Borel Summation and 3-D Navier Stokes Existence

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Research supported in part by

- **Institute for Math Sciences (IC), EPSRC & NSF.**

Background and main idea

Existence of smooth 3-D Navier-Stokes solution an important open problem, though some sufficient conditions exist (Beale,Kato & Majda, Constantin-Fefferman,...)

Classical Results involving Sobolev space methods give smooth solutions (Leray) for smooth data, only locally in time. Energy estimates not good enough to push further.

Borel summation is a summation procedure for divergent series, under some conditions that generates an isomorphism between series and functions they represent (Ecalle, Costin,..).

Borel summation may be for large x, large t, small t

Formal expansion of N-S solution possible for small t:
$v(x,t) = v_0(x) + \sum_{m=1}^{\infty} t^m v_m(x)$.

Borel Summability ensures that this represents an actual to N-S.

Background-II

Sobolev methods are based on *a priori* **energy estimates. Without good global estimates, only local classical solutions are amenable. In that case, beyond some time determined by initial data, one is usually unable to say anything about the solution, except perhaps that weak solutions exist.**

An important feature of the Borel-based method that we introduce here is that the solution is expressed as a Laplace transform of a known solution to an integral equation. If the growth rate of this known solution at ∞ is sub-exponential in p, then global existence in t follows. So, global existence question can be related to an asymptotic problem.

Small t expansion for evolution PDEs

$$v_t = \mathcal{N}[v] \ , \ v(x,0) = v_0(x)$$

$$v(x,t) = v_0(x) + t v_1(x) + t^2 v_2(x) \ + \ ..,$$

where $v_1(x) = \mathcal{N}[v_0](x)$, $v_2 = \frac{1}{2}\left\{\mathcal{N}_v(v_0)[v_1]\right\}(x)$,..

Such expansions divergent if the order of space derivative in $\mathcal{N}$ is higher than 1.

Borel Summability ensures that

$$v(x,t) = v_0(x) + \int_0^\infty U(x,p) e^{-p/t} dp,$$

where $U(x,p)$ is analytic in p for $p \geq 0$ and $e^{-\alpha p}|U(x,p)|$ bounded, where crude bounds on α so far obtained involves v_0.

Solution representation valid for $t \in [0, \alpha^{-1})$. With subexponential growth, solution exists on $[0, \infty)$.

Borel Summation Illustrated in a Simple Linear ODE

$$y' - y = \frac{1}{x^2}$$

Want solution $y \to 0$, as $x \to +\infty$

Dominant Balance (or formally plugging a series in $1/x$):

$$y \sim -\frac{1}{x^2} + \frac{2}{x^3} + \ \ldots \frac{(-1)^k k!}{x^{k+1}} + .. \equiv \tilde{y}(x)$$

Borel Transform:

$$\mathcal{B}[x^{-k}](p) = \frac{p^{k-1}}{\Gamma(k)} = \mathcal{L}^{-1}[x^{-k}](p) \ \ \text{for} \ \ Re\, p > 0$$

$$\mathcal{B}\left[\sum_{k=1}^{\infty} a_k x^{-k}\right](p) = \sum_{k=1}^{\infty} \frac{a_k}{\Gamma(k)} p^{k-1}$$

Borel Summation for linear ODE -II

$$Y(p) \equiv \mathcal{B}[\tilde{y}](p) = \sum_{k=1}^{\infty} (-1)^k p^k = -\frac{p}{1+p}$$

$$y(x) \equiv \int_0^{\infty} e^{-px} Y(p) dp = \mathcal{L}\mathcal{B}[\tilde{y}]$$

is the linear ODE solution we seek. Borel Sum defined as $\mathcal{L}\mathcal{B}$.

Note once solution is found, it is not restricted to large x.

Necessary properties for Borel Sum to exist:

1. **The Borel Transform $\mathcal{B}[\tilde{y}_0](p)$ analytic for $p \geq 0$,**
2. **$e^{-\alpha p} |\mathcal{B}[\tilde{y}_0](p)|$ bounded so that Laplace Transform exists.**

Remark: Difficult to check directly for non-trivial problems

Borel sum of nonlinear ODE solution

Instead, directly apply $\mathcal{L}^{-1}$ to equation; for instance

$$y' - y = \frac{1}{x^2} + y^2; \quad \text{with} \lim_{x\to\infty} y = 0$$

Inverse Laplace transforming, with $Y(p) = [\mathcal{L}^{-1}y](p)$:

$$-pY(p) - Y(p) = p + Y * Y \quad \text{implying}\; Y(p) = -\frac{1}{1+p} - \frac{Y * Y}{1+p} \tag{1}$$

For functions Y analytic for $p \geq 0$ and $e^{-\alpha p}Y(p)$ bounded, it can be shown that (1) has unique solution for sufficiently large α.

Implies ODE solution $y(x) = \int_0^\infty Y(p)e^{-px}dp$ for $Re\, x > \alpha$

The above is a special case of results available for generic nonlinear ODEs (Costin, 1998)

Borel sum of nonlinear ODE solution-II

Define $\chi_j(p)$ characteristic function, equalling 1 for $p \in [j,(j+1))$ and zero otherwise.

Define $Y_j(p) = Y(p)\chi_j(p)$. Then from property of Laplace convolution $*$ for $p \in [j, j+1)$: $Y * Y = \sum_{l=0}^{j} Y_l * Y_{j-l}$

Therefore, integral equation for $p \in [j, j+1)$ becomes:

$$Y_j + \frac{2Y_0 * Y_j}{1+p} = -\frac{p}{1+p} - \frac{1}{1+p}\sum_{l=1}^{j-1} Y_l * Y_{j-l}$$

Nonlinear ODE problem transformed to a sequence of linear problems beyond $[0,1)$ interval. If a convergent series or other representation is available in $[0,1)$, the rest involves a sequence of linear problem. This feature generalizes to nonlinear PDEs as well.

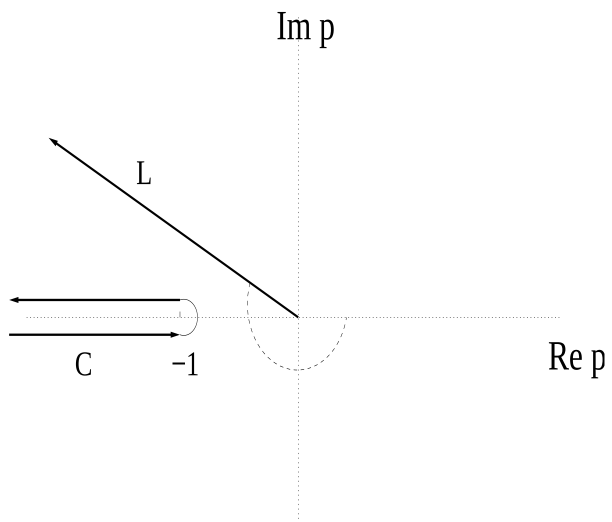
Singularities in Borel Plane–Stokes Phenomena

No singularities in the p-plane so far; but can arise.

For instance, we had solution to $y' - y = \frac{1}{x^2}$:

$$y(x) = -\int_0^\infty \frac{pe^{-px}}{1+p}dp,$$

If rotate counter-clockwise in the complex x-plane past $\arg x = \pi$, the Laplace contour in the p-plane needs to be rotated clockwise past $\arg p = -\pi$. From contour deformation, get additional contribution (Stokes phenomenon) from contour C below:



Integral equation for N-S in Borel plane

$$\hat{\mathrm{v}}_t + |\mathrm{k}|^2\hat{\mathrm{v}} = -ik_j P_k\left[\hat{v}_j \hat{*} \hat{\mathrm{v}}\right] + \hat{\mathrm{f}}(\mathrm{k}) \quad , \quad \hat{\mathrm{v}}(\mathrm{k},0) = \hat{\mathrm{v}}_0(\mathrm{k}),$$

$$P_k = \left[1 - \frac{\mathrm{k}\,(\mathrm{k}\cdot)}{|\mathrm{k}|^2}\right],$$

where $\hat{*}$ denotes Fourier convolution. Introducing $\tau = \frac{1}{t}$:

$$-\tau^2\hat{\mathrm{v}}_\tau + |\mathrm{k}|^2\hat{\mathrm{v}} = -ik_j P_k\left[\hat{v}_j \hat{*} \hat{\mathrm{v}}\right] + \hat{\mathrm{f}}(\mathrm{k})$$

Decompose $\hat{\mathrm{v}} = \hat{\mathrm{v}}_0 + \hat{\mathrm{u}}$ so that $\hat{\mathrm{u}} \to 0$ as $\tau \to \infty$. Applying $\mathcal{L}^{-1}$ in the τ variable, obtain equation for $\mathrm{U}(\mathrm{k},p) = \mathcal{L}^{-1}[\mathrm{u}(\mathrm{k},.)](p)$:

$$\partial_{pp}\left[p\hat{\mathrm{U}}\right] + |\mathrm{k}|^2\mathrm{U} = \hat{\mathrm{R}}$$

$$\mathrm{U}(\mathrm{k},p) = \int_0^p K(p,p')\hat{\mathrm{R}}(\mathrm{k},p')dp'$$

Integral Equation formulation for 3-D NS -II

$$\hat{\mathrm{R}}(k,p) = -ik_j P_k \left[\hat{v}_{0,j} \hat{*} \hat{\mathrm{U}} + \hat{U}_j \hat{*} \hat{\mathrm{v}}_0 + \hat{U}_j \overset{*}{*} \hat{\mathrm{U}} \right] + \hat{\mathrm{v}}_1 \delta(p)$$

where $\overset{*}{*}$ denote Laplace convolution, followed by Fourier convolution. $K(p,p')$, $\hat{\mathrm{v}}_1(\mathrm{k})$ given by:

$$K(p,p') = \frac{\pi}{z} \left(z' J_1(z) Y_1(z') - z' Y_1(z) J_1(z') \right) \ , z = 2|\mathrm{k}|\sqrt{p},$$

$$z' = 2|\mathrm{k}|\sqrt{p'} \ \ , \ \ \hat{\mathrm{v}}_1(k) = -|\mathrm{k}|^2 \mathrm{v}_0 - ik_j P_k \left[\hat{v}_{0,j} \hat{*} \hat{\mathrm{v}}_0 \right] + \hat{\mathrm{f}}(\mathrm{k})$$

Introduce norm $\|.\|_{\mu,\beta}$, with $\mu > 3$, $\beta \geq 0$ so that

$$\|v_0\|_{\mu,\beta} = \sup_{k \in \mathbb{R}^3} e^{\beta|k|} (1+|k|)^{\mu} |\hat{v}_0(k)|$$

Results on 3-D Navier-Stokes

Theorem 1: If $\|\hat{v}_0\|_{\mu+2,\beta} < \infty$*,* $\mu > 3$*,* $\beta \geq 0$*, NS has a unique solution in the form with* $\|\hat{v}(\cdot,t)\|_{\mu,\beta} < \infty$ *for* $\mathrm{Re}\,\frac{1}{t} > \alpha$*, where* α *depends on* $\hat{v}_0$*. Furthermore,* $\hat{v}(\cdot,t)$ *is analytic for* $\mathrm{Re}\,\frac{1}{t} > \alpha$ *and* $\|\hat{v}(\cdot,t)\|_{\mu+2,\beta} < \infty$ *for* $t \in [0,\alpha^{-1})$*. If* $\beta > 0$*, this implies that* v *is analytic in* x *with the same analyticity width as* v_0 *and* f*.*

Theorem 2: For $\beta > 0$ *(analytic initial data) and* $\mu > 3$*, the solution* v *is Borel summable in* $1/t$*, i.e. there exists* $U(x,p)$*, analytic in a neighborhood of* $\mathbb{R}^+$*, exponentially bounded, and analytic in* x *for* $|\,\mathrm{Im}\ x| < \beta$ *so that*

$$v(x,t) = v_0(x) + \int_0^\infty U(x,p)e^{-p/t}dp$$

Therefore, in particular as $t \longrightarrow 0$*,*

$$v(x,t) \sim v_0(x) + \sum_{m=1}^{\infty} t^m v_m(x)$$

Remarks on Theorem

Remark: Borel summability and classical Gevrey-asymptotic results imply, for small t:

$$\left| v(x,t) - v_0(x) - \sum_{m=1}^{m(t)} v_m(x) t^m \right| \leq A_0 [m(t)]^{1/2} e^{-m(t)}$$

where $m(t) = \lfloor B_0^{-1} t^{-1} \rfloor$. Our bounds on B_0 are likely suboptimal. Formal arguments in the recurrence relation of v_{m+1} in terms of v_m, v_{m-1},...,v_1, indicate that B only depend on β, but not on $\|\hat{v}_0\|_{\mu,\beta}$.

Remark: While most of the results in Theorem 1 already known through classical methods, the problem of continuation of solution beyond $t = \frac{1}{\alpha}$ can be related to an asymptotics problem for known solution $\hat{\mathrm{U}}(k,p)$ of an integral equation. No equivalent characterization known before.

Conclusions

We have shown how Borel summation methods provides an alternate existence theory for N-S equation

By converting a nonlinear PDE problem into an integral equation involving convolutions, global existence is implied if the known solution to an integral equation has subexponential growth.

Beyond an initial interval, the integral equation breaks up into a sequence of linear problems, making it computationally and analytically attractive.

Borel methods give precise asymptotic estimates for small time dynamics.

Key points in the proof

$$\text{Define norm}: \ \|\hat{f}(\mathrm{k},p)\| = \sup_{p\ge 0} e^{-\alpha p}(1+p^2)\|\hat{f}(.,p)\|_{\mu,\beta}$$

Because of properties

$$\frac{e^{\alpha p}}{(1+p^2)} * \frac{e^{\alpha p}}{(1+p^2)} = e^{\alpha p}\int_0^p \frac{ds}{(1+s^2)[1+(p-s)^2]} \le \frac{M_0 e^{\alpha p}}{1+p^2}$$

$$\left[e^{-\beta|\mathrm{k}|}(1+|\mathrm{k}|)^{-\mu}\right] \hat{*} \left[e^{-\beta|\mathrm{k}|}(1+|\mathrm{k}|)^{-\mu}\right] \le \frac{C_0(\mu)e^{-\beta|\mathrm{k}|}}{(1+|\mathrm{k}|)^{-\mu}},$$

the following algebraic properties follow:

$$\|[\hat{f}(\mathrm{k},p)]\hat{*}[\hat{g}(\mathrm{k})]\|_{\mu,\beta} \le C_0\|\hat{f}(\cdot,p)\|_{\mu,\beta}\|\hat{g}\|_{\mu,\beta}$$

$$\|\hat{u} \overset{*}{*} \hat{v}\| \le M_0 C_0 \|\hat{u}\|\|\hat{v}\| \ , \ \|\int_0^p |\hat{u}(\mathrm{k},s)|ds\| \le C\alpha^{-1}\|\hat{u}\|$$

Small t Expansion-II

Note: $1/t$ not always correct choice for Borel-Transform

For instance, for Kuramato-Sivishinsky equation

$$u_t + u_{xx} + uu_x + u_{xxxx} = 0 \ , \ \ u(x,0) = u_0(x),$$

correct variable is $T = t^{-1/3}$ for Borel-Transform:

$$-\frac{T^4}{3} u_T + u_{xx} + uu_x + u_{xxxx} = 0$$

Writing $u(x,T) = u_0(x) + v(x,T)$ and Borel-Transforming in T:

$$\frac{1}{3} [pV]_{pppp} + V_{xx} + V_{xxxx} + u_0 V_x + V u_{0x} + V * V_x = R(x)$$

Fourth-order in both p and x; Cauchy-Kowalewski ideas may be applied. Solution $u(x,t) = u_0(x) + \int_0^\infty V(x,p) e^{-pt^{-1/3}} dp$