 = 1$. In terms of the initial dependent variable u, it means that we are considering the one-parameter family of solutions with initial condition $u(1) = \epsilon k$. But (if $k \neq 0$), the constant k can be absorbed in ϵ, hence we may take $k = 1$.

We note the significance of the "artificially" introduced parameter ϵ: in fact, in our example ϵ is now an initial condition and the expansion (2.3) is an expansion in the initial data.

The reduced equation has the solution $U_0 = x^\mu$ hence $U \equiv U(x;\epsilon) = x^\mu + O(\epsilon)$. The solutions are multivalued (if μ is not an integer). The power series in ϵ converges for $|\epsilon|$ small enough and for x belonging to a closed path γ starting and ending at 1, encircling the origin (as will be shown in Lemma 1).

Assume there is a single-valued function F constant on the solutions of Eq. (2.1): $F(x, \epsilon U(x;\epsilon)) = \text{const}$. Then if $U^{ac}(x;\epsilon)$ denotes some analytic continuation of $U(x;\epsilon)$ (on a closed path) we must have

$$F(x, \epsilon U(x;\epsilon)) = F(x, \epsilon U^{ac}(x;\epsilon)) \tag{2.4}$$

or, using the (convergent) ϵ-expansion,

$$F(x, \epsilon U_0(x) + \epsilon^2 U_1(x) + ...) = F(x, \epsilon U_0^{ac}(x) + \epsilon^2 U_1^{ac}(x) + ...) \tag{2.5}$$

For heuristic purposes, suppose we keep only the first term in the series above and discard the rest; we get

$$F(x, \epsilon U_0(x)) \approx F(x, \epsilon U_0^{ac}(x)) \tag{2.6}$$

or

$$F(x, \epsilon x^{\mu}) \approx F(x, \epsilon x^{\mu} e^{2n\pi i\mu})$$

for all integers n.

If μ is real, irrational, the set $\{e^{2n\pi i\mu}; n \in \mathbf{Z}\}$ is dense in the unit circle. Therefore F cannot depend on the second variable. But then F must be constant, so the reduced equation has no single-valued first integrals. When the poly-Painlevé test is done in practice, one immediately concludes that the original equation (2.2) has no first integrals as well. This last step is intuitively appealing, but the question is: is it always correct?

To formulate more precisely the question, let us consider (2.4) for a fixed value of x, say $x = 1$ (for other values of x we may simply rescale ϵ) and denote $\phi(\epsilon) = \epsilon U^{ac}(1;\epsilon)$ where the analytic continuation is considered along a closed path encircling the origin once, counterclockwise. So ϕ is holomorphic in a neighborhood of the origin and $\phi(\epsilon) = \epsilon e^{2\pi i\mu} + O(\epsilon^2)$. With this notation equality (2.4) can be written as

$$F(1, \epsilon) = F(1, \phi(\epsilon))$$

which can be iterated and gives

$$F(1, \epsilon) = F(1, \phi(\epsilon)) = F(1, (\phi \circ \phi)(\epsilon)) = ... = F(1, \phi^{n\circ}(\epsilon))$$

where

$$\phi^{n\circ}(\epsilon) \equiv (\phi \circ ... \circ \phi)(\epsilon) = \epsilon e^{2n\pi i\mu} + O(\epsilon^2)$$

Therefore, $F(1, \cdot)$ must have the same values on all the iterates $\phi^{n\circ}(\epsilon)$ of $\phi(\epsilon)$. While the iterates $\epsilon e^{2n\pi i\mu}$ of the linear part of ϕ have values dense on a closed curve (a circle) it is not clear that the same will be true for the iterates of $\phi(\epsilon)$. We can not directly infer nonintegrability of the original equation based on the same property of its linear approximation, and the α-method needs more justification.

A common feature of most nonintegrability tests is the presence of supplementary assumptions (playing an important role in the proofs) on the first integrals (regularity, meromorphicity, or algebraic character). But these are somewhat artificial: in fact, a first integral with essential singularities (like, say, the integral $F(x,u) = x\exp(u^{-1})$ for the equation $du/dx = u^2/x$) is, for most purposes, just as good as a meromorphic one—it implies that the trajectories are confined to a lower dimensional manifold.

On the other hand, in many cases meromorphicity is a quite strong condition: it implies that the reduced equation must have a *rational* (in u or/and in x) first integral; analyticity, as an assumption, implies the existence of an integral which is polynomial in U_0. To illustrate this, consider the example of equation (2.1). Suppose that there is a first integral F which is regular at $u = 0$. We may then expand in power series:

$$\begin{aligned}
&F(x,u(x)) = F(x,\epsilon U_0(x) + \epsilon^2 U_1(x) + ...) = \\
&\quad F(x,0) + F_u(x,0)\left(\epsilon U_0(x) + \epsilon^2 U_1(x) + ...\right) + \\
&\quad \frac{1}{2}F_{uu}(x,0)\left(\epsilon U_0(x) + \epsilon^2 U_1(x) + ...\right)^2 + ... = \\
&\quad F(x,0) + \epsilon F_u(x,0)U_0(x) + \epsilon^2\left(F_u(x,0)U_1(x) + \frac{1}{2}F_{uu}(x,0)U_0(x)^2\right) + ...
\end{aligned}$$

and if F is constant on the solutions, it follows that $F(x,0) \equiv \text{const}$ and that $\partial_u^k F(x,0)U_0^k$ is a first integral for the reduced equation (where k is the smallest number such that $\partial_u^k F(x,0) \not\equiv 0$). Similarly, if we only require meromorphicity of F, we get that a first integral of the reduced equation must have the form $f(x)U_0^k$ where k is an integer.

We thus find that for the equation (2.1) to have a first integral which is meromorphic at $u = 0$ it is necessary that μ be rational.

The same holds, of course, for the equation $xu' = \mu u$. However, this equation is in fact integrable for almost all the values of μ, namely for any μ not real: there is a single-valued first integral which has essential singularity at $u = 0$. Indeed, $ux^{-\mu}$ is constant on the solutions, and so is $\ln u - \mu\ln x$. The multivaluedness of the last expression is $\ln u - \mu\ln x + 2n\pi i - 2m\mu\pi i$ where n, m are arbitrary integers. If μ is not real, let $\mathcal{F}$ be a doubly periodic function, with periods $2\pi i$ and $2\mu\pi i$. Then the function $F(x,u) = \mathcal{F}(\ln u - \mu\ln x)$ is a *single-valued first integral* of the equation. Clearly, $F(x,\cdot)$ has essential singularity at $u = 0$ (accumulation of poles), for all x for which the function is defined. [1]

Setting $h(x,u) = u^2$ in equation (2.1) (with μ not real) we get another example of integrable equations with singular first integrals. Indeed, the general solution is $u = -\mu/(1+cx^{-\mu})$ (and there is an additional solution $u=0$). A single-valued first integral can be obtained as in the case $h=0$. Namely, from $c = -(\mu u^{-1}+1)x^{\mu}$ we find that the function $F(x,u) = \mathcal{F}(\ln(\mu u^{-1}+1)+\mu\ln x)$ is a single valued first integral if $\mathcal{F}$ is a doubly periodic function of periods $2\pi i$ and $2\pi i\mu$. As before, $F(x,\cdot)$ has essential singularity at $u=0$.

We have thus found in the first order nonlinearly perturbed Euler equations (2.1) a class of examples on which the first integrals must be thought of as typically not meromorphic, and not algebraic. The question is then to find a rigorous method, or argument, for proving nonintegrability (if that is the case). The present paper aims at answering this question on a (more general) class of equations.

2.1.2 Description of the Main Results

The present work is a study of the integrability properties, in the complex domain, of a class of differential equations: the nonlinearly perturbed Euler equation

$$x^n u^{(n)} + c_{n-1}x^{n-1}u^{n-1} + ... + c_0 u = h(x,u,u',...,u^{(n-1)}) \tag{2.7}$$

where $c_j \in \mathbf{C}$, h is a holomorphic function of $n+1$ variables in a neighborhood of the origin in $\mathbf{C}^{n+1}$ and has a zero of order 2 at $(u,u',...,u^{(n-1)}) = (0,0,...,0)$, and more generally, for systems

$$xu'(x) = Mu(x) + a(x,u) \qquad u(x), a(x,u) \in \mathbf{C}^n,\ x \in \mathbf{C} \tag{2.8}$$

where M is a constant matrix, $a = (a_1,...,a_n)$, with $a_j(x,u)$ holomorphic for x in an annulus centered at 0 and u small, and having a zero of order 2 at $u=0$. (The substitution $(x\frac{d}{dx})^j u = u_{j+1}, j = 0,...,n-1$ transforms the equation (2.7) into a system of the type (2.8) where $a_n = h(x,u_1,u_2/x,...)$ so a might be singular at $x=0$.)

---

[1] There is more than one definition for an "essential singularity" for functions of *several* complex variables. We will not deal with the subtle aspects of this issue, but merely prove "nonmeromorphicity."

In order to perform the poly-Painlevé test in the region where the vector u is small, and x is of order 1, one can proceed as in the one-dimensional case: the substitution $u = \epsilon U$ in (2.8) yields

$$xU'(x) = MU(x) + \epsilon^{-1}a(x, \epsilon U) \quad , \quad \text{where } \epsilon^{-1}a(x, \epsilon U) = O(\epsilon) \quad (\epsilon \to 0) \tag{2.9}$$

The reduced ($\epsilon = 0$) equation (which is also the linearized equation) is studied in Section 2.2.1. We consider the question of existence of first integrals which are holomorphic functions of $n+1$ variables on a domain $x \in D_x \subset \mathbf{C}$, $u \in D_u \subset \mathbf{C}^n$, where D_x contains a loop surrounding the point $x = 0$ (i.e. sufficiently general to permit analytic continuation of solutions around the branch point). Sufficient conditions for nonintegrability (within this class) are found. Also, when first integrals do exist, they are shown not to be meromorphic near the variety $u = 0$ (in generic cases).

The main issue is to prove that nonintegrability of the reduced equation implies nonintegrability of the original equation.

We need to specify the class of regularity for the first integrals considered for equations (2.8). We are not assuming that the integrals are meromorphic near $u = 0$ (and thus, do not necessarily have power series in ϵ). Instead, we deal with a more general class of functions:

Requirement R: We require that the integrals $F = F(x, u)$ be holomorphic on a domain of the form $D_x \times D_u$ where $0 \in \partial D_u$ and $D_x \subset \mathbf{C}$ is such that in any neighborhood of $x = 0$ there is a closed loop around the origin contained in D_x. (For example, functions F holomorphic on an open set of the form $\mathbf{C}^{n+1} \setminus V$, where V is a holomorphic variety consisting of singularities of F, satisfy the requirement.)

In Section 2.2.2 it is shown that, for almost all matrices M, equation (2.8) is holomorphically equivalent to its linear part (2.10), on a certain domain in the phase space. This type of result is usually referred to, in the literature, as "equivalence to a normal form" [20].

There are exceptional cases when the result of Section 2.2.2 does not hold and the following subsections are devoted to the study of the integrability properties in such cases. To this end, the multivaluedness of solutions is studied.

In order to find the values of solutions on closed paths surrounding the origin, we first show, in section 2.2.3, that the solutions are defined on such paths. Then, in section 2.2.4, the form of the solutions after analytic continuation is given.

Section 2.2.6 gives sufficient conditions for the multivaluedness of the solutions of (2.9) to form a set holomorphically equivalent to the one corresponding to the reduced equation (2.10). As a consequence, whenever the reduced equation is not integrable (and the sufficiency conditions fulfilled) in the class of first integrals specified above , so is the original equation, in the class of first integrals satisfying Requirement R.

There are still cases when the results of neither Section 2.2.2 nor Section 2.2.6 apply. Section 2.2.5 deals with them in the one-dimensional case. Further investigation is needed for higher dimensions.

2.2 Main Results

Consider the equation (2.8). We will assume in what follows that the characteristic exponents of its linear part, i.e. the eigenvalues $\mu_1, ..., \mu_n$ of M, are distinct.

We may therefore assume that the matrix M is diagonal:
$M = \text{diag}\{\mu_1, ..., \mu_n\}$.

2.2.1 Integrability Properties of the Reduced Equation

We will first study the reduced equation

$$xU'(x) = MU(x) \tag{2.10}$$

We will distinguish several cases.

Remark 1 *If at least one of the characteristic exponents μ_j is not real, then the equation (2.10) has n independent single-valued first integrals.*

Proof

A fundamental system of solutions is $u_j = c_j x^{\mu_j}, j = 1, ..., n$. Hence $c_j = u_j x^{-\mu_j}, j = 1, ..., n$ are (typically multivalued) first integrals. We will use them to produce single-valued ones.

For $j = 1, ..., n$ such that μ_j is not real, consider the function $F_j(x,u) = \mathcal{F}_j(\ln u_j - \mu_j \ln x)$ (where $\mathcal{F}_j$ is a doubly periodic function, of periods $2\pi i$ and $2\pi i \mu_j$).

Suppose that μ_1 is not real. For j such that μ_j is real, the function $F_j(x,u) = \mathcal{F}_j(\mu_1 \ln u_j - \mu_j \ln u_1)$ (where $\mathcal{F}_j$ is a doubly periodic function, of periods $2\pi i \mu_1$ and $2\pi i \mu_j$) is a single-valued first integral. ∎

The integrals that we found are, clearly, not meromorphic (near $u = 0$). The next proposition shows that (for generic equations) this is the case for any single-valued first integrals.

Proposition 1 *Consider the equation (2.10). Assume that the numbers* $1, \mu_1, ..., \mu_n$ *are linearly independent over* $\mathbf{Z}$.

Denote by $\mu_1, ..., \mu_q$ *the real characteristic exponents, and* $\mu_{q+1}, ..., \mu_n$ *the non-real ones.*

If the numbers $\Im\mu_{q+1}, ..., \Im\mu_n$ *are linearly independent over* $\mathbf{Z}$, *then there is no single-valued first integral meromorphic on a domain* $D = D_x \times D_u$ *with* $0 \in D_u$.

Proof

Step I: we first show (as we stated in the introduction) that the existence of a meromorphic local first integral implies the existence of a rational homogeneous one.

Suppose that the equation (2.10) admits a single-valued first integral

$F(x,u)$, meromorphic on D. Then F is a function of the n constants of integration $c_k = u_k x^{-\mu_k}$:

$$F(x,u) = \Phi(u_1 x^{-\mu_1}, ..., u_n x^{-\mu_n}) \tag{2.11}$$

Fix $x \neq 0$ for which the function $F(x, \cdot)$ is defined (as a meromorphic function on D_u). Then (2.11) clearly implies that Φ is meromorphic on a neighborhood B of the origin.

Set

$$\lambda_j = e^{2\pi i \mu_j} \tag{2.12}$$

Since F is single-valued, Φ must satisfy

$$\Phi(c_1, ..., c_n) = \Phi(\lambda_1^{-1} c_1, ..., \lambda_n^{-1} c_n) \tag{2.13}$$

for all c in a suitable neighborhood of the origin.

We introduce a parameter α in the problem to split the first integral into terms of different degrees; contibutions of different degrees will be first integrals.

Therefore, substitute $U = \alpha V$. The new variable V satisfies the same differential equation as U and there is a meromorphic first integral, namely $F(x, \alpha V) = \Phi(\alpha C_1, ..., \alpha C_n)$ where $C_k = V_k x^{-\mu_k}$. Since Φ is meromorphic, there are two functions f, g, analytic near 0, such that $\Phi = f/g$. By expanding in Taylor series and collecting the terms in α we get

$$\Phi(\alpha C_1, ..., \alpha C_n) = \frac{f(\alpha C_1, ..., \alpha C_n)}{g(\alpha C_1, ..., \alpha C_n)} = \frac{\sum_{r \geq r_0} \alpha^r P_r(C_1, ..., C_n)}{\sum_{s \geq s_0} \alpha^s Q_s(C_1, ..., C_n)}$$

where P_r, Q_s are polynomials (homogeneous of degrees r and s respectively) and by assumption $P_{r_0}, Q_{s_0} \not\equiv 0$. By expanding further as a (convergent) power series in α, we get

$$\Phi(\alpha C_1, ..., \alpha C_n) = \alpha^{r_0 - s_0} \left(\frac{P_{r_0}(C_1, ..., C_n)}{Q_{s_0}(C_1, ..., C_n)} + O(\alpha) \right) \quad (\alpha \to 0) \tag{2.14}$$

Introducing (2.14) in the equation $\Phi = \text{const}$ it is easy to see that P_{r_0}/Q_{s_0} is a first integral.

So we may assume that the function Φ is rational.

Step II: we show that the relation (2.13) implies that Φ does not depend on $c_{q+1}, ...,$ c_n.

We may assume that $|\lambda_j| \geq 1$ for all j. Indeed, we can substitute $c_j = d_j$ if $|\lambda_j| \geq 1$ and $c_j = d_j^{-1}$ if $|\lambda_j| < 1$ and the function Φ, in the new variables $(d_1, ..., d_n)$, is also rational.

Since the numbers $1, \mu_1, ..., \mu_q$ are real and nonresonant, there exists a sequence $\{n_s\}_{s \in \mathbf{N}}$ of natural numbers such that (cf. (2.12))

$$\lim_{s \to \infty} \lambda_j^{-n_s} = 1 \quad , \quad j = 1, ..., q \tag{2.15}$$

Iterating (2.13) we get

$$\Phi(c_1, ..., c_n) = \Phi(\lambda_1^{-n_s} c_1, ..., \lambda_n^{-n_s} c_n) \quad , \quad \text{for all } s \tag{2.16}$$

Denote $\Phi = P/Q$, where P, Q are polynomials.

Also denote by $c = (c', c'')$ the splitting of the vector $c = (c_1, ..., c_n)$ with $c' = (c_1, ..., c_q)$ and $c'' = (c_{q+1}, ..., c_n)$.

Several cases are possible.

Case (i): $Q(c', 0) \neq 0$. Then, by taking the limit $s \to \infty$ in (2.16) we get (by (2.15) and since $|\lambda_j| > 1$ for $j = q+1, ..., n$)

$$\Phi(c', c'') = \Phi(c', 0)$$

so Φ does not depend on c''.

Case (ii): $Q(c', 0) = 0, P(c', 0) \neq 0$. In this case replacing Φ by Φ^{-1} we get the same conclusion as above.

Case (iii): $Q(c', 0) = P(c', 0) = 0$. In order to find the limit in relation (2.16) we write $P(c', c'') = \sum_K P_K(c')(c'')^K$ (where $P_K \not\equiv 0$ are polynomials and K are multi-indices) and similarly for Q. Then

$$\Phi(\lambda_1^{-n_s} c_1, ..., \lambda_n^{-n_s} c_n) = \frac{\sum_K P_K(\lambda_1^{-n_s} c_1, ..., \lambda_q^{-n_s} c_q) \left((\lambda'')^K\right)^{-n_s} (c'')^K}{\sum_L Q_L(\lambda_1^{-n_s} c_1, ..., \lambda_q^{-n_s} c_q) \left((\lambda'')^L\right)^{-n_s} (c'')^L} \tag{2.17}$$

Note that because the numbers $\Im\mu_{q+1}, ..., \Im\mu_n$ are linearly independent over $\mathbf{Z}$ we have

$$|(\lambda'')^{K_1}| \neq |(\lambda'')^{K_2}| \quad \text{if } K_1 \neq K_2 \tag{2.18}$$

We order the real numbers $N_K = |(\lambda'')^K|, D_L = |(\lambda'')^L|$ which appear at the numerator, respectively denominator, in (2.17) with nonzero coefficients.

By (2.18), there are only three possible cases:

Case (a): there is only one largest term, namely $D_{L_0} = |(\lambda'')^{L_0}|$. Then taking the limit $s \to \infty$ in (2.16) we get

$$\Phi(c', c'') = 0$$

Case (b): there is only one largest term, namely $N_{K_0} = |(\lambda'')^{K_0}|$. Replacing Φ by Φ^{-1}, the same argument as above shows that the assumed first integral must in fact be a constant.

Case (c): the largest terms are $N_{K_0} = |(\lambda'')^{K_0}|$ and $D_{K_0} = |(\lambda'')^{K_0}|$. Then from (2.16) we get, in the limit $s \to \infty$,

$$\Phi(c_1, ..., c_n) = \frac{P_{K_0}(c')(c'')^{K_0}}{Q_{K_0}(c')(c'')^{K_0}}$$

hence Φ does not depend on c''.

Step III: we show that Φ is actually constant.

We use the fact that the set $\{(\lambda_1^n, ..., \lambda_q^n) \ ; \ n \in \mathbf{Z}\}$ is dense in the q-dimensional torus. Iterating (2.16) we get

$$\Phi(c_1, ..., c_q) = \Phi(\alpha_1 c_1, ..., \alpha_q c_q) \ \text{ for all } \alpha_j \text{ with } |\alpha_j| = 1$$

so Φ must be a constant. ∎

Note Results similar to Proposition 1 above can be obtained even in some cases when the numbers $\Im\mu_{q+1}, ..., \Im\mu_n$ are linearly dependent over $\mathbf{Z}$. Consider for example the 2-dimensional case with complex-conjugate characteristic exponents: $\mu_{1,2} = \alpha \pm i\beta$ $(\beta > 0)$. Two constants of motion are $c_1 = u_1 x^{-\alpha - i\beta}$ and $c_2 = u_2 x^{-\alpha + i\beta}$. It is more convenient to consider the constants $d_1 = c_1 c_2 = u_1 u_2 x^{-2\alpha}$ and $d_2 = c_2/c_1 = u_2/u_1 x^{2i\beta}$. If α is rational, then there is a rational first integral (a power of d_1). If, on the other hand, α is not rational, then any rational first integral has the form $\Phi(d_1, d_2)$ and the same reasoning as in Proposition 1 shows that Φ must be constant.

Remark 2 *Assume that all the exponents $\mu_1, ..., \mu_n$ are real. Let q be the number of irrational exponents linearly independent over $\mathbf{Z}$. Then on any domain $D = D_x \times D_u$ with D_x containing a closed loop surrounding the point $x = 0$ the equation (2.10) has exactly $n - q$ independent holomorphic first integrals.*

Proof

Suppose that $\mu_1, ..., \mu_N$ are rational, $\mu_{N+1}, ..., \mu_{N+q}$ are irrational, linearly independent over $\mathbf{Z}$, and $\mu_{N+q+1}, ..., \mu_n$ are irrational, linearly dependent over $\mathbf{Z}$ on the previous $N + q$ exponents.

Suppose $F(x, u_1, ..., u_n)$ is a single-valued first integral. After analytic continuation along a closed path around the origin $x = 0$, the solutions $u_j = c_j x^{\mu_j}$ have the new values $c_j\lambda_j x^{\mu_j}$ (where $\lambda_j = e^{2\pi i\mu_j}$). Therefore we must have

$$F(x, u_1, ..., u_n) = F(x, \lambda_1 u_1, ..., \lambda_n u_n) = ... = F(x, \lambda_1^p u_1, ..., \lambda_n^p u_n)$$

for all integers p. Since $\mu_{N+1}, ..., \mu_{N+q}$ are independent over the integers, it follows that the set $\{(\lambda_{N+1}^p, ..., \lambda_{N+q}^p) \ ; \ p \in \mathbf{Z}\}$ is dense in the torus $(S^1)^q$, therefore F cannot depend on $u_{N+1}, ..., u_{N+q}$. So there are at most $n-q$ independent first integrals.

But there are obviously $n-q$ of them: if $\mu_j = n_j/m_j$ with n_j, m_j integers, then the integrals $u_j^{m_j} x^{-n_j}$, $j = 1, ..., N$ are single-valued and if $p_j\mu_j = p_1\mu_1 + ... + p_{N+q}\mu_{N+q}$, with p_k integers, then $u_j^{p_j} u_1^{-p_1} ... u_{N+q}^{-p_{N+q}}$ is a single-valued first integral. ∎

2.2.2 Normal Form for the Flow

We now turn to the basic question, namely the significance of the integrability properties of the reduced equation (2.10) for the original equation (2.8).

From an intuitive point of view, since the reduced equation is obtained as an asymptotic approximation of the original equation (in a certain region of the phase space) we might expect that the two equations have, qualitatively, the same behavior. Moreover, in the class of equations that we study, the reduced equation is also the linear part of the original equation. The natural question is then to see whether the two equations are equivalent. Theorem 1 addresses this question.

We do not expect the equivalence to hold in a region of the phase space containing arbitrarily small values of x (because $a(x, u)$ might be singular at $x = 0$, so nonlinear terms might not be negligible for x small).

We give the following definition, similar to the ones found in [20],[22]:

Definition 1 *Let $C > 0, \nu > 0$. We say that $\{\mu_1, ..., \mu_n\}$ is a collection of type $(C, \nu)_1$ if for all multi-indices $K \in \mathbf{N}^n$ with $|K| \geq 2$, and all $l \in \mathbf{Z}$, $s \in \{1, ..., n\}$ we have*

$$|K \cdot \mu + l - \mu_s| > C(|K| + |l|)^{-\nu}$$

(where $K \cdot \mu \equiv K_1\mu_1 + ... + K_n\mu_n$, $|K| = K_1 + ... + K_n$).

The set of multiplets $(\mu_1, ..., \mu_n)$ forming collections of type $(C, \nu)_1$ for some C, ν have full Lebesgue measure in $\mathbf{C}^n$ [20].

Theorem 1 (Normal Form of Equations in an Annulus Surrounding a Singular Point)

Consider equation (2.8) with $a(x,u)$ *holomorphic for* $|u| < r'$, $r'' < |x| < r'''$. *Assume that the characteristic exponents* $\{\mu_1, ..., \mu_n\}$ *form a collection of type* $(C, \nu)_1$.

Then the equation is biholomorphically equivalent to its linear part in a region $|u| < \rho'$, $\rho'' < |x| < \rho'''$ *of the phase space.*

The theorem is in the same spirit as the results known in the literature as "reduction to normal forms" [22] (e.g. normal linear form, normal form of an equation with periodic coefficients).

The proofs of these results (as well as of analogous ones for maps, and also of KAM theorem) share common characteristics: the holomorphic sought-for change of coordinates can be calculated as a formal series, term by term, but its convergence might be very difficult to prove in some cases because of the presence of small denominators. Instead, a sequence of approximations based on a generalization of Newton's method is a more efficient approach.

The proof of Theorem 1 resembles the proof of the analogous result for systems with periodic coefficients. We give it in Section 2.4.

The upshot is that, for almost all matrices M, *the integrability of the original equation (2.8) is equivalent to the integrability of the reduced equation (2.10)* (in the sense of existence of first integrals satisfying *Requirement R* of Section 2.1.1), in the domain of the (extended) phase space where the two equations are holomorphically equivalent.

In particular, the number of single-valued first integrals of (2.8) on the whole phase space (less the variety $u = 0$) is *at most equal* to the number of first integrals of its linear part (some integrals of (2.8) might have branch points outside the domain where the two equations are equivalent).

For example, if the characteristic exponents are real, linearly independent over $\mathbf{Z}$, and form a collection of type $(C, \nu)_1$, then equation (2.8) has no first integrals satisfying *Requirement R*.

Indeed, let $u = h(U, x) = \sum_{k \geq 0} h_k(x) U^k$ be the transformation of Theorem 1 which takes equation (2.8) into its linear part (2.10), holomorphic for small U and x in an annulus around 0. Assume equation (2.8) has a meromorphic first integral $F(u,x) = f(u,x)/g(u,x)$ with f, g regular for small u. Then

$$F(h(U,x),x) = \frac{\sum_{n\geq 0} f_n(x)\left(\sum_{k\geq 0} h_k(x)U^k\right)^n}{\sum_{n\geq 0} g_n(x)\left(\sum_{k\geq 0} h_k(x)U^k\right)^n}$$

is a meromorphic first integral for (2.10).

Also, under the conditions of the theorem, if the equation (2.8) does have first integrals satisfying *Requirement R* (so necessarily the characteristic exponents are resonant, or one of them is not real), they will almost always (cf. Proposition 1) be nonmeromorphic near $u = 0$.

A natural question that arises here is whether there are cases (if the $(C,\nu)_1$ condition is not fulfilled) when the system and its linear part are not equivalent. This may indeed happen in the case of normal linear form at a regular point ([20], Ch.4, 1.3). This fact strongly suggests that the same could be true in our case.

2.2.3 Analytic Continuation of Solutions

The next question is to find the integrability properties of (2.8) when the condition $(C,\nu)_1$ does not hold. To this end, we study the multivaluedness of the solutions. Since they could be branched at the origin, we study their analytic continuations for paths, in the x-plane, encircling the point $x = 0$. But we first have to show that the solutions are defined on such paths. The following lemma shows that the solutions of the equation (2.8) can be analytically continued on paths winding a finite number of times around the origin in the complex x-plane.

Lemma 1 *Consider the equation (2.8) with $a(x,u)$ holomorphic in a neighborhood of the set*

$$\mathcal{D}_{\rho_0} = \{(x,u) \in \mathbf{C} \times \mathbf{C}^n \ ; \ \rho'' \leq |x| \leq \rho''', \ |u| \leq \rho_0'\}$$

$(0 < \rho'' < \rho''')$.

Let $\mathcal{R}$ be a relatively compact subdomain of the universal covering of $\mathbf{C} \setminus \{0\}$ (i.e. the Riemann surface of the logarithm) that lies above the annulus $\rho'' < |x| < \rho'''$ in $\mathbf{C}$.

Then the solution u of the equation corresponding to the initial condition $u(x_0) = u_0$, where (x_0, u_0) belongs to the set

$$\mathcal{D}_\rho = \{(x,u) \in \mathbf{C} \times \mathbf{C}^n \ ; \ \rho'' \le |x| \le \rho''', \ |u| \le \rho'\}$$

is holomorphic on $\mathcal{R}$ provided that the number ρ' is small enough.

For our purposes we only need to consider two-sheeted coverings $\mathcal{R}$.

Proof

We will rewrite (2.8) as an integral equation and use a fixed point theorem to show that the equation has a solution.

The linear part of the equation has the solutions $U_k(x) = c_k x^{\mu_k}, k = 1, ..., n$ (c_k are arbitrary constants). In vector notation, $U(x) = U_0(x)c$, where $U_0 = \mathrm{diag}(x^{\mu_1}, ..., x^{\mu_n})$ and $c = (c_1, ..., c_n)^T$ is the general solution of (2.10). Using the variation of constants (i.e. writing $u(x) = U_0(x)c(x)$), the solutions u of (2.8) will satisfy the system

$$u(x) = U_0(x)c + U_0(x)\int_{x_0}^{x} t^{-1}U_0(t)^{-1}a(t,u(t))dt \ \equiv N(u)(x) \tag{2.19}$$

We will need the following estimate for the function $a = (a_1, ..., a_n)^T$. We assumed that $a_k(x,u)$ have a zero of order 2 at $u = 0$, i.e. a_k can be written as

$$a_k(x,u) = \sum_{j,l=1}^{n} u_j u_l a_{kjl}(x,u) \tag{2.20}$$

with a_{kjl} holomorphic near $\mathcal{D}_{\rho_0}$. Let M be an upper bound for a_{kjl} $(k,j,l = 1, ..., n)$ on $\mathcal{D}_{\rho_0}$ and M' be an upper bound for $\partial_{s+1}a_{kjl}$ $(s,k,j,l = 1, ..., n)$.

Let $\rho' \le \rho'_0$. Consider the Banach space $\mathcal{B}$ of the n-tuples of analytic functions $u = (u_1, ..., u_n)$ on (the piece of a Riemann surface) $\mathcal{R}$, continuous on $\overline{\mathcal{R}}$, with the sup norm:

$$\| u \| = \max_k \ \sup_{x \in \overline{\mathcal{R}}} |u_k(x)|$$

Then $a, (a_{1jl}, ..., a_{njl}) \equiv a_{jl} \in \mathcal{B}$.

Let $N = (N_1, ..., N_n)^T$ be the operator defined by (2.19). Then

$$N_k(u)(x) = x^{\mu_k} c_k + x^{\mu_k}\int_{x_0}^{x} t^{-1-\mu_k} a_k(t,u(t))dt \tag{2.21}$$

Clearly, if $u \in \mathcal{B}$ then also $N(u) \in \mathcal{B}$.

Let $B_{\rho'}$ be the ball of radius ρ' in the Banach space $\mathcal{B}$.

We first show that if ρ' is small enough, and for small initial conditions u_0, we have $N(\overline{B_{\rho'}}) \subset \overline{B_{\rho'}}$.

Indeed, let $(u_1, ..., u_n) \in \overline{B_{\rho'}}$. Let Γ be an upper bound for the functions $|x^{\mu_k}|$, $|x^{-1-\mu_k}|$, $k = 1, ..., n$ on $\mathcal{R}$. Let L be an upper bound for the lengths of paths on $\mathcal{R}$ (up to homotopic equivalence).

Then from (2.20)

$$|N_k(u)(x)| \ \le \ \Gamma\,|c_k| \ + L\,\Gamma^2 \sup|a_k(x,u)| \le \Gamma\,|c_k| \ + L\,\Gamma^2\, n^2 M \rho'^2$$

For c_k and ρ' small enough (depending only on ρ'', ρ''', n) we have $|N_k(u)(x)| \le \rho'$.

The constants c_k are related to the initial conditions u_0 by a linear invertible transformation:

$$u_0 = U_0(x_0)C \tag{2.22}$$

We conclude that for u_0 and ρ' small, the operator N maps the ball $\overline{B_{\rho'}}$ into itself.

Also, N is a contraction. Indeed,

$$|N_k(v)(x) - N_k(w)(x)| \le L\,\Gamma^2\, M''$$

where

$$M'' = \max\left\{|a_k(x,v(x)) - a_k(x,w(x))|\ ;\ x \in \overline{\mathcal{R}}\right\}$$

Using the Taylor expansion with integral remainder and the relation (2.20)

$$|a_k(x,v) - a_k(x,w)| \le$$

$$|\int_0^1 \sum_{j=1}^n (v_j - w_j)\partial_{j+1} a_k(x, w_0, ..., w_{j-1}, w_j + (v_j - w_j)t, v_{j+1}, ..., v_n)\, dt| \le$$

$$\max_{j=1,...,n} |v_j - w_j| \quad \max_{j=2,...,n+1} \quad \max_{(x,v)\in\mathcal{D}_\rho} \ |\partial_j a_k(x,v)| \le$$

$$\max_j |v_j - w_j|(2nM|v| + n^2M'|v|^2) \leq \max_j |v_j - w_j|(2nM\rho' + n^2M'(\rho')^2)$$

Applying the above inequality to estimate $|a_k(x, v(x)) - a_k(x, w(x))|$ we get

$$|N_k(v)(x) - N_k(w)(x)| \leq L\,\Gamma^2\,(2nM\rho' + n^2M'(\rho')^2)\max_j \max_{x\in\overline{\mathcal{R}}} |v_j(x) - w_j(x)|$$

which shows that N is a contraction for ρ' small.

Therefore the operator N has a fixed point in $\overline{B_\rho}$ and the lemma is proved. ∎

2.2.4 The Monodromy Map

We now study the multivaluedness of the solutions.

Let $\mathcal{R}, \mathcal{D}_\rho$ be as in Lemma 1 and let $(x_0, u_0) \in \mathcal{D}_\rho$. Denote by $u(x; x_0, u_0)$ the solution of the equation (2.8) with the initial condition u_0 at $x = x_0$. Consider a path γ in $\mathcal{R}$, starting and ending above x_0, whose projection on the complex plane encircles the origin once, counterclockwise. After analytic continuation along γ, the solution has a new value, $u^+(x_0; x_0, u_0)$. The map Φ_{x_0} given by

$$\Phi_{x_0}(u_0) = u^+(x_0; x_0, u_0) \tag{2.23}$$

is the monodromy map at x_0 on the annulus covered by $\mathcal{R}$, and is defined for $u_0 \in \mathbf{C}^n$, $|u_0| < \rho'$.

Lemma 2 *Let $(x, u) \in \mathcal{D}_\rho$, $u = (u_1, ..., u_n)$.*

The monodromy map has the form

$$\Phi_x(u_1, ..., u_n) = (\Phi_{x,1}(u_1, ..., u_n), ..., \Phi_{x,n}(u_1, ..., u_n))$$

where $\Phi_{x,j}$ are holomorphic in a neighborhood of $u = 0$ and

$$\Phi_{x,j}(u_1, ..., u_n) = \lambda_j u_j + O(u_k u_l) \qquad (u \to 0)$$

Proof

Note first that the solutions with initial conditions in $\mathcal{D}_\rho$ depend holomorphically on the constants $c_1, ..., c_n$ (since they are holomorphic in the initial value of u (for u small), and the constants $c_1, ..., c_n$ are related to the initial conditions by a linear relation (2.22)).

From relations (2.19),(2.20) we see that

$$u_k(x) = c_k x^{\mu_k} + \tilde{u}_k(x) \quad , \quad k = 1, ..., n \tag{2.24}$$

where $\tilde{u}_k(x)$ is holomorphic in $c_1, ..., c_n$ for c_k small and $x \in \mathcal{R}$, and $\tilde{u}_k(x) = O(c_j c_l)$, (as $c \to 0$).

After analytic continuation on a closed path γ in the x-plane the u_k become

$$u_k^+(x) = c_k \lambda_k x^{\mu_k} + \tilde{u}_k^+(x) \quad , \quad k = 1, ..., n \tag{2.25}$$

where $\tilde{u}_k^+(x) = O(c_j c_l)$ $(c \to 0)$.

Using the holomorphic implicit function theorem in equations (2.24) to solve for c in terms of u (for small enough c) and plugging in the expression of c into (2.25) we get

$$u_j^+ = u_j \lambda_j + \tilde{\tilde{u}}_k^+ \quad , \quad j = 1, ..., n \tag{2.26}$$

where $\tilde{\tilde{u}}_k^+$ is holomorphic for small u and $\tilde{\tilde{u}}_k^+ = O(u_k u_l)$. ∎

2.2.5 The Local Form of the First Integrals

Since the solutions u depend holomorphically on the constants c for small c and $x \neq 0$ fixed, and satisfy (2.24), we can solve for c in terms of u, for c and u small, and obtain $c_k = c_k(x, u), k = 1, ..., n$. These are in fact n independent first integrals. They are holomorphic in u and the coefficients of the power series in u are holomorphic functions of x on $\mathcal{R}$ (so they may be branched at $x = 0$).

2.2.6 Normal Form for the Monodromy Map

Some Classic Results

We will first reproduce some basic facts following [22].

The numbers $\lambda_1, ..., \lambda_n$ are called *nonresonant* if there are no integers $k_1, ..., k_n$ such that $k_l \geq 0$, $\sum k_l \geq 2$ and $\lambda_j = \lambda_1^{k_1} ... \lambda_n^{k_n}$.

A collection of eigenvalues $(\lambda_1, ..., \lambda_n)$ *belongs to the Poincaré domain* if the moduli of the eigenvalues are all smaller or all greater than 1.

The complement of the Poincaré domain is *the Siegel domain.*

Let C, ν be positive constants. A collection of eigenvalues $(\lambda_1, ..., \lambda_n)$ is called *collections of multiplicative type* (C, ν) if

$$|\lambda_j - \lambda_1^{k_1} ... \lambda_n^{k_n}| \geq C(k_1 + ... + k_n)^{-\nu}$$

for all $j = 1, ..., n$, $k_l \geq 0$, $\sum k_l \geq 2$.

The set of collections $(\lambda_1, ..., \lambda_n)$ which are not of multiplicative type (C, ν) for any C has measure 0 if $\nu > (n-1)/2$.

Poincaré's Theorem *If at a fixed point the collection of the eigenvalues of the linear part of a holomorphic diffeomorphism belongs to the Poincaré domain, and they are nonresonant, then in a neighborhood of the fixed point the mapping can be reduced to its linear part by means of a biholomorphic diffeomorphism.*

Siegel's Theorem *If the collection of eigenvalues of the linear part of a holomorphic diffeomorphism at a fixed point has multiplicative type (C, ν) for some C, ν , then the diffeomorphism is biholomorphically equivalent to its linear part at the fixed point.*

Eigenvalues Belonging to the Poincaré Domain

Assume that at least one of the μ_k is not real. The reduced equation (2.10) is then integrable (cf. Remark 1). In some cases, this implies integrability of the original equation in a region of the phase space.

Proposition 2 *Consider the equation (2.8) with the characteristic exponents $\mu_1, ..., \mu_n$, not all real.*

Assume that $\lambda_1, ..., \lambda_n$ (where $\lambda_k = e^{2\pi i \mu_k}$) satisfy the hypothesis of Poincaré's theorem (i.e. $\Im \mu_k > 0, k = 1, ..., n$ or $\Im \mu_k < 0, k = 1, ..., n$).

Let $\mathcal{R}, \mathcal{D}_\rho$ be as in Lemma 1.

Then the equation (2.8) has n independent first integrals, holomorphic on the (interior of the) domain $\mathcal{D}_\rho \setminus V$, where V is the variety $u = 0$ in $\mathbf{C} \times \mathbf{C}^n$.

There are no first integrals meromorphic near V.

Note

The first integrals guaranteed by Proposition 2 above may not be single-valued outside the domain $\mathcal{D}_\rho$.

Proof

We study the multivaluedness of the first integrals c_k and show that we can find uniformizing functions.

Let u be a solution starting at an initial point in the interior of $\mathcal{D}_\rho$. Then u has the form (2.24) for some constants c_k. After analytic continuation on a closed path in the x-plane the functions u_k have the form (2.25).

Since u^+ is again a solution of the equation, there are new constants c_k^+ such that

$$u_k^+(x) = c_k^+ x^{\mu_k} + \tilde{u}_k^+(x) \quad , \quad k = 1, ..., n \tag{2.27}$$

where $\tilde{u}_k^+(x)$ depend holomorphically on c^+ (near the origin) and
$\tilde{u}_k^+(x) = O(c_j^+ c_l^+)$ $(c^+ \to 0)$.

(In fact, c_k^+ is the value of the first integral c_k after analytic continuation in x around the origin.)

We consider the equations (2.25), (2.27) and solve for c_k^+, $k = 1, ..., n$ in terms of c_k, $k = 1, ..., n$: $c_k^+ = f_k(c_1, ..., c_n)$, $k = 1, ..., n$ where $f = (f_1, ..., f_n)$ is a biholomorphism in a neighborhood of the origin and

$$f_k(c_1, ..., c_n) = \lambda_k c_k + O(c_j c_l) \quad , \quad k = 1, ..., n \tag{2.28}$$

(Note that the functions f_k do not depend on x because c_k and c_k^+ are independent of x.)

By the Poincaré theorem, there exists a biholomorphic change of coordinates near the origin, $c_k = \phi_k(d_1, ..., d_n)$, $k = 1, ..., n$ such that

$$f_k \circ (\phi_1, ..., \phi_n)(d_1, ..., d_n) = \phi_k(\lambda_1 d_1, ..., \lambda_n d_n), \quad k = 1, ..., n$$

We look for n independent first integrals; they are functions of the constants of integration $\Phi(c_1, ..., c_n)$. Since we are looking for single-valued first integrals, we must have

$$\Phi(c_1, ..., c_n) = \Phi(c_1^+, ..., c_n^+)$$

or, equivalently,

$$\Phi(c_1, ..., c_n) = \Phi(f_1(c_1, ..., c_n), ..., f_n(c_1, ..., c_n))$$

so

$$\Phi \circ (\phi_1, ..., \phi_n)(d_1, ..., d_n) = \Phi \circ (\phi_1, ..., \phi_n)(\lambda_1 d_1, ..., \lambda_n d_n)$$

Therefore $\Phi_1, ..., \Phi_n$ are independent first integrals for (2.8) if and only if $\Phi_1 \circ (\phi_1, ..., \phi_n), ..., \Phi_n \circ (\phi_1, ..., \phi_n)$ are independent first integrals for (2.10). Remark 1 and Proposition 1 conclude the proof. ∎

Eigenvalues of Absolute Value 1

Assume now that all the μ_k are real.

As in Proposition 2, one may use Siegel's theorem (for each fixed x) to find sufficient conditions for the monodromy map to be holomorphically conjugate to its linear part (i.e. to the monodromy map of the linearized equation).

However, the condition that the characteristic exponents $\mu_1, ..., \mu_n$ form a collection of type $(C, \nu)_1$ for some $C, \nu > 0$ is equivalent to the condition that $\lambda_1, ..., \lambda_n$ form a collection of multiplicative type (C, ν) for some $C, \nu > 0$ necessary for Siegel's Theorem.

Therefore the conditions for nonintegrability based on Siegel's theorem are equivalent to those assumed in Theorem 1.

Stronger results may be obtained if one could use (stronger) sufficient conditions for topological equivalence of maps. One such result is available in the case $n = 1$.

Proposition 3 *Consider the equation (2.1) with μ irrational.*

Let $\mathcal{D}_\rho$ be such that Lemma 1 holds.

Then there are no first integrals, holomorphic on $\mathcal{D}_\rho$.

Proof

Suppose there exists F, single-valued first integral, holomorphic on $\mathcal{D}_\rho$. Fix $x \in \mathcal{R}$ such that $F(x,\cdot)$ is defined a.e. on $|u| < \rho'$.

Since F is single-valued, $F(x,u(x))$ must have the same value after the analytic continuation of the solution $u(x)$:

$$F(x,u) = F(x,\Phi_x(u)) \tag{2.29}$$

for all u in the domain of $F(x,\cdot)$.

Since the monodromy map is holomorphic in a neighborhood of the origin, and satisfies $\Phi_x(u) = \lambda u + O(u^2)$ with $\lambda = \exp(2\pi i\mu)$, μ irrational, by Denjoy's theorem [20] Φ_x is topologically conjugate to its linear part: there exists a local homeomorphism ϕ such that $\Phi_x(\phi(v)) = \phi(\lambda v)$.

Relation (2.29) becomes

$$F(x,\phi(v)) = F(x,\phi(\lambda v)))$$

which can now be iterated

$$F(x,\phi(v)) = F(x,\phi(\lambda v))) = F(x,\phi(\lambda^2 v))) = ... = F(x,\phi(\lambda^n v)))$$

for all n integer.

Since the set $\{\lambda^n \ ; \ n \in \mathbf{Z}\}$ is dense on the unit circle, and F,ϕ are continuous, it follows that

$$F(x,\phi(v)) = F(x,\phi(\alpha v))$$

for all α on the unit circle. So $F(x,\cdot)$ has the same value on a closed curve in the complex u plane, hence cannot depend on u. It follows that F must be constant. ∎

2.3 Conclusions

We considered the nonlinearly perturbed Euler equation (2.8) in the case when the characteristic exponents $\mu_1,...,\mu_n$ of the linear part are distinct and nonresonant (i.e. if $k_1\mu_1+...+k_n\mu_n-\mu_j \notin \mathbf{Z}$ for any nonnegative integers $k_1,...,k_n$ with $k_1+...+k_n \geq 2$ and $j \in 1,...,n$.)

If the numbers $\mu_1, ..., \mu_n$ form a collection of type $(C,\nu)_1$ then the equation is holomorphically equivalent to its linear part for x in an annulus and u small. (The set of multiplets $(\mu_1, ..., \mu_n)$ for which the $(C,\nu)_1$-type condition fails for all $C, \nu > 0$ has measure zero in $\mathbf{C}^n$.)

As a consequence, the integrability properties of equation (2.8) and of its linear part are the same, on the above domain of the phase space, for almost all matrices M. Furthermore:

(i) if in addition, all the exponents are real, equation (2.8) has no single-valued first integrals.

(ii) if one of the exponents is not real, and a stronger nonresonance conditions fulfilled, then any single-valued first integral of (2.8) (if there is any) is not meromorphic near the variety $u = 0$.

If the condition $(C,\nu)_1$ is not fulfilled, but $\Im\mu_j > 0, j = 1, ..., n$, or $\Im\mu_j < 0, j = 1, ..., n$ then the monodromy map is holomorphically conjugate with its linear part; therefore all first integrals are not meromorphic near $u = 0$.

The case $n = 1$ can be more completely analyzed: if the characteristic exponent is irrational, then the equation is nonintegrable.

In the resonant case, the equivalence of the equation with its linear part cannot be in general obtained, not even by formal power series changes of coordinates. Instead, some terms of the nonlinear part are asymptotically significant and should be kept when considering a normal form for the equation.

It would be interesting to see under which conditions topological equivalence of the monodromy map with its linear part implies equivalence of the integrability properties (since it apparently is a weaker condition than that of holomorphic equivalence). For the case of *regular* vector fields (i.e. equations of the form $v'(z) = Mv(z) + a(v, z)$, with a holomorphic at the origin) we note, however, the following statement in [20] (Ch.5, 6.1): "A vector field germ with nonresonant linear part in the complex plane is either analytically equivalent to its linear part [...], or is not even topologically equivalent to it (V.A. Nayshul)."

2.4 Normal Form In the Presence of a Regular Singular Point

We consider the system

$$zv'(z) = Mv(z) + a(v,z) \quad , \quad v \in \mathbf{C}^n \ , \ z \in \mathbf{C} \tag{2.30}$$

We assume that:

(i) M is a constant $n \times n$ matrix, with the spectrum consisting of n distinct eigenvalues $\mu_1, ..., \mu_n$;

(ii) $a = (a^1, ..., a^n)$ is a holomorphic vector-valued function, defined on a domain

$$\Pi_r = \{(v,z) \in \mathbf{C}^n \times \mathbf{C} \ ; \ |v| < r', \ r'' < |z| < r'''\} \tag{2.31}$$

(where $r \equiv (r', r'', r''')$, $r' > 0, 0 \leq \ r'' < r'''$ and $|v| \equiv \max\{|v_1|, ..., |v_n|\}$) and a is continuous on the closure $\overline{\Pi_r}$ of the domain;

(iii) a has a zero of order 2 at $v = 0$:

$$a^j(v,z) = \sum_{|K|\geq 2, l \in \mathbf{Z}} a^j_{K,l} v^K z^l \quad , \quad j = 1, ..., n$$

(where K is a multiindex $K \equiv (K_1, ..., K_n) \in \mathbf{N}^n$, $|K| \equiv K_1 + ... + K_n$ and $v^K \equiv v_1^{K_1} ... v_n^{K_n}$);

(iv) we may also assume that M is diagonal (which can be arranged through a linear substitution).

The main result is the following:

Theorem 2 *If the eigenvalues of M form a collection of type $(C, \nu)_1$ then the system (2.30) is holomorphically equivalent, in a domain $\Pi_\rho \subset \Pi_r$, to its linear part.*

The following subsections of the Appendix contain the proof of Theorem 2. It follows the main steps of the proofs in [22], Sec.12 (for the analytic reduction of analytic circle diffeomorphisms to rotations) and Sec.28 (for the local normal form of mappings at a fixed point).

2.4.1 The Homological Equation

Consider the following equation for the unknown vector-valued function h:

$$Mh(y,z) + a(y,z) - zh_z(y,z) - D_y h(y,z) My = 0 \quad , \quad y, h(y,z) \in \mathbf{C}^n \tag{2.32}$$

where a, M satisfy the conditions (i),...,(iv).

A direct calculation shows that the homological equation (2.32) has a formal power series solution

$$h^j(y,z) = \sum_{|K| \geq 2, l \in \mathbf{Z}} h^j_{K,l} y^K z^l \tag{2.33}$$

where

$$h^j_{K,l} = \frac{a^j_{K,l}}{K \cdot \mu + l - \mu_j} \tag{2.34}$$

We introduce the notation

$$\Pi_{r,\delta} = \left\{ (y,z) \in \mathbf{C}^n \times \mathbf{C} \ ; \ |y| < r' e^{-\delta}, \ r'' e^{\delta} < |z| < r''' e^{-\delta} \right\} \subset \Pi_r$$

Lemma 3 *The series (2.33) converges and there exists a constant $\alpha > 0$ (depending only on C, ν, n) such that*

$$\max_{\overline{\Pi_{r,\delta}}} |h^j(y,z)| \leq \delta^{-\alpha} \max_{\overline{\Pi_r}} |a^j(y,z)|$$

for every $\delta \in (0, 1/2)$ and every r $(r', r''' > 0, r'' \geq 0)$.

Proof

We use Cauchy estimates for a^j:

$$|a^j_{K,l}| \leq (r')^{-K} (r''')^{-l} \max_{\overline{\Pi_r}} |a^j(y,z)| \ , \quad \text{for } l \geq 0 \tag{2.35}$$

and

$$|a^j_{K,l}| \leq (r')^{-K} (r'')^{-l} \max_{\overline{\Pi_r}} |a^j(y,z)| \ , \quad \text{for } l < 0 \tag{2.36}$$

So for $(y,z) \in \overline{\Pi_{r,\delta}}$

$$h_+^j := |\sum_{|K|\geq 2, l\geq 0} h_{K,l}^j y^K z^l| \leq \sum_{|K|\geq 2, l\geq 0} \frac{|a_{K,l}^j|}{|K\cdot\mu + l - \mu_j|} |y^K z^l| \leq$$

$$\max_{\overline{\Pi_r}} |a^j(y,z)| \sum_{|K|\geq 2, l\geq 0} \frac{1}{|K\cdot\mu + l - \mu_j|} e^{-\delta(|K|+l)} \leq$$

$$C^{-1}\ \max_{\overline{\Pi_r}} |a^j(y,z)| \sum_{|K|\geq 2, l\geq 0} (|K|+l)^\nu e^{-\delta(|K|+l)}\ <$$

$$C^{-1}\ \max_{\overline{\Pi_r}} |a^j(y,z)|\ \sum_{p\geq 2} \mathcal{N}_{n,p}\ p^\nu e^{-\delta p}$$

where

$$\mathcal{N}_{n,p} = \#\{(K_1,...,K_n,l)\ ;\ K_j \geq 0, l\geq 0,\ |K|+l = p\} =$$

$$\begin{pmatrix} n+p \\ n \end{pmatrix} \leq c_n p^n$$

Since $p^{n+\nu} e^{-\delta p/2} \leq C_1 \delta^{-(n+\nu)}$ for $p \geq 2$ (where C_1 depends only on $n+\nu$, and not on δ) then

$$h_+^j \leq C^{-1} c_n C_1 \delta^{-(n+\nu)} \frac{e^{-\delta}}{1-e^{-\delta/2}} \max_{\overline{\Pi_r}} |a^j(y,z)|\ \leq$$

$$2C^{-1} c_n C_1 \delta^{-(n+1+\nu)} \max_{\overline{\Pi_r}} |a^j(y,z)|$$

Similar estimates hold for

$$|\sum_{|K|\geq 2, l<0} h_{K,l}^j y^K z^l|$$

The lemma is proved. ▮

2.4.2 Norms

We will use the following norm: for functions $\phi = (\phi^1, ..., \phi^n)$ holomorphic on Π_r and continuous on $\overline{\Pi_r}$, with $\phi(0, z) = 0$ define

$$||\phi||_r = \sup_{\overline{\Pi_r}} \frac{|\phi(y,z)|}{|y|}$$

We denote by $||.||_{r,\delta}$ the corresponding norm on the domain $\Pi_{r,\delta}$.

Of course, a similar definition can be used for scalar functions.

Let f be a scalar function, with the regularity as above:

$$f(y,z) = \sum_{|K|\geq 1, l\in \mathbf{Z}} f_{K,l} y^K z^l$$

Remark 3

$$|f_{K,l}| \leq \begin{cases} ||f||_r (r')^{-|K|+1} (r''')^{-l} & \textit{for } l \geq 0 \\ \\ ||f||_r (r')^{-|K|+1} (r'')^{-l} & \textit{for } l < 0 \end{cases}$$

Proof

Follows from (2.35), (2.36) and

$$\max_{\overline{\Pi_r}} |f(y,z)| = r' \max_{\overline{\Pi_r}} \frac{|f(y,z)||}{r'} \leq r' ||f||_r$$

∎

Remark 4 *There exists a positive constant α, depending only on n, such that, for all $\delta \in (0, 1/2)$*

$$\max_{\overline{\Pi_{r,\delta}}} |f(y,z)| \leq r' \delta^{-\alpha} ||f||_r$$

Proof

Using Remark 3 we have

$$|f(y,z)| \leq \sum_{|K|\geq 1, l\geq 0} ||f||_r (r')^{-|K|+1} (r''')^{-l} |y^K z^l|$$

$$\sum_{|K|\geq 1, l<0} ||f||_r (r')^{-|K|+1}(r'')^{-l}|y^K z^l| \leq$$

for $(y, z) \in \Pi_{r,\delta}$

$$\leq r'||f||_r \sum_{|K|\geq 1, l\in \mathbf{Z}} e^{-\delta(|K|+|l|)} \leq 2r'||f||_r \sum_{p\geq 1} \mathcal{N}_{n,p} e^{-\delta p} \leq$$

using the same estimates as in Lemma 3,

$$\leq 2r'||f||_r c_n \sum_{p\geq 1} p^n e^{-\delta p} \leq 2r'||f||_r c_n C_2 \delta^{-n} \sum_{p\geq 1} e^{-\delta p/2} =$$

$$2r'||f||_r c_n C_2 \frac{\delta^{-n}}{e^{\delta/2}-1} \leq 4r'||f||_r c_n C_2 \delta^{-n-1}$$

$\blacksquare$

Remark 5 *Lemma 3 holds in the norm $||.||_r$. More precisely:*

$$||h^j||_{r,\delta} \leq \delta^{-\alpha}||a^j||_r$$

Proof

The inequality is obtained following the same steps as in the proof of Lemma 3 to estimate $|h^j(y,z)|/|y|$. $\blacksquare$

Remark 6 *Let f be holomorphic on Π_r, continuous on $\overline{\Pi_r}$, with $f(0,z)=0$. Then*

$$\max_{\overline{\Pi_{r,\delta}}} \left|\frac{\partial f}{\partial y_j}\right| \leq (1-e^{-\delta})^{-1}||f||_r$$

Proof

Using the Cauchy formula for $(y,z) \in \Pi_{r,\delta}$

$$\left|\frac{\partial f}{\partial y_j}(y,z)\right| \leq \frac{1}{2\pi}\left|\oint_{|\xi-y_j|=r'-r'e^{-\delta}} \frac{f(y_1,...,y_{j-1},\xi,y_{j+1},...,y_n,z)}{(\xi-y_j)^2} d\xi\right| \leq$$

$$(1-e^{-\delta})^{-1}(r')^{-1}\sup_{|\xi|\leq r}|f(y_1,...,y_{j-1},\xi,y_{j+1},...,y_n,z)| \leq (1-e^{-\delta})^{-1}||f||_r$$

$\blacksquare$

2.4.3 The Remainder

Given the equation (2.30) we change the dependent variable v to the new variable y using the substitution $v = y + h(y,z)$, where h is the solution of the homological equation (2.32). The equation for y is

$$zy'(z) = My(z) + \mathcal{R}(y,z) \tag{2.37}$$

where

$$\mathcal{R}(y,z) = (I + D_y h)^{-1}\left[a(y + h(y,z), z) - a(y,z)\right]$$

Lemma 4 *Let α be as in Lemma 3 and Remarks 4,5.*

There exists a constant $\mathcal{K} \geq 2$ such that for all $\beta > 2\alpha + 2$, $\delta \in (0, \mathcal{K}^{-1})$ and r: if $||a||_r \leq \delta^\beta$, then the matrix $I + D_y h(y,z)$ is invertible for $(y,z) \in \Pi_{r,\delta}$ and the following estimates hold:

$$\max_{(y,z)\in\overline{\Pi_{r,\delta}}} ||D_y h(y,z)|| \leq C_4 \delta$$

and

$$\max_{(y,z)\in\overline{\Pi_{r,\delta}}} ||\left(I + D_y h(y,z)\right)^{-1}|| \leq C_3$$

where the constants C_3, C_4 do not depend on r or δ.

Proof

Using Remarks 6 and 5

$$\max_{\overline{\Pi_{r,\delta}}} ||D_y h(y,z)|| = \max_{\overline{\Pi_{r,\delta}}} \max_j \left(\sum_i \left|\frac{\partial h^j}{\partial y_i}(y,z)\right|^2\right)^{1/2} \leq$$

$$(1 - e^{-\delta/2})^{-1}\sqrt{n}\max_j ||h^j||_{r,\delta/2} \leq (1 - e^{-\delta/2})^{-1}\sqrt{n}(\delta/2)^{-\alpha}||a||_r \leq$$

$$(1 - e^{-\delta/2})^{-1}\sqrt{n}\delta^{\beta-\alpha}2^\alpha \leq (1 - e^{-\delta/2})^{-1}\sqrt{n}\delta^2 2^\alpha \leq$$

$$2e^{1/4}\sqrt{n}\delta 2^\alpha \equiv C_4\delta$$

Let $\mathcal{K} \geq 2$ be large enough, so that $C_3' \equiv C_4 \mathcal{K}^{-1} < 1$. Then, for $\delta \in (0, \mathcal{K}^{-1})$ we have

$$\max_{\Pi_{r,\delta}} ||D_y h(y,z)|| < C_3' < 1 \tag{2.38}$$

Therefore

$$|| \left(I + D_y h(y,z)\right)^{-1} || \leq \left(1 - ||D_y h(y,z)||\right)^{-1} \leq (1 - C_3')^{-1} \equiv C_3$$

∎

Lemma 5 *Assume $\delta > 0$ is such that the map $(y,z) \to (y + h(y,z), z)$ is well defined on $\Pi_{r,\delta}$, and its image J is relatively compact in Π_r.*

Then

$$||a(y + h(y,z), z) - a(y,z)||_{r,\delta} \leq n(1 - e^{-\delta})^{-1} \delta^{-\alpha} ||a||_r^2$$

Proof

$$||a(y + h(y,z), z) - a(y,z)||_{r,\delta} \leq$$

$$\max_j \sup_{\Pi_{r,\delta}} \frac{|a^j(y + h(y,z), z) - a^j(y,z)|}{|y|} \leq$$

$$n \max_j \sup_J \left|\left| \frac{\partial a^j}{\partial y^i} \right|\right| \, ||h||_{r,\delta} \leq n(1 - e^{-\delta})^{-1} \delta^{-\alpha} ||a||_r^2$$

where the last inequality follows from Remarks 5 and 6. ∎

Lemma 6 *Let $\mathcal{K}$ be the constant in Lemma 4 and $\beta > 2\alpha + 2$.*

There exists a constant $\mathcal{K}_0 \geq \mathcal{K}$ with the following property: assuming that $\delta \in (0, \mathcal{K}_0^{-1})$ and r are such that the image of the domain $\Pi_{r,\delta}$ under the map $(y + h(y,z), z)$ is relatively compact in Π_r, we have: if $||a||_r < \delta^\beta$, then

$$||\mathcal{R}||_{r,\delta} \leq \delta^{-1-2\alpha} ||a||_r^2$$

Proof

Combining the results of Lemma 4 and Lemma 5

$$||\mathcal{R}||_{r,\delta} \leq \sup_{(y,z)\in\Pi_{r,\delta}} \left|\left|(I+D_y h(y,z))^{-1}\right|\right| \; ||[a(y+h(y,z),z)-a(y,z)]||_{r,\delta} \leq$$

$$C_3 n(1-e^{-\delta})^{-1}\delta^{-\alpha}||a||_r^2 \equiv C_5(1-e^{-\delta})^{-1}\delta^{-\alpha}||a||_r^2$$

But

$$C_5(1-e^{-\delta})^{-1}\delta^{-\alpha} \leq C_5\delta^{-1-\alpha}(1-\delta/2)^{-1} \leq \delta^{-1-2\alpha}$$

where the last inequality holds for δ small enough. ∎

2.4.4 The Iteration

We construct a sequence of holomorphic substitutions $v = y + h_k(y,z)$ which take equations of form (2.30) (with the nonlinear term a_k instead of a, defined on Π_{r_k}) into equations of the same form, but with the nonlinear terms a_{k+1} closer to 0 than a_k, and defined on smaller domains $\Pi_{r_{k+1}} \subset \Pi_{r_k}$.

We first choose the numbers $r_0 = (r_0', r_0'', r_0'''), \delta_0, N$. Using them as starting points, we define inductively, for $k \geq 0$ the sequences:

$$\delta_{k+1} = \delta_k^{3/2}\;, \quad r_{k+1}' = r_k' e^{-\delta_k}, \quad r_{k+1}'' = r_k'' e^{\delta_k}, \quad r_{k+1}''' = r_k''' e^{-\delta_k}$$

We then construct a sequence of functions as follows. We start with the system (2.30) satisfying (i) to (iv). Suppose $\Pi_{r_0} \subset \Pi_r$. Let $a_0 = a$. We solve the homological equation (2.32) and we get the solution $h \equiv h_0$. After the substitution $v = y + h_0(y,z)$ the system for y is (2.37); it has the form (2.30) where the nonlinearity is now the remainder $\mathcal{R}$. Set $a_1 = \mathcal{R}$, defined on Π_{r_1}, and solve the homological equation (with a_1 in place of a) to get the solution h_1, and so on.

Fix r_0 so that $\Pi_{r_0} \subset \Pi_r$.

Let $0 < \delta_0 < \mathcal{K}_0^{-1}$. Then

$$r_{k+1}' = r_0' e^{-(\delta_0+\delta_1+\ldots+\delta_k)} = r_0' e^{-(\delta_0+\delta_0^{3/2}+\ldots+\delta_0^{(3/2)^k})} \equiv r_0' e^{-\Delta_k(\delta_0)}$$

and similarly for r''_{k+1}, r'''_{k+1}.

The sequence Δ_k of continuous functions on $[0, \mathcal{K}_0^{-1}]$ converges uniformly to a continuous function Δ. Clearly, $\Delta(0) = 0$ and for $\delta_0 > 0$, $\Delta(\delta_0) > 0$. We choose δ_0 small enough so that $\rho'' < \rho'''$, where

$$r''_k < r''_0 e^{\Delta(\delta_0)} \equiv \rho'' \quad , \quad \rho''' \equiv r'''_0 e^{-\Delta(\delta_0)} < r'''_k$$

We also choose δ_0 sufficiently small, so that the following condition holds:

$$e^{C_4 \Delta(\delta_0)} < \frac{3}{2}$$

where C_4 is the constant of Lemma 4.

We impose a new restriction to r'_0 (besides $r'_0 \leq r$). The function $a(y,z)$ has a zero of order 2 at $y = 0$ and is continuous on $\overline{\Pi_r}$. Therefore, there exists a constant C_a such that $|a(y,z)| < C_a |y|^2$ on $\overline{\Pi_r}$. So $||a||_{r_0} < C_a r'_0$. Therefore, for r'_0 small we will have $||a||_{r_0} \leq \delta_0^N$.

Denote $r'_0 e^{-\Delta(\delta_0)} \equiv \rho'$.

Fix $\beta > 2\alpha + 2$. We finally choose $N \geq \beta$ so that if the image of $\Pi_{r_{k+1}}$ under the map

$$V_k(y,z) = (y + h_k(y,z), z) \tag{2.39}$$

is relatively compact in Π_{r_k}, and if $||a_k||_{r_k} < \delta_k^N$ then $||a_{k+1}||_{r_{k+1}} < \delta_{k+1}^N$: for $||a_k||_{r_k} < \delta_k^N \leq \delta_k^\beta$ we have, by Lemma 6,

$$||a_{k+1}||_{r_{k+1}} \leq \delta_k^{-1-2\alpha} ||a_k||_{r_k}^2 \leq \delta_k^{2N-1-2\alpha} \leq \delta_{k+1}^N$$

where the last inequality holds provided that $3N/2 \leq 2N - 1 - 2\alpha$. So we must have $N \geq \max\{\beta, 2 + 4\alpha\}$.

2.4.5 Convergence of the Iteration

Remark 7 *1) The image of $\Pi_{r_{k+1}}$ under the map V_k given by (2.39) is relatively compact in Π_{r_k}; a_{k+1} is holomorphic on $\Pi_{r_{k+1}}$, continuous on $\overline{\Pi_{r_{k+1}}}$.*

2) The map V_k is a biholomorphism of $\Pi_{r_{k+1}}$ onto its image.

Proof

1) The proof is by induction on k.

For $k = 0$: by construction, a_0 is holomorphic on Π_{r_0}, continuous on $\overline{\Pi_{r_0}}$. Using Lemma 3, the function h_0 is holomorphic on every subdomain of Π_{r_0}, and for $(y, z) \in \Pi_{r_1} = \Pi_{r_0,\delta_0}$

$$|y + h_0(y, z)| < r'_1 + \delta_0^{-\alpha} \max_{\Pi_{r_0}} |a_0| = r'_0 e^{-\delta_0} + \delta_0^{-\alpha} r'_0 \max_{\Pi_{r_0}} \frac{|a_0|}{r'_0} \leq$$

$$r'_0 e^{-\delta_0} + \delta_0^{-\alpha} r'_0 ||a_0||_{r_0} \leq r'_0 e^{-\delta_0} + \delta_0^{N-\alpha} r'_0 < r'_0 \left(e^{-\delta_0} + \delta_0^2\right) < r'_0$$

As a consequence, $a_0(y + h_0(y, z), z)$ is well defined on a neighborhood of $\overline{\Pi_{r_1}}$, hence $a_1 = \mathcal{R}$ is holomorphic on Π_{r_1}, continuous on $\overline{\Pi_{r_1}}$.

Assuming the claim of part 1) of our Remark for V_k, a_k, one shows exactly as above that the claim is true for V_{k+1}, a_{k+1}.

2) The map $(y + h_k, z)$ is one-to-one.Indeed, suppose that $y_1 + h_k(y_1, z) = y_2 + h_k(y_2, z)$ for some points $(y_1, z), (y_2, z) \in \Pi_{r_{k+1}}$; then by (2.38)

$$|y_2 - y_1| = |h_k(y_1, z) - h_k(y_2, z)| \leq C'_3 |y_2 - y_1|$$

with $C'_3 < 1$. So $y_1 = y_2$. ∎

We may therefore consider the mapping $H_k = (y + h_0) \circ (y + h_1) \circ ... \circ (y + h_k)$, which is a biholomophism of $\Pi_{r_{k+1}}$ into Π_{r_0}.

Since $||a_k||_{r_k} \leq \delta_k^{\beta}$, we may apply Lemma 4 and get $||D_y h_k|| \leq C_4 \delta_k$ on $\Pi_{r_{k+1}}$. It follows that the derivative of H_k satisfies the estimates

$$\max_{\Pi_{r_{k+1}}} ||D_y H_k|| \leq \max_{\Pi_{r_1}} ||I + D_y h_0|| \max_{\Pi_{r_2}} ||I + D_y h_1|| ... \max_{\Pi_{r_{k+1}}} ||I + D_y h_k|| \leq$$

$$(1 + C_4\delta_0)(1 + C_4\delta_1)...(1 + C_4\delta_k) < e^{C_4 \Delta(\delta_0)} \tag{2.40}$$

The sequence $H_k(y, z)$ is Cauchy in the norm $||.||_\rho$ on the domain Π_ρ. Indeed,

$$||H_k - H_{k+1}||_\rho = \sup_{\Pi_\rho} \frac{1}{|y|} |H_k(y, z) - H_k(y + h_{k+1}(y), z)| \leq$$

$$\sup_{\Pi_{r_{k+1}}} ||D_y H_k|| \, ||h_{k+1}||_\rho \leq$$

by Remark 5 and (2.40)

$$\leq e^{C_4 \Delta(\delta_0)} \delta_{k+1}^{-\alpha} ||a_{k+1}||_{r_{k+1}} \leq e^{C_4 \Delta(\delta_0)} \delta_{k+1}^{\beta-\alpha}$$

Since the series $\sum \delta_{k+1}^{\beta-\alpha}$ converges, it follows that H_k is Cauchy.

Using Remark 4, H_k converges to an analytic function H.

The limit H is one-to-one. The proof is the same as of Remark 6, since

$$\max_{\Pi_{r_{k+1}}} ||D_y H_k - I|| = \max_{\Pi_{r_{k+1}}} ||(I + D_y h_0)(I + D_y h_1)...(I + D_y h_k) - I|| =$$

(where the derivatives $I + D_y h_j$ are evaluated at some suitable points)

$$\max_{\Pi_{r_{k+1}}} || \sum_{s=1,...,k+1} \sum_{j_1 < j_2 < ... < j_s} D_y h_{j_1} ... D_y h_{j_s} || \leq$$

$$\sum_{s=1,...,k+1} \sum_{j_1 < j_2 < ... < j_s} C_4^s \delta_{j_1} ... \delta_{j_s} =$$

$$(1 + C_4 \delta_0)(1 + C_4 \delta_1)...(1 + C_4 \delta_k) - 1 < e^{C_4 \Delta(\delta_0)} - 1 < \frac{1}{2}$$

Finally, the substitution $v = H(y, z)$ reduces the system (2.30) to its linear part because $a_k \longrightarrow 0$ (since $||a_k||_\rho \leq \delta_k^N$).

Chapter 3

Integrability Properties of a Generalized Lamé Equation; Applications to Polynomial Systems

The present research uses the poly-Painlevé test to find sufficient conditions for nonintegrability of a large class of differential systems (which includes polynomial Hamiltonian systems). In certain limits (studied in Section 3.2), such systems reduce to second order linear equations of the form

$$(x^k - 1)\frac{d^2u}{dx^2} + \frac{k}{2}x^{k-1}\frac{du}{dx} + \mu x^{k-2}u = 0 \tag{3.1}$$

(μ is a real parameter and $k \geq 3$ is an integer). Equation (3.1) is (except for omission of a constant term in a coefficient) a generalization of the Lamé equation, which, in its algebraic form, is [11]

$$(x^3 - 1)\frac{d^2u}{dx^2} + \frac{3}{2}x^2\frac{du}{dx} + \left(h + \frac{n(n+1)}{4}x\right)u = 0 \tag{3.2}$$

The Lamé equation is known to have a uniform solution if n is a positive integer (h is a constant, taken to be 0 in (3.1)).

The study of equation (3.1) is interesting in itself. Owing to a discrete symmetry of the equation, its monodromy group can be explicitly found. Therefore, the multivaluedness of the solutions can be obtained and the existence of first integrals studied. It is shown that there is only one real-analytic first integral, namely a hermitian form.

It was Riemann who introduced the monodromy group, understood its significance for the problem of global description of the solutions of the differential equation in question, and computed it explicitly for the hypergeometric differential equation

$$z(z-1)\frac{d^2F}{dz^2}+[(a+b+1)z-c]\frac{dF}{dz}+abF=0$$

Recently, the monodromy of the generalized hypergeometric function ${}_nF_{n-1}$ (solution of an n–th order differential equation with three regular singular points) has been computed [16]. It is remarkable that, just as in the case of the hypergeometric equation, there is a hermitian form which is invariant under the action of the monodromy group. We prove that this is also the case for another generalization (3.1) of the hypergeometric equation, obtained by increasing the number of singular points. Moreover, we show that under additional conditions, this is the only real-analytic invariant function. The result permits finding the differential Galois group of (3.1).

The results in the present chapter generalize those of [14].

3.1 The Generalized Lamé Equation

Equation (3.1) being linear, we can easily write down two local constants of the motion, as follows.

The points $x=\omega^m$, $m=0,...,k-1$ (where $\omega=\exp(2\pi i/k)$) and $x=\infty$ are regular singular points for the equation. In a neighborhood of $x=1$ there are two independent solutions

$$\phi(x),\ \gamma(x)=\sqrt{x-1}\psi(x) \tag{3.3}$$

with $\phi,\ \psi$ analytic.

All the solutions in the complex plane can be written in a neighborhood of 1 as $u=c_1\phi+c_2\gamma$ for some constants $c_1,\ c_2$. Two independent local integrals for the equation can be immediately obtained:

$$\begin{aligned} c_1&=\frac{u\gamma'(x)-u'\gamma(x)}{\phi(x)\gamma'(x)-\phi'(x)\gamma(x)}\\ c_2&=\frac{u'\phi(x)-u\phi'(x)}{\phi(x)\gamma'(x)-\phi'(x)\gamma(x)}\end{aligned} \tag{3.4}$$

The fundamental solutions $\phi,\ \gamma$ are multivalued in the complex plane, having branch points at the singular points of the equation (and so are the local integrals (3.4)).

We address the question of existence of global holomorphic and real-analytic first integrals for the equation.

3.1.1 The Monodromy Group

Consider a fundamental set of solutions near $x = 1$,

$$V_0(x) = \begin{pmatrix} \phi(x) \\ \gamma(x) \end{pmatrix}$$

as in (3.3).

Then $V_j(x) = V_0(x/\omega^j),\ j = 1, ..., k-1$ are fundamental sets of solutions near $x = \omega^j$ for each $j = 0, ..., k-1$, and $V_0, V_1, ..., V_{k-1}$ are analytic in a neighborhood of the origin. For x in that neighborhood, define the transition matrices M_j by $V_j(x) = M_j V_{j-1}(x),\ j = 1, ..., k\ , V_0(x) = M_k V_{k-1}(x)$. These relations also hold if x is replaced by x/ω. It follows that $M_1 = M_2 = ... = M_k$ and that $M_1^k = I$. Denote $M_1 = M$.

The point at infinity is also a regular singular point and a fundamental set of solutions near infinity has the form

$$V_\infty(x) = \begin{pmatrix} \phi_\infty(x) \\ \gamma_\infty(x) \end{pmatrix}$$

where

$$\phi_\infty(x) = x^{r_-}\tilde{\phi}(x)\ ,\ \gamma_\infty(x) = x^{r_+}\tilde{\gamma}(x)$$

with $\tilde{\phi},\ \tilde{\gamma}$ analytic functions at infinity ; $r_\pm$ are the characteristic exponents at infinity:

$$r_\pm = -\frac{k-2}{4} \pm \nu \ \ ,\text{where}\ \ \nu = \left(\left(\frac{k-2}{4}\right)^2 - \mu\right)^{1/2} \tag{3.5}$$

Define the transition matrix Q by $V_\infty = QV_0$.

Let

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

denote the monodromy matrix around a finite singular point (i.e. the matrix which defines the change of the fundamental set $V_j(x)$ upon analytic continuation on a closed path near the point $x = \omega^j$ which encircles the point once, counterclockwise).

Let

$$B = \begin{pmatrix} e^{2\pi i r_-} & 0 \\ 0 & e^{2\pi i r_+} \end{pmatrix}$$

denote the monodromy matrix at infinity (i.e. the change of the fundamental set $V_\infty(x)$ upon analytic continuation on a path encircling all the finite singular points, counterclockwise).

To find the monodromy group of the equation (3.1) we consider the change of the fundamental set $V_0(x)$ upon analytic continuation on closed paths starting near $x = 1$ and encircling only one singular point. On such a path encircling $x = 1$ the monodromy matrix is $X_0 = A$. On a path encircling $x = \omega$ the matrix is $X_1 = M^{-1}AM$ (since we first write V_0 in terms of V_1, then we continue V_1 on a closed path around $x = \omega$ and finally we go back to the vector V_0). Generally, on a path encircling $x = \omega^j$, the monodromy matrix is $X_j = M^{-j}AM^j$. The monodromy group $\mathcal{G}$ is generated by $X_j, j = 0, ..., k-1$.

We have more information about the group $\mathcal{G}$: analytic continuation around all the singular (finite) points is in fact analytic continuation on a path around ∞, therefore $X_{k-1}X_{k-2}...X_0 = Q^{-1}BQ$.

Therefore

$$(MA)^k = Q^{-1}BQ \quad , \quad M^k = I \tag{3.6}$$

Further information about the matrix M can be obtained as follows. Firstly, taking $x = 0$ in the relation $V_0(x/\omega) = MV_0(x)$ we get that $V_0(0)$ is an eigenvector of M, corresponding to the eigenvalue 1. Secondly, the Wronskian of the solutions satisfies the equation $W'(x) + kx^{k-1}/2(x^k-1)^{-1}W(x) = 0$, so $W(x) = c(x^k-1)^{-1/2}$. Since

$$W(x/\omega) = \begin{vmatrix} \phi(x/\omega) & \gamma(x/\omega) \\ \phi'(x/\omega) & \gamma'(x/\omega) \end{vmatrix} = \begin{vmatrix} \phi(x/\omega) & \gamma(x/\omega) \\ \omega\frac{d}{dx}\phi(x/\omega) & \omega\frac{d}{dx}\gamma(x/\omega) \end{vmatrix}$$

$$= \omega \det M \begin{vmatrix} \phi(x) & \gamma(x) \\ \phi'(x) & \gamma'(x) \end{vmatrix} = \omega(\det M)W(x)$$

we have $\det(M) = \omega^{-1}$.

Simple algebra can be now used to calculate the matrix M, with spectrum consisting of 1 and ω^{-1}, which satisfies (3.6). However, the matrix M cannot be uniquely defined by these conditions. The line of reasoning used here leaves the liberty of considering any scalar multiple of ϕ or γ. To see how the matrix M is affected by such a change, let

$$\tilde{V}_0(x) = \begin{pmatrix} c_1\phi(x) \\ c_2\gamma(x) \end{pmatrix}$$

be another fundamental set of solutions, and denote the elements of the transition matrix M and of the transition matrix $\tilde{M}$ of the new set of fundamental solutions by

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \tilde{M} = \begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{pmatrix}$$

From the relation $\tilde{V}_0(x/\omega) = \tilde{M}\tilde{V}_0(x)$ we get

$$\begin{cases} \phi(x/\omega) = \tilde{a}\phi(x) + \tilde{b}c_2/c_1\gamma(x) \\ \\ c_2/c_1\gamma(x/\omega) = \tilde{c}\phi(x) + \tilde{d}c_2/c_1\gamma(x) \end{cases}$$

which shows that $a = \tilde{a}, d = \tilde{d}, bc = \tilde{b}\tilde{c}$ and b is arbitrary. Therefore, we may take $b = 1$ in what follows.

Since the spectrum of M consists of 1 and ω^{-1} and the spectrum of MA consists of roots of order k of the eigenvalues of B, say $\beta_+ = e^{2\pi i(r_+ + n_+)/k}, \beta_- = e^{2\pi i(r_- + n_-)/k}$

$(0 \le n_\pm \le k-1)$ then

$$a + d = 1 + \omega^{-1}$$

$$a - d = \beta_+ + \beta_-$$

$$ad - c = \omega^{-1}$$

$$-ad + c = \exp\left[2\pi i(-(k-2)/2 + n_+ + n_-)\right]$$

So

$$-\omega^{-1} = \exp\left[2\pi i(-(k-2)/2 + n_+ + n_-)\right]$$

therefore $n_+ + n_- = k-2$ and

$$a = \tfrac{1}{2}\left(1 + \omega^{-1} + \beta_+ + \beta_-\right) = e^{il}(\cos l - i\cos p)$$

$$d = \tfrac{1}{2}\left(1 + \omega^{-1} - \beta_+ - \beta_-\right) = e^{il}(\cos l + i\cos p)$$

$$c = \tfrac{1}{4}\left[\left(1 + \omega^{-1}\right)^2 - (\beta_+ + \beta_-)^2\right] - \omega^{-1} = e^{2il}(\cos^2 l + \cos^2 p - 1)$$

$$\text{where} \quad l = \tfrac{\pi(k-1)}{k} \quad , \quad p = \tfrac{2\pi}{k}\left(\nu + \tfrac{n_+ - n_-}{2}\right)$$

The matrix Q can now be calculated:

$$Q = \frac{1}{\beta_+ - \beta_-}\begin{pmatrix} a - \beta_- & -1 \\ -(a - \beta_+) & 1 \end{pmatrix}$$

($\beta_+ \ne \beta_-$ because ν is irrational.) The number n_- remains undetermined.

3.1.2 The Invariant Function

Proposition

Consider the equation (3.1) with $k \ge 3$.

Let Ω be a domain in the phase space $\mathbf{C}^3$ whose projection on the x-coordinate contains closed paths around the points $x = 1,\ x = \omega, ..., x = \omega^{k-1}$ starting and ending at one common point.

If μ is such that $\sqrt{\left(\frac{k-2}{4}\right) - \mu}$ is irrational, then any holomorphic (in 6 variables) first integral $F(u, u', x, \bar{u}, \bar{u'}, \bar{x})$ (i.e. real-analytic in the real and imaginary parts of u, u', x) defined on $\Omega \times \bar{\Omega}$ is a function of

$$|c_1|^2 + \tau_\mu |c_2|^2$$

where $c_1,\ c_2$ are given by (3.4) and τ_μ is a real number.

In particular, there are no holomorphic first integrals $F(u, u', x)$ on Ω.

Let

$$u(x) = \begin{pmatrix} c_1 & c_2 \end{pmatrix} \begin{pmatrix} \phi(x) \\ \gamma(x) \end{pmatrix} \tag{3.7}$$

be a solution of (3.1) near $x = 1$. After analytic continuation along a closed path around the singular points, the value of solution becomes

$$\begin{pmatrix} c_1 & c_2 \end{pmatrix} G \begin{pmatrix} \phi(x) \\ \gamma(x) \end{pmatrix} \tag{3.8}$$

where $G \in \mathcal{G}$ is the corresponding monodromy matrix.

Note that, if $u(x)$ satisfies (3.7), then

$$\overline{u(x)} = \begin{pmatrix} \overline{c_1} & \overline{c_2} \end{pmatrix} \begin{pmatrix} \overline{\phi(x)} \\ \overline{\gamma(x)} \end{pmatrix}$$

After analytic continuation in the x-plane, if the new value of the solution $u(x)$ is given by (3.8), then the new value of $\overline{u(x)}$ is

$$\begin{pmatrix} \overline{c_1} & \overline{c_2} \end{pmatrix} \overline{G} \begin{pmatrix} \overline{\phi(x)} \\ \overline{\gamma(x)} \end{pmatrix}$$

Assume that there is a holomorphic first integral: $F(u, u', x, \overline{u}, \overline{u'}, \bar{x})$ defined on the domain $\Omega \times \bar{\Omega}$.

Let (u_0, u_0', x_0) be a point in Ω, where x_0 is not one of the singular points. Depending on the position of x_0 in the complex plane, we can write any solution with initial conditions at x_0 as $u(x) = (c_1, c_2)V_j(x)$ where $j = 0, 1, ...k-1$ or ∞. We assume $j = 0$ (for all other cases we change the constants using a transition matrix).

Let $u(x)$ be the solution with the initial conditions $u(x_0) = u_0, u'(x_0) = u_0'$. Then (3.7) holds, for some constants c_1, c_2. Denote, for short, $c = (c_1, c_2)$.

We will consider closed paths in Ω, around the singular points of the equation, starting and ending at x_0. The first integral F must have the same value when the solution is analytically continued along such paths. In particular, for the values of the solution at the begining and at the ending of paths:

$$\begin{aligned} &F\left(cV_0(x_0), cV_0'(x_0), x_0, \bar{c}\overline{V_0(x_0)}, \bar{c}\overline{V_0(x_0)'}, \overline{x_0}\right) \\ &\quad = F\left(cGV_0(x_0), cGV_0'(x_0), x_0, \bar{c}\bar{G}\overline{V_0(x_0)}, \bar{c}\bar{G}\overline{V_0(x_0)'}, \overline{x_0}\right) \end{aligned} \tag{3.9}$$

for all $G \in \mathcal{G}$, if the point $\left(cV_0(x_0), cV_0'(x_0), x_0, \bar{c}\overline{V_0(x_0)}, \bar{c}\overline{V_0(x_0)'}, \overline{x_0}\right)$ is in the domain of F.

After a linear change of coordinates, we may omit the fundamental solutions in (3.9) (and we also omit the dependence on the point x_0 which will be fixed hereafter) and simply write

$$F(cG, \bar{c}\bar{G}) = F(c, \bar{c}) \tag{3.10}$$

where F is holomorphic in a domain $U \times \bar{U}$.

Consider the following element of $\mathcal{G}$: $X = X_{k-1}X_{k-2}...X_0 = (AM)^k = Q^{-1}BQ$. Since ν is irrational, the set $\{4nr_+ (\text{mod } 1)\ ;\ n \in \mathbf{Z}\}$ is dense in the interval $[0,1]$ hence $\{X^{4n}\ ;\ n \in \mathbf{Z}\}$ is dense in the set

$$\{Q^{-1}D_\alpha Q\ ;\ D_\alpha = \text{diag}(e^{i\alpha}, e^{-i\alpha})\ ,\ \alpha \in \mathbf{R}\}$$

It follows that

$$F(c, \bar{c}) = F(cQ^{-1}D_\alpha Q, \bar{c}\overline{Q^{-1}}\overline{D_\alpha Q})$$

must also be true, for all α for which $cQ^{-1}D_\alpha Q \in U$, therefore for α small enough.

Denote $d = cQ^{-1}$ and let $\tilde{F}$ be defined by $\tilde{F}(d, \bar{d}) = F(dQ, \overline{dQ})$. Then $\tilde{F}$ satisfies $\tilde{F}(d, \bar{d}) = \tilde{F}(dD_\alpha, \bar{d}D_{-\alpha})$ for α small, or,

$$\tilde{F}(d_1, d_2, \overline{d_1}, \overline{d_2}) = \tilde{F}(e^{i\alpha}d_1, e^{-i\alpha}d_2, e^{-i\alpha}\overline{d_1}, e^{i\alpha}\overline{d_2})$$

Taking the derivative in α and evaluating it at $\alpha = 0$ we get the equation

$$d_1\frac{\partial\tilde{F}}{\partial d_1} - \bar{d}_1\frac{\partial\tilde{F}}{\partial\bar{d}_1} - d_2\frac{\partial\tilde{F}}{\partial d_2} + \bar{d}_2\frac{\partial\tilde{F}}{\partial\bar{d}_2} = 0$$

which has as solutions arbitrary functions of $|d_1|,\ |d_2|,\ \Re(d_1 d_2)$.

Thus any integral has the form

$$F(c, \bar{c}) = \Phi\left(|(cQ^{-1})_1|^2, |(cQ^{-1})_2|^2, \Re((cQ^{-1})_1(cQ^{-1})_2)\right)$$

Pick now another element in $\mathcal{G}$ having the spectrum on the unit circle, namely $Y = X_1X_2...X_{k-1}X_0 = (M^{-1}A)^k = R^{-1}B^{-1}R$ where $R = QX_0 = QA$.

By the above argument, there is a function Ψ such that

$$F(c, \bar{c}) = \Psi\left(|(cR^{-1})_1|^2, |(cR^{-1})_2|^2, \Re((cR^{-1})_1(cR^{-1})_2)\right) =$$
$$\Phi\left(|(cQ^{-1})_1|^2, |(cQ^{-1})_2|^2, \Re((cQ^{-1})_1(cQ^{-1})_2)\right)$$

Denote $cQ^{-1} = z$ and $P = QR^{-1}$:

$$P = QR^{-1} = \begin{pmatrix} \lambda & 1+\lambda \\ 1-\lambda & -\lambda \end{pmatrix} \quad \text{where} \quad \lambda = \frac{\cos l}{\sin p}$$

Then

$$\Phi\left(|z_1|^2, |z_2|^2, \Re(z_1 z_2)\right) = \Psi\left(|(zP)_1|^2, |(zP)_2|^2, \Re((zP)_1(zP)_2)\right) \tag{3.11}$$

for all pairs of complex numbers $z = (z_1, z_2)$.

It is convenient to write (3.11) in polar coordinates: if $z = (z_1, z_2)$ with $z_k = r_k \exp(i\theta_k)$ and if we denote $\theta = \theta_1 + \theta_2$, $\xi = \theta_1 - \theta_2$, then the condition (3.11) becomes

$$\tilde{\Phi}(r_1^2, r_2^2, \theta) = \tilde{\Psi}(A, B, C)$$

where

$$A = \lambda^2 r_1^2 + (1-\lambda)^2 r_2^2 + 2\lambda(1-\lambda) r_1 r_2 \cos(\xi)$$
$$B = (1+\lambda)^2 r_1^2 + \lambda^2 r_2^2 - 2\lambda(1+\lambda) r_1 r_2 \cos(\xi)$$
$$C = \lambda(1+\lambda) r_1^2 \cos(\theta+\xi) - \lambda(1-\lambda) r_2^2 \cos(\theta-\xi) + (1-2\lambda^2) r_1 r_2 cos(\theta)$$

Since the relation (3.11) must hold for all ξ, by differentiating with respect to ξ and evaluating at $\xi = 0$ we get that $\lambda \Psi_C(A, B, C)|_{\xi=0} = 0$. For $k \geq 3$ we have $\lambda \neq 0$ (while for $k = 2$, $\lambda = 0$). Since the functions A, B, C evaluated at $\xi = 0$ are functionally independent, it follows that Ψ does not depend on the third variable. Similarly, neither does Φ. So we may write $\Phi(r_1^2, r_2^2) = \Psi(A, B)$. Differentiating this relation with respect to ξ and evaluating at $\xi = \pi/2$ we get

$$\left(\Psi_A + \frac{\lambda+1}{\lambda-1}\Psi_B\right)|_{\xi=\pi/2} = 0$$

and since the functions $A|_{\xi=\pi/2}, B|_{\xi=\pi/2}$ are independent it follows that $\Psi(A, B)$ must be a function of $A + \frac{1-\lambda}{1+\lambda}B$.

Therefore, any first integral must be a function of

$$\mathcal{B}(z, \bar{z}) = |z_1|^2 + \frac{1-\lambda}{1+\lambda}|z_2|^2$$

Since $z = cQ^{-1}$, returning to the variable c, it follows that all first integrals are functions of

$$\mathcal{B}(cQ^{-1}, \overline{cQ^{-1}}) = |c_1|^2 + \tau|c_2|^2 \quad \text{,where} \quad \tau = \sin^2 p - \cos^2 l$$

On the other hand, direct calculation shows that this expression is invariant to M. As is also invariant to A, it follows that is invariant to all matrices in $\mathcal{G}$. ∎

3.2 Applications

3.2.1 Examples

1. A simple higher order system

If for first order equations nonintegrability is due, in general, to accumulation of movable branch points of solutions, new phenomena appear in the higher order case. We can understand this by examining a very simple and well understood example: harmonic oscillators with irrationally related frequencies:

$$\ddot{x} = -\omega_1^2 x \quad , \quad \ddot{y} = -\omega_2^2 y \quad , \ \omega_1/\omega_2 \notin \mathbf{Q}$$

We know that there are two constants of the motion (the energy of each particle). The trajectories, in the real phase space, are dense in a torus, fact which can be equivalently stated as: there is no (continuous) third constant of the motion.

Let us re-examine these properties using analysis in the complex domain. We look for first integrals, i.e. for functions of $(x, y, \dot{x}, \dot{y})$, constant on the trajectories

$$\begin{aligned} x(t) &= A_1 \sin(\omega_1 t + B_1) \\ y(t) &= A_2 \sin(\omega_2 t + B_2) \\ \dot{x}(t) &= A_1 \omega_1 \cos(\omega_1 t + B_1) \\ \dot{y}(t) &= A_2 \omega_2 \cos(\omega_2 t + B_2) \end{aligned}$$

Two constants of motion (related to the energies of the two particles) are $A_1^2 = x^2 + \omega_1^{-2}\dot{x}^2$ and $A_2^2 = y^2 + \omega_2^{-2}\dot{y}^2$. To find an independent third integral (which should depend simultaneously on both particles), we eliminate the time in the formulae for $x(t)$ and $y(t)$ and get:

$$\begin{aligned} K &\equiv B_2 - \frac{\omega_2}{\omega_1} B_1 \\ &= \arcsin \frac{y}{\sqrt{y^2 + \omega_2^{-2}\dot{y}^2}} - \frac{\omega_2}{\omega_1} \arcsin \frac{x}{\sqrt{x^2 + \omega_1^{-2}\dot{x}^2}} \end{aligned} \tag{3.12}$$

$$\equiv F\left(\frac{x}{A_1}, \frac{y}{A_2}\right)$$

The first integral (3.12) is multivalued. If there is a single-valued integral, independent of the two energies, then there is an analytic function $\Phi(K, A_1, A_2)$ such that $\Phi(F(\frac{x}{A_1}, \frac{y}{A_2}), A_1, A_2)$ is single-valued. But the values of F on different branches are

$$F(X,Y) = (-1)^n \arcsin(Y) + 2n\pi - (-1)^m \frac{\omega_2}{\omega_1} \arcsin X + \frac{\omega_2}{\omega_1} 2m\pi$$

for any $n, m \in \mathbf{Z}$. If ω_1/ω_2 is irrational, then the set

$$\left\{ n + \frac{\omega_2}{\omega_1} m \ ; \ n, m \in \mathbf{Z} \right\}$$

is dense in the real line. It follows that the holomorphic function Φ cannot depend on the first argument, hence there is no first integral independent on the two energies.

The same argument shows that there is no additional analytic first integral for the system considered in the real domain.

This example shows that for higher order equations, nonintegrability can occur (not only due to accumulation of branch points but also) due to the presence of oscillations with irrationally related frequencies.

In the examples and classes of equations studied in this chapter we use this idea to set up the poly-Painlevé test and find criteria for nonintegrability.

2. The Hénon-Heiles system

is given by the Hamiltonian having the potential

$$V(q_1, q_2) = a q_1^2 q_2 + \frac{b}{3} q_2^3$$

The equations of motion are

$$\ddot{q}_1 = -2a q_1 q_2 \quad , \quad \ddot{q}_2 = -a q_1^2 - b q_2^2 \tag{3.13}$$

For generic values of the parameters a, b, numerical experiments show that the system exhibits chaotic behavior.

The poly-Painlevé test for first order equations is usually done by an expansion around points in the phase space. Its natural generalization to higher order systems is expansion around manifolds.

The simplest example of manifold is the linear one. In order to obtain a regularly perturbed system, this manifold should be an orbit. We therefore look for particular solutions of (3.13) of the form $q_1(t) = \alpha_1\phi(t)$, $q_2(t) = \alpha_2\phi(t)$. Substituting in the equations we get

$$\ddot{\phi} = \phi^2 \quad , \quad \alpha_1 = -2a\alpha_1\alpha_2 \quad , \quad \alpha_2 = -a\alpha_1^2 - b\alpha_2^2$$

We obtain the following values for (α_1, α_2): $(0,0)$, $(0, -1/b)$, and $(-(2a)^{-1}, \pm[(2a^2)^{-1} - b(4a^3)^{-1}]^{1/2}$.

The first value for the parameters does not seem to produce a simpler system under scaling and we will study the other three values. The corresponding particular solutions are doubly periodic functions, and we expect (in view of Example 1) that generic perturbations might exhibit nonintegrability.

Therefore, we introduce a new dependent variable u by changing q_1:

$$q_1 = \frac{\alpha_1}{\alpha_2}q_2 + \epsilon u \tag{3.14}$$

Denote for short $q_2 = q$. The system (3.13) becomes

$$\ddot{u} = 2a(\alpha_1^2\alpha_2^{-2} - 1)qu + \epsilon a\alpha_2\alpha_2^{-1}u^2 \tag{3.15}$$

$$\ddot{q} = (-a\alpha_1^2\alpha_2^{-2} - b)q^2 - \epsilon 2a\alpha_2\alpha_2^{-1}qu - \epsilon^2 au^2 \tag{3.16}$$

We first analyze the integrability properties of the reduced ($\epsilon = 0$) system. With the notation $\lambda = 2a\alpha_2(\alpha_1^2\alpha_2^{-2} - 1)$ (to be consistent with the notations of Section 3.2.2), and since $-a\alpha_1^2\alpha_2^{-2} - b = \alpha_2^{-1}$ the reduced system is

$$\ddot{u} = \lambda\alpha_2^{-1}qu \quad , \quad \ddot{q} = \alpha_2^{-1}q^2 \tag{3.17}$$

The last equation can be integrated once and gives

$$\dot{q}^2 = \frac{2}{3\alpha_2}q^3 - C$$

We now eliminate the time in (3.17) by turning q into the independent variable. Then rescaling q by the substitution $q = (3C\alpha_2/2)^{1/3}x$ we get the linear second order equation

$$(x^3 - 1)\frac{d^2u}{dx^2} + \frac{3}{2}x^2\frac{du}{dx} - \frac{3}{2}\lambda x u = 0 \tag{3.18}$$

The integrability properties of this equation were studied in Section 3.1.1. It has no holomorphic first integrals (on a domain sufficiently large) if the number $\nu = \sqrt{1/16 + 3\lambda/2}$ is irrational. For the three values of (α_1, α_2) we obtain two values for the number ν:

$$\nu_1 = \sqrt{(1 + 48ab^{-1})/16} \ \ , \ \ \nu_2 = \sqrt{(24ba^{-1} - 23)/16} \tag{3.19}$$

We now address the question of how the non-existence of a holomorphic first integral of the equation (3.18) relates to the non-existence of additional first integrals (independent of the Hamiltonian) for the Henon-Heiles system (3.13).

Note that the constant of integration C corresponds to the Hamiltonian at $\epsilon = 0$.

Assume that (3.13) has a first integral F which is independent of the Hamiltonian. We assume that F is defined on a domain $D \subset \mathbf{C}^4$ sufficiently large: such that the projection D_{q_2} of D on the q_2-coordinate contains closed paths around the roots of the polynomial (in q) $q^3 - 3C\alpha_2/2$, for some C (e.g. contains an annulus centered at the origin). We choose C such that this property holds.

The first step is to show that the integral F produces a first integral for (3.17). In order to prove this step we make a restrictive assumption: that F is meromorphic near the linear manifold $q_1 = \alpha_1\alpha_2^{-1}q_2,\ \dot{q}_1 = \alpha_1\alpha_2^{-1}\dot{q}_2$. Under this assumption there is an integer p such that

$$F(\epsilon u, \epsilon\dot{u}, q, \dot{q}) = \epsilon^p G(\epsilon, u, \dot{u}, q, \dot{q}) = \epsilon^p \sum_{n=0}^{\infty} \epsilon^n G_n(u, \dot{u}, q, \dot{q})$$

For ϵ small the functions G_n have the same regularity as F and the series converges for $(u, \dot{u}, q, \dot{q})$ in the domain of F.

Then G_0 is a first integral for the reduced system (3.17) (or is constant). If G_0 depends on u or $\dot{u}$, then it is independent of $H|_{\epsilon=0}$, hence

$G(0, u, \dot{u}, q, \sqrt{\frac{4\lambda}{3}q^3 - C})$ is a first integral for (3.17).

If G_0 depends only on $(q, \dot{q})$ or is constant, we can reduce to the preceding case in the following way. The function G_0 will be an integral for the reduced system (3.17) (or a constant); therefore $\epsilon^{-p-1}(F - G_0)$ is either a first integral for (3.15) or a constant.

We repeat the procedure, until, for some n_0, G_{n_0} depends on u or $\dot{u}$. (There is such a number n_0 because (3.15) obviously does not admit integrals depending only on $(q, \dot{q})$.) The upshot is that there exists a first integral $G(u, \dot{u}, q, \dot{q})$ for (3.17), having the same regularity as F.

The next step is deducing a first integral for the system with time eliminated, namely the equation (3.18).

It is clear that $G\left(u, \frac{du}{dq}\sqrt{\frac{4}{3}\lambda q^3 - C}, q, \sqrt{\frac{4}{3}\lambda q^3 - C}\right)$ is a first integral for (3.18). Denote it by $\tilde{G}\left(u, u', q, \sqrt{\frac{4}{3}\lambda q^3 - C}\right)$ (here u' stands for the derivative of u with respect to q). By restricting $\tilde{G}$ to a sub-domain $\tilde{D}_u \times D_q \subset D$, we may assume that $\tilde{G}$ is analytic in $(u, u', q, \dot{q})$.

After the rescaling of q

$$\tilde{G}(u, u', q, \sqrt{\frac{4}{3}\lambda q^3 - C}) \equiv \hat{G}(u, u', x, \sqrt{x^3 - 1})$$

where $\hat{G}$ is analytic in four variables on $\tilde{D}_u \times D_x$ (where, clearly, D_x is a dilation of D_q: $D_x = (3C\lambda/2)^{-1/3} D_q$).

In Section 3.1.1 it was shown that if $\sqrt{1/16 - \mu}$ is irrational then there are no first integrals for (3.1). In that argument, only paths surrounding the singular points an even number of times are used. Therefore, the argument also applies to first integrals of the type $\hat{G}(u, u', x, \sqrt{x^3 - 1})$. Hence $r_\pm$ given by (3.5) must be rational.

To summarize, if any of the numbers ν_1, ν_2 of (3.19) is not rational, then the system (3.13) has no additional holomorphic first integrals.

However, the existence of real-analytic first integrals is not ruled out by the present analysis, and neither is the integrability in the real domain. The study of the next orders for the solutions in (3.15) might strengthen the result.

It is interesting to compare the results of the present analysis with the results of the Painlevé test on the Henon-Heiles system.

The investigation of the Painlevé property shows [13] the existence of solutions with behavior

$$x(t) \sim A(t-t_0)^{s_1} \tag{3.20}$$

$$y(t) \sim B_0(t-t_0)^{-2}, \ (t \to t_0) \tag{3.21}$$

where

$$s_1 = \frac{1}{2} \pm \frac{1}{2}\sqrt{1+48ab^{-1}} \tag{3.22}$$

(A is arbitrary, $B_0 = -6/b$), as well as resonances, some being given by

$$x(t) \sim A_1(t-t_0)^{-2} + ... + C(t-t_0)^{s_2} \tag{3.23}$$

$$y(t) \sim B_1(t-t_0)^{-2} + ... + D(t-t_0)^{s_2}, \ (t \to t_0) \tag{3.24}$$

where

$$s_2 = \frac{1}{2} \pm \frac{1}{2}\sqrt{24a^{-1}b-23} \tag{3.25}$$

(A_1, B_1 have definite values, C, D are arbitrary).

Therefore, the system has the Painlevé property only if the numbers s_1, s_2 are integers, which is the case for values of $a^{-1}b$ equal to $1, \ 2, \ 6$.

We note that the behaviors (3.20)...(3.25) of solutions (when t is eliminated) are obtained in the poly-Painlevé analysis as behavior of solutions of (3.18) at infinity.

3. Nonhomogeneous Henon-Heiles System

is the Hamiltonian system with the potential

$$V(q_1, q_2) = \frac{1}{2}(rq_1^2 + sq_2^2) + aq_1^2q_2 + \frac{b}{3}q_2^3$$

The equations of motion are

$$\ddot{q}_1 = -rq_1 - 2aq_1q_2 \tag{3.26}$$

$$\ddot{q}_2 = -sq_2 - aq_1^2 - bq_2^2 \tag{3.27}$$

Rescaling both the variables and the time we can reduce the equations (3.26) to (3.13): set

$$t = \delta T,\ q_1 = \gamma X,\ q_2 = \gamma Y \tag{3.28}$$

(since we want to obtain the homogeneous second order part of the system (3.26) the variables q_1 and q_2 will have the same scaling factor $\gamma \gg 1$). Then the equations are

$$\begin{aligned}\ddot{X} &= -\delta^2 rX - 2\delta^2\gamma aXY\\ \ddot{Y} &= -\delta^2 sY - \delta^2\gamma(aX^2 + bY^2)\end{aligned}$$

Therefore, if $\gamma = \delta^{-2},\ \delta \ll 1$ the system at $\delta = 0$ is (3.13). Combining (3.14) and (3.28) for $\delta = \epsilon$ amounts to the change of variables in (3.26)

$$q_1(t) = \epsilon^{-2}\frac{\alpha_1}{\alpha_2}Y\left(\frac{t}{\epsilon}\right) + \epsilon^{-1}u\left(\frac{t}{\epsilon}\right)$$

$$q_2(t) = \epsilon^{-2}Y\left(\frac{t}{\epsilon}\right)$$

which yields the system

$$\begin{aligned}u'' &= 2a(\alpha_1^2\alpha_2^{-2} - 1)Yu + \epsilon a\alpha_1\alpha_2^{-1}u^2 - \epsilon r\left(\frac{\alpha_1}{\alpha_2}Y + \epsilon u\right)\\ Y'' &= \alpha_2^{-1}Y^2 - 2\epsilon a\alpha_1\alpha_2^{-1}Yu - \epsilon^2 au^2 - \epsilon^2 sY\end{aligned}$$

(the sign “prime” stands for d/dT).

The zero order system is the same as in Example 2 and so are the nonintegrability conditions.

The relevance of this result on the nonintegrability of the original system is proven as in Example 4, with the difference that, in order to produce a rigorous result, we assume here that the sought-for first integrals F are meromorphic near the linear manifold $q_1 = \alpha_1\alpha_2^{-1}q_2,\ \dot{q}_1 = \alpha_1\alpha_2^{-1}\dot{q}_2$ for large values of the variables.

4. Systems with a zero of high order

Consider, for example, a Hamiltonian system with the potential

$$V(q_1, q_2) = aq_1^2q_2 + \frac{b}{3}q_2^3 + \frac{c}{4}q_1^4 + \frac{d}{4}q_2^4$$

Proceeding as in Example 3, a scaling

$$t = \delta T,\ q_1 = \gamma X,\ q_2 = \gamma Y \tag{3.29}$$

with $\gamma\delta^2 = 1, \gamma \ll 1$ gives a perturbation of (3.13). Combining (3.14) and (3.29) for $\gamma = \epsilon^2$ one obtains the following substitution

$$\begin{aligned} q_1 &= \epsilon^2 \frac{\alpha_1}{\alpha_2} Y\left(\epsilon t\right) + \epsilon^3 u\left(\epsilon t\right) \\ q_2(t) &= \epsilon^2 Y\left(\epsilon t\right) \end{aligned}$$

The system becomes

$$\begin{aligned} u'' &= 2a(\alpha_1^2\alpha_2^{-2} - 1)Yu + \epsilon(a\alpha_1\alpha_2^{-1}u^2 + dY^3) - \epsilon c(\alpha_1\alpha_2^{-1}Y + \epsilon u)^3 \\ Y'' &= \alpha_2^{-1}Y^2 - 2\epsilon a\alpha_1\alpha_2^{-1}Yu - \epsilon^2(au^2 + dY^3) \end{aligned}$$

(the sign “prime” stands for d/dT).

The zero order system is the same as in Example 2 and so are the nonintegrability conditions.

The relevance of this result on the nonintegrability of the original system is proven as in Example 2, with the difference that, in order to produce a rigorous result, we assume here that the sought-for first integrals F are meromorphic near the linear manifold $q_1 = \alpha_1\alpha_2^{-1}q_2,\ \dot{q}_1 = \alpha_1\alpha_2^{-1}\dot{q}_2$ for small values of the variables.

3.2.2 Generalizations

Homogeneous systems

Consider the system of second order differential equations:

$$\ddot{q}_m = P_m(\mathbf{q})\ ,\quad m = 1, ..., n,\quad \mathbf{q} = (q_1, ..., q_n) \tag{3.30}$$

where P_m are homogeneous polynomials of degree $k-1$. In particular, (3.30) can be a Hamiltonian system with potential $V(\mathbf{q})$ homogeneous polynomial of degree k, if $P_m = \frac{\partial V}{\partial q_m},\ m = 1, ..., n$.

We generalize the treatment of the Henon-Heiles system (3.13) and look for particular solutions which are multiples of each other (see also Ito [12]): $q_m(t) = \alpha_m\phi(t),\ m =$

$1, ..., n$. Substituting in (3.30) we get the following equations for the function ϕ and the constants α_m:

$$\ddot{\phi} = \phi^{k-1}, \ \alpha_m = P_m(\alpha_1, ..., \alpha_m), \ m = 1, ..., n \tag{3.31}$$

Let $(\alpha_1, ..., \alpha_n)$ be one non-zero solution of (3.31). Suppose $\alpha_n \neq 0$. We introduce new variables $u_1, ..., u_{n-1}$ and Q, and a small parameter ϵ, by the substitution

$$q_m(t) = \frac{\alpha_m}{\alpha_n} Q(t) + \epsilon u_m(t), \ m = 1, ..., n-1 \tag{3.32}$$

$$q_n(t) = Q(t) \tag{3.33}$$

Expanding P_m in Taylor series and using the homogeneity, the system (3.30) takes the form

$$\ddot{u}_m = \sum_{j=1}^{n-1} \left(\frac{\partial P_m}{\partial q_j}(\alpha) - \frac{\alpha_m}{\alpha_n} \frac{\partial P_n}{\partial q_j}(\alpha) \right) Q^{k-2} \alpha_n^{-k+2} u_j + \epsilon R_m(\mathbf{u}, Q, \epsilon), \qquad m = 1, ..., n-1 \tag{3.34}$$

$$\ddot{Q} = Q^{k-1} \alpha_n^{-k+1} P_n(\alpha) + \epsilon R_n(\mathbf{u}, Q, \epsilon) \tag{3.35}$$

where R_j are polynomials, $\mathbf{u} = (u_1, ..., u_{n-1})$; in vector notation, (3.34) can be written as

$$\mathbf{u} = Q^{k-2} \alpha_n^{-k+2} N_P \mathbf{u} + \epsilon R(\mathbf{u}, Q, \epsilon) \tag{3.36}$$

where

$$N_P = \left[\frac{\partial P_m}{\partial q_j}(\alpha) - \frac{\alpha_m}{\alpha_n} \frac{\partial P_n}{\partial q_j}(\alpha) \right]_{m,j=1,...,n-1}$$

We assume that the matrix N_P has distinct eigenvalues $\lambda_1, ..., \lambda_{n-1}$, hence it can be diagonalized: $S^{-1} N_P S = \operatorname{diag}(\lambda_1, ..., \lambda_{n-1})$. We change variables linearly by $\mathbf{v} = S^{-1}\mathbf{u}$. The reduced equations (i.e. the system (3.36), (3.35) for $\epsilon = 0$) is then

$$\ddot{v}_m = \lambda_m Q^{k-2} \alpha_n^{-k+2} v_m, \ m = 1, ..., n-1 \tag{3.37}$$

$$\ddot{Q} = Q^{k-1} \alpha_n^{-k+2} \tag{3.38}$$

Integrating once the equation for Q we get

$$\dot{Q}^2 = \frac{2}{k} \alpha_n^{-k+2} Q^k - C \tag{3.39}$$

We formally eliminate the time in the system (3.37), (3.39) by turning Q into independent variable. We obtain a decoupled system of $n-1$ linear equations:

$$\left(Q^k - C\frac{k}{2}\alpha_n^{k-2}\right)\frac{d^2v_m}{dQ^2} + Q^{k-1}\alpha_n^{-k+2}\frac{dv_m}{dQ} - Q^{k-2}\alpha_n^{-k+2}\lambda_m v_m = 0$$
$$m = 1, ..., n-1$$

A rescaling of the variable Q by

$$Q = \left(C\frac{k}{2}\alpha_n^{-k+2}\right)^{1/k} x$$

yields

$$(x^k - 1)\frac{d^2v_m}{dx^2} + \frac{k}{2}x^{k-1}\frac{dv_m}{dx} - \frac{k}{2}\lambda_m x^{k-2}v_m = 0 \quad, m = 1, ..., n-1 \tag{3.40}$$

Assume that N numbers among

$$\nu_n = \sqrt{(k-2)^2/16 + \frac{k}{2}\lambda_m}, \qquad m = 1, ..., n-1 \tag{3.41}$$

are irrational. Then using the results of section 3.1.1 the corresponding N linear equations (3.40) have no holomorphic first integrals. Therefore, the reduced system (3.37), (3.38) has at most $2n-1-2N$ first integrals. The same proof as in Example 2 shows that also the system (3.30) can not have more than $2n-1-2N$ independent integrals which are meromorphic near the linear manifold $q_m = \alpha_m/\alpha_n q_n$, $\dot{q}_m = \alpha_m/\alpha_n \dot{q}_n$, $m = 1, ..., n-1$.

For particular models the result can be used for different n-tuples $(\alpha_1, ..., \alpha_n)$ to possibly further reduce the maximal number of first integrals.

Nonhomogeneous polynomial systems

Generalizing Example 3 of section 3.2.1 we now consider the system of second order differential equations:

$$\ddot{q}_m = P_m(\mathbf{q}) + f_m(\mathbf{q}) \ , \quad m = 1, ..., n, \quad \mathbf{q} = (q_1, ..., q_n) \tag{3.42}$$

where P_m are homogeneous polynomial of degree $k-1$ and each $f_m(\mathbf{q})$ is a sum of a polynomial of degree at most $k-2$ and a function holomorphic at infinity, i.e.

$$f_m(\mathbf{q}) = \sum_{i_1+\ldots+i_n \leq k-2,\ i_j \in \mathbf{Z}} f_{m,i_1,\ldots,i_n} q_1^{i_1} \ldots q_n^{i_n}$$

and the series converges. In particular, (3.42) can be a Hamiltonian system with the potential $V(\mathbf{q})$ of the form polynomial of degree k plus a function holomorphic at infinity.

To reduce the system to its highest order homogeneous part, we consider a substitution of the form

$$q_m(t) = \epsilon^c \frac{\alpha_m}{\alpha_n} Q(\epsilon^b t) + \epsilon^a u_m(\epsilon^b t),\ m = 1, \ldots, n-1 \tag{3.43}$$

$$q_n(t) = \epsilon^c Q(\epsilon^b t) \tag{3.44}$$

where $(\alpha_1, \ldots, \alpha_n)$ is a non-zero solution of the system $\alpha_m = P_m(\alpha_1, \ldots, \alpha_n)$, $m = 1, \ldots, n$, and we assume $\alpha_n \neq 0$.

Substituting (3.43), (3.44) in (3.42) one obtains

$$u_m'' = \sum_{j=1}^{n-1} \left(\frac{\partial P_m}{\partial q_j}(\alpha) - \frac{\alpha_m}{\alpha_n} \frac{\partial P_n}{\partial q_j}(\alpha) \right) (\epsilon^c Q \alpha_m \alpha_n)^{k-2} \epsilon^{-2b} u_j + R_m$$

$$\text{for } m = 1, \ldots, n-1$$

$$\epsilon^{c+2b} Q'' = (\epsilon^c Q/\alpha_n)^{k-1} P_n(\alpha) + (\epsilon^c Q/\alpha_n)^{k-2} \epsilon^a \nabla P_n(\alpha) \cdot \mathbf{u} + R_n$$

where

$$R_m = O(\epsilon^{ci-a-2b}, i \leq k-2) + O(\epsilon^{c(k-1-j)+ja-a-2b}, j \geq 2)$$

$$\text{for } \ m = 1, \ldots, n-1$$

$$R_n = O(\epsilon^{ci}, i \leq k-2) + O(\epsilon^{c(k-1)+2(a-c)})$$

In order to obtain a regular perturbation, and that the $\epsilon = 0$ system is the same as for the homogeneous case we must have $c(k-2) - 2b = 0,\ c < a < 0$. (We note that this is not the maximal balance, which occurs for $a = 0$ and gives one more term in the reduced equation.) For example, we may choose

$$a = -1,\ c = -2,\ b = -(k-2) \tag{3.45}$$

From this point on the system can be treated as in the homogeneous case and Example 3: the system (3.42) can not have more than $2n-1-2N$ independent integrals which are meromorphic near the linear manifold $q_m = \alpha_m/\alpha_n q_n,\ \dot{q}_m = \alpha_m/\alpha_n \dot{q}_n,\ m = 1,...,n-1$ for large values of the variables (N is the number of irrationals among $\nu_1,...,\nu_{n-1}$ defined by (3.41)).

Systems with high order zeroes

Generalizing Example 4 of section 3.2.1 we consider the system of second order differential equations:

$$\ddot{q_m} = P_m(\mathbf{q}) + g_m(\mathbf{q})\ , \quad m = 1,...,n, \quad \mathbf{q} = (q_1,...,q_n) \tag{3.46}$$

where P_m are homogeneous polynomials of degree $k-1$ and $g_m(\mathbf{q})$ are functions holomorphic at the origin, whose Taylor series contain only terms of degree at least k. In particular, (3.46) can be a Hamiltonian system with potential having a zero of order k at the origin.

Proceeding in a way similar to the preceding case, one can easily establish that the substitution

$$q_m(t) = \epsilon \frac{\alpha_m}{\alpha_n} Q(\epsilon^{k-2}t) + \epsilon^2 u_m(\epsilon^{k-2}t),\ m = 1,...,n-1$$

$$q_n(t) = \epsilon Q(\epsilon^{k-2}t)$$

gives, in the zero order, the same system as in the homogeneous case. The upshot is that the system (3.46) can not have more than $2n-1-2N$ independent integrals which are meromorphic near the linear manifold $q_m = \alpha_m/\alpha_n q_n,\ \dot{q}_m = \alpha_m/\alpha_n \dot{q}_n,\ m = 1,...,n-1$ for small values of the variables (N is defined as at point *1*).

Chapter 4

Various Applications

4.1 A density Lemma

The following result is needed to establish dense branching for examples in the following sections.

Lemma 7 *Let z_1, z_2, z_3 be complex numbers, $z_k = x_k + iy_k,\ k = 1, 2, 3.$*

The necessary and sufficient condition that the lattice formed of integer combinations of z_1, z_2, z_3 be dense in the complex plane is the following: the areas of the parallelograms in the complex plane determined by any two pairs formed with z_1, z_2, z_3 are linearly independent over the integers, i.e. if for some integers M, N, P

$$\det \begin{pmatrix} M & N & P \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix} = 0 \tag{4.1}$$

then $M = N = P = 0$.

Proof

The condition is sufficient

We will identify the complex plane $\mathbf{C}$ with the real plane $\mathbf{R}^2$. The proof proceeds in two steps.

Step 1

We note that the condition of independence (4.1) implies that z_1, z_2, z_3 are pairwise linearly independent over the reals.

Let $\mathcal{L}$ be the lattice

$$\mathcal{L} = \{Nz_1 + Mz_2\ ;\ N, M \in \mathbf{Z}\}$$

Consider the 2-dimensional torus $\mathbf{T}^2$ defined as $\mathbf{R}^2/\mathcal{L}$ i.e. classes of equivalence $[z]$ of vectors $z \in \mathbf{R}^2$ modulo elements in $\mathcal{L}$.

Consider the translation on the torus:

$$g : \mathbf{T}^2 \longrightarrow \mathbf{T}^2 \quad , \quad g([z]) = [z + z_3]$$

Claim: For each z the iterates $\{g^n([z]); n \in \mathbf{Z}\}$ are dense in the torus.

Proof

The problem being linear (modulo $\mathcal{L}$), it is enough to prove the statement for one value of z, say $z = 0$.

Let $\epsilon_k > 0$, $\epsilon_k \to 0$. From the Poincaré recurrence theorem, there are integers $n_k \geq 1$ such that the distance (on the torus) of the point $g^{n_k}(0)$ to 0 is less than ϵ_k. Since $g^{n_k}(0) = [n_k z_3]$, there are integers N_k, M_k such that

$$n_k z_3 = N_k z_1 + M_k z_2 + Z_{n_k}$$

where $||Z_{n_k}|| < \epsilon_k$. Note that $Z_{n_k} \neq 0$, since the vectors z_1, z_2, z_3 are not dependent over the integers. Denote $Z_{n_k} = (a_k, b_k)$; it belongs to the parallelogram $\mathcal{P}$ with two sides z_1, z_2.

Remark: There are two numbers $k.l$ such that the vectors Z_{n_k} and Z_{n_l} are linearly independent.

Indeed, assume the contrary. Then for all k there are real numbers λ_k such that $a_k = \lambda_k a_1$ and $b_k = \lambda_k b_1$. Therefore $a_k b_1 - a_1 b_k = 0$. But

$$0 = a_k b_1 - a_1 b_k = \det \begin{pmatrix} a_k & a_1 \\ b_k & b_1 \end{pmatrix} =$$

$$\det \begin{pmatrix} n_k x_3 - N_k x_1 - M_k x_2 & n_1 x_3 - N_1 x_1 - M_1 x_2 \\ n_k y_3 - N_k y_1 - M_k y_2 & n_1 y_3 - N_1 y_1 - M_1 y_2 \end{pmatrix} =$$

$$A_k \det \begin{pmatrix} x_1 & x_3 \\ y_1 & y_3 \end{pmatrix} + B_k \det \begin{pmatrix} x_2 & x_3 \\ y_2 & y_3 \end{pmatrix} + C_k \det \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}$$

where $A_k = N_1 n_k - N_k n_1,\ B_k = M_1 n_k - M_k n_1,\ C_k = N_1 B_k - M_1 A_k$. It follows that $A_k = B_k = C_k = 0$ which implies

$$\frac{n_k}{n_1} = \frac{N_k}{N_1} = \frac{M_k}{M_1}$$

Since

$$x_3 = \frac{N_k}{n_k}x_1 + \frac{M_k}{n_k}x_2 + \frac{a_k}{n_k} = \frac{N_1}{n_1}x_1 + \frac{M_1}{n_1}x_2 + \frac{a_1}{n_1}$$

then

$$\frac{a_k}{n_k} = \frac{a_1}{n_1} \text{ for all } k$$

and since $\lim_{k\to\infty} a_k/n_k = 0$ then $a_1 = 0$. Similarly $b_1 = 0$, which contradicts the fact that $Z_1 \neq 0$. The *Remark* is proven.

Therefore, for all $\epsilon > 0$ there are two numbers m, n such that the vectors Z_m, Z_n (equivalent to $g^n(0),\ g^m(0)$ modulo $\mathcal{L}$) are linearly independent and of length less than ϵ.

Consider the parallelogram $\mathcal{P}_\epsilon$ in $\mathcal{P}$ of vertices $0,\ Z_n,\ Z_m,\ Z_{m+n}$. (Note that $[Z_{m+n}] = g^{m+n}(0)$.) The images of $\mathcal{P}_\epsilon$ under the maps $g^k,\ k \in \mathbf{Z}$ cover all $\mathcal{P}$. Therefore, any point in $\mathcal{P}$ lies in $g^k(\mathcal{P}_\epsilon)$ for some k, and thus is at a distance less than $\epsilon\sqrt{2}$ from some Z_N, proving density.

Step 2

Let z be a point in plane. Write $z = n_1 z_1 + n_2 z_2 + Z$, with $Z \in \mathcal{P}$ and n_1, n_2 integers. From *Step 1* it follows that for any $\epsilon > 0$ there is an integer n such that the distance on the torus $d(g^n(0), Z)$ is less than ϵ. Since $nz_3 = Mz_1 + Nz_2 + Z_n$ with $Z_n \in \mathcal{P}$, we have $d(g^n(0), Z) = |Z_n - Z| < \epsilon$ so that

$$\epsilon > d(g^n(0), Z) = |Z_n - Z| = |(n_1 - M)z_1 + (n_2 - N)z_2 + nz_3 - z|$$

The condition is necessary

After a linear change of coordinates in the plane, we may assume that $z_1 = 1,\ z_2 = i$.

Suppose that the condition of independence of the lemma fails. For the particular form of the lattice assumed here, it means that there are integers r_1, r_2, p such that $r_2 y_3 = r_1 x_3 + p$ where at least one of the numbers r_1, r_2 is not 0. Suppose $r_2 \neq 0$.

Let B_ϵ be a ball of radius ϵ on the torus: $B_\epsilon = \{[z]; d([z],[0]) < \epsilon\}$. We show that there exists $\epsilon > 0$ so that $[g^n(0)] \in B_\epsilon$ implies that the image Z_n of $g^n(0)$ in $\mathcal{P}$ is parallel to the vector (r_1, r_2). Thus the set $\{g^n(0); n \in \mathbf{Z}\}$ is not dense in B_ϵ.

For all n with $g^n(0) \in B_\epsilon$ we write

$$nx_3 = M_n + \delta_n \quad , \quad ny_3 = N_n + \gamma_n \quad , \quad \text{with } \delta_n, \gamma_n < \epsilon$$

so $(\delta_n, \gamma_n) \in \mathcal{P}$.

Then

$$n\left(\frac{r_1}{r_2} x_3 + \frac{p}{r_2}\right) = N_n + \gamma_n$$

or

$$r_1 M_n + np - N r_2 = \gamma_n r_2 - \delta_n r_1$$

The left side of the equality is an integer, while the right side has modulus less than 1 (if $\epsilon < (|r_1| + |r_2|)^{-1}$). It follows that $\delta_n/\gamma_n = r_1/r_2$, proving the claim. ∎

4.2 A model in statistical mechanics

The following model was proposed by Joel Lebowitz

$$\begin{aligned} \ddot{x} &= a - \lambda(\dot{x},\dot{y})\dot{x} - h\dot{y} - \omega x \\ \ddot{y} &= b - \lambda(\dot{x},\dot{y})\dot{y} + h\dot{x} - \omega y \end{aligned} \tag{4.2}$$

where

$$\lambda(\dot{x},\dot{y}) = \frac{a\dot{x} + b\dot{y}}{\dot{x}^2 + \dot{y}^2}$$

and

$$a, b, h, \omega \in \mathbf{R} \ , \ \omega > 0$$

It represents one particle in the plane, of position (x, y) moving in a harmonic potential, and subjected to an electric field (a, b) and a magnetic field of magnitude h perpendicular to the xy-plane; $\lambda(\dot{x},\dot{y})$ is a thermostat keeping the energy fixed.

There is one conserved quantity, namely the energy

$$K = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2 + \omega x^2 + \omega y^2\right) \tag{4.3}$$

so the motion takes place on a 3-dimensional surface $K = \text{const}$. The question is to find out if the trajectories fill densely this surface or if the motion takes place on a smaller dimensional manifold. We will therefore study the existence of aditional first integrals. For example, if the system has two more conserved quantities, say F_1 and F_2 (such that K, F_1, F_2 are functionally independent), then on each trajectory we have

$$K(x, y, \dot{x}, \dot{y}) = K_0, \quad F_1(x, y, \dot{x}, \dot{y}) = F_{1,0}, \quad F_2(x, y, \dot{x}, \dot{y}) = F_{2,0}$$

and this system of equations defines a curve. Intuitive considerations, based on physical arguments, suggested that the system does not have any aditional integrals. We use the poly-Painlevé test to show this.

Setting up for the poly-Painlevé test

We do the analysis near a singular manifold of the system, namely $\dot{x}^2 + \dot{y}^2 = 0$. We first set $\dot{y} = i\dot{x} + v$ (substituting $\dot{y}$ by v). The system is now

$$\dot{y} = i\dot{x} + v \tag{4.4}$$

$$\dot{v} = b - ia - \frac{(a+ib)\dot{x} + bv}{2i\dot{x}v + v^2}v + ihv - \omega(y - ix) \tag{4.5}$$

$$\ddot{x} = a - -\frac{(a+ib)\dot{x} + bv}{2i\dot{x}v + v^2}\dot{x} - ih\dot{x} - hv - \omega x \tag{4.6}$$

An interesting limit is obtained for x, y large. We see in (4.4) that the quantity $b - ia - \omega(y - ix)$ is of the same order, or much less than v and $\dot{v}$, and that $\ddot{x} - a + ih\dot{x} + \omega x$ is of the same order as $\dot{x}/v$. These conditions are fulfilled if we require $x, y \gg 1$, $v, \dot{v} = O(1)$ and $t = O(1)$ for maximum balance in the second equation. Therefore we set

$$x = X/\epsilon \;\;,\;\; y = iX/\epsilon + U \;\;,\;\; X, U = O(1) \;\; (\epsilon \to 0)$$

We get a system in X, U, regularly perturbed in $\epsilon \ll 1$:

$$\ddot{X} = -\Lambda\dot{X} - ih\dot{X} - \omega X + \epsilon(a - h\dot{U}) \tag{4.7}$$

$$\ddot{U} = (b - ia) + ih\dot{U} - \Lambda\dot{U} - \omega U \tag{4.8}$$

$$\text{where} \quad \Lambda = \frac{(a+ib)\dot{X} + \epsilon b\dot{U}}{2i\dot{X}\dot{U} + \epsilon\dot{U}^2} \tag{4.9}$$

The conserved quantity is

$$K = \frac{i}{\epsilon}\left(\dot{X}\dot{U} + \omega XU\right) + \left(\dot{U}^2 + \omega U^2\right)$$

We write the solutions of (4.7), (4.8) as series in ϵ: $X = X_0 + \epsilon X_1 + ...$ and $U = U_0 + \epsilon U_1 + ...$ and we get for the zero order

$$\ddot{U}_0 - ih\dot{U}_0 + \omega U_0 = \frac{b - ia}{2} \tag{4.10}$$

$$\ddot{X}_0 + ih\dot{X}_0 + \omega X_0 = \frac{ia - b}{2}\frac{\dot{X}_0}{\dot{U}_0} \tag{4.11}$$

The conserved quantity K has a power series expansion in ϵ of the form

$$\begin{aligned} K = {} & \epsilon^{-1} 2i\left(\dot{U}_0\dot{X}_0 + \omega U_0 X_0\right) + \\ & 2i\left(\dot{U}_0\dot{X}_1 + \dot{U}_1\dot{X}_0 + \omega U_0 X_1 + \omega U_1 X_0 + \dot{U}_0^2 + \omega U_0^2\right) + O(\epsilon) \end{aligned} \tag{4.12}$$

Each term of the series (4.12) in ϵ is a conserved quantity.

Remark 1

Suppose that the system (4.2) has a conserved quantity (function of $(x, \dot{x}, y, \dot{y})$) which is independent of the energy (4.3). After the substitution $y = ix + U$, $\dot{y} = i\dot{x} + \dot{U}$, it will be a function of $(x, \dot{x}, U, \dot{U})$. But the variable $\dot{x}$ can be eliminated using the conserved quantity $K = i\dot{x}\dot{U} + \dot{U}^2/2 + i\omega xU + \omega u^2/2$, hence it may be assumed that the additional first integral does not depend on $\dot{x}$.

The zero order

The conserved quantity for the reduced system (4.10), (4.11) is (cf. (4.12)):

$$\dot{U}_0\dot{X}_0 + \omega U_0 X_0 = K_0 = \text{const} \tag{4.13}$$

We solve the system formed of (4.13) and (4.10). The equation for U_0 is linear, with the general solution

$$U_0 = C_1 e^{ir_1 t} + C_2 e^{ir_2 t} + \frac{b - ia}{2\omega} \quad , \quad \text{where} \;\; r_{1,2} = \frac{h}{2} \pm \sqrt{\omega + \frac{h^2}{4}} \in \mathbf{R} \tag{4.14}$$

and C_1, C_2 are arbitrary constants.

We use the expression (4.14) of U_0 in (4.13) and get

$$\dot{X}_0 + F(t)X_0 = G(t)$$

where

$$F(t) = \omega\frac{U_0(t)}{\dot{U}_0(t)} \quad , \quad G(t) = \frac{1}{\dot{U}_0(t)}$$

Then X_0 is given by the formula

$$X_0(t) = Ce^{-\int_{t_*}^{t} F(s)ds} + K_0 e^{-\int_{t_*}^{t} F(s)ds} \int_{t_*}^{t} G(\tau) e^{\int_{t_*}^{\tau} F(s)ds} d\tau \tag{4.15}$$

(where t_* is some fixed point). We study the multivaluedness of X_0. The function $\dot{U}_0(t)$ has infinitely many zeroes, at the points

$$t_n = \frac{1}{i(r_1 - r_2)} \left[\ln\left(-\frac{C_2 r_2}{C_1 r_1} \right) + 2n\pi i \right] \quad , \quad n \in \mathbf{Z}$$

equally spaced on a line parallel to the real axis.

Fix the integers $M < N$ and we now decompose the functions F, G into poles and holomorphic parts in the strip $S_{M,N} = \{x \; ; \; \Re t_{M-1} < \Re x < \Re t_{N+1}\}$. Since

$$\dot{U}_0(t) = (t - t_M)...(t - t_N)u(t)$$

$$\frac{1}{\dot{U}_0(t)} = \sum_{n=M}^{N} \frac{a_n}{t - t_n} + h(t)$$

with u and h holomorphic in $S_{M,N}$, u having no zeroes in $S_{M,N}$, and

$$a_n = \lim_{t\to t_n} \frac{t-t_n}{\dot{U}_0(t)} = \frac{1}{\ddot{U}_0(t_n)}$$

we can then write

$$F(t) = \sum_{n=M}^{N} \frac{F_n}{t-t_n} + f(t)$$

where

$$F_n = \omega a_n U_0(t_n) = 1 + \beta e^{-in\theta} \tag{4.16}$$

with

$$\beta = \frac{b-ia}{2\omega^2} C_2^{-1} \left(1 - \frac{r_2}{r_1}\right)^{-1} \left(-\frac{r_1 C_1}{r_2 C_2}\right)^{r_2/(r_1-r_2)} \quad , \quad \theta = 2\pi \frac{r_2}{r_1 - r_2} \tag{4.17}$$

(where some branch of the power is fixed).

Note that $F_n \neq 0$ for $n = M, ..., N$ and generic values of the constants C_1, C_2.

Let $t_* \in S_{M,N}$; then for $\tau \in S_{M,N}$

$$Z(\tau) \equiv G(\tau) \exp\left[\int_{t_*}^{\tau} F(s)ds\right] =$$

$$\frac{1}{(\tau - t_M)...(\tau - t_N)} \exp\left[\int_{t_*}^{\tau} \left(\frac{F_M}{s-t_M} + ... + \frac{F_N}{s-t_N} + f(s)\right) ds\right] =$$

$$(\tau - t_M)^{F_M - 1} ... (\tau - t_N)^{F_N - 1} g(\tau) \tag{4.18}$$

with g holomorphic in $S_{M,N}$.

Let γ_{n,l_n} be a path in $S_{M,N}$, starting at t_* and ending at t, which winds only around the singular point t_n, a number l_n of times (where $n = M, ..., N$) and which does not pass through any singular point.

Claim: *If $Z(\tau)$ is integrable at $\tau = t_n$ then*

$$\int_{\gamma_{n,l_n}} Z(\tau)d\tau = \left(1 - e^{2\pi i l_n F_n}\right) \int_{t_*}^{t_n} Z(\tau)d\tau + \int_{t_*}^{t} Z(\tau)d\tau$$

where the last integral is calculated on a path not winding around any singular point and not passing through any of them.

Proof

Let $\epsilon > 0$ be small and A_ϵ be a point at distance ϵ from t_n. Then

$$\int_{\gamma_{n,l_n}} Z(\tau)d\tau = \int_{t_*}^{A_\epsilon} Z(\tau)d\tau +$$

$$\oint_{|\tau - t_n|=\epsilon,\ l_n \text{ times}} Z(\tau)d\tau + e^{2\pi i l_n F_n}\int_{A_\epsilon}^{t_*} Z(\tau)d\tau + \int_{t_*}^{t} Z(\tau)d\tau$$

where the integrals between t_* and A_ϵ, and t_* and t, respectively, are calculated on a paths not winding around any singular point and not passing through them. For $\epsilon \to 0$ we obtain the formula claimed. ∎

Denote

$$T_n = \int_{t_*}^{t_n} Z(\tau)d\tau$$

Therefore, after analytic continuation along the path γ_{n,l_n}, the value of $X_0(t)$ (given by (4.15)) becomes

$$X_0^{ac}(t) = \left[Ce^{-2\pi i l_n F_n} + K_0 T_n\left(e^{-2\pi i l_n F_n} - 1\right)\right] e^{-\int_{t_*}^{t} F(s)ds} +$$

$$K_0 e^{-\int_{t_*}^{t} F(s)ds}\int_{t_*}^{t} G(\tau)e^{\int_{t_*}^{\tau} F(s)ds}d\tau \tag{4.19}$$

and the corresponding monodromy matrix, which transforms the integration constants K_0, C of (4.15) into the constants $K_0, C' = Ce^{-2\pi i l_n F_n} + K_0 T_n(e^{-2\pi i l_n F_n} - 1)$ of (4.19) is

$$G_{n,l_n} = \begin{pmatrix} 1 & 0 \\ T_n\left(\beta_n^{l_n} - 1\right) & \beta_n^{l_n} \end{pmatrix} \quad \text{where } \beta_n = e^{-2\pi i F_n} \tag{4.20}$$

The monodromy group contains matrices of the form (4.20) for n such that $Z(t)$ is integrable at $t = t_n$.

Claim: *Assume $\theta \notin \pi\mathbf{Q}$ (i.e. θ is an irrational multiple of π). Then the set $\mathcal{N}_\theta$ of the integers n such that $Z(\tau)$ is integrable at $\tau = t_n$ is infinite.*

Proof: The singularity of $Z(\tau)$ at $\tau = t_n$ is given by (4.18), (4.16). For the function to be integrable, is enough to choose n such that $\Re\left(\beta e^{-in\theta}\right) > 0$. If $\theta \notin \pi\mathbf{Q}$, then the vectors of the form $e^{-in\theta},\ n \in \mathbf{Z}$ are dense in the unit circle, hence there are infinitely many of them which will rotate the complex number β into the right half-plane. ▮

Our approach for studying the multivaluedness of $X_0(t)$ depends on whether all the numbers $T_n,\ n \in \mathcal{N}_\theta$ are equal or not. This question could be answered, in principle, by a careful study. However, the final result is qualitatively the same in both cases (dense branching for generic values of θ).

Case 1:

Assume that there are $n, m \in \mathcal{N}_\theta$ such that $T_n \neq T_m$.

Let G_{n,l_n} and G_{m,l_m} be two matrices in the monodromy group. Then the matrix

$$H_{l_m,l_n} = G_{m,l_m} G_{n,l_n} G_{m,l_m}^{-1} G_{n,l_n}^{-1} = \begin{pmatrix} 1 & 0 \\ z_{l_m,l_n} & 1 \end{pmatrix}$$

where

$$z_{l_m,l_n} = (T_n - T_m)(\beta_n^{l_n} - 1)(\beta_m^{l_m} - 1)$$

is also an element of the monodromy group, as well as arbitrary products of such matrices:

$$H_{k_1,k_2,k_3} = H_{l_m,l_n}^{k_1} H_{l'_m,l'_n}^{k_2} H_{l''_m,l''_n}^{k_3} = \begin{pmatrix} 1 & 0 \\ k_1 z_{l_m,l_n} + k_2 z_{l'_m,l'_n} + k_3 z_{l''_m,l''_n} & 1 \end{pmatrix}$$

for any $k_1, k_2, k_3 \in \mathbf{Z}$.

Claim: *Assume $\theta \notin \pi\mathbf{Q}$. Then for almost all pairs of constants (C_1, C_2) (cf. (4.14)), the numbers $z_{1,1}, z_{2,1}, z_{1,2}$ satisfy the conditions of the density lemma (7) of section 4.1.*

Proof: We first note that $z_{1,1} \neq 0$ for almost all complex numbers β. Indeed, if $\beta_n = 1$, then $\beta e^{-in\theta} \in \mathbf{Z}$. For fixed n and θ the set of all complex numbers β with the above property is at most countable.

We then remark that, given the complex numbers $w \neq 0, w_1, w_2$, the area of the parallelogram determined by the numbers ww_1 and ww_2 equals $|w|^2$ multiplied by the area determined by w_1, w_2. Therefore, for $w \neq 0$, the condition of Lemma 7 is satisfied by $ww_1,\ ww_2,\ ww_3$ if and only if it is satisfied by $w_1,\ w_2,\ w_3$.

Therefore, $z_{1,1}, z_{2,1}, z_{1,2}$ satisfy the conditions of Lemma 7 if and only if the numbers $1,\ \beta_n + 1,\ \beta_m + 1$ satisfy it.

Denote $\beta = \rho e^{i\phi}$ (with $\rho \geq 0, \phi \in [0, 2\pi)$). Then $\beta_n = \rho\cos(\phi - n\theta) + i\rho\sin(\phi - n\theta)$, $\beta_m = \rho\cos(\phi - m\theta) + i\rho\sin(\phi - m\theta)$ and assume that there are integers M, N, P such that

$$0 = \det \begin{pmatrix} M & N & P \\ 1 & \Re\beta_n + 1 & \Re\beta_m + 1 \\ 0 & \Im\beta_n & \Im\beta_m \end{pmatrix} =$$

$$M\rho^2 \sin[(m-n)\theta] + (N+M)\sin(\phi - n\theta) + (P-M)\sin(\phi - m\theta)$$

With the notation

$$m - n = k,\ \phi - m\theta = \psi,\ N + M = N',\ P - M = P'$$

it follows that for $\rho \neq 0$

$$M\rho\sin(k\theta) + N'\sin(k\theta + \psi) + P'\sin(\psi) = 0$$

Sub-claim: The set

$$\mathcal{S}_\theta = \{(\rho, \psi) \in [0, \infty) \times (0, 2\pi)\ ;$$
$$M\rho\sin(k\theta) + N'\sin(k\theta + \psi) + P'\sin\psi = 0,\ M, N', P' \in \mathbf{Z}\}$$

has measure zero.

Indeed, for $(\rho, \psi) \in \mathcal{S}_\theta$ we have, if $M \neq 0$,

$$\rho \equiv \rho_{\theta,M,N',P'}(\psi) = -\frac{N'\sin(k\theta + \psi) + P'\sin(\psi)}{M\sin(k\theta)} \tag{4.21}$$

therefore

$$\mathcal{S}_\theta = \bigcup_{M\neq 0, N', P'} \{(\rho_{\theta,M,N',P'}(\psi), \psi)\ ;\ \psi \in [0, 2\pi)\} \bigcup$$

$$\bigcup_{N', P'} [0, \infty) \times \{\psi \in [0, 2\pi)\ ;\ N'\sin(k\theta + \psi) + P'\sin(\psi) = 0\}$$

which is a countable union of smooth graphs.

Therefore, if $\theta \notin \pi\mathbf{Q}$, then for almost all β, the values of $X_0(t)$ on all branches is dense in the complex plane. Since β is given by (4.17) the same holds for almost all the pairs (C_1, C_2) ∎

Case 2: $T_n = T$ for all $n \in \mathcal{N}_\theta$

Let $m, n, p \in \mathcal{N}_\theta$. In this case the products of matrices generating the monodromy group (cf. (4.20)) have a simple form:

$$G_{n,l_n} G_{m,l_m} G_{p,l_p} = \begin{pmatrix} 1 & 0 \\ T\left(\beta_n^{l_n}\beta_m^{l_m}\beta_p^{l_p} - 1\right) & \beta_n^{l_n}\beta_m^{l_m}\beta_p^{l_p} - 1 \end{pmatrix} \tag{4.22}$$

In order to prove dense branching for X_0 it is enough, cf. (4.20), (4.16), to show that the numbers $\beta e^{-in\theta}$, $\beta e^{-im\theta}$, $\beta e^{-ip\theta}$ satisfy the hypothesis of Lemma 7. Equivalently, for $\beta \neq 0$, is enough to show that the numbers 1, $e^{-i(m-n)\theta}$, $e^{-i(p-n)\theta}$ satisfy the hypothesis of the lemma.

Suppose it was not true. Then there are integers M, N, P such that

$$\det \begin{pmatrix} M & N & P \\ 1 & \cos(n-m)\theta & \cos(n-p)\theta \\ 0 & \sin(n-m)\theta & \sin(n-p)\theta \end{pmatrix} =$$

$$M \sin(m-p)\theta - N \sin(n-p)\theta + P \sin(n-m)\theta = 0 \tag{4.23}$$

Equation (4.23) implies that $\sin\theta$ is an algebraic number. Indeed, one can expand (4.23) and obtain that a polynomial of two variables $Q(X, Y)$, with integer coefficients, equals zero at the point $(X, Y) = (\sin\theta, \cos\theta)$. Then the polynomial $Q(X, Y)Q(X, -Y)$ evaluated at $(X, Y) = (\sin\theta, \cos\theta)$ is a polynomial expression in $\sin\theta$ with integer coefficients, hence $\sin\theta$ is algebraic.

The upshot of the analysis of *Case 2* is that for generic (C_1, C_2) (i.e. such that $\beta \neq 0$) and generic θ (i.e. such that $\sin\theta$ is transcendental) the solutions $X_0(t)$ have dense branching.

We now return to the study of existence of first integrals for the reduced system (4.10),(4.11). We consider the generic case, when $\sin\theta$ is transcendental (note that this implies that $\theta \notin \pi\mathbf{Q}$); then $X_0(t)$ has dense branching.

Let $\Phi(X_0, U_0, \dot{U}_0)$ be a holomorphic first integral. (Cf. *Remark 1* we may assume that Φ does not depend on $\dot{X}_0$). If Φ is single-valued, then, in view of the fact that generic solutions $X_0(t)$ have dense branching, it can not depend on X_0. Therefore, $\Phi = \Phi(U_0, \dot{U}_0)$ and we investigate the existence of such first integrals.

Using the explicit formulae for $U_0(t)$ and $\dot{U}_0(t)$ we eliminate the time and get

$$U_0 = CY^{r_1/r_2} + Y + \alpha$$

where

$$Y = \frac{\dot{U}_0 - ir_1U_0 + ir_1\alpha}{i(r_2 - r_1)} \ , \ \alpha = \frac{b - ia}{2\omega}$$

and C is an arbitrary constant of integration (related to C_1, C_2 by the formula $C = C_1C_2^{-r_1/r_2}$).

We may assume that Φ is a function of U_0 and Y. Since any first integral is a function of the constant of integration C, it follows that Φ is a function of the expression

$$F(U_0, Y) = (U_0 - Y - \alpha)Y^{-r_1/r_2}$$

The function $F(U_0, Y)$ is multivalued, and its value on different branches is

$$F(U_0, Y) = (U_0 - Y - \alpha)Y^{-r_1/r_2}e^{-2n\pi ir_1/r_2}$$

Since r_1/r_2 is irrational, the set $\{\exp(-2n\pi ir_1/r_2 \ ; \ n \in \mathbf{Z}\}$ is dense in the unit circle, hence Φ must be a function of the absolute value of $F(U_0, Y)$, so it must be a constant. This implies that there are no holomorphic first integrals for (4.10), (4.11) other than (4.13).

However, the reduced system has a real-analytic first integral (independent of the energy (4.13)), namely $|F(U_0, Y)|$. Therefore, the zero order analysis cannot rule out integrability in the real domain; analysis of the next orders might rule out the existence of such first integrals.

4.3 A First Order Equation with Two Singular Points

4.3.1 Introduction

One of the first examples given by Kruskal of using the poly-Painlevé test for proving nonintegrability is an Abel equation [5], [17]:

$$\frac{dx}{dt} = x^3 + t \tag{4.24}$$

More generally, in [17] is discussed the question of finding the values of the parameter p for which the equation

$$\frac{dx}{dt} = x^3 + t^p \tag{4.25}$$

is integrable. The analysis proceeds as follows. First, one looks for an asymptotic region which is both significant for the branching properties of the solutions (namely captures several branch points in the dominant order) and such that the equation in the dominant order is relatively easy to analyze. For maximal dominant balance for large t we must have

$$\frac{x}{t} \sim x^3 \sim t^p$$

therefore $x \sim t^{-1/2}$ and $p = -3/2$. Hence, in (4.25) the number $p = -3/2$ is special: in this case the equation has a scaling invariance, hence it can be integrated by quadratures.

For $p > -3/2$ a small parameter ϵ is introduced in (4.25) by the substitution $x = e^{-p/3}$, $t = \epsilon^{-1} + \epsilon^{2p/3}T$. The analysis is therefore carried on for large t: in a patch near ∞. The equation is equivalent, for all $\epsilon \neq 0$ to

$$\frac{dX}{dT} = X^3 + (1+\tilde{\epsilon}T)^p \quad , \quad \text{where } \tilde{\epsilon} = \epsilon^{1+2p/3} \tag{4.26}$$

For $p < -3/2$ the small parameter is introduced in (4.25) by the substitution $x = e^{p/3}$, $t = \epsilon + \epsilon^{-2p/3}T$, hence the poly-Painlevé analysis is done in a patch near the singular point $t = 0$; equation (4.25) is equivalent to

$$\frac{dX}{dT} = X^3 + (1+\hat{\epsilon}T)^p \quad , \quad \text{where } \hat{\epsilon} = \epsilon^{1-2p/3} \tag{4.27}$$

The solutions of (4.26), (4.27) are calculated by considering T as a function of X and finding the perturbation series in $\tilde{\epsilon}$, respectively in $\hat{\epsilon}$. Each term of the series has three branch points at the cubic roots of -1 and their calculation involves integration on a corresponding the Riemann surface. Bad branching is found for all values of p excepting $p = 0, p = -3/2$, for which the equation is, in fact, integrable.

Further remarks in [17] indicate that the same ideas can be used for equations of the form

$$\frac{dx}{dt} = x^3 + R(t)$$

if $R(t)$ is a rational function. However, the analysis of the equation

$$\frac{dx}{dt} = x^3 + \frac{k}{t^2+1} \tag{4.28}$$

could not be obtained in this way and was left as an open question.

The purpose of the present section is to carry on the integrability analysis for the equation (4.28) using the poly-Painlevé test and to identify possible values for the parameter k for which the equation might be integrable.

4.3.2 The Integrability Analysis

Setting up for the Poly-Painlevé Test

We consider the equation

$$\frac{dx}{dt} = -\frac{1}{2}x^3 + \frac{k}{t(1-t)} \tag{4.29}$$

(which differs from (4.28) by a linear substitution). We investigate the branching properties of the solutions near the singular point $t = 0,\ x = \infty$. We try to introduce a small parameter ϵ by a substitution of the form $t = \epsilon^a T,\ x = \epsilon^{-b} X$ with $a, b > 0$. The equation (4.29) becomes

$$\frac{dX}{dT} = \epsilon^{a-2b} X^3 + \frac{k\epsilon^b}{T(1-\epsilon^a T)}$$

and we see that for $a = 2b > 0$ (which is not the maximal balance) we obtain a regularly perturbed equation.

It is more convenient for the calculations to use ϵ/k instead of ϵ; therefore we substitute $t = \epsilon^2/k^2 z,\ x = ke^{-1}u$. The equation (4.29) becomes

$$\frac{du}{dz} = -\frac{1}{2}u^3 + \frac{\epsilon}{z(1-\epsilon^2 k^{-2} z)} \tag{4.30}$$

It is also convenient for the calculations to consider u the independent variable, and z as a function of u:

$$\frac{dz}{du} = \left(-\frac{1}{2}u^3 + \frac{\epsilon}{z(1-\epsilon^2 k^{-2} z)}\right)^{-1} \tag{4.31}$$

We note that in the differential equation for $z(u)$ the point $u = \infty$ is a regular point. Indeed, by substituting $u = 1/v$ in (4.31) one gets

$$\frac{dz}{dv} = \frac{2v}{1 - 2v^3\epsilon\left[z(1-\epsilon^2 k^{-2} z)\right]^{-1}}$$

We prefer to work in the variable u rather than v in order to avoid more complicated calculations. We will therefore consider u belonging to the extended complex plane minus the origin (point which will play the role of the point at infinity).

We consider the one-parameter family of solutions given by the initial data $z(\infty) = z_0$. Since an arbitrary initial value z_0 can be absorbed in ϵ, we may assume that $z_0 = 1$ (a similar discussion was done for equation (2.1) in the Introduction to Chapter 2).

The solutions of (4.31) are given as a power series in ϵ:

$$z(u) = z_0(u) + \epsilon z_1(u) + \epsilon^2 z_2(u) + \ldots \tag{4.32}$$

which converges uniformly for u in a neighborhood of ∞ and for ϵ small (since $u = \infty$ is a regular point for the equation).

The zero order term $z_0(u)$ satisfies the equation

$$\frac{dz_0}{du} = -2z_0^{-3}$$

therefore $z_0 = 1 + u^{-2}$.

Substituting the series (4.32) in (4.30) and the expression for $z_0(u)$ one gets the equations for the first terms of the series:

$$z_1' = -\frac{4}{u^4\,(1+u^2)} \tag{4.33}$$

$$z_2' = \frac{4\,z_1}{u^2\,(1+u^2)^2} - \frac{8}{u^5\,(1+u^2)^2} \tag{4.34}$$

$$z_3' = \frac{4\,z_2}{u^2\,(1+u^2)^2} - \frac{4\,{z_1}^2}{(1+u^2)^3} - \frac{4}{u^6 k^2} + \frac{16\,z_1}{u^3\,(1+u^2)^3} - \frac{16}{u^6\,(1+u^2)^3} \tag{4.35}$$

Calculations

Note that z_1 is the integral of a rational function, analytic on the extended complex plane minus the points $u=0,\ u=i,\ u=-i$. Therefore, z_1 will be defined on the covering space $\mathcal{R}$ of the extended complex plane minus the three singular points. Since we solve equation (4.31) with the initial condition $z(\infty)=1$, we need $z_k(\infty)=0$ for $k\geq 1$. From (4.34), z_2 will also be analytic on $\mathcal{R}$ and the same is true for all $z_k(u)=0$.

Let us choose the following fundamental contours (i.e. non-homotopic) on $\mathcal{R}$: γ and $\bar{\gamma}$, where γ is a closed path in the extended complex plane, starting and ending at infinity, winding once around the point $u=i$, and not winding around $u=0$ or $u=-i$, and $\bar{\gamma}$ its complex conjugate.

We will integrate on curves on $\mathcal{R}$ built up by concatenation of $\gamma,\bar{\gamma}$ and their inverses; the set of such paths forms a group which will be denoted by $\mathcal{F}_2$ (it is the free group with two generators).

Note that all the rational functions involved in the calculation of the z_k's have real coefficients; therefore, the value of any of the integrals on $\bar{\gamma}$ is the complex conjugate of the value of the same integral on γ.

Let us introduce a shorthand notation. Let ξ be a contour on $\mathcal{R}$, starting and ending above the point $u=\infty$ (say, $\xi(t)$ defined for $0\leq t\leq 1$) . Let $R(u),\ S(u),\ T(u)$ be rational functions analytic on $\mathcal{R}$ (as are the ones needed for the calculation of the terms z_k). We will write "the integral $\int R\int S$ calculated on ξ" and denote for

$$\int_\xi R\int S := \int_0^1 \xi'(t)dt\ R(\xi(t)) \int_0^t \xi'(s)ds\ S(\xi(s))$$

We will also use

$$\int_\xi R(\int S)(\int T) :=$$

$$\int_0^1 \xi'(t)dt\ R(\xi(t)) \left(\int_0^t \xi'(s)ds\ S(\xi(s))\right) \left(\int_0^t \xi'(s)ds\ T(\xi(s))\right)$$

By integrating z_k' on the contours in $\mathcal{F}_2$ we find values at the point at infinity for the (multivalued) functions z_k.

Using MAPLE for a direct, but lengthy integration, and using integration by parts to reduce the number of repeated integrals, we got the expressions for the first 3 terms of the series (4.32) :

$$z_1(u) = \frac{4}{3\,u^3} - \frac{4}{u} - 4\int_\infty^{\cdot} \frac{1}{1+u^2}du \tag{4.36}$$

$$z_2(u) = +12\left(\int_\infty^{\cdot} \frac{1}{1+u^2}du\right)^2 + 4\int_\infty^{\cdot} \frac{2}{(1+u^2)\,u}du + \frac{32}{3+3\,u^2} + \frac{2}{3\,u^4}$$

$$+\frac{16}{3\,u^2} + \frac{8\,(2+3\,u^2)}{(1+u^2)\,u}\int_\infty^{\cdot} \frac{1}{1+u^2}du \tag{4.37}$$

$$z_3(u) = -\frac{48\,u}{1+u^2}\left(\int_\infty^{\cdot} \frac{1}{1+u^2}du\right)^2 - \frac{16\,u}{(1+u^2)^2}\left(\int_\infty^{\cdot} \frac{1}{1+u^2}du\right)^2 - \frac{152}{3\,u}$$

$$+\frac{68}{3}\int_\infty^{\cdot} \frac{1}{1+u^2}du - \frac{80}{3\,(1+u^2)^2}\int_\infty^{\cdot} \frac{1}{1+u^2}du - \frac{320}{3+3\,u^2}\int_\infty^{\cdot} \frac{1}{1+u^2}du$$

$$+\frac{220\,u}{3+3\,u^2} + \frac{64\,u}{9\,(1+u^2)^2} - 32\left(\int_\infty^{\cdot} \frac{1}{1+u^2}du\right)^3 - \frac{48}{u}\left(\int_\infty^{\cdot} \frac{1}{1+u^2}du\right)^2$$

$$-8\left(\frac{2}{u} + \frac{u}{1+u^2}\right)\left(\int_\infty^{\cdot} \frac{2}{u(1+u^2)}du\right) - \frac{8}{45\,u^5}$$

$$-\frac{64}{3\,u^2}\int_\infty^{\cdot} \frac{1}{1+u^2}du - 32\left(\int_\infty^{\cdot} \frac{1}{1+u^2}du\right)\left(\int_\infty^{\cdot} \frac{2}{u(1+u^2)}du\right)$$

$$+\frac{4}{5\,u^5 k^2} + \int_\infty^{\cdot} \frac{8}{1+u^2}\left(\int_\infty^{u} \frac{2}{(1+{u_1}^2)\,u_1}du_1\right)du \tag{4.38}$$

We will calculate values of $z_j(\infty),\ j = 1,2,3$ on different branches; this amounts to calculating the integrals which appear in (4.36), (4.37), (4.38) on different paths in $\mathcal{F}_2$. The terms of (4.36), (4.37), (4.38) not involving integration will not contribute to the values of $z_j(\infty),\ j = 1,2,3$. Neither will terms consisting of (convergent) integrals multiplied by functions null at infinity contribute to these values. Hence the values of $z_1(\infty)$ on a curve ξ in $\mathcal{F}_2$ are the same as the values of

$$S_1(\xi) = -4\int_\xi \frac{1}{1+u^2}du$$

the values of $z_2(\infty)$ are the same as for

$$S_2(\xi) = 12\left(\int_\xi \frac{1}{1+u^2}du\right)^2 + 4\int_\xi \frac{2}{(1+u^2)\,u}du$$

and for $z_3(\infty)$

$$S_3(\xi) = \frac{68}{3}\int_\xi \frac{1}{1+u^2}du - 32\left(\int_\xi \frac{1}{1+u^2}du\right)^3$$

$$-32\left(\int_\xi \frac{1}{1+u^2}du\right)\left(\int_\xi \frac{2}{u\,(1+u^2)}du\right)$$

$$+ \quad \int_\xi \frac{8}{1+u^2}\left(\int \frac{2}{u_1\,(1+{u_1}^2)}du_1\right)du$$

Denote, for simplicity,

$$A = \frac{1}{1+u^2} \qquad B = \frac{2}{u(1+u^2)}$$

Then

$$S_1(\xi) = -4\int_\xi A$$

$$S_2(\xi) = 4\int_\xi B + 12\left(\int_\xi A\right)^2$$

$$S_3(\xi) = \frac{68}{3}\int_\xi A \ - \ 32\left(\int_\xi A\right)^3 \ - \ 32\left(\int_\xi A\right)\left(\int_\xi B\right) \ + \ 8\int_\xi A\int B$$

We study the multivaluedness of the quantity

$$S_{[3]} = z_0 + \epsilon S_1 + \epsilon^2 S_2 + \epsilon^3 S_3$$

A path ξ can be formally written as a concatenation of integer powers of γ and $\bar{\gamma}$. Let $m \equiv m(\xi)$ be sum of all the powers of γ in ξ (i.e. is the number of occurrences of γ in ξ minus the number of occurrences of γ^{-1}) and $n \equiv n(\xi)$ the sum of the powers of $\bar{\gamma}$. Clearly

$$S_1(\xi) = S_1(\gamma^m \bar{\gamma}^n) = mS_1(\gamma) + nS_1(\bar{\gamma})$$

$$S_2(\xi) = S_2(\gamma^m \bar{\gamma}^n)$$

and, since

$$\int_\gamma A = \pi \quad , \quad \int_\gamma B = -2\pi i$$

then

$$S_1(\xi) = -4(m+n)\pi$$

$$S_2(\xi) = 8\pi i(n-m) + 12\pi^2(m+n)^2$$

In order to study S_3, first note that if ζ is another path in $\mathcal{F}_2$ then

$$J(\xi\zeta) := \int_{\xi\zeta} A \int B = \int_\xi A \int B + \int_\zeta A \int B + \int_\xi A \int_\zeta B$$

so

$$J(\xi\gamma\bar{\gamma}\zeta) - J(\xi\bar{\gamma}\gamma\zeta) = J(\xi\gamma\bar{\gamma}) - J(\xi\bar{\gamma}\gamma) + \int_{\xi\gamma\bar{\gamma}} A \int_\zeta B - \int_{\xi\bar{\gamma}\gamma} A \int_\zeta B$$

$$= J(\gamma\bar{\gamma}) - J(\bar{\gamma}\gamma) + \int_\xi A \int_{\gamma\bar{\gamma}} B - \int_\xi A \int_{\bar{\gamma}\gamma} B$$

$$= \int_\gamma A \int_{\bar{\gamma}} B - \int_{\bar{\gamma}} A \int_\gamma B = 4\pi^2 i$$

Hence

$$S_3(\xi\gamma\bar{\gamma}\zeta) - S_3(\xi\bar{\gamma}\gamma\zeta) = 8\left(J(\xi\gamma\bar{\gamma}\zeta) - J(\xi\bar{\gamma}\gamma\zeta)\right) = 32\pi^2 i$$

Therefore

$$S_3(\xi\gamma\bar{\gamma}^n) = S_3(\xi\bar{\gamma}\gamma\bar{\gamma}^{n-1}) + 32\pi^2 i = S_3(\xi\bar{\gamma}^2\gamma\bar{\gamma}^{n-2}) + 2(32\pi^2 i) =$$

$$... = S_3(\xi\bar{\gamma}^n\gamma) + 32\pi^2 ni$$

then

$$S_3(\xi\gamma^m\bar{\gamma}^n) = S_3(\xi\gamma^{m-1}\bar{\gamma}^n\gamma) + 32\pi^2 ni = S_3(\xi\gamma^{m-2}\bar{\gamma}^n\gamma^2) + 2n(32\pi^2 i) =$$

$$... = S_3(\xi\bar{\gamma}^n\gamma^m) + 32\pi^2 nmi$$

Now take an arbitrary contour $\xi = \gamma^{m_1}\bar{\gamma}^{n_1}...\gamma^{m_k}\bar{\gamma}^{n_k}$

Then

$$S_3(\xi) = S_3(\gamma^{m_1}\bar{\gamma}^{n_1}...\gamma^{m_k}\bar{\gamma}^{n_k})$$

$$= S_3(\gamma^{m_1}\bar{\gamma}^{n_1}...\gamma^{m_{k-1}}\bar{\gamma}^{n_{k-1}+n_k}\gamma^{m_k}) + 32\pi^2 m_k n_k i$$

$$= S_3(\gamma^{m_1}\bar{\gamma}^{n_1}...\gamma^{m_{k-2}}\bar{\gamma}^{n_{k-2}+n_{k-1}+n_k}\gamma^{m_{k-1}+m_k})$$

$$+\left[m_{k-1}(n_{k-1}+n_k) + m_k n_k\right] 32\pi^2 i =$$

$$... = S_3(\bar{\gamma}^n\gamma^m) + [m_1(n_1 + ... + n_{k-1} + n_k) + ... + m_{k-2}(n_{k-2}+$$

$$n_{k-1} + n_k) + m_{k-1}(n_{k-1}+n_k) + m_k n_k]\, 32\pi^2 i := S_3(\bar{\gamma}^n\gamma^m) + 32\pi^2 ip =$$

(where $m = m_1 + ... + m_k$, $n = n_1 + ... + n_k$, $p \equiv p(\xi)$ are integers)

$$= S_3(\bar{\gamma}^n) + S_3(\gamma^m) + 8\int_{\bar{\gamma}^n} A \int_{\gamma^m} B + 32\pi^2 ip =$$

$$= \frac{68}{3}(m+n)\pi - 32(m+n)^3\pi^3 - 64i(n^2 - m^2)\pi^2$$

$$+8\left[J(\bar{\gamma}^n) + J(\gamma^m) - 2i\pi^2 mn\right] + 32\pi^2 ip$$

To find the expression for $J(\gamma^m)$ we use the recurrence relation:

$$J(\gamma^m) = J(\gamma^{m-1}) + J(\gamma) + (m-1)\pi(-2\pi i)$$

and if we denote $\omega \equiv J(\gamma)$ then

$$J(\gamma^m) = m\omega - im(m-1)\pi^2$$

Hence:

$$\begin{aligned} z(\infty) = 1 - 4\epsilon(m+n)\pi + \epsilon^2 \left(8\pi i(n-m) + 12\pi^2(m+n)^2\right) \\ +\epsilon^3 \Big(8m\omega - 8im(m-1)\pi^2 + 8n\bar{\omega} + 8in(n-1)\pi^2 \\ +\frac{68}{3}(m+n)\pi - 32(m+n)^3\pi^3 - 64i(n^2-m^2)\pi^2 \\ -16i\pi^2 mn + 32\pi^2 ip\Big) + O(\epsilon^4) \end{aligned} \tag{4.39}$$

where

$$m = m_1 + ... + m_k \tag{4.40}$$

$$n = n_1 + ... + n_k \tag{4.41}$$

$$\begin{aligned} p = m_1(n_1 + ... + n_{k-1} + n_k) + ... + m_{k-2}(n_{k-2} + n_{k-1} + n_k) + \\ m_{k-1}(n_{k-1} + n_k) + m_k n_k \end{aligned} \tag{4.42}$$

It is easy to see that m, n, p can be any real integers.

Denote

$$m + n = M \quad , \quad n - m = N \quad , \quad 2p - mn = P$$

In this notation

$$z(\infty) = 1 - 4\pi M\epsilon + \epsilon^2 \left(8\pi iN + 12\pi^2 M^2\right)$$

$$+\epsilon^3 \Big(16\pi^2 iP - 64\pi^2 iMN - 32\pi^3 M^3 + \frac{68}{3}\pi M + 8M\Re(\omega) + 8iN\Im(\omega)$$

$$+8\pi^2 iN(M-1)\Big) + O(\epsilon^4)$$

Arguments similar to those in [17] can be made now to deduce that the three arbitrary uncertainties $M, N, P \in \mathbf{Z}$ determine a dense set of values for $z(\infty)$.

If the initial condition is set at an arbitrary point (rather than at ∞), the values of the solution is obtained by adding a constant (series in ϵ) to the series found above, hence this solution also exhibits dense branching.

In conclusion, equation (4.29) is not integrable for any non-zero value of the parameter k.

Further Considerations

It is interesting to note the group structure of the multivaluedness (4.39) of the solutions of (4.31). For this, it is better to consider solutions $z(u) \equiv z(u;w)$ with arbitrary values at ∞: $z(\infty;w) = w$. As we noted before, the solution with initial value w is obtained from the solution with initial value 1 by replacing ϵ^2 with $\epsilon^2 w$ in the expression of $\epsilon^2 z$. Therefore, the multivaluedness of $z(\infty;w)$ is obtained from (4.39) as

$$\begin{aligned} z(\infty;w) = {} & w - 4\epsilon w^{3/2}(m+n)\pi + \epsilon^2 w^2 \left(8\pi i(n-m) + 12\pi^2(m+n)^2\right) \\ & +\epsilon^3 w^{5/2}\Big(8m\omega - 8im(m-1)\pi^2 + 8n\bar{\omega} + 8in(n-1)\pi^2 \\ & +\frac{68}{3}(m+n)\pi - 32(m+n)^3\pi^3 - 64i(n^2-m^2)\pi^2 \\ & -16i\pi^2 mn + 32\pi^2 ip\Big) + O(\epsilon^4) \end{aligned} \tag{4.43}$$

Denote $\omega_1 = 8\omega + 68/3\pi$.

Letting $k=1,\ m_1=1,\ n_1=0$ in (4.43) (cf. also (4.40), (4.41), (4.42)), we get

$$\begin{aligned} f(w) \equiv {} & w - 4\epsilon w^{3/2}\pi + \epsilon^2 w^2\left(-8\pi i + 12\pi^2\right) \\ & +\epsilon^3 w^{5/2}\left(\omega_1 - 32\pi^3 + 56i\pi^2\right) + O(\epsilon^4) \end{aligned} \tag{4.44}$$

Letting $n=1,\ m=p=0$ in (4.43) we get

$$\begin{aligned} g(w) \equiv {} & w - 4\epsilon w^{3/2}\pi + \epsilon^2 w^2\left(8\pi i + 12\pi^2\right) \\ & +\epsilon^3 w^{5/2}\left(\bar{\omega_1} - 32\pi^3 - 56i\pi^2\right) + O(\epsilon^4) \end{aligned} \tag{4.45}$$

We note that the series in (4.44), (4.45) are convergent. The functions $f(w)$, $g(w)$ give the monodromy along the path γ, and $\bar{\gamma}$, respectively. Then the general monodromy (4.43) is obtained by compositions of the functions $f(w)$, $g(w)$ and their inverses. The inverse of f is obtained by letting $k=1, m_1=-1, n_1=0$ in (4.43), and the inverse of g for $k=1, n_1=-1, m_1=0$. The function f composed with itself m times is given by (4.43) for $k=1$, $m_1=m$, $n_1=0$:

$$f^{\circ m}(w) = w - 4\epsilon w^{3/2}\pi m + \epsilon^2 w^2\left(-8\pi i m + 12\pi^2 m^2\right)$$
$$+\epsilon^3 w^{5/2}\left(-32\pi^3 m^3 + 56 i\pi^2 m^2 + \omega_1 m\right) + O(\epsilon^4)$$

and the function g composed with itself n times is given by (4.43) for $k=1$, $m_1=0$, $n_1=n$:

$$g^{\circ n}(w) = w - 4\epsilon w^{3/2}\pi n + \epsilon^2 w^2\left(8\pi i n + 12\pi^2 n^2\right)$$
$$+\epsilon^3 w^{5/2}\left(-32\pi^3 n^3 - 56 i\pi^2 n^2 + \bar{\omega_1} n\right) + O(\epsilon^4)$$

The third arbitrary integer p in (4.43) is present due to the non-commutativity of the group generated by f, g under composition:

$$h(w) \equiv \left(f\circ g\circ f^{-1}\circ g^{-1}\right)(w) = w + \epsilon^3 w^{5/2} 32\pi^2 i + O(\epsilon^4) \tag{4.46}$$

and

$$h^{\circ p}(w) = w + \epsilon^3 w^{5/2} 32\pi^2 i p + O(\epsilon^4)$$

So the monodromy group of equation (4.31) at the point $u=\infty$ is generated by the functions (4.44), (4.45). It consists of functions expressed as a convergent power series in ϵ of the form (4.43) (the radius of convergence may depend on n, m, p). Equivalently, one may consider two other generators, namely f and $g_1 = g^{-1}\circ f$; we have

$$g_1^{\circ n}(w) = w - 16 i\epsilon^2 w^2 p\pi n + \epsilon^3 w^{5/2}\left(16 i\pi^2 n^2 + (\omega_1 - \overline{\omega_1})n\right) + O(\epsilon^4) \tag{4.47}$$

We may compare the monodromy group obtained for the equation (4.31) —which is generated by two non-commuting elements— to the monodromy groups of the non-linearly perturbed Euler equation, studied in Chapter 2—which are generated by one element.

In the examples of Chapter 2 we were able to find a rigorous proof for nonintegrability by showing that the generator of the monodromy group is analytically equivalent (in generic cases) to its linear part. The natural question is whether a similar idea can work for the equation (4.31). In this case, one may ask if there is a holomorphic function of ϵ and w, which is invertible, and takes the functions f, g into their truncation to the third order in ϵ. However, compositions of polynomials of degree 3 in ϵ yields polynomials of higher degree, hence the form of the monodromy group would be almost as complicated in these new variables. Furthermore, the existence of such a transformation in variables ϵ, w does not seem to be possible: it appears (in the calculation using repeated integrals) that, just as the arbitrary integer p appear in the third order in (4.43), new independent integers appear at all the higher order in ϵ. Another possibility would be to use a substitution which is not invertible. Even if this works, it still remains to be proven the fact that the set

$$\begin{aligned}\mathcal{S}_{\epsilon,w} = \Big\{ & w - 4\epsilon w^{3/2}(m+n)\pi + \epsilon^2 w^2 \left(8\pi i(n-m) + 12\pi^2(m+n)^2\right) \\ & +\epsilon^3 w^{5/2}\Big(8m\omega - 8im(m-1)\pi^2 + 8n\bar{\omega} + 8in(n-1)\pi^2 + \\ & \frac{68}{3}(m+n)\pi - 32(m+n)^3\pi^3 - 64i(n^2-m^2)\pi^2 - 16i\pi^2 mn + \\ & +32\pi^2 ip\Big) \ ; \ m,n,p \in \mathbf{Z}\Big\}\end{aligned}$$

(which is the truncation of $z(\infty, w)$ to the third order in ϵ) is dense in the complex plane.

4.4 Using the Poly-Painlevé Test for Obtaining Asymptotic Expansions of First Integrals

A natural question which arises in connection to the poly-Painlevé test is the following: suppose that, for a certain differential equation, the test does not rule out integrability for that particular equation (i.e. non-dense branching is found). What kind of information is contained in the perturbation series that one obtains by doing the test?

We discussed in the earlier chapters the signification of the small parameter ϵ which is introduced in the poly-Painlevé test: it defines a region in the phase space, and in that region an asymptotic expansion for the solutions is found. Suppose now that the equation on which the test is performed was, in fact, integrable. Then the asymptotic expansion for the solutions should yield an expansion for a first integral. Of course, a first integral is a function of (at least) two variables, and a priori, the notion of asymptotic expansion does not make sense in general. However, the small parameter ϵ defines a region of the phase space where such an expansion may have a precise sense (derived from the expansion in the variable ϵ). Another issue is that a single-valued function may have a multivalued asymptotic expansion (e.g. the Airy functions at ∞). In the examples we treat below we will always look for expansions with single-valued terms.

We chose two simple first order differential equations to illustrate how asymptotic expansions for first integrals can be obtained using the poly-Painlevé test. We looked for examples in which the integrals can be explicitly obtained, in order to compare our results with the actual solutions. We chose to study equations in regions where the integrals have essential singularities.

4.4.1 Example 1

Consider the equation

$$\frac{dy}{dx} = y^2 - \frac{y}{x} \tag{4.48}$$

It is a Riccati equation which can be solved by a standard procedure; it has the first integral

$$F(x,y) = xe^{\frac{1}{xy}} \tag{4.49}$$

But we assume that we have no knowledge of the first integral and we apply the poly-Painlevé test. Since the point $x = 0$ is a singular point for equation (4.48) we may choose to study the solutions for x small. One possibility is to substitute

$$x = \epsilon X \quad , \quad y = \epsilon Y \quad , \quad \epsilon \ll 1 \tag{4.50}$$

which gives

$$\frac{dY}{dX} = \epsilon^2 Y^2 - \frac{Y}{X} \tag{4.51}$$

The equation (4.51) has a series solution of the form

$$Y = Y_0(X) + \epsilon^2 Y_1(X) + \epsilon^4 Y_2(X) + ... \tag{4.52}$$

Since any initial condition at, say, $X = 1$ (or at any other non-singular point) such that $Y(1) = O(\epsilon^0)$ can be absorbed in ϵ, we may assume $Y(1) = 1$, which gives $Y_0(1) = 1$ and $Y_n(1) = 0$ for $n \geq 1$. In this situation, ϵ should be interpreted as one parameter indexing the family of solutions of (4.48).

Substituting (4.52) in (4.51) and solving order by order, one gets

$$Y_0(X) = \frac{1}{X}$$

$$Y_1(X) = \frac{\ln X}{X}$$

$$Y_2(X) = \frac{\ln^2 X}{X}$$

Consider the first approximation for the solution:

$$Y(X) = \frac{1}{X} + O(\epsilon^2)$$

Using (4.50) to go back to the original variables x, y we get

$$y = \frac{\epsilon^2}{x} + O(\epsilon^3) \tag{4.53}$$

where the terms considered $O(\epsilon^3)$ in (4.53) are expressions in x, y, ϵ, and x, y are assumed to be $O(\epsilon)$ (cf. (4.50)).

From (4.53) we get

$$xy = \epsilon^2 + O(\epsilon^4) \tag{4.54}$$

which means that the function $F_0(x,y) = xy$ is constant mod $O(x^2, xy, y^2)$.

It is interesting to note how the expression F_0 relates to the actual first integral (4.49): for small x and y the main asymptotic contribution in (4.49) is made by the exponential, which is a function of F_0.

We now consider a better approximation for the solutions:

$$Y(X) = \frac{1}{X} + \epsilon^2 \frac{\ln X}{X} + O(\epsilon^4)$$

which, in original variables x, y is

$$y = \frac{\epsilon^2}{x} + \epsilon^4 \frac{\ln x}{x} - \epsilon^4 \frac{\ln \epsilon}{x} + O(\epsilon^5)$$

or, equivalently,

$$xy - \epsilon^4 \ln x = \epsilon^2 - \epsilon^4 \ln \epsilon + O(\epsilon^6) \tag{4.55}$$

which means that the function given by the left side of (4.55) is constant modulo $O(\epsilon^6)$. However, this expression is not a satisfactory first integral: on one hand, approximations for first integrals of (4.48) should not contain the parameter ϵ, and on the other hand, we are looking for single-valued approximations. We may eliminate ϵ from the expression using (4.54): since $\epsilon^4 = x^2y^2 + O(\epsilon^6)$ (cf. (4.50))) the expression (4.55) becomes

$$xy - x^2y^2 \ln x = \epsilon^2 - \epsilon^4 \ln \epsilon + O(\epsilon^6) \tag{4.56}$$

We then look for a uniformization for the left side of (4.56), i.e. for a function $\phi(z)$ which satisfies

$$\phi\left(z - z^2 \ln x + O(\epsilon^6)\right) = \phi\left(z - z^2(2n\pi i + \ln x) + O(\epsilon^6)\right)$$

for all $n \in \mathbf{Z}$,and for all z of order $O(\epsilon^2)$. Since

$$\left(z - z^2 \ln x\right)^{-1} = \frac{1}{z} + \ln x + O(z \ln^2 x)$$

we may apply the function $\phi(z) = \exp(1/z)$ in (4.56) and get

$$e^{\frac{1}{xy}}\left(x+O(\epsilon^6\ln^2\epsilon)\right)=e^{\epsilon^{-2}}\left(1+O(\epsilon^6\ln^2\epsilon)\right) \tag{4.57}$$

Hence, the next approximation for the first integral of (4.48) is

$$F_1(x,y)=xe^{\frac{1}{xy}}$$

We may also consider higher order truncations for the solution $Y(X)$ and reason as above; obviously, in the present simple example we obtain that the function F_1 is first integral of (4.48) with better and better precision. The procedure of recovering an asymptotic expansion for the first integral ends in a finite number of steps because the first integral has an asymptotic expansion consisting of a finite number of terms.

4.4.2 Example 2

We next consider an example of integrable equation which has a first integral with an expansion consisting of an infinite number of terms in the region analyzed.

Consider the equation

$$\frac{dy}{dx}=y^3+xy \tag{4.58}$$

This equation looks similar to the example

$$\frac{dy}{dx}=y^3+x \tag{4.59}$$

on which Kruskal illustrated and applied the ideas of the poly-Painlevé test [5], [17] and deduced nonintegrability. However, equation (4.58) can be integrated and has the single-valued first integral

$$F(x,y)=2\int_1^x e^{t^2}\,dt+\frac{e^{x^2}}{y^2} \tag{4.60}$$

(of course, the lower limit of integration in (4.60), here taken to be 1, can be any fixed point).

We will not use the first integral (4.60). Instead, we perform the poly-Painlevé test on (4.58) and find an asymptotic expansion for a first integral.

In order to set up the equation for the poly-Painlevé test, one can reason as in [17] and do the analysis for large x, in a patch near ∞. Therefore we look for a substitution of the form $x = \alpha + \beta X,\ y = \epsilon^{-1} Y$ with $\alpha \gg 1,\ \alpha \gg \beta,\ \epsilon \ll 1$ (X, Y are finite variables). Equation (4.58) becomes

$$\frac{1}{\epsilon\beta}\frac{dY}{dX} = \frac{1}{\epsilon^3}Y^3 + \frac{\alpha}{\epsilon}Y + \frac{\beta}{\epsilon}XY$$

which is constitutes a regular perturbation for the maximal balance $\beta = \epsilon^2,\ \alpha\beta = 1,\ \beta \ll 1$. Hence the substitution has the form

$$x = \frac{1}{\epsilon^2} + \epsilon^2 X \quad , \quad y = \frac{1}{\epsilon}Y \tag{4.61}$$

and (4.58) becomes

$$\frac{dY}{dX} = Y^3 + Y + \epsilon^4 XY \tag{4.62}$$

To simplify the calculations, it is preferable to invert the variables; the equation for $X(Y)$ is, obviously,

$$\frac{dX}{dY} = \left(Y^3 + Y + \epsilon^4 XY\right)^{-1} \tag{4.63}$$

It is interesting to note that, up to this point, there are no significant differences between the treatment of (4.59) (which is nonintegrable) and of (4.58). However, the calculations on the present example will not give densely branched solutions.

We calculate a series solution for (4.63): substituting the series $X(Y) = X_0(Y) + \epsilon^4 X_1(Y) + ...$ in the equation one gets

$$\frac{dX_0}{dY} = \frac{1}{Y(Y^2+1)}$$

$$\frac{dX_1}{dY} = -\frac{X_0(Y)}{Y(Y^2+1)}$$

with the solutions

$$X_0(Y) = -\frac{1}{2}\ln\left(1+\frac{1}{Y^2}\right) + C_0 \tag{4.64}$$

$$X_1(Y) = -\frac{1}{2}X_0(Y)^2 + \frac{1}{2}X_0(Y)\frac{Y^2}{1+Y^2} - \frac{1}{2}\frac{1}{1+Y^2} + C_1 \tag{4.65}$$

Consider the first approximation for the solutions $X(Y)$:

$$X = -\frac{1}{2}\ln\left(1+\frac{1}{Y^2}\right) + C_0 + O(\epsilon^4) \tag{4.66}$$

It is tempting at this point to apply the uniformizing function $z \to \exp(2z)$ to the equality (4.66). However, one should go back to the variables x, y [cf. (4.61)] before the uniformization (and we will see shortly that a different uniformizing function will be needed); relation (4.66) then becomes

$$x + \frac{\epsilon^2}{2}\ln\left(1+\frac{1}{\epsilon^2 y^2}\right) = \frac{1}{\epsilon^2} + \epsilon^2 C_0 + O(\epsilon^6) \tag{4.67}$$

The left side of the equality (4.67) is therefore constant up to order six in ϵ [where $x = O(\epsilon^{-2}),\ y = O(\epsilon^{-1})$ cf. (4.61)]. An approximate first integral for (4.58) should not contain the parameter ϵ. Therefore, we eliminate it using (4.61):

$$\epsilon^2 = (x - \epsilon^2 X)^{-1} = \frac{1}{x} + O(\epsilon^6) \tag{4.68}$$

and relation (4.67) becomes

$$x + \frac{1}{2x}\ln\left(1+\frac{x}{y^2}\right) = \frac{1}{\epsilon^2} + \epsilon^2 C_0 + O(\epsilon^6) \tag{4.69}$$

Finally, we look for a function $\phi(z)$ which applied to the left side of (4.69) yields a single-valued function. Such a function should satisfy

$$\phi(x + \frac{1}{2x}z + O(\epsilon^6)) = \phi(x + \frac{1}{2x}(z + 2n\pi i) + O(\epsilon^6))$$

for generic $x = O(\epsilon^{-2}),\ z = O(\epsilon^0)$, and all integers n. We note that the logarithm can be made to appear as an additive factor by taking the square on the left side of (4.69):

$$\left[x + \frac{1}{2x}\ln\left(1+\frac{x}{y^2}\right) + O(\epsilon^6)\right]^2 = x^2 + \ln\left(1+\frac{x}{y^2}\right) + O(\epsilon^4)$$

hence we may take $\phi(z) = \exp(z^2)$. When applied to (4.69) it gives

$$\exp(x^2)\left(1+\frac{x}{y^2}\right)\left(1+O(\epsilon^4)\right) = \exp(\epsilon^{-4}+2C_0)\left(1+O(\epsilon^4)\right) \tag{4.70}$$

and the left side is an expansion for a first integral of (4.58) in the region of the phase space defined by the relations (4.61).

To obtain more precision for the first integral, we use one more term in the expansion of the solutions: $X = X_0(Y) + \epsilon^4 X_1(Y) + O(\epsilon^8)$, where $X_0(Y), X_1(Y)$ are given by (4.64), (4.65). Using the fact that $X_0 = X + O(\epsilon^4)$, the series for $X(Y)$ can be written as

$$X = -\frac{1}{2}\ln\left(1+\frac{1}{Y^2}\right) + C_0 \tag{4.71}$$

$$+\epsilon^4\left(-\frac{1}{2}X^2 + \frac{1}{2}X\frac{Y^2}{1+Y^2} - \frac{1}{2}\frac{1}{1+Y^2} + C_1\right) + O(\epsilon^8) \tag{4.72}$$

We now use (4.61) to write the expression (4.72) in the original variables x, y, and, using the expansion

$$\ln\left(1+\frac{1}{Y^2}\right) = \ln\left(1+\frac{1}{\epsilon^2 y^2}\right) = \ln\left(1+\frac{x-\epsilon^2 X}{y^2}\right)$$

$$= \ln\left(1+\frac{x}{y^2}\right) - \epsilon^2\frac{X}{x+y^2} + O(\epsilon^8)$$

we get

$$x^2 + \ln\left(1+\frac{x}{y^2}\right) - 2C_0 - \frac{\epsilon^2 X}{x} - \frac{1}{2}\frac{1}{x(y^2+x)} - 2\epsilon^4 C_1$$
$$= \frac{1}{\epsilon^4} + O(\epsilon^8) \tag{4.73}$$

We could eliminate the factor $\epsilon^2 X$ in the fourth term of (4.74) in favor of x, y, since

$$\frac{\epsilon^2 X}{x} = \frac{\epsilon^2 X_0}{x} + O(\epsilon^8) = \frac{1}{x^2}\left[-\frac{1}{2}\ln\left(1+\frac{x}{y^2}\right) + C_0\right] + O(\epsilon^8)$$

but this form is unnecessarily complicated, and makes less obvious the fact that we can use a simple exponential as uniformizing function; instead, we write

$$\frac{\epsilon^2 X}{x} = \epsilon^4 X + O(\epsilon^8)$$

Before applying an exponential to both sides of the equality (4.74) to obtain a single-valued first integral, we may consider whether the constants of integration $2C_0 + 2\epsilon^4 C_1$ should be placed to the right side of the equality or to the left-side. The place of the constant $2C_0$ does not affect the final result—after applying an exponential, it produces a multiplicative factor on one side or the other of the equality. However, the place of the constant $2\epsilon^4 C_1$ affects qualitatively the result, and in order to see this we write $C_1 = C_1' - k/2$ with C_1', k constants (C_1' is thought as a constant of integration, indexing the solutions, and k as a fixed number, the same for all the solutions) and rewrite (4.74) as

$$x^2 + \ln\left(1+\frac{x}{y^2}\right) - \epsilon^4 X - \frac{1}{2}\frac{1}{x(y^2+x)} + \frac{k}{x^2}$$

$$= \frac{1}{\epsilon^4} + 2C_0 + 2\epsilon^4 C_1' + O(\epsilon^8) \tag{4.74}$$

where we used $\epsilon^4 k = k/x^2 + O(\epsilon^8)$. After applying an exponential, (4.74) becomes

$$e^{x^2}\left(1+\frac{x}{y^2}\right)\left(1-\epsilon^4 X + \frac{k}{x^2} + O(\epsilon^8)\right)\left(1-\frac{1}{2}\frac{1}{x(y^2+x)} + O(\epsilon^8)\right) =$$

$$e^{\epsilon^{-4}+2C_0}\left(1 + 2\epsilon^4 C_1' + O(\epsilon^8)\right) \tag{4.75}$$

and, since

$$1-\epsilon^4 X = \frac{1}{1+\epsilon^4 X} + O(\epsilon^8) = \frac{1}{\epsilon^2 x} + O(\epsilon^8) = \frac{1}{\epsilon^2}\left(\frac{1}{x} + O(\epsilon^{10})\right)$$

equation (4.75) can be written as

$$e^{x^2}\left(\frac{1}{x}+\frac{1}{y^2}+O(\epsilon^{10})\right)\left(1-\frac{1}{2}\frac{1}{x(y^2+x)}+\frac{k}{x^2}+O(\epsilon^8)\right) =$$

$$\epsilon^2 e^{\epsilon^{-4}+2C_0}\left(1+2\epsilon^4 C_1' + O(\epsilon^8)\right) \tag{4.76}$$

or, after simplifications,

$$e^{x^2}\left[\frac{1}{x}+\frac{1}{y^2}-\frac{1}{2x^2y^2}+\frac{k}{x^3}\left(1+\frac{x}{y^2}\right)+O(\epsilon^{10})\right] =$$

$$\epsilon^2 e^{\epsilon^{-4}+2C_0}\left(1+2\epsilon^4 C_1' + O(\epsilon^8)\right) \tag{4.77}$$

Equality (4.77) gives the next approximation for the first integral of (4.58). We remark that there is no contradiction between (4.77) and (4.70), since (4.70) can be rewritten

$$\epsilon^2 e^{x^2}\left(1+\frac{x}{y^2}\right)\left(1+O(\epsilon^4)\right) = \epsilon^2 e^{\epsilon^{-4}+2C_0}\left(1+O(\epsilon^4)\right) \tag{4.78}$$

and, using the fact that $\epsilon^2 = 1/x + O(\epsilon^6)$, the left side of (4.78) is a truncation of the left side of (4.77).

The number k in (4.77) is undetermined, at least to this order of calculation (and there will be other numbers appearing in higher order approximations). A possible interpretation of these undeterminations is the following. A first integral of a differential equation is not unique, since any function of a first integral is, again, an integral. The undetermined constants which appear in our procedure may correspond to an undetermined function applied to a first integral. Also, another liberty in our proposed procedure is in the choice of the uniformizing function: a single-valued function composed with the exponential will work as well.

At this point, we may digress and ask what happens if, in Example 1, we do the same procedure: instead of imposing an initial condition for Y independent of ϵ, we now impose the initial condition as a power series: $1+\epsilon^2 k_1+\epsilon^4 k_2+....$ In Example 1 none of the constants $k_1, k_2, ...$ affects the final results (they merely give multiplicative factors), and the liberty in the integral lies only in the choice of the uniformizing function (any single-valued function, composed with the exponential, composed with an inversion is uniformizing).

We may want to find the value of k which corresponds to the integral (4.60). Since

$$2\int_1^x e^{t^2}\,dt+\frac{e^{x^2}}{y^2}\sim e^{x^2}\left(\frac{1}{y^2}+\frac{1}{x}+\frac{1}{2x^3}+O(x^{-5})\right)\quad,\quad x\gg 1 \tag{4.79}$$

we see that this is the same as the expansion in the left side of (4.77) for $k=1/2$.

4.4.3 Further Remarks

We illustrated on two examples a procedure of recovering an asymptotic expansion for a first integral using the poly-Painlevé test (in cases when it does not predict nonintegrability). The method proceeds in several steps:

1) given a differential equation, we first set up for the poly-Painlevé test (by introducing a small parameter) and doing an expansion in a region of the phase space which includes singular points of the equation;

2) we calculate the power series expansion (up to a certain order) for the solution of the equation in the new variables;

3) we write the expansion in the original variables and eliminate the small parameter;

4) we write the constants, and respectively, the variables, on different sides of the equality; the outcome is a (generally) multivalued (expansion for a) first integral;

5) we apply a uniformizing function to the multivalued integral (if it is possible).

The last two steps include a certain arbitrariness, as discussed previously.

We should point out that, even if the procedure works, there is, a priori, no guarantee, that the equation is integrable. (Counterexamples can be obtained in cases when one expands solutions of a nonintegrable equation around a regular point.)

An open question in this procedure is the following. Using the α-method, one can (at least in principle) find series solutions. Then, solving for the constants of integration, one finds first integrals (which are multivalued in general). If the equation has a single-valued first integral (in the region of the phase space which is analyzed), and if the first integral has essential singularities there, it is not clear that the integral necessarily has a single-valued expansion.

Also, another question is to understand the information carried by such calculations in cases of nonintegrable equations. There are equations (such as (1.9), or the one in Section 4.3) for which dense branching occurs only for higher order approximations of solutions (in the cited examples at least 3), but the first few terms have discrete branching. Then one may follow the procedure outlined in Section 4.4 and find a single-valued quantity which is conserved, up to a certain approximation. Such a function may carry some information about the trajectories and it would be interesting to understand what this is.

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Vita

Rodica D. Costin

1979 Graduated from Mihai Viteazul High School, Bucharest, Romania.

1979-83 Attended University of Bucharest. Majored in Mathematics (with emphasis on Operator Theory).

1983 B.A., University of Bucharest.

1983-84 Attended University of Bucharest, Romania (specialization year in Algebra-Geometry).

1984 M.S. in Mathematics, University of Bucharest.

1984-91 Researcher in applied mathematics, INCREST (later IMFDZ), Bucharest

1990 O. Costin, R.D. Costin, C.P. Grünfeld. Infinite Order Phase Transition in a Classical Spin System, J.Stat.Phys. 59, 5/6, 1531-1546

1991 O.Costin, R.D. Costin. Limit Probability Distributions for an Infinite Order Phase Transition Model, J. Stat. Phys. 64, 1/2, 193-205

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1992-97 Graduate work in Mathematics, Rutgers, The State University of New Jersey, New Brunswick, New Jersey.

1992-96 Teaching Assistant, Department of Mathematics.

1996-97 Louis Bevier University Fellowship.

1996 O. Costin, R.D. Costin. Rigorous WKB for Discrete Schemes with Smooth Coefficients , SIAM J. of Math. Analysis, Vol.27,No 1,pp.110-134, Jan.

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1997 Ph.D. in Applied Mathematics.