A new approach to Regularity and Singularity Questions in some PDEs including 3-D Navier-Stokes

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Regularity and Singularity in PDEs-background

- PDEs modeling physical phenomena typically include some effects while ignoring others.

- Existence and uniqueness questions of smooth solutions fundamental to relevance of a PDE model, as is blow-up.

- Global existence of evolutionary PDE solutions typically rely on "Energy" methods. Control over sufficiently higher order Sobolev norm often necessary.

- Numerical discretization not rigorously controllable, generally. Further, numerical resolution becomes an issue in higher dimensions.

Navier-Stokes existence–background

· Global Existence of smooth 3-D Navier-Stokes solution is an important open problem.

· Deviation from linear stress-strain relation or incompressibility is potentially important if N-S solutions are singular

· Globally smooth solutions known only when Reynolds number small

· Generally, smooth solutions for smooth data on $[0, T]$ known to exist, for T scaling inversely with initial data/forcing.

· Global weak solutions known since Leray, but not known whether they are unique. For unforced problem in $\mathbb{T}^3$, such a solution becomes smooth again for $t > T_c$, T_c depends on IC

Borel Summation–background and main idea

· **Borel summation generates an isomorphism between formal series and actual functions with respect to all usual algebraic operations (Ecalle, Costin,..). Borel summation used in exponential asymptotics (Dingle, Berry,..).**

· **Borel sum can involve large or small variable(s)/ parameter(s).**

· **Formal expansion for $t << 1$: $v(x,t) = v_0(x) + \sum_{m=1}^{\infty} t^m v_m(x)$ obtained algorithmically by plugging into $v_t = \mathcal{N}[v]$, where $\mathcal{N}$ being some differential operator. Series usually divergent**

· **Borel Sum of this series gives actual solution, which transcends restriction $t << 1$**

· **For NS or Burger's equation, Borel sum given by:**

$$v(x,t) = v_0(x) + \int_0^{\infty} U(x,p) e^{-p/t} dp$$

Borel Summation Illustrated in a Simple Linear ODE

$$y' - y = \frac{1}{x^2}$$

Want solution $y \to 0$, as $x \to +\infty$

Dominant Balance (or formally plugging a series in $1/x$):

$$y \sim -\frac{1}{x^2} + \frac{2}{x^3} + \ ...\frac{(-1)^k k!}{x^{k+1}} + .. \equiv \tilde{y}(x)$$

Borel Transform:

$$\mathcal{B}[x^{-k}](p) = \frac{p^{k-1}}{\Gamma(k)} = \mathcal{L}^{-1}[x^{-k}](p) \ \text{ for } \ Re\, p > 0$$

$$\mathcal{B}\left[\sum_{k=1}^{\infty} a_k x^{-k}\right](p) = \sum_{k=1}^{\infty} \frac{a_k}{\Gamma(k)} p^{k-1}$$

Borel Summation for linear ODE -II

$$Y(p) \equiv \mathcal{B}[\tilde{y}](p) = \sum_{k=1}^{\infty} (-1)^k p^k = -\frac{p}{1+p}$$

$$y(x) \equiv \int_0^{\infty} e^{-px} Y(p) dp = \mathcal{L}\mathcal{B}[\tilde{y}]$$

is the linear ODE solution we seek. Borel Sum defined as $\mathcal{L}\mathcal{B}$.

Note once solution is found, it is not restricted to large x.

Necessary properties for Borel Sum to exist:

1. **The Borel Transform $\mathcal{B}[\tilde{y}_0](p)$ analytic for $p \geq 0$,**
2. **$e^{-\alpha p}|\mathcal{B}[\tilde{y}_0](p)|$ bounded so that Laplace Transform exists.**

Remark: Difficult to check directly for non-trivial problems

Borel sum of nonlinear ODE solution

Instead, directly apply $\mathcal{L}^{-1}$ to equation; for instance

$$y' - y = \frac{1}{x^2} + y^2; \ \ \text{with} \lim_{x\to\infty} y = 0$$

Inverse Laplace transforming, with $Y(p) = [\mathcal{L}^{-1}y](p)$:

$$-pY(p) - Y(p) = p + Y * Y \ \ \text{implying}\ Y(p) = -\frac{1}{1+p} - \frac{Y * Y}{1+p}$$

For functions Y analytic for $p \geq 0$ and $e^{-\alpha p}Y(p)$ integrable, it can be shown above has unique solution for sufficiently large α.

Implies ODE solution $y(x) = \int_0^\infty Y(p) e^{-px} dp$ for $Re\ x > \alpha$

The above is a special case of nonlinear ODEs (Costin, 1998).

Generalized to sectorial PDE solutions (Costin & T., '07)

Borel sum of nonlinear ODE solution-II

Define $\chi_j(p)$ characteristic function, equalling 1 for $p \in [j, (j+1))$ and zero otherwise.

Define $Y_j(p) = Y(p)\chi_j(p)$. Then from property of Laplace convolution $*$ for $p \in [j, j+1)$: $Y * Y = \sum_{l=0}^{j} Y_l * Y_{j-l}$

Therefore, integral equation for $p \in [j, j+1)$ becomes:

$$Y_j + \frac{2Y_0 * Y_j}{1+p} = -\frac{p}{1+p} - \frac{1}{1+p} \sum_{l=1}^{j-1} Y_l * Y_{j-l}$$

Nonlinear ODE problem transformed to a sequence of linear problems beyond $[0, 1)$ interval. If a convergent series or other representation is available in $[0, 1)$, the rest involves a sequence of linear problem. This feature generalizes to nonlinear PDEs as well.

Integral Equation corresponding to Burger's equation

Plug in $v = v_0(x) + u(x,t)$ into 1-D Burger's to obtain

$$u_t - u_{xx} = -v_0 u_x - u v_{0,x} - u u_x + v_1(x) \ , \ \ v_1(x) = v_0'' - v_0 v_{0,x}$$

with $u(x,0) = 0$

Inverse Laplace Transform in $1/t$ and Fourier-Transform in x:

$$p\hat{U}_{pp} + 2U_p + k^2 \hat{U} = -ik\hat{v}_0 \hat{*} \hat{U} - ik\hat{U} \hat{*} \hat{U} \equiv \hat{G}(k,p)$$

Inverting left side using $\hat{U}(k,0) = 0$ gives:

$$\hat{U}(k,p) = \int_0^p \mathcal{K}(p,p';k)\hat{G}(k,p')dp' + \hat{U}^{(0)}(k,p) \equiv \mathcal{N}\left[\hat{U}\right](k,p)$$

$$\mathcal{K}(p,p';k) = \frac{ik\pi}{z}\left\{z'Y_1(z')J_1(z) - z'Y_1(z)J_1(z')\right\}$$

$$z = 2|k|\sqrt{p} \ , \ \ z' = 2|k|\sqrt{p'} \ , \ \hat{U}^{(0)}(k,p) = 2\frac{J_1(z)}{z}\hat{v}_1(k)$$

Solution to integral equation $\hat{U} = \mathcal{N}[\hat{U}]$

$$|\mathcal{K}(p,p';k)| \le \frac{C}{\sqrt{p}} \quad , \quad C \text{ a constant}$$

$$\|\hat{F}(.,p)\hat{*}\hat{G}(.,p)\|_{L^1(\mathbb{R}^3)} \le C\|\hat{F}(.,p)\|_{L^1(\mathbb{R}^3)}\|\hat{G}(.,p)\|_{L^1(\mathbb{R}^3)}$$

Define for functions of $F(p,k)$ the norm:

$$\|F\|^{(\alpha)} = \int_0^\infty e^{-\alpha p}\|F(.,p)\|_{L^1(\mathbb{R}^3)}\, dp \quad \text{, then can show}$$

$$\|F \overset{*}{*} G\|^{(\alpha)} \le C\|F\|^{(\alpha)}\|G\|^{(\alpha)}$$

Using above, can show $\mathcal{N}$ contractive for large α; implies integral equation has unique solution and so Burger PDE has continuous solution for $\mathrm{Re}\,\frac{1}{t} > \alpha$ as $v(x,t) = v_0(x) + \int_0^\infty e^{-p/t}U(x,p)dp$

Global PDE solution if $\|\hat{U}(.,p)\|_{L^1(\mathbb{R}^3)}$ does not grow as $p \to \infty$

Incompressible 3-D Navier-Stokes in Fourier-Space

Consider 3-D N-S in infinite geometry or periodic box. Similar results expected for finite domain with no-slip BC using eigenfunctions of Stokes operator as basis. In Fourier-Space

$$\hat{v}_t + \nu |k|^2 \hat{v} = -i k_j P_k \left[\hat{v}_j \hat{*} \hat{v} \right] + \hat{f}(k)$$

$$P_k = \left(I - \frac{k (k \cdot)}{|k|^2} \right) \quad , \quad \hat{v}(k,0) = \hat{v}_0(k)$$

where P_k is the Hodge projection in Fourier space, $\hat{f}(k)$ is the Fourier-Transform of forcing $f(x)$, assumed divergence free and t-independent. Subscript j denotes the j-th component of a vector. $k \in \mathbb{R}^3$ or $\mathbb{Z}^3$. Einstein convention for repeated index followed. $\hat{*}$ denotes Fourier convolution.

Decompose $\hat{v} = \hat{v}_0 + \hat{u}(k,t)$, inverse-Laplace Transform in $1/t$ and invert the differential operator on the left side

Integral equation associated with Navier-Stokes

We obtain:

$$\hat{U}(k,p) = \int_0^p \mathcal{K}_j(p,p';k)\hat{H}_j(k,p')dp' + \hat{U}^{(0)}(k,p) \equiv \mathcal{N}\left[\hat{U}\right](k,p) \tag{1}$$

$$\mathcal{K}_j(p,p';k) = \frac{ik_j\pi}{z}\left\{z'Y_1(z')J_1(z) - z'Y_1(z)J_1(z')\right\}$$

$$z = 2|k|\sqrt{\nu p}\,,\; z' = 2|k|\sqrt{\nu p'}\,,\; \hat{H}_j = P_k\left\{\hat{v}_{0,j}\hat{*}\hat{U} + \hat{U}_j\hat{*}\hat{v}_0 + \hat{U}_j \overset{*}{*} \hat{U}\right\}$$

$$\hat{U}^{(0)}(k,p) = 2\frac{J_1(z)}{z}\hat{v}_1(k) \;\; , \;\; P_k = \left(I - \frac{k(k\cdot)}{|k|^2}\right)$$

$$\hat{v}_1(k) = \left(-\nu|k|^2\hat{v}_0 - ik_j\mathcal{P}_k\left[\hat{v}_{0,j}\hat{*}\hat{v}_0\right]\right) + \hat{f}(k),$$

$\hat{*}$, denotes Fourier Convolution, $*$ denotes Laplace convolution, while $\overset{*}{*}$ denotes Fourier followed by Laplace convolution. J_1 and Y_1 are the usual Bessel functions.

Results for Integral equation and Navier-Stokes-1

Theorem: If $\|\hat{v}_0\|_{l^1(\mathbb{Z}^3)}, \ \|\hat{f}\|_{l^1(\mathbb{Z}^3)} < \infty$ *then there exists some* α *so that integral equation* $\hat{U} = \mathcal{N}\left[\hat{U}\right]$ *has a unique solution for* $p \in \mathbb{R}^+$ *in the space of functions* $\left\{\hat{U} : \|\hat{U}\|^{(\alpha)} < \infty\right\}$*. Further,* $\hat{v}(k,t) = \hat{v}_0(k) + \int_0^\infty \hat{U}(k,p) e^{-p/t} dp$ *solves 3-D Navier-Stokes in Fourier-Space; the corresponding* $v(x,t)$ *is a classical Navier-Stokes solution for* $t \in (0, \alpha^{-1})$*.*

Remark 1: Local existence results in Theorem 1 already known through classical methods. In the present formulation, global PDE existence is a question of asymptotics of known solution to integral equation in the sense that a sub-exponential growth of $\hat{U}$ **as** $p \to \infty$ **implies global existence of PDE solution.**

More Remarks on Theorem 1 for 3-D Navier-Stokes

Remark 2: Errors in Numerical solutions rigorously controlled. Discretization in p and Galerkin approximation in k results in:

$$\hat{U}_\delta(k, m\delta) = \delta \sum_{m'=0}^{m} \mathcal{K}_{m,m'} \mathcal{P}_N \mathcal{H}_\delta(k, m'\delta) + \hat{U}^{(0)}(k, m\delta)$$

$$\equiv \mathcal{N}_\delta \left[\hat{U}_\delta\right] \quad \text{for} \quad k_j = -N, ...N, \quad j = 1, 2, 3$$

$\mathcal{P}_N$ is the Galerkin Projection into N-Fourier modes. $\mathcal{N}_\delta$ has properties similar to $\mathcal{N}$. The continuous solution $\hat{U}$ satisfies $\hat{U} = \mathcal{N}_\delta\left[\hat{U}\right] + E$, where E is the truncation error. Thus, $\hat{U} - \hat{U}_\delta$ can be estimated using same tools as in Theorem 1.

Note: Similar control over discretized solutions to PDEs not available since truncation errors involve derivatives of PDE solution which are not known to exist beyond a short-time.

Numerical Solutions to integral equation

We choose the Kida initial conditions and forcing

$$\mathbf{v}_0(\mathbf{x}) = (v_1(x_1, x_2, x_3, 0), v_2(x_1, x_2, x_3, 0), v_3(x_1, x_2, x_3, 0))$$

$$v_1(x_1, x_2, x_3, 0) = v_2(x_3, x_1, x_2, 0) = v_3(x_2, x_3, x_1, 0)$$

$$v_1(x_1, x_2, x_3, 0) = \sin x_3 \left(\cos 3x_2 \cos x_3 - \cos x_2 \cos 3x_3\right)$$

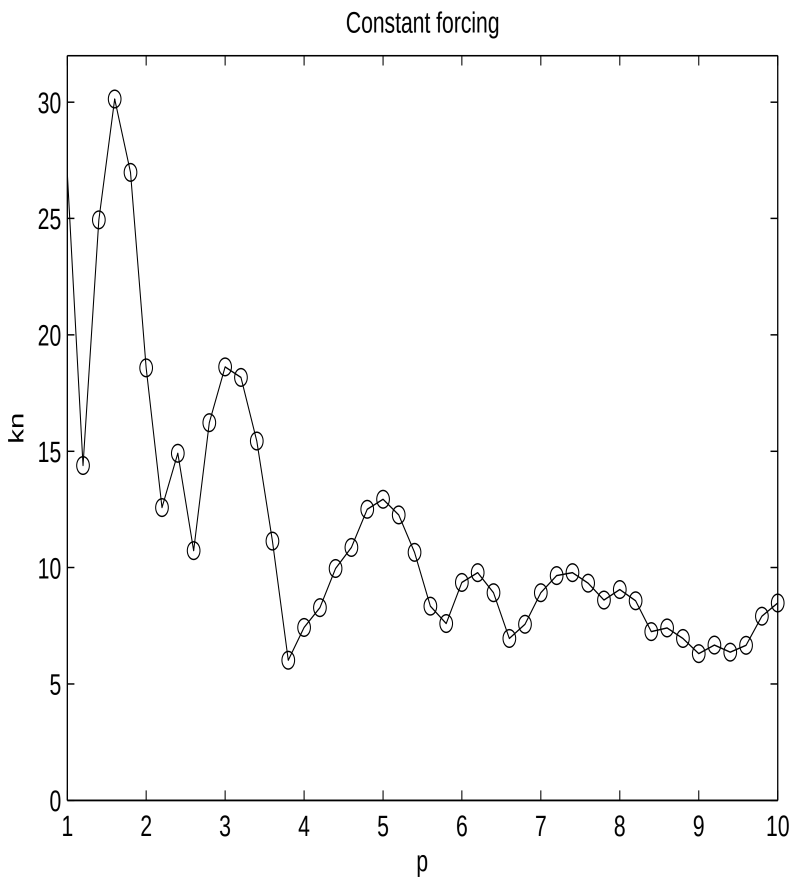
$$f_1(x_1, x_2, x_3) = \frac{1}{5} v_1(x_1, x_2, x_3, 0)$$

High Degree of Symmetry makes computationally less expensive

Corresponding Euler problem believed to blow up in finite time; so good candidate to study viscous effects

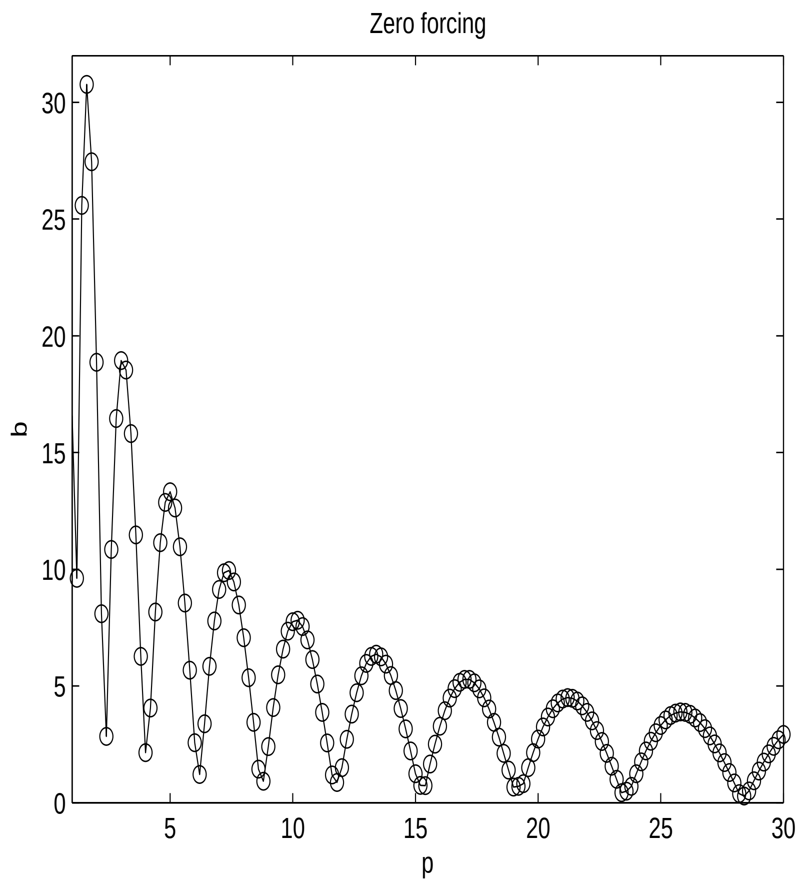
In the plots, "constant forcing" corresponds to $f = (f_1, f_2, f_3)$ as above, while zero forcing refers to $f = 0$. Recall sub-exponential growth in p corresponds to global N-S solution.

Numerical solution to integral equation-plot-1



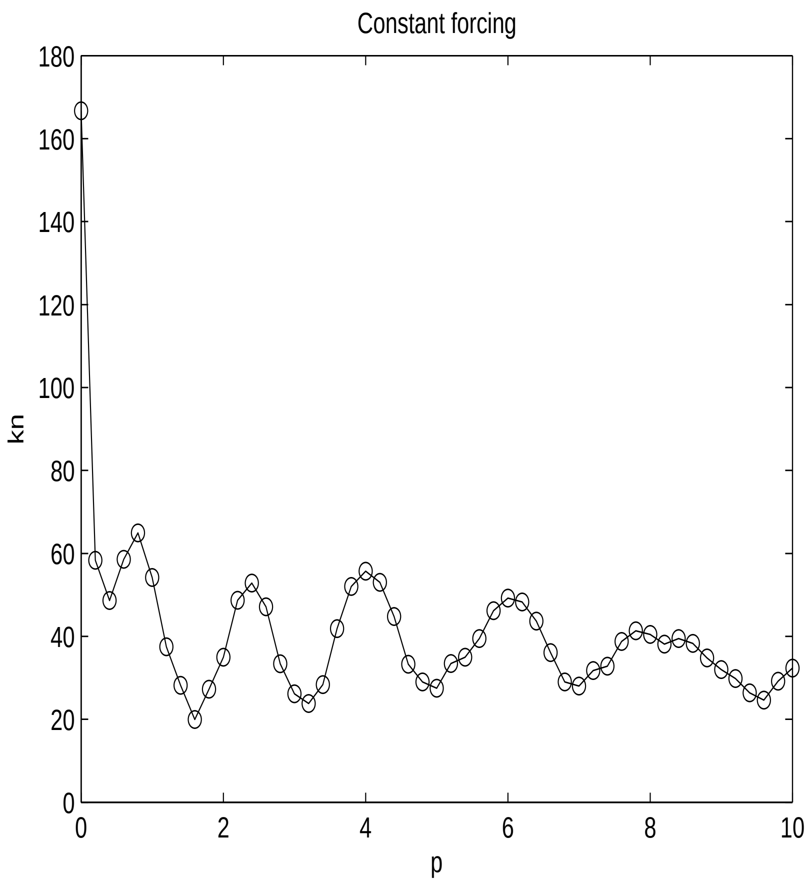
$\|\hat{U}(.,p)\|_{l^1}$ **vs.** p **for** $\nu = 1$**, constant forcing.**

Numerical solution to integral equation-plot-2



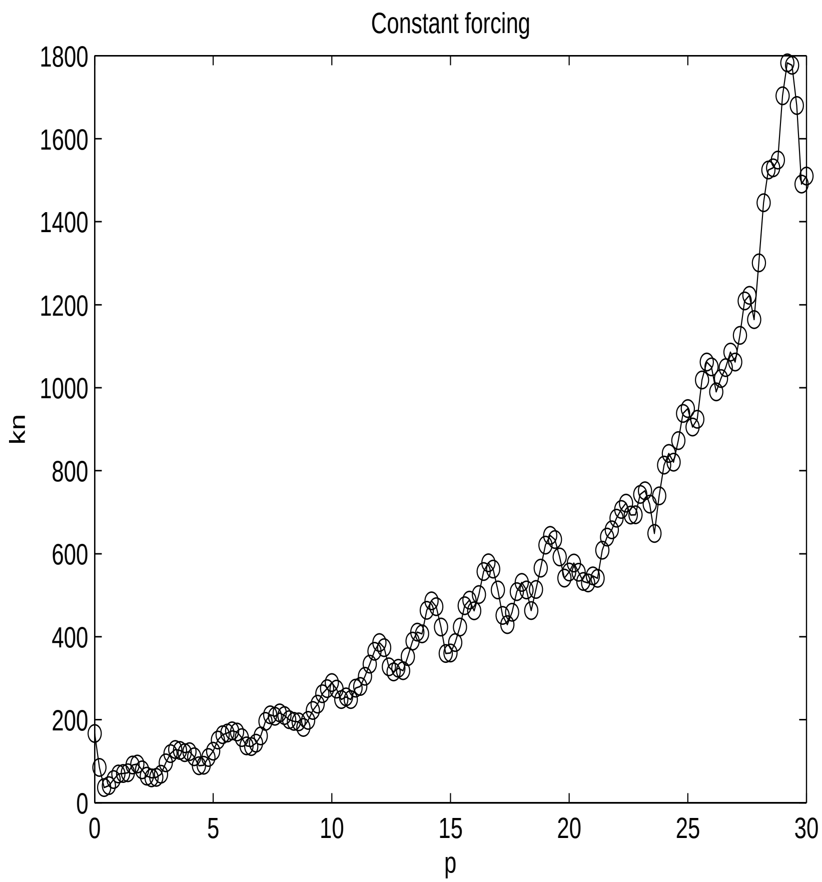
$\|\hat{U}(., p)\|_{l^1}$ **vs.** p **for** $\nu = 1$**, no forcing**

Numerical solution to integral equation-plot-3



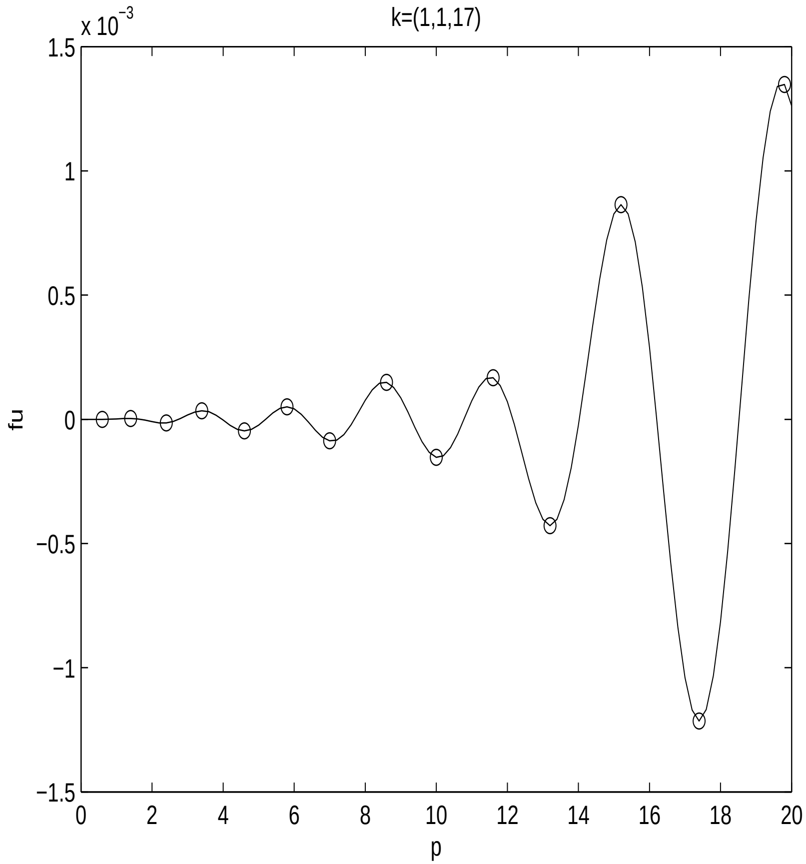
$\|\hat{U}(.,p)\|_{l^1}$ **vs.** p **for** $\nu = 0.16$**, constant forcing**

Numerical solution to integral equation-plot-4



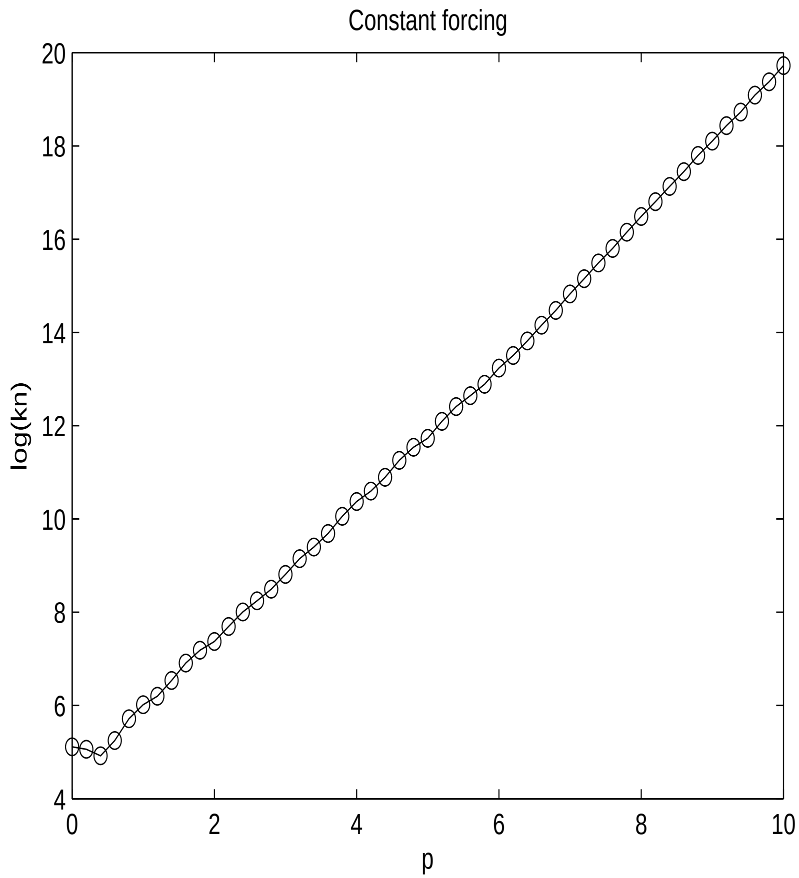
$\|\hat{U}(.,p)\|_{l^1}$ **vs.** p **for** $\nu = 0.1$**, constant forcing**

Numerical solution to integral equation-plot-5



$\hat{U}(k, p)$ **vs.** p **for** $k = (1, 1, 17)$**,** $\nu = 0.1$**, no forcing.**

Numerical solution to integral equation-plot-6



$\log \|\hat{U}(., p)\|_{l^1}$ **vs.** $\log p$ **for** $\nu = 0.001$**, constant forcing**

Issues raised by numerical computations

Numerical solutions to integral equation available on finite interval $[0, p_0]$, yet N-S solution requires $[0, \infty)$ interval since $\hat{v}(k,t) = \hat{v}_0 + \int_0^\infty e^{-p/t} \hat{U}(k,p) dp$

Actually, the integral over $\int_0^{p_0}$ gives an approximate N-S solution, with errors that can be bounded for a time interval $[0, T]$, if computed solution to integral equation eventually decreases with p on a sufficiently large interval $[0, p_0]$.

Further, a non-increasing $\hat{U}$ over a sufficiently large interval $[0, p_0]$ gives smaller bounds on growth rate α as $p \to \infty$. Therefore, in such cases smooth NS solution exists over a long interval $[0, \alpha^{-1})$.

Recall for unforced problem in $\mathbb{T}^3$, even weak solution to NS becomes smooth for $t > T_c$, with T_c estimated from initial data. Hence global existence follows under some conditions.

Extending Navier-Stokes interval of existence

For $\alpha_0 \geq 0$, define

$$\epsilon = \nu^{-1/2} p_0^{-1/2} \;, \; a = \|\hat{v}_0\|_{l^1} \;, \; c = \int_{p_0}^{\infty} \|\hat{U}^{(0)}(.,p)\|_{l^1} e^{-\alpha_0 p} dp$$

$$\epsilon_1 = \nu^{-1/2} p_0^{-1/2} \left(2 \int_0^{p_0} e^{-\alpha_0 s} \|\hat{U}(.,s)\|_{l^1} ds + \|\hat{v}_0\|_{l^1} \right)$$

$$b = \frac{e^{-\alpha_0 p_0}}{\sqrt{\nu p_0} \alpha} \int_0^{p_0} \|\hat{U} \hat{*} \hat{U} + \hat{v}_0 \cdot \hat{U}\|_{l^1} ds$$

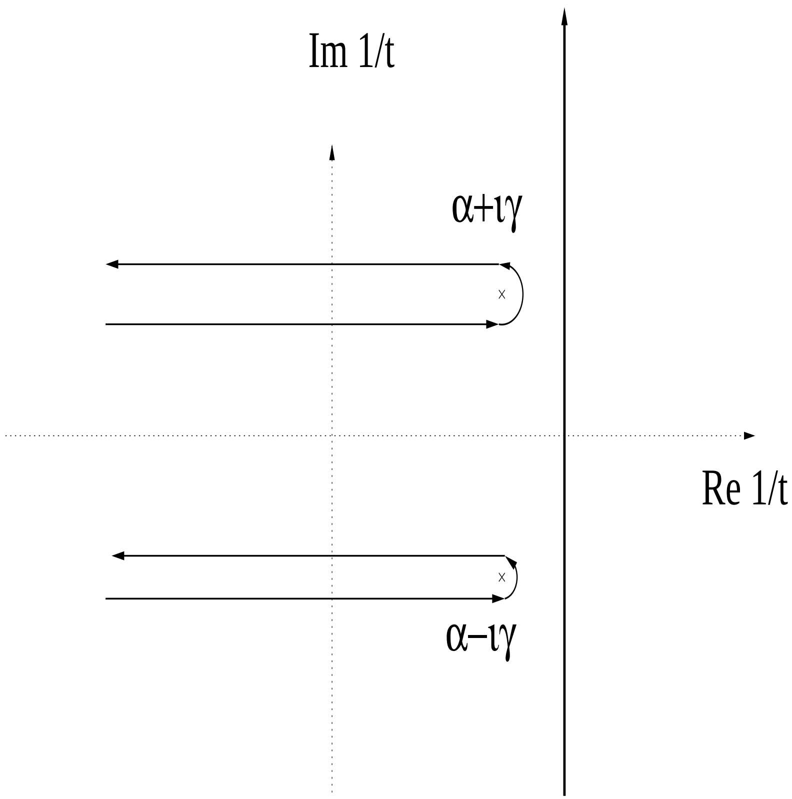
Theorem 3: A smooth solution to 3-D Navier-Stokes equation exists on the interval $[0, \alpha^{-1})$, when $\alpha \geq \alpha_0$ is chosen to satisfy

$$\alpha > \epsilon_1 + 2\epsilon c + \sqrt{(\epsilon_1 + 2\epsilon c)^2 + 4b\epsilon - \epsilon_1^2}$$

Remark: If p_0 is chosen large enough, ϵ, ϵ_1 is small when computed solution in $[0, p_0]$ decays with q. Then α can be chosen rather small.

Relation of Optimal α to Navier-Stokes singularities

$$\hat{U}(k,p) = \frac{1}{2\pi i} \int_{c_0 - i\infty}^{c_0 + i\infty} e^{p/t} \left[\hat{v}(k,t) - \hat{v}_0(k) \right] d\left[\frac{1}{t} \right]$$



Rightmost singularity(ies) of NS solution $\hat{v}(k,t)$ in the $1/t$ plane determines optimal α. γ gives dominant oscillation frequency.

Laplace-transform and accelerated representation

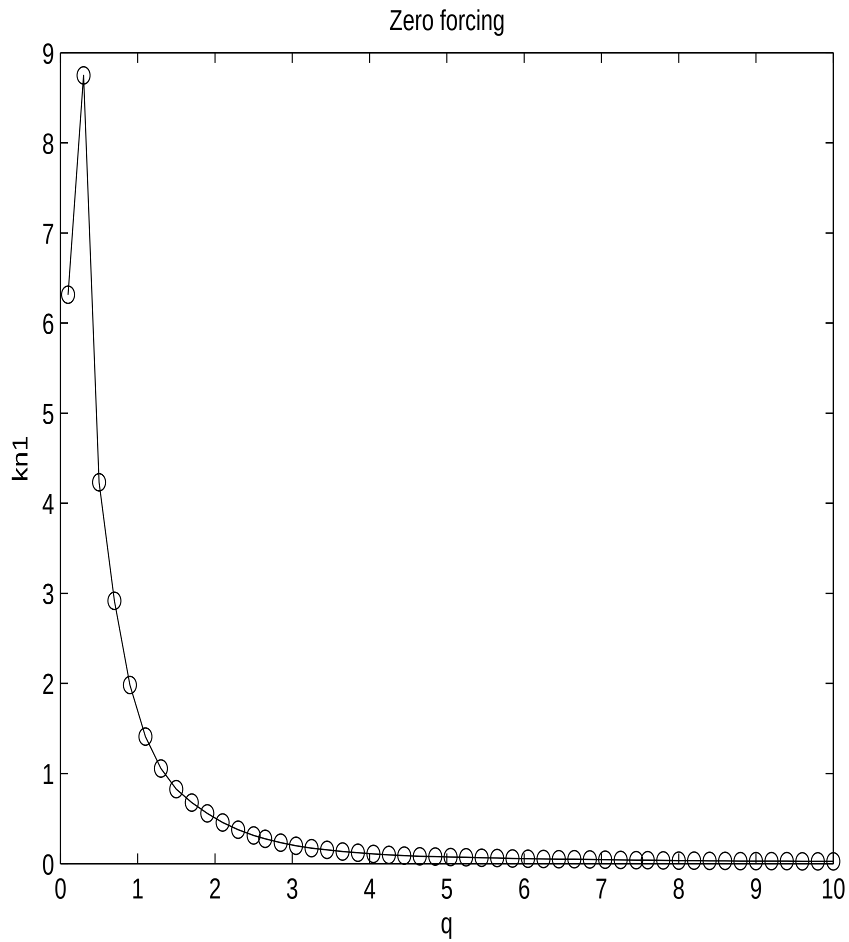
To get rid of the effect of complex singularity, it is prudent to seek a more general Laplace-transform involves

$$\hat{v}(k,t) = \hat{v}_0(k) + \int_0^\infty e^{-q/t^n} \hat{U}(k,q) dq$$

We have proved that for the unforced problem, if there are complex singularities t_s in the right-half plane, but not on the real axis, then a a nonzero lower bound for $|\arg t_s|$ exists. Then, for sufficiently large n, no singularities in the $\tau = t^{-n}$ plane in the right-half plane. Hence, $\hat{U}(k,q)$ will not grow with q

$\hat{U}(k,q)$ satisfies an integral equation similar to the one satisfied by $\hat{U}(k,p)$ and Theorems similar to Theorem 1 follow. In the context of ODEs, change of variable $p \to q$ is called acceleration (Ecalle)

$\|\hat{U}(.,q)\|_{l^1}$ vs. q, $n = 2$, $\nu = 0.1$



Kida I.C. $v_1^{(0)} = \sin x_1 (\cos 3x_2 \cos x_3 - \cos x_2 \cos 3x_3)$

Other components from cyclic relation:

$$v_1^{(0)}(x_1, x_2, x_3) = v_1^{(0)}(x_3, x_1, x_2) = v_3^{(0)}(x_2, x_3, x_1)$$

Extending Navier-Stokes interval of existence

For $\alpha_0 \geq 0$, define

$$\epsilon_1 = \nu^{-1/2} q_0^{-1+1/(2n)} \quad , \quad c = \int_{q_0}^{\infty} \|\hat{U}^{(0)}(.,q)\|_{l^1} e^{-\alpha_0 q} dq$$

$$\epsilon_1 = \nu^{-1/2} q_0^{-1+1/(2n)} \left(2 \int_0^{q_0} e^{-\alpha_0 s} \|\hat{U}(.,s)\|_{l^1} ds + \|\hat{v}_0\|_{l^1} \right)$$

$$b = \frac{e^{-\alpha_0 q_0}}{\sqrt{\nu} q_0^{1-1/(2n)} \alpha} \int_0^{q_0} \|\hat{U} \hat{*} \hat{U} + \hat{v}_0 \cdot \hat{U}\|_{l^1} ds$$

Theorem 4: A smooth solution to 3-D Navier-Stokes equation exists in the $\|.\|_{l^1}$ space on the interval $[0, \alpha^{-1/n})$, when $\alpha \geq \alpha_0$ is chosen to satisfy

$$\alpha > \epsilon_1 + 2\epsilon c + \sqrt{(\epsilon_1 + 2\epsilon c)^2 + 4b\epsilon - \epsilon_1^2}$$

Remark: If q_0 is chosen large enough, ϵ, ϵ_1 is small when computed solution in $[0, q_0]$ decays with q. Then α can be chosen rather small.

Example problems where approach is applicable

- **Navier-Stokes with temperature field (Boussinesq approximation)**

- **Fourth order Parabolic equations of the type:**

$$u_t + \Delta^2 u = N[u, Du, D^2u, D^3u]$$

- **KDV and related equations.**

- **Magneto-hydrodynamic equation with certain approximations.**

- **For some PDE problems with finite-time blow-up, blow-up time related to exponent α of exponential growth of IE solution, provided there is no-oscillation even with $p \to q$ acceleration.**

Conclusions

We have shown how Borel summation methods provides an alternate existence theory for PDE Initial value problems like N-S.

With this integral equation (IE) approach, the PDE global existence is implied if known solution to IE has subexponential growth at ∞.

The solution to integral equation in a finite interval can be computed numerically with rigorously controlled errors.

Integral equation in a suitable accelerated variable q will decay exponentially for unforced N-S equation, unless there is a real time singularity of PDE solution.

The computation over a finite $[0, q_0]$ interval gives a refined bound on exponent α at ∞, and hence a longer existence time $[0, \alpha^{-1/n})$ to 3-D Navier-Stokes.

Approach should be useful in both regularity and singularity studies of more general PDE initial value problems.