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Finite products sets and minimally almost periodic groups



V. Bergelson^{a,1}, J.C. Christopherson^a, D. Robertson^{a,*},
P. Zorin-Kranich^{b,2}

^a Department of Mathematics, The Ohio State University, 231 West 18th Avenue,
Columbus, OH 43210-1174, USA

^b Institute of Mathematics, Hebrew University, Givat Ram, Jerusalem, 91904,
Israel

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ABSTRACT

We characterize those locally compact, second countable, amenable groups in which a density version of Hindman's theorem holds and those countable, amenable groups in which a two-sided density version of Hindman's theorem holds. In both cases the possible failure can be attributed to an abundance of finite-dimensional unitary representations, which allows us to construct sets with large density that do not contain any shift of a set of measurable recurrence, let alone a shift of a finite products set. The possible success is connected to the ergodic-theoretic phenomenon of weak mixing via a two-sided version of the Furstenberg correspondence principle. We also construct subsets with large density that are not piecewise syndetic in arbitrary non-compact amenable groups. For countably infinite amenable groups, the symbolic systems

* Corresponding author.

E-mail addresses: vitaly@math.ohio-state.edu (V. Bergelson), cory@math.ohio-state.edu (J.C. Christopherson), robertson@math.ohio-state.edu (D. Robertson), pzorin@math.huji.ac.il (P. Zorin-Kranich).

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associated to such sets admit invariant probability measures that are not concentrated on their minimal subsystems.

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1. Introduction

Given a sequence x_n in $\mathbb{N} = \{1, 2, \dots\}$ the *finite sums set* or *IP set* generated by x_n is defined by

$$FS(x_n) = \left\{ \sum_{n \in \alpha} x_n : \emptyset \neq \alpha \subset \mathbb{N} \text{ finite} \right\}.$$

The following theorem, proved by N. Hindman in 1974, confirmed a conjecture of Graham and Rothschild.

Theorem 1.1. (See Hindman [21].) *For any finite partition $\mathbb{N} = C_1 \cup \dots \cup C_r$ there is some $i \in \{1, \dots, r\}$ and some sequence x_n in \mathbb{N} such that $FS(x_n) \subset C_i$.*

Given a partition result, it is natural to ask whether it has a density version. For example, van der Waerden’s theorem states that, for any finite partition of \mathbb{N} there is a cell of the partition containing arbitrarily long arithmetic progressions. The density version of van der Waerden’s theorem, Szemerédi’s theorem, states that if $E \subset \mathbb{N}$ has *positive upper density*, meaning that

$$\bar{d}(E) = \limsup_{N \rightarrow \infty} \frac{|E \cap \{1, \dots, N\}|}{N}$$

is positive, then E contains arbitrarily long arithmetic progressions. If the above limit exists then its value is called the *density* of E and denoted $d(E)$.

In an attempt to discover a density version of Hindman’s result, Erdős asked whether $d(E) > 0$ implies that E contains a shift of some finite sums set. In other words, does every $E \subset \mathbb{N}$ satisfying $d(E) > 0$ contain a set of the form $FS(x_n) + t$ for some sequence x_n in \mathbb{N} and some t in \mathbb{N} ? It is necessary to allow for a shift because having positive density is a shift-invariant property, whereas being a finite sums set is not. Indeed $2\mathbb{N} + 1$ has positive density but contains no finite sums set. The following theorem of E. Straus provided “a counterexample to all such attempts” [14, p. 105].

Theorem 1.2. (E. Straus, unpublished; see [2, Theorem 2.20] or [22, Theorem 11.6].) *For every $\varepsilon > 0$ there is a set $E \subset \mathbb{N}$ having density $d(E) > 1 - \varepsilon$ such that no shift of E contains a finite sums set.*

This is proved by removing the tails of sparser and sparser infinite progressions from \mathbb{N} : for each n in \mathbb{N} one removes a set of the form $\{a_n k + n : k \geq b_n\}$. Since every finite sums set $FS(x_n)$ intersects every set of the form $t\mathbb{N}$ (because infinitely many of the x_n are congruent modulo t , say) after these removals one is left with a subset of \mathbb{N} no shift of which contains a finite sums set. Moreover, by carefully choosing the values of a_n and b_n , one can ensure that the density of the resulting set exists and is as close to 1 as we like. Since any subset F of \mathbb{N} with $\bar{d}(F) = 1$ contains arbitrarily long intervals, and hence a finite sums set, $d(E) > 1 - \varepsilon$ is the best we can expect.

This paper is concerned with the natural task of finding and characterizing groups in which an analogue of [Theorem 1.2](#) holds. More specifically, we wish to describe those groups in which there are arbitrarily large sets no shift of which contains some sort of structured set. To make our results precise we need to decide what we mean by “large” and “structured”. We will consider only locally compact, Hausdorff topological groups, hereafter called simply “locally compact groups”. Within this class of groups, we can define a notion of largeness whenever the group has a Følner sequence.

Definition 1.3. Let G be a locally compact group with a left Haar measure m . A sequence $N \mapsto \Phi_N$ of compact, positive-measure subsets of G is called a *left Følner sequence* if

$$\lim_{N \rightarrow \infty} \frac{m(\Phi_N \cap g\Phi_N)}{m(\Phi_N)} = 1 \tag{1.4}$$

uniformly on compact subsets of G , and a *right Følner sequence* if

$$\lim_{N \rightarrow \infty} \frac{m(\Phi_N g \cap \Phi_N)}{m(\Phi_N)} = 1$$

uniformly on compact subsets of G . By a *two-sided Følner sequence* we mean a sequence that is both a left Følner sequence and a right Følner sequence.

A locally compact group G with a left Haar measure m is said to be *amenable* if $L^\infty(G, m)$ has a left-invariant mean. Every locally compact, second countable, amenable group has a left Følner sequence [[27, Theorem 4.16](#)]. This was originally proved by Følner for countable groups in [[16](#)].

Definition 1.5. Let Φ be a left, right, or two-sided Følner sequence in a locally compact, second countable, amenable group G with a left Haar measure m and let E be a measurable subset of G . Define the *upper density* of E with respect to Φ to be

$$\bar{d}_\Phi(E) = \limsup_{N \rightarrow \infty} \frac{m(E \cap \Phi_N)}{m(\Phi_N)},$$

the *lower density* of E with respect to Φ by

$$\underline{d}_\Phi(E) = \liminf_{N \rightarrow \infty} \frac{m(E \cap \Phi_N)}{m(\Phi_N)},$$

and the *density* of E with respect to Φ as $d_\Phi(E) = \bar{d}_\Phi(E) = \underline{d}_\Phi(E)$ whenever the limit exists.

Thus the question we wish to answer is the following one: in which locally compact, second countable, amenable groups is it possible to find a subset with arbitrarily large density no shift of which contains a set of some specified type? In this paper we concern ourselves with two types of sets: piecewise syndetic sets and sets of measurable recurrence.

Definition 1.6. Let G be a topological group. A subset S of G is called *syndetic* if there exists a compact set K such that $KS = G$. A subset T of G is called *thick* if for every compact set K there exists $g \in G$ such that $Kg \subset T$. A subset P of G is called *piecewise syndetic* if there exists a compact set K such that KP is thick.

To be precise, we should speak of “left thick”, “left syndetic” and “left piecewise syndetic” sets. However, since we will not need the corresponding right-sided notions, we will continue to omit “left” from the terminology.

Definition 1.7. Let (X, \mathcal{B}, μ) be a separable probability space. By a *measure-preserving action* of a topological group G on (X, \mathcal{B}, μ) we mean a jointly measurable map $T : G \times X \rightarrow X$ such that the induced maps $T^g : X \rightarrow X$ preserve μ and satisfy $T^g T^h = T^{gh}$ for all g, h in G .

Definition 1.8. A subset R of a topological group G is a set of *measurable recurrence* if for every compact set K in G , every measure-preserving action T of G on a separable probability space (X, \mathcal{B}, μ) , and every non-null, measurable subset A of X there exists g in $R \setminus K$ such that $\mu(A \cap T^g A) > 0$.

For piecewise-syndetic sets we have a version of [Theorem 1.2](#) in every locally compact, second countable, amenable group that is not compact. The proof is given in [Section 6](#).

Theorem 1.9. *For any locally compact, second countable, amenable group G that is not compact, any left Følner sequence Φ in G , and any $\varepsilon > 0$ there is a closed subset Q of G with $\underline{d}_\Phi(Q) > 1 - \varepsilon$ that is not piecewise syndetic.*

Whether a version of [Theorem 1.2](#) holds for sets of measurable recurrence depends on how many finite-dimensional representations the group has.

Definition 1.10. By a *representation* of a topological group G we mean a continuous homomorphism from G to the unitary group of a complex Hilbert space. A group is a *WM group* if it has no non-trivial finite-dimensional representations. A group is a

virtually WM group if it has a subgroup of finite index that is a WM group. Lastly, a group is WM-by-compact if it has a closed, normal, WM subgroup that is cocompact.

The terminology WM is from ergodic theory. In [4], a topological group is said to be WM if any ergodic, measure preserving action is weakly mixing. By Theorem 3.4 in [30], this agrees with our terminology. It follows immediately from von Neumann’s work on almost-periodic functions (discussed in Section 2) that a group is a WM group if and only if it is minimally almost-periodic (which means that the only almost-periodic functions on the group are the constant functions).

One can measure how far from being WM a topological group G is by considering the closed, normal subgroup G_0 of G , obtained by intersecting the kernels of all finite-dimensional representations: G is WM if and only if $G_0 = G$. We have a version of Theorem 1.2 for sets of measurable recurrence whenever G/G_0 is not compact. We give the proof in Section 5.

Theorem 1.11. *Let G be a locally compact, second countable, amenable group such that G/G_0 is not compact. Then for any left Følner sequence Φ in G and any $\varepsilon > 0$ there is a measurable subset E of G with $d_\Phi(E) > 1 - \varepsilon$ such that no set of the form KEK with $K \subset G$ compact contains a set of measurable recurrence.*

When $G = \mathbb{Z}$ the group G_0 is trivial (G_0 is trivial for every countable, abelian group) so Theorem 1.11 strengthens Theorem 1.2 by exhibiting a set E with $d_\Phi(E) > 1 - \varepsilon$ for any prescribed Følner sequence Φ . Since every finite sums set is a set of measurable recurrence (see Example 2.22) it also strengthens Theorem 1.2 by widening the class of sets that cannot be shifted into E .

As in the proof of Theorem 1.2, we will prove Theorem 1.11 by removing sparser and sparser sets from G . To prohibit shifts of sets of measurable recurrence, we will construct E by removing from G shifts of tails of return time sets arising from certain G actions. That is, for certain measure-preserving actions $T : G \times X \rightarrow X$ we will remove from G shifts of tails of sets with small density of the form $\{g \in G : T^g x \in U\}$. These actions will have a unique invariant measure, which will imply that the density of the sets we remove will exist. Combining this with a version of the monotone convergence theorem for density (Lemma 5.1) will allow us to prove that the density $d_\Phi(E)$ exists. It is only when G/G_0 is not compact that we can produce sufficiently many actions suitable for this approach.

The two-sided finite product sets, which we now define, are what prevent Theorem 1.11 from holding in the WM case.

Definition 1.12. Let $\mathcal{F} = \{\alpha \subset \mathbb{N} : 0 < |\alpha| < \infty\}$. For any sequence g_n in a topological group G such that $g_n \rightarrow \infty$ (in the sense that it eventually leaves any compact subset of G) and any $\alpha = \{k_1 < \dots < k_m\}$ in \mathcal{F} define $I_\alpha(g_n) = g_{k_1} \cdots g_{k_m}$ and $D_\alpha(g_n) = g_{k_m} \cdots g_{k_1}$. Then $\text{FPI}(g_n) := \{I_\alpha(g_n) : \alpha \in \mathcal{F}\}$ is the increasing finite products set

determined by the sequence g_n and $\text{FPD}(g_n) := \{D_\alpha(g_n) : \alpha \in \mathcal{F}\}$ is the *decreasing finite products set* determined by the sequence g_n . Put $I_\emptyset(g_n) = D_\emptyset(g_n) = \text{id}_G$. Lastly, define by

$$\text{FP}(g_n) = \{I_\alpha(g_n)D_\beta(g_n) : \alpha, \beta \in \mathcal{F} \cup \{\emptyset\}, \alpha \cap \beta = \emptyset, \alpha \cup \beta \neq \emptyset\}$$

the *two-sided finite products set* determined by the sequence g_n .

We insist that the sequence g_n determining a finite products set escapes to infinity in order to produce sets of measurable recurrence, cf. [Example 2.22](#). Note that $\text{FP}(g_n)$ contains both $\text{FPI}(g_n)$ and $\text{FPD}(g_n)$ because we can take either β or α to be empty.

Definition 1.13. Let G be a locally compact, second countable, amenable group. A subset S of G is called *left substantial* if $S \supset UW$ for some non-empty, open subset U of G containing id_G and some measurable subset W of G having positive upper density with respect to some left Følner sequence in G .

By [\[4, Theorem 2.4\]](#), a locally compact, second countable, amenable group G is WM if and only if every left substantial subset of G contains an increasing finite products set. It follows that [Theorem 1.11](#) fails badly when G is WM.

Definition 1.14. Let G be a locally compact, second countable, amenable, unimodular group. We say that a subset S of G is *substantial* if $S \supset UWU$ for some non-empty, open subset U of G containing id_G and some measurable subset W of G having positive upper density with respect to some two-sided Følner sequence in G .

Our next result, a two-sided version of [\[4, Theorem 2.4\]](#), strengthens the degree to which a version of [Theorem 1.2](#) is unavailable in WM groups. (Note that WM groups are always unimodular by [Lemma 2.1](#).)

Theorem 1.15. *Let G be a locally compact, second countable, amenable, WM group. Then every substantial subset of G contains a two-sided finite products set.*

We would be interested to know whether two-sided finite products sets are partition regular.

Question 1.16. Is it true that, for any finite partition $C_1 \cup \dots \cup C_r$ of an infinite group G one can find $1 \leq i \leq r$ such that C_i contains a two-sided finite products set?

As in [\[4\]](#), we deduce [Theorem 1.15](#) from results about measurable recurrence of WM groups using a version of the Furstenberg correspondence principle. This is why we restrict our attention to substantial subsets: as explained in [Section 3](#), there is in general no correspondence principle for arbitrary positive-density subsets.

Since two-sided finite products sets involve multiplication on the left and on the right, we will need a two-sided version of the correspondence principle to prove [Theorem 1.15](#). We develop such a principle in [Section 3](#). When G has a two-sided Følner sequence the correspondence principle can be stated as follows.

Theorem 1.17. *Let G be a locally compact, second countable, amenable group with a left Haar measure m and a two-sided Følner sequence Φ . Let W be a Borel subset of G , let $U \subset G$ be an open neighborhood of the identity and let $S \supset UWU$ be measurable. Then there is a compact metric space X with a continuous $G \times G$ action $L \times R$, an $L \times R$ invariant Borel probability measure μ on X and a non-negative, continuous function ξ on X such that*

$$\bar{d}_\Phi(g_1^{-1}Sh_1 \cap \dots \cap g_n^{-1}Sh_n) \geq \int L^{g_1}R^{h_1}\xi \dots L^{g_n}R^{h_n}\xi \, d\mu$$

for any $g_1, \dots, g_n, h_1, \dots, h_n$ in G .

See [Theorem 3.2](#) for the general version, which also shows it is possible to assume μ is ergodic for $L \times R$ at the cost of the Følner sequence.

To prove [Theorem 1.15](#) we need to exhibit multiple recurrence for the measure preserving actions L and R produced by our correspondence principle. Using results from [\[9\]](#), which are based on ideas introduced in [\[23\]](#) and [\[1\]](#), we will obtain the following strong form of recurrence (cf. [Corollary 4.9](#)).

Theorem 1.18. *Let G be a locally compact, second countable, amenable, WM group and let T_1, T_2 be measure-preserving actions of G on a probability space (X, \mathcal{B}, μ) such that $(g, h) \mapsto T_1^g T_2^h$ is an ergodic $G \times G$ action on (X, \mathcal{B}, μ) . Then for any $0 \leq f \leq 1$ in $L^\infty(X, \mathcal{B}, \mu)$ and any $\varepsilon > 0$ the set*

$$\left\{ g \in G : \int f \cdot T_1^g f \cdot T_1^g T_2^g f \, d\mu \geq \left(\int f \, d\mu \right)^4 - \varepsilon \right\}$$

has density 1 with respect to every left Følner sequence on G .

We will show (see [Example 4.12](#)) that it is not sufficient to assume the set in [Theorem 1.15](#) is left substantial. In fact we will construct, in a countable group, a set that has density 1 with respect to a left Følner sequence and yet cannot contain a decreasing finite products set. By [\[4, Theorem 2.4\]](#) this set must have zero density with respect to every right Følner sequence. The existence of such a set answers the question, raised in [\[12\]](#), of whether a set having positive upper density with respect to some left Følner sequence must have positive upper density for some right Følner sequence.

Our next result, a consequence of [Theorem 1.15](#), shows that a version of [Theorem 1.2](#) fails to hold whenever G is WM-by-compact.

Theorem 1.19. *Let G be a locally compact, second countable, amenable group that is WM-by-compact but not WM. Then every substantial subset of G contains a left-shift of some two-sided finite products set and a right-shift of some (possibly different) two-sided finite products set. On the other hand, for every two-sided Følner sequence Φ in G there exists a subset of G having positive upper density with respect to Φ containing no set of measurable recurrence.*

In view of the fact (see Section 2.3 below) that a countable group G is virtually WM if and only if G_0 has finite index, Theorems 1.11, 1.15 and 1.19 together yield the following trichotomy for countable, amenable groups.

Theorem 1.20. *Let G be a countable, infinite, amenable group. Then exactly one of the following holds.*

1. G is WM. Then every subset of G having positive upper density with respect to some two-sided Følner sequences contains a two-sided finite products set.
2. G is virtually WM, but not WM. Then every subset of G having positive upper density with respect to some two-sided Følner sequences contains a left shift of some two-sided finite products set and a right shift of some two-sided finite products set. However, for any two-sided Følner sequence there is a subset with positive upper density that does not contain a set of measurable recurrence.
3. G is not virtually WM. Then for any two-sided Følner sequence Φ and any $\varepsilon > 0$ there is a subset E of G with $d_\Phi(E) > 1 - \varepsilon$ such that no set of the form KEK , where $K \subset G$ is finite, contains a set of measurable recurrence.

In general, however, the situation is more complicated. As Example 2.16 shows, there are locally compact, second countable, amenable groups G which have cocompact von Neumann kernel and yet fail to be WM-by-compact. Nevertheless, for such groups we still have a one-sided version of Theorem 1.19.

Theorem 1.21. *Let G be a non-compact, locally compact, second countable, amenable group such that G/G_0 is compact. Let S be a substantial subset of G . Then S contains a (left or right) shift of an increasing finite products set.*

Question 1.22. Does a two-sided version of Theorem 1.19 hold?

Thus we have the following one-sided trichotomy for locally compact, second countable, amenable groups that are not compact.

Theorem 1.23. *Let G be a locally compact, second countable, amenable group that is not compact. Then exactly one of the following holds.*

1. G is WM. Then every substantial subset of G contains an increasing finite products set.
2. G is not WM, but G/G_0 is compact. Then every substantial subset of G contains a left shift of an increasing finite products set and a right shift of an increasing finite products set. However, there are substantial sets that do not contain a set of measurable recurrence.
3. G/G_0 is not compact. Then there are substantial subsets with density arbitrarily close to 1 no shift of which contains a set of measurable recurrence.

The rest of the paper runs as follows. In the next section we recall some necessary facts about locally compact groups, Følner conditions and almost periodic functions, and what we need about topological dynamics, sets of measurable recurrence, and certain classes of large subsets of groups. In Section 3 we discuss a two-sided Furstenberg correspondence principle that relates ergodic theory to combinatorics. Section 4 starts with the necessary background on magic extensions and the multiple recurrence results these imply. After using these results to prove Theorem 1.15, we give an example to show that the result is not always true for left substantial sets. We conclude the section with a proof of Theorem 1.19. In Section 5 we prove Theorem 1.11 using a version of the monotone convergence theorem for density and describe some combinatorial properties of the sets the theorem produces. Lastly, in Section 6, we prove Theorem 1.9 and use this result to exhibit a topological dynamical system (X, G) in which the only minimal closed, invariant subset of X is a singleton, but having an invariant measure that is non-atomic.

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2. Preliminaries

Throughout this paper, we write “locally compact group” as a shorthand for “locally compact, Hausdorff topological group”.

2.1. Locally compact groups

We will prove most of our results for locally compact, second countable, amenable groups. Working at this level of generality exposes more completely the differences between left multiplication and right multiplication.

Throughout this section G denotes a locally compact, second countable group and m is a fixed left Haar measure on G . Given g in G and a function $f : G \rightarrow \mathbb{R}$ we denote by $l_g f$ and $r_g f$ the functions $G \rightarrow \mathbb{R}$ defined by $(l_g f)(x) = f(g^{-1}x)$ and $(r_g f)(x) = f(xg)$ respectively. With the exception of these l and r being lowercase, we follow the notational conventions in [15]. For instance, $\|\cdot\|_\infty$ denotes the supremum norm on bounded functions, and given a function f on G , we define \check{f} by $\check{f}(g) = f(g^{-1})$ for all g in G .

The *modular function* of G is the unique continuous homomorphism $\Delta : G \rightarrow (0, \infty)$ such that $\Delta(g)\|r_g f\|_1 = \|f\|_1$ for every $g \in G$ and $f \in L^1(G, m)$. Recall that G is said to be *unimodular* if $\Delta = 1$. We record the following for completeness.

Lemma 2.1. *Any locally compact, WM group G is unimodular.*

Proof. Composing the modular function Δ of G with any character of $(0, \infty)$ gives a representation of G on \mathbb{C} . Since G is WM this representation is trivial, so the modular function takes values in the kernel of the character. Choosing characters whose kernels intersect trivially shows that $\Delta = 1$. \square

The space $L^1(G, m)$ becomes an involutive Banach algebra upon defining an isometric involution $f \mapsto f^*$ by $f^*(x) = \overline{f(x^{-1})}\Delta(x^{-1})$ and taking the convolution

$$(f \star h)(x) = \int f(y)h(y^{-1}x) \, dm(y) = \int f(xy)h(y^{-1}) \, dm(y)$$

as the multiplication. We have $l_g(f \star h) = (l_g f) \star h$ and $r_g(f \star h) = f \star (r_g h)$ for any $g \in G$ and any functions f, h in $L^1(G, m)$.

Definition 2.2. A function $f : G \rightarrow \mathbb{R}$ is *left uniformly continuous* if, for any $\varepsilon > 0$, there is a neighborhood V of id_G in G such that $\|l_v f - f\|_\infty < \varepsilon$ for all v in V and *right uniformly continuous* if, for any $\varepsilon > 0$, there is a neighborhood V of id_G in G such that $\|r_v f - f\|_\infty < \varepsilon$ for all v in V . A function is *uniformly continuous* if it is both left and right uniformly continuous.

Note that the space of uniformly continuous functions is closed in the supremum norm. Since l is a continuous action of G on $L^1(G, m)$, the convolution $f \star h$ is left uniformly continuous and the convolution $h \star \check{f}$ is right uniformly continuous whenever $f \in L^1(G, m)$ and $h \in L^\infty(G, m)$.

We conclude the discussion of general locally compact, second countable groups with the following basic fact that we will use often.

Lemma 2.3. *Let G be a locally compact, second countable group. Then there exists an increasing sequence $K_1 \subset K_2 \subset \dots$ of compact subsets of G such that every compact subset of G is contained in some K_N . In particular $G = \cup\{K_N : N \in \mathbb{N}\}$.*

Proof. Let g_1, g_2, \dots be a dense sequence in G and let U be a compact neighborhood of id_G . Let $K_N = Ug_1 \cup \dots \cup Ug_N$ for each N in \mathbb{N} . Let V be the interior of U . For any compact subset K of G we can find N in \mathbb{N} such that $Vg_1 \cup \dots \cup Vg_N \supset K$. Thus $K \subset K_N$. Since G is locally compact the sets K_N cover G . \square

2.2. Følner conditions

We will make use of two Følner conditions: Følner sequences and Reiter sequences.

We remark that it suffices to consider countably many compact sets K when proving uniformity in (1.4). This follows from Lemma 2.3. In fact, it even suffices to verify (1.4) for almost every g , since the local uniformity then follows from Egorov’s theorem and continuity of convolutions.

Every locally compact, second countable, amenable group has a left Følner sequence [27, Theorem 4.16]. Since we fixed a left Haar measure m to begin with, it is not immediately clear that such groups have right Følner sequences.

Lemma 2.4. *Let G be a locally compact, second countable, amenable group with a left Haar measure m . Then G has a right Følner sequence if and only if G is unimodular.*

Proof. When G is unimodular the right Haar measure \tilde{m} defined by $\tilde{m}(E) = m(E^{-1})$ agrees with m on Borel sets. It follows from this that $N \mapsto \Phi_N^{-1}$ is a right Følner sequence whenever Φ is a left Følner sequence. On the other hand, if G is not unimodular then Δ is unbounded so there is some g in G with $\Delta(g) \geq 3$. Thus $m(Eg\Delta E) \geq 2m(E)$ for all Borel sets E , precluding the existence of a right Følner sequence. \square

Thus it is impossible to find a two-sided Følner sequence in a general locally compact, second countable, amenable group. However, by relaxing the requirement that Φ be a sequence of sets we can overcome this problem.

Definition 2.5. A sequence Φ of non-negative functions in $L^1(G, m)$ each having integral 1 is called a left Reiter sequence if

$$\|l_g \Phi_N - \Phi_N\|_1 \rightarrow 0 \tag{2.6}$$

for every $g \in G$, a right Reiter sequence if

$$\|\Delta(g)r_g \Phi_N - \Phi_N\|_1 \rightarrow 0 \tag{2.7}$$

for every $g \in G$, and a two-sided Reiter sequence if it is both left and right Reiter.

Note that if Φ is a left Følner sequence in a locally compact, second countable group G then

$$N \mapsto \frac{1}{m(\Phi_N)} \cdot \mathbf{1}_{\Phi_N} \tag{2.8}$$

is a left Reiter sequence. Thus in particular every locally compact, second countable, amenable group G has a left Reiter sequence. Moreover, given a left Følner sequence Φ the sequence

$$N \mapsto \frac{1}{m(\Phi_N)^2} \cdot \mathbf{1}_{\Phi_N} \star \mathbf{1}_{\Phi_N}^*$$

is a two-sided Reiter sequence, so every locally compact, second countable, amenable group has such a sequence. Note also that, when G is unimodular, every left/right/two-sided Følner sequence Φ induces, via (2.8), a left/right/two-sided Reiter sequence.

Reiter sequences give rise to a notion of density that agrees with the usual notion of density when the sequence happens to arise from a Følner sequence as in (2.8).

Definition 2.9. Let Φ be a left/right/two-sided Reiter sequence in $L^1(G, m)$. Given a Borel subset E of G , denote by

$$\bar{d}_\Phi(E) = \limsup_{N \rightarrow \infty} \int \mathbf{1}_E \cdot \Phi_N \, dm$$

the *upper density* of E with respect to Φ . The *lower density* of E with respect to Φ , which is denoted $\underline{d}_\Phi(E)$, is defined as the corresponding liminf, and the *density* of E with respect to Φ is the value $d_\Phi(E)$ of the limit, if it exists.

In unimodular groups, the standard slicing argument allows one to construct a two-sided Følner sequence from a given two-sided Reiter sequence. We will make use of a relativized version of this construction, in which the resulting Følner sequence assigns to a given subset a lower density not undercutting the upper density with respect to the given two-sided Reiter sequence.

Proposition 2.10. *Let G be a locally compact, second countable, amenable group that is unimodular, and let Φ be a two-sided Reiter sequence in $L^1(G, m)$. Then for every Borel subset $E \subset G$ there exists a two-sided Følner sequence Ψ such that $\underline{d}_\Psi(E) \geq \bar{d}_\Phi(E)$.*

This result implies in particular that every locally compact, second countable unimodular amenable group admits a two-sided Følner sequence. For a shorter proof of this fact see [26, I.§1, Proposition 2].

Proof of Proposition 2.10. Let $r = \bar{d}_\Phi(E)$. Passing to a subsequence we may assume that $d_\Phi(E) = r$.

Claim. *Let $K \subset G$ be a compact set and $\varepsilon > 0$. Then there exists a subset $K' \subset K$ with $m(K \setminus K') < \varepsilon$ and a nonnull compact set F in G such that $m(F \Delta xF)/m(F) < \varepsilon$ and $m(Fx \Delta F)/m(F) < \varepsilon$ for every $x \in K'$ and $m(E \cap F)/m(F) > r - \varepsilon$.*

We have $\Phi_N = \int_0^\infty \mathbf{1}_{A_{N,h}} \, dh = \int_0^\infty \chi_{N,h} \, d\lambda_N(h)$, where $A_{N,h} = \{\Phi_N > h\}$ are the superlevel sets, $\chi_{N,h} = m(A_{N,h})^{-1} \mathbf{1}_{A_{N,h}}$ are the normalized characteristic functions, and λ_N is the probability measure on $(0, \infty)$ with density $h \mapsto m(A_{N,h})$ with respect to the Lebesgue measure. We have

$$r - \varepsilon < \int \Phi_N \mathbf{1}_E \, dm = \int_0^\infty \int_E \chi_{N,h} \, dm \, d\lambda_N(h)$$

for sufficiently large N . It follows that the set $H = \{h : \int_E \chi_{N,h} \, dm > r - 2\varepsilon\}$ satisfies $\lambda_N(H) \geq \frac{\varepsilon}{1-(r-2\varepsilon)}$.

Consider

$$|l_x \Phi_N - \Phi_N| = \left| \int_0^\infty \mathbf{1}_{xA_{N,h}} - \mathbf{1}_{A_{N,h}} \, dh \right|.$$

The central observation in the slicing argument is that the sets $A_{N,h}$ are nested, so the integrand on the right-hand side cannot take both strictly positive and strictly negative values at any given point. Therefore the right-hand side equals

$$\int_0^\infty |\mathbf{1}_{xA_{N,h}} - \mathbf{1}_{A_{N,h}}| \, dh.$$

It follows that

$$\|l_x \Phi_N - \Phi_N\|_1 = \int_0^\infty m(xA_{N,h} \Delta A_{N,h}) \, dh = \int_0^\infty \frac{m(xA_{N,h} \Delta A_{N,h})}{m(A_{N,h})} \, d\lambda_N(h).$$

If N is sufficiently large, then by (2.6) and the dominated convergence theorem

$$\begin{aligned} \frac{\varepsilon}{1-(r-2\varepsilon)} \frac{\varepsilon^2}{2} &> \int_K \|l_x \Phi_N - \Phi_N\|_1 \, dm(x) \\ &= \int_0^\infty \int_K \frac{m(xA_{N,h} \Delta A_{N,h})}{m(A_{N,h})} \, dm(x) \, d\lambda_N(h). \end{aligned}$$

Analogously, by (2.7), the dominated convergence theorem and, crucially, the fact that G is unimodular, for sufficiently large N we have

$$\frac{\varepsilon}{1-(r-2\varepsilon)} \frac{\varepsilon^2}{2} > \int_0^\infty \int_K \frac{m(A_{N,h}x \Delta A_{N,h})}{m(A_{N,h})} \, dm(x) \, d\lambda_N(h).$$

It follows that

$$\int_K \frac{m(xA_{N,h} \Delta A_{N,h})}{m(A_{N,h})} + \frac{m(A_{N,h}x \Delta A_{N,h})}{m(A_{N,h})} \, dm(x) < \varepsilon^2$$

for some $h \in H$. Therefore $\frac{m(xA_{N,h}\Delta A_{N,h})}{m(A_{N,h})} + \frac{m(A_{N,h}x\Delta A_{N,h})}{m(A_{N,h})} < \varepsilon$ for x in a subset $K' \subset K$ with $m(K \setminus K') < \varepsilon$, proving the claim.

Claim. Let $K \subset G$ be a compact set and $\varepsilon > 0$. Then there exists a nonnull compact set F in G such that $m(xF\Delta F)/m(F) < \varepsilon$ and $m(Fx\Delta F)/m(F) < \varepsilon$ for every $x \in K$ and $m(E \cap F)/m(F) > r - \varepsilon$.

We may assume that $m(K) > \varepsilon$. We apply the previous claim to the set $\tilde{K} := K \cup K^2$ with $\varepsilon/2$ in place of ε . We obtain a subset $K' \subset \tilde{K}$ with $m(K \setminus K') < \varepsilon/2$ and a compact set F in G such that $m(xF\Delta F)/m(F) < \varepsilon/2$ and $m(Fx\Delta F)/m(F) < \varepsilon/2$ for every $x \in K'$ and $m(E \cap F)/m(F) > r - \varepsilon$.

Let now $k \in K$. Then we have $k\tilde{K} \cap \tilde{K} \supseteq kK$, so that $m(k\tilde{K} \cap \tilde{K}) > \varepsilon$. It follows that $kK' \cap K' \neq \emptyset$, so that $k = k_1k_2^{-1}$ for some $k_1, k_2 \in K'$. Therefore

$$\begin{aligned} m(kF\Delta F) &= m(k_2^{-1}F\Delta k_1^{-1}F) \leq m(k_2^{-1}F\Delta F) + m(F\Delta k_1^{-1}F) \\ &\leq m(F\Delta k_2F) + m(k_1F\Delta F) < \varepsilon \end{aligned}$$

and, since G is unimodular,

$$m(Fk\Delta F) = m(Fk_1\Delta Fk_2) \leq m(Fk_1\Delta F) + m(F\Delta Fk_2) < \varepsilon.$$

This proves the claim.

The conclusion of the theorem follows quickly using the observation that it suffices to verify (1.4) for the increasing sequence of compact sets $K_1 \subset K_2 \subset \dots$ given by Lemma 2.3: let the sets Ψ_N be given by the last claim with the compact sets K_N and $\varepsilon = 2^{-N}$. \square

Definition 2.11. Let G be a locally compact, second countable, amenable group. Define by

$$d^*(E) = \sup \{ \bar{d}_\Phi(E) : \Phi \text{ a two-sided Reiter sequence} \}$$

the two-sided upper Banach density of a Borel subset E of G .

Note that, by Proposition 2.10, when G is unimodular the definition of two-sided upper Banach density is unchanged if one considers the supremum over only the two-sided Følner sequences.

2.3. Almost periodic functions

We will now recall the notion of an almost-periodic function on a locally compact, second countable group G and its relationship with the finite-dimensional representations of G . Denote by $C_b(G)$ the Banach space of all bounded continuous functions

$f : G \rightarrow \mathbb{C}$ equipped with the supremum norm. The G -actions l and r on $C_b(G)$ are isometric. A function $f \in C_b(G)$ is called *almost periodic* if one of the following equivalent conditions holds.

1. The subset $\{l_g f : g \in G\}$ of $C_b(G)$ is relatively compact,
2. the subset $\{r_g f : g \in G\}$ of $C_b(G)$ is relatively compact, or
3. f is the pullback of a continuous function under the maximal topological group compactification $\iota : G \rightarrow \tilde{G}$ (also called the *almost periodic* or *Bohr* compactification).

For the equivalence (1) \iff (2) see [7, Theorem 9.2] and for the equivalence (2) \iff (3) see [7, Remark 9.8]. Denote by $AP(G)$ the space of all almost periodic functions on G . It follows from the characterization (3) that almost periodic functions are uniformly continuous.

The *matrix coefficients* of a finite-dimensional continuous representation give rise to almost-periodic functions on G . Specifically, given a continuous representation ϕ of G on a finite-dimensional, complex Hilbert space V and vectors x, y in V we can form the almost-periodic function $f(g) = \langle \phi(g)x, y \rangle$.

Theorem 2.12. (See [34, Theorems 30 and 31].) *Matrix coefficients span a dense subspace of $AP(G)$.*

The constant functions are always almost-periodic. There are groups having no other almost-periodic functions (see [35] and Examples 2.13 and 2.14). Such groups are said to be *minimally almost-periodic*. Theorem 2.12 implies that a group is minimally almost-periodic if and only if it is a WM group. In view of the Peter–Weyl theorem, non-trivial compact groups are never WM groups.

Example 2.13. Recall that a group is called *periodic* if each of its elements has finite order. Let G be a countably infinite periodic group that is the union of an increasing sequence of simple subgroups G_n (a constant sequence $G_n = G$ is allowed). Then G is WM. For contradiction assume that G admits a non-trivial finite-dimensional unitary representation π . Then π is faithful on every simple subgroup on which it is non-trivial, so it is faithful on G . The Jordan–Schur theorem [11, Theorem 36.14] now implies that G has a finite index normal abelian subgroup H . In particular, $H \cap G_n \subset G_n$ is a normal abelian subgroup for each n , and it is non-trivial for sufficiently large n , a contradiction.

This applies, for instance, to the finite alternating group of the integers $A(\mathbb{N})$, which is the subgroup of the finite symmetric group of the integers

$$S(\mathbb{N}) = \{\sigma : \mathbb{N} \rightarrow \mathbb{N} : \sigma \text{ is a bijection and } \{n \in \mathbb{N} : \sigma(n) \neq n\} \text{ is finite}\}$$

consisting of the even permutations. Indeed, $A(\mathbb{N})$ is the union $\cup A_n$ where A_n is the alternating group on n points. Now $[S(\mathbb{N}) : A(\mathbb{N})] = 2$, so $S(\mathbb{N})$ is a virtually WM group.

On the other hand, $\phi(\sigma) = (-1)^{\text{sgn}(\sigma)}$ defines a non-trivial representation of $S(\mathbb{N})$, so $S(\mathbb{N})$ is not a WM group.

This also applies to the projective linear group $\text{PSL}_n(F)$ over an infinite algebraic field F of finite characteristic for any $n \geq 2$. Indeed, F can be written as an increasing union of finite subfields $F^{(k)}$, and $\text{PSL}_n(F)$ is the increasing union of copies of $\text{PSL}_n(F^{(k)})$. It is a classic result that $\text{PSL}_n(F^{(k)})$ is simple unless $n = 2$ and $|F^{(k)}| = 2, 3$.

In these examples the groups G are amenable since they are locally finite.

Example 2.14. Recall that a group is called *residually finite* if its points are separated by the homomorphisms into finite groups. By [25, Theorem 7] any finitely generated group that admits a faithful representation over a field of characteristic zero is residually finite. It follows that any countably infinite, finitely generated, simple group G is WM. Indeed, suppose that G is not WM. Since G is simple, this implies that G has a faithful finite-dimensional representation. By the result cited above this forces G to be residually finite, which contradicts simplicity.

It has been recently shown that there exist (many) countably infinite, finitely generated, simple, amenable groups, see [24, Corollary B].

It follows from Theorem 2.12 that the subset

$$\{g \in G : f(g) = f(\text{id}_G) \text{ for all } f \in \text{AP}(G)\}$$

of G , sometimes called the *von Neumann kernel* of G , is precisely G_0 . In fact, G_0 is the kernel of the almost periodic compactification ι of G . Note, however, that G_0 need not be a WM group.

Example 2.15. Let

$$G = \left\{ \begin{pmatrix} u & v \\ 0 & 1 \end{pmatrix} : u, v \in \mathbb{Q}, u \neq 0 \right\}$$

be the affine group of \mathbb{Q} . It is shown in [35] that

$$G_0 = \left\{ \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix} : v \in \mathbb{Q} \right\}.$$

Clearly G_0 , which is isomorphic to $(\mathbb{Q}, +)$, is not WM.

The locally compact, second countable groups G can be classified according to the properties of their almost periodic compactification $\iota : G \rightarrow \bar{G}$ as follows.

1. The almost periodic compactification $\iota : G \rightarrow \bar{G}$ is not a surjective map. Equivalently, G/G_0 is not compact. Groups with this property are the subject of Theorem 1.11.

In this case G is not WM-by-compact, since for any WM subgroup $H \leq G$ we have $H \leq G_0$, and if G/H is compact, then G/G_0 would also be compact.

2. The almost periodic compactification $\iota : G \rightarrow \bar{G}$ is surjective. In this case our knowledge depends on the cardinality of \bar{G} .
 - (a) \bar{G} consists of one point, or equivalently $G = G_0$, or G is WM. In this case we have [Theorem 1.15](#).
 - (b) \bar{G} consists of more than one point but is finite, or equivalently $1 < [G : G_0] < \infty$. In this case G_0 is WM, since any non-trivial finite-dimensional representation of G_0 would induce a finite-dimensional representation of G that does not vanish identically on G_0 . [Corollary 1.19](#) holds.
 - (c) \bar{G} is infinite. In this case G_0 need not be WM and, more generally, G need not be WM-by-compact (see [Example 2.16](#)). [Theorem 1.21](#) holds. Note that this case cannot occur for discrete G .

Example 2.16 (*K. Schmidt*). The von Neumann kernel of a connected, locally compact group admits an explicit description, see [\[33, Theorem 3\]](#). A special case of this description shows that the von Neumann kernel of a solvable, connected, locally compact group is the closed commutator subgroup (this can be seen as a version of Lie–Kolchin theorem, cf. [\[32, Theorem 3\]](#)). Thus the von Neumann kernel G_0 of the semidirect product $G = \text{SO}(2) \ltimes \mathbb{R}^2$ (taken with respect to the defining action of $\text{SO}(2)$ on \mathbb{R}^2) equals \mathbb{R}^2 . This shows that G_0 need not be WM when G/G_0 is compact in the case that G is not discrete.

In order to prove [Theorem 1.11](#) we will need a supply of almost periodic functions that vary sufficiently slowly.

Lemma 2.17. *Let G be a locally compact, second countable, amenable group such that the almost periodic compactification $\iota : G \rightarrow \bar{G}$ is not surjective. Then for every $\varepsilon > 0$ and every compact set K in G there exists an almost periodic function $f : G \rightarrow [0, 1]$ such that $\|l_g f - f\|_\infty < \varepsilon$ for every $g \in K$ and the range of f is ε -dense in $[0, 1]$.*

Proof. Since G is amenable, it has a left Følner sequence, so there exists a nonnull compact set $F \subset G$ such that $m(F\Delta gF)/m(F) < \varepsilon$ for every $g \in K$. In particular, $\iota(F^{-1}) \subset \iota(G)$ is a compact subset. On the other hand, by the assumption that ι is not surjective, $\iota(G)$ is a proper dense subgroup of the compact group \bar{G} , so it is not compact.

Therefore there exist $g_i \in G, i = 0, \dots, [1/\varepsilon]$, such that the sets $\iota(F^{-1}g_i)$ are pairwise disjoint. Let $\tilde{f} : \bar{G} \rightarrow [0, 1]$ be a continuous function that equals $i\varepsilon$ on $\iota(F^{-1}g_i)$. (Such a function can be constructed using the Urysohn lemma.)

Consider now the continuous function $f = m(F)^{-1} \int_F l_g \tilde{f} \, dm(g)$ on \bar{G} . Then $f(g_i) = i\varepsilon$, so that $f(G)$ is ε -dense in $[0, 1]$. Moreover, it follows from the Følner condition that $\|l_g f - f\|_\infty < \varepsilon$ for every $g \in K$. \square

2.4. Topological dynamics

We now recall some facts from topological dynamics. A *topological dynamical system* (X, G) is a compact metric space (X, d) together with a jointly continuous left action $(g, x) \mapsto gx$ of G on (X, d) . We say that (X, G) is *topologically transitive* if there is some $x \in X$ with dense orbit and *minimal* if every point in X has dense orbit. A system (X, G) is *equicontinuous* if the collection of homeomorphisms $x \mapsto gx, g \in G$, is equicontinuous. A Borel measure μ on X is *invariant* with respect to a left action of G if, for all Borel sets $A \subset X$ and all $g \in G$, one has $\mu(gA) = \mu(A)$. A version of the Bogolioubov–Krylov theorem (see [36, Corollary 6.9.1]) for amenable groups guarantees that if G is amenable, then any topological dynamical system (X, G) admits an invariant Borel probability measure. The system (X, G) is said to be *uniquely ergodic* if there is only one invariant Borel probability measure on X . The *support* of a Borel probability measure on a compact metric space (X, d) is the intersection of all closed sets with full measure. Since X is second countable, the support has full measure. Given a topological dynamical system (X, G) , a point $x \in X$, and a subset A of X , write $R_A(x) = \{g \in G : gx \in A\}$. Sets of the form $R_A(x)$ are called *return time sets*.

Theorem 2.18. (See [31, Theorem 7].) *Let (X, G) be a topological dynamical system that is topologically transitive and equicontinuous. Then (X, G) is minimal and uniquely ergodic. If X is infinite and μ is the unique G -invariant probability measure, then μ is non-atomic.*

Lemma 2.19. *Let (X, G) be a topological dynamical system. If (X, G) is uniquely ergodic with invariant Borel probability measure μ , A is a Borel set with $\mu(\partial A) = 0$, x is a point in X , and Φ is any left Reiter sequence, then $d_\Phi(R_A(x)) = \mu(A)$.*

Proof. Since (X, G) is uniquely ergodic, for every f in $C(X)$ and every $x \in X$ we have

$$\lim_{N \rightarrow \infty} \int \Phi_N(g) f(gx) \, d\mu(g) = \int f \, d\mu.$$

Fix $\varepsilon > 0$. Since $\mu(\partial A) = 0$ we can find f_1, f_2 in $C(X)$ so that $f_1 \leq \mathbf{1}_A \leq f_2$ and $\int f_2 - f_1 \, d\mu < \varepsilon$. Thus

$$\begin{aligned} \mu(A) - \varepsilon &\leq \int f_1 \, d\mu = \lim_{N \rightarrow \infty} \int \Phi_N(g) f_1(gx) \, d\mu(g) \\ &\leq \liminf_{N \rightarrow \infty} \int \Phi_N(g) \mathbf{1}_A(gx) \, d\mu(g) \leq \limsup_{N \rightarrow \infty} \int \Phi_N(g) \mathbf{1}_A(gx) \, d\mu(g) \\ &\leq \lim_{N \rightarrow \infty} \int \Phi_N(g) f_2(gx) \, d\mu(g) = \int f_2 \, d\mu \leq \mu(A) + \varepsilon, \end{aligned}$$

giving $d_\Phi(R_A(x)) = \mu(A)$ as desired. \square

Lemma 2.20. *Let (X, d) be a compact metric space with a non-atomic probability measure μ . For any $\varepsilon > 0$ and any point x in the support of μ there is an open set A containing x such that $\mu(A) < \varepsilon$ and $\mu(\partial A) = 0$.*

Proof. Let x be a point in the support of μ . For every $t > 0$ the open ball B_t centered at x with radius t has positive measure. Their boundaries ∂B_t are disjoint, so only countably many of the sets ∂B_t have positive measure. Let t_n be a sequence decreasing to 0 such that $\mu(\partial B_{t_n}) = 0$ for all n . We have $\mu(B_{t_n}) \rightarrow \mu(\{x\}) = 0$ because μ is non-atomic. Put $A = B_{t_n}$ with n so large that $\mu(B_{t_n}) < \varepsilon$. \square

2.5. Sets of measurable recurrence

In this subsection we discuss sets of measurable recurrence in locally compact, second countable groups. Note that the joint measurability condition in Definition 1.7 implies (see [20, 22.20(b)], for example) that the induced action of G on $L^2(X, \mathcal{B}, \mu)$ is strongly continuous. We begin by pointing out that, when G is countable and infinite, Definition 1.8 coincides with the usual definition.

Proposition 2.21. *Let G be a countable, infinite group. Then a subset R of G is a set of measurable recurrence if and only if, for every measurable action of G on a separable probability space (X, \mathcal{B}, μ) and every B in \mathcal{B} with positive measure, we can find r in $R \setminus \{\text{id}_G\}$ such that $\mu(B \cap T^r B) > 0$.*

Proof. It is clear that every set of measurable recurrence has the stated property. Conversely, suppose that G is a countable group and that R is a subset of G with the stated property. Let K be a compact subset of G , let (X, \mathcal{B}, μ) be a separable probability space equipped with a measure-preserving G action, and let A be a nonnull measurable subset of X . Consider the space $Y = \{0, 1\}^G$ with the $(\frac{1}{2}, \frac{1}{2})$ -Bernoulli measure and the shift action of G . Let $F = K \setminus \{\text{id}_G\}$, which is a finite set, and put

$$B = \{y \in Y : y(\text{id}_G) = 1, y(f) = 0 \text{ for all } f \in F\}.$$

Then $A \times B$ is a nonnull measurable subset of $X \times Y$, and by the hypothesis there exists a $g \in R \setminus \{\text{id}_G\}$ such that $(A \times B) \cap (gA \times gB)$ is nonnull. By construction of B it follows that $g \in R \setminus K$ and $A \cap gA$ is nonnull. Hence R is a set of measurable recurrence. \square

We remark that the measure of the set B in the above proof can be improved considerably. Indeed, above B has measure 2^{-n-1} , where $n = |F|$, but if one uses instead the $(\frac{n}{n+1}, \frac{1}{n+1})$ -Bernoulli measure then the measure of B will become $\frac{1}{n+1}(\frac{n}{n+1})^n > e^{-1}/(n + 1)$.

Example 2.22. Let G be a locally compact, second countable group. Let P be a subset of G such that for every neighborhood U of id_G the set UP contains a set of the form $\text{FPI}(g_n)$ with $g_n \rightarrow \infty$. Then P is a set of measurable recurrence.

Proof. Fix a compact set K in G and a measure-preserving action of G on a separable probability space (X, \mathcal{B}, μ) . Let A be a non-null, positive-measure subset of X . Since the induced action on $L^2(X, \mathcal{B}, \mu)$ is strongly continuous, there is a relatively compact, symmetric neighborhood U of the identity such that $\mu(A \cap gA) > \mu(A) - \mu(A)^2/2$ for any $g \in U$.

Let $n \mapsto g_n$ be a sequence such that $g_n \rightarrow \infty$ and $\text{FPI}(g_n) \subset UP$. Let $n \mapsto h_n$ be a subsequence of $n \mapsto g_n$ such that $\text{FPI}(h_n) \subset G \setminus \overline{U}K$. Such a sequence can be constructed by choosing h_{n+1} from

$$\{g_i : i \in \mathbb{N}\} \setminus \cup \{I_\alpha(h_i)^{-1}\overline{U}K : \emptyset \neq \alpha \subset \{1, \dots, n\}\}$$

inductively. By a standard argument (originally due to Gillis [19]) we have

$$\mu(h_1 \cdots h_m A \cap h_1 \cdots h_n A) > \mu(A)^2/2$$

for some $m > n$. Then $h_{n+1} \cdots h_m \in \text{FPI}(g_n) \setminus \overline{U}K$ and $\mu(A \cap h_{n+1} \cdots h_m A) > \mu(A)^2/2$. Let $g \in U$ be such that $gh_{n+1} \cdots h_m \in P$. Then $\mu(A \cap gh_{n+1} \cdots h_m A) > 0$. \square

In any given group G we denote the class of sets of measurable recurrence by \mathcal{R} .

Lemma 2.23. *The class \mathcal{R} is partition regular.*

Proof. Suppose $R_1 \cup \cdots \cup R_r \in \mathcal{R}$. We need to show that $R_i \in \mathcal{R}$ for some $i \in \{1, \dots, r\}$. Assume $R_i \notin \mathcal{R}$ for all $i = 1, \dots, r$. Thus for each $1 \leq i \leq r$ there is a compact set K_i in G , a measure-preserving action T_i of G on a separable probability space, and a positive measure set B_i in the probability space witnessing the fact that R_i is not a set of measurable recurrence. The positive-measure set $B_1 \times \cdots \times B_r$ in the product probability space equipped with the product action $T_1 \times \cdots \times T_r$, and the compact set $K_1 \cup \cdots \cup K_r$ now witness the fact that $R_1 \cup \cdots \cup R_r$ is not a set of measurable recurrence, which is a contradiction. \square

Given a class of subsets it is natural (see [3] for an extensive discussion) to consider its *dual class*, which consists of all subsets whose intersection with every member of the given class is non-empty. Denote by \mathcal{R}^* the dual class of \mathcal{R} . It is clear that if $A \in \mathcal{R}^*$, then every superset of A is also a member \mathcal{R}^* . It follows from partition regularity of \mathcal{R} that the intersection of any two members of \mathcal{R}^* is again a member of \mathcal{R}^* . Since every cocompact subset of a member of \mathcal{R} is clearly a member of \mathcal{R} , this has the following consequence.

Corollary 2.24. Any cocompact subset of an \mathcal{R}^* set is \mathcal{R}^* .

We will need later the fact that return time sets in certain topological dynamical systems belong to \mathcal{R}^* .

Lemma 2.25. Let (X, G) be a minimal topological dynamical system where G acts by isometries. For any x in X and any neighborhood U of x the set $R_U(x)$ is in \mathcal{R}^* .

Proof. Let $\delta > 0$ be such that the ball $B_\delta(x)$ is contained in U . Let μ be an invariant Borel probability measure on (X, G) . By minimality $V := B_{\delta/2}(x)$ has positive measure. Since G acts by isometries, the \mathcal{R}^* set $\{g : V \cap gV \neq \emptyset\}$ is contained in $R_U(x)$. \square

Finally, we note that the classes \mathcal{R} and \mathcal{R}^* are closed under conjugation.

2.6. *Syndetic, thick, and piecewise-syndetic sets*

Given a topological group G , denote by \mathcal{S} , \mathcal{T} , and \mathcal{P} , the classes of syndetic, thick, and piecewise syndetic subsets, respectively.

Lemma 2.26. $\mathcal{S}^* = \mathcal{T}$ and $\mathcal{T}^* = \mathcal{S}$.

Proof. First, let S be syndetic in G and let T be thick in G . By syndeticity there exists a compact set K such that $KS = G$. By thickness there exists $g \in G$ such that $K^{-1}g \subset T$. Write $g = ks$ with $k \in K$, $s \in S$. Thus the intersection of any syndetic set with any thick set is non-empty. This implies $\mathcal{T} \subset \mathcal{S}^*$ and $\mathcal{S} \subset \mathcal{T}^*$.

We now prove that if $G = P \cup Q$ then either P is thick or Q is syndetic. If P is not thick then there exists a compact set K such that for each $g \in G$ we have $Kg \cap Q \neq \emptyset$. Thus for any $g \in G$ we can find $k \in K$ and $q \in Q$ such that $kg = q$. This implies that $K^{-1}Q = G$ as desired.

Now, if P does not belong to \mathcal{T} then its complement is syndetic so P does not belong to \mathcal{S}^* . Similarly, if Q does not belong to \mathcal{S} then its complement is thick so Q does not belong to \mathcal{T}^* . Thus $\mathcal{T} \supset \mathcal{S}^*$ and $\mathcal{S} \supset \mathcal{T}^*$. \square

Lemma 2.27. Let G be a group that is not compact. Then for any thick subset T of G and any compact subset K of G there is $g \in G \setminus K$ such that $Kg \subset T$.

Proof. Let T and K be thick and compact subsets of G respectively. Let $h \in G \setminus KK^{-1}$. By thickness we have $(K \cup Kh)g \subset T$ for some $g \in G$. Suppose that $g \in K$ and $hg \in K$. Then $h = (hg)g^{-1} \in KK^{-1}$, a contradiction. Hence $g \notin K$ or $hg \notin K$, as required. \square

3. The Furstenberg correspondence principle

Since the configurations we are interested in (see Definition 1.12) are two-sided in nature, we need a correspondence principle that is sensitive to multiplication on the left and

on the right. Roughly speaking, a correspondence principle should relate recurrence for sets of positive upper density in our group to recurrence in a certain measure-preserving action of the group on a probability space. It turns out that this is not possible for arbitrary Borel sets of positive upper density even in \mathbb{R} . Indeed, setting $\Phi_N = [0, N]$ in \mathbb{R} , [5, Theorem D] exhibits for all but countably many $\alpha > 0$ a subset E of \mathbb{R} with $d_\Phi(E) = 1/2$ such that $d_\Phi(E - n^\alpha \cap E) = 0$ for all n in \mathbb{N} . On the other hand, whenever $\alpha > 0$ is irrational, the set $\{n^\alpha : n \in \mathbb{N}\}$ is a set of measurable recurrence because (as first proved in [10]) the sequence $n^\alpha t$ is uniformly distributed mod 1 for any real, non-zero t .

An appropriate one-sided correspondence principle for general locally compact, second countable groups was given in [4, Theorem 1.1]. We now describe a two-sided version adequate for our needs. It is for substantial subsets, which we now define in general, that we can prove a correspondence principle.

Definition 3.1. We say that a subset S of G is *substantial* if one can find a measurable subset W of G with $\bar{d}_\Phi(W) > 0$ for some two-sided Reiter sequence Φ in $L^1(G, m)$ and a symmetric, open subset U of G containing id_G such that $S \supset UWU$.

In unimodular groups the notion of being substantial does not change if one demands $\bar{d}_\Phi(W) > 0$ for some two-sided Følner sequence Φ in the definition; this follows from Proposition 2.10. Thus Definition 3.1 agrees with Definition 1.14 when both apply.

Theorem 3.2 (Correspondence principle). *Let G be a locally compact, second countable, amenable group with a left Haar measure m and a two-sided Reiter sequence Φ . Let W be a Borel subset of G , let U be an open neighborhood of id_G , and let $S \supseteq UWU$ be measurable. Then there is a compact metric space X with a continuous $G \times G$ -action $L \times R$ and a non-negative, continuous function ξ on X such that the following hold.*

1. *There exists an $L \times R$ -invariant Borel probability measure μ on X such that $\int \xi \, d\mu = \bar{d}_\Phi(W)$ and*

$$\bar{d}_\Phi(g_1^{-1}Sh_1 \cap \dots \cap g_n^{-1}Sh_n) \geq \int L^{g_1}R^{h_1}\xi \dots L^{g_n}R^{h_n}\xi \, d\mu$$

for any $g_1, \dots, g_n, h_1, \dots, h_n$ in G .

2. *There exist an ergodic $L \times R$ -invariant Borel probability measure ν on X and a two-sided Reiter sequence Ψ such that $\int \xi \, d\nu \geq \bar{d}_\Phi(W)$ and*

$$\bar{d}_\Psi(g_1^{-1}Sh_1 \cap \dots \cap g_n^{-1}Sh_n) \geq \int L^{g_1}R^{h_1}\xi \dots L^{g_n}R^{h_n}\xi \, d\nu \tag{3.3}$$

for any $g_1, \dots, g_n, h_1, \dots, h_n$ in G .

If G is discrete, we can take ξ to be a characteristic function of a clopen subset of X .

In the second part of [Theorem 3.2](#) we obtain an ergodic measure-preserving system at the cost of modifying the Reiter sequence. Thus if one is only interested in the two-sided upper Banach density, it suffices to consider ergodic measure-preserving systems, see [Corollary 3.4](#). This was already observed for the group \mathbb{Z} in [\[6, Proposition 3.1\]](#).

The probability space that our correspondence yields will be built on the Gelfand spectrum of a C^* -algebra of functions on G . Recall that one can think of the Gelfand spectrum as either the space of maximal ideals, or as the space of all non-trivial multiplicative linear forms. As in [\[4\]](#), the C^* -algebra we use will consist of uniformly continuous functions.

Proof of [Theorem 3.2](#). We may assume without loss of generality that U is symmetric. Let ψ be a continuous non-negative function supported in U such that $\psi = \psi^*$ and $\int \psi \, dm = 1$. The function $\xi := \psi \star \mathbf{1}_W \star \check{\psi}$ is non-negative, uniformly continuous, and dominated by $\mathbf{1}_S$. We now consider the minimal closed $*$ -subalgebra \mathfrak{A} of bounded functions on G that contains $\mathbf{1}_G$, contains ξ , and is invariant under l and r . By the Gelfand–Naimark theorem \mathfrak{A} is canonically isomorphic to $C(X)$, where X is the Gelfand spectrum of \mathfrak{A} , that is, the space of all non-trivial multiplicative linear forms on \mathfrak{A} with the weak* topology.

Since \mathfrak{A} consists of uniformly continuous functions, the $G \times G$ -action $l \times r$ on \mathfrak{A} is jointly continuous. Since the action $l \times r$ is by C^* -algebra automorphisms, it follows that it induces a continuous $G \times G$ action $L \times R$ on X such that $l_g r_g f = f \circ L_{g^{-1}} R_{g^{-1}}$ for every $f \in \mathfrak{A}$. Since \mathfrak{A} is separable, it follows that X is metrizable.

Part (1). Passing to a subsequence of Φ we may assume that $d_\Phi(W)$ exists. Passing to a further subsequence of Φ we may assume that

$$\mu(f) := \lim_{N \rightarrow \infty} \int \Phi_N f \, dm$$

exists for every $f \in \mathfrak{A}$. This is a positive unital linear functional on \mathfrak{A} , so by the Riesz–Markov–Kakutani representation theorem μ corresponds to a Borel probability measure on X . The Reiter property of Φ implies that μ is $L \times R$ -invariant. We have

$$\begin{aligned} \mu(\xi) &= \lim_{N \rightarrow \infty} \int \Phi_N(\psi \star \mathbf{1}_W \star \check{\psi}) \, dm \\ &= \lim_{N \rightarrow \infty} \iiint \Phi_N(x)\psi(y)\mathbf{1}_W(z)\check{\psi}(z^{-1}y^{-1}x) \, dm(x) \, dm(y) \, dm(z) \\ &= \lim_{N \rightarrow \infty} \iint \psi(y)\check{\psi}(x) \int \Phi_N(yzx)\mathbf{1}_W(z) \, dm(z) \, dm(x) \, dm(y) \\ &= \lim_{N \rightarrow \infty} \iint \psi(y)\psi(x)\Delta(x) \int (l_{y^{-1}}r_x\Phi_N)(z)\mathbf{1}_W(z) \, dm(z) \, dm(x) \, dm(y) \\ &= \lim_{N \rightarrow \infty} \int \Phi_N(z)\mathbf{1}_W(z) \, dm(z) = d_\Phi(W) \end{aligned}$$

by the Reiter property of Φ and the dominated convergence theorem.

Lastly we have

$$\begin{aligned} & \overline{d}_\Phi(g_1^{-1}Sh_1 \cap \dots \cap g_n^{-1}Sh_n) \\ & \geq \lim_{N \rightarrow \infty} \int \Phi_N \cdot l_{g_1^{-1}r_{h_1^{-1}}}(\psi \star \mathbf{1}_W \star \check{\psi}) \cdots l_{g_n^{-1}r_{h_n^{-1}}}(\psi \star \mathbf{1}_W \star \check{\psi}) \, d\mathbf{m} \\ & = \int L^{g_1}R^{h_1}\xi \cdots L^{g_n}R^{h_n}\xi \, d\mu \end{aligned}$$

for any $g_1, \dots, g_n, h_1, \dots, h_n$ in G as desired.

Part (2). Let ν be an ergodic component of μ under the action $L \times R$ such that $\int \xi \, d\nu \geq \int \xi \, d\mu$. Let $e : \mathfrak{A} \rightarrow \mathbb{C}$ be the evaluation map at id_G . Then $e \in X$. Moreover, the orbit of e under L (or R) consists of the evaluation morphisms at all points of G . Therefore both $L^G e$ and $R^G e$ separate points in \mathfrak{A} , so that $\overline{L^G e} = \overline{R^G e} = X$ by the Urysohn lemma. A version of [17, Proposition 3.9] now implies that the point e is generic for ν under the action $L \times R$ with respect to some left Følner sequence of the form $(\Theta_N s_N \times \Theta_N)$ on $G \times G$. Since the function $L^{g_1}R^{h_1}\xi \cdots L^{g_n}R^{h_n}\xi$ is continuous, we have

$$\begin{aligned} & \int_X L^{g_1}R^{h_1}\xi \cdots L^{g_n}R^{h_n}\xi \, d\nu \\ & = \lim_{N \rightarrow \infty} \frac{1}{\mathbf{m}(\Theta_N)\mathbf{m}(\Theta_N s_N)} \int_{(l,r) \in \Theta_N s_N \times \Theta_N} (l_{g_1^{-1}r_{h_1^{-1}}}\xi \cdots l_{g_n^{-1}r_{h_n^{-1}}}\xi)(L^l R^r e) \, d\mathbf{m}(l) \, d\mathbf{m}(r) \\ & \leq \limsup_{N \rightarrow \infty} \frac{1}{\mathbf{m}(\Theta_N)^2 \Delta(s_N)} \int_{(l,r) \in \Theta_N s_N \times \Theta_N} (l_{g_1^{-1}r_{h_1^{-1}}}\mathbf{1}_S \cdots l_{g_n^{-1}r_{h_n^{-1}}}\mathbf{1}_S)(lr^{-1}) \, d\mathbf{m}(l) \, d\mathbf{m}(r) \\ & = \limsup_{N \rightarrow \infty} \frac{1}{\mathbf{m}(\Theta_N)^2 \Delta(s_N)} \int_{g_1^{-1}Sh_1 \cap \dots \cap g_n^{-1}Sh_n} \mathbf{1}_{\Theta_N s_N} \star \mathbf{1}_{\Theta_N}^* \, d\mathbf{m}(l), \end{aligned}$$

and we obtain the conclusion with the two-sided Reiter sequence $\Psi_N = \mathbf{m}(\Theta_N)^{-2} \times \Delta(s_N)^{-1} \mathbf{1}_{\Theta_N s_N} \star \mathbf{1}_{\Theta_N}^*$.

Finally, when G is discrete we can take $U = \{\text{id}_G\}$, so $\xi = \mathbf{1}_W = \xi^2$ is an indicator function of some clopen set B . \square

We remark that, in Part 2 of Theorem 3.2, we need to pass from the given two-sided Reiter sequence to one for which e is generic in order to get an ergodic measure-preserving system. This can be masked if one is willing to use two-sided upper Banach density.

Corollary 3.4. *Let G be a locally compact, second countable, amenable group with a left Haar measure \mathbf{m} . Let W be a Borel subset of G , let U be an open neighborhood of id_G , and let $S \supseteq UWU$ be measurable. Then there is a compact metric space X with a continuous*

$G \times G$ -action $L \times R$, an ergodic $L \times R$ -invariant Borel probability measure ν on X , and a non-negative, continuous function ξ on X such that $\int \xi \, d\nu = d^*(W)$ and

$$d^*(g_1^{-1}Sh_1 \cap \dots \cap g_n^{-1}Sh_n) \geq \int L^{g_1} R^{h_1} \xi \dots L^{g_n} R^{h_n} \xi \, d\nu$$

for any $g_1, \dots, g_n, h_1, \dots, h_n$ in G .

If G is discrete, we can take ξ to be a characteristic function of a clopen subset of X .

Proof. Let Φ be a two-sided Reiter sequence such that $d_\Phi(W) = d^*(W)$. Let $(X, \nu, L \times R)$ be the regular probability measure-preserving system and ξ the continuous function given by Theorem 3.2(2). It remains to show that $\int \xi \, d\nu \leq d^*(W)$. By construction we have

$$\begin{aligned} \int_X \xi \, d\nu &= \lim_{N \rightarrow \infty} \frac{1}{m(\Theta_N)m(\Theta_N s_N)} \int_{(l,r) \in \Theta_N s_N \times \Theta_N} \xi(lr^{-1}) \, dm(l) \, dm(r) \\ &= \lim_{N \rightarrow \infty} \frac{1}{m(\Theta_N)m(\Theta_N s_N)} \\ &\quad \times \int_{(l,r) \in \Theta_N s_N \times \Theta_N} \int_{y,z} \psi(y) \mathbf{1}_W(y^{-1}lr^{-1}z) \check{\psi}(z^{-1}) \, dm(y) \, dm(z) \, dm(l) \, dm(r) \\ &= \limsup_{N \rightarrow \infty} \frac{1}{m(\Theta_N)m(\Theta_N s_N)} \\ &\quad \times \int_{y,z} \psi(y)\psi(z) \int_{(l,r) \in \Theta_N s_N \times \Theta_N} \mathbf{1}_W(y^{-1}lr^{-1}z) \, dm(l) \, dm(r) \, dm(y) \, dm(z). \end{aligned}$$

By the Fatou lemma this is bounded above by

$$\begin{aligned} &\int_{y,z} \psi(y)\psi(z) \limsup_{N \rightarrow \infty} \frac{1}{m(\Theta_N)m(\Theta_N s_N)} \\ &\quad \times \int_{(l,r) \in \Theta_N s_N \times \Theta_N} \mathbf{1}_W(y^{-1}l(z^{-1}r)^{-1}) \, dm(l) \, dm(r) \, dm(y) \, dm(z), \end{aligned}$$

and the lim sup above equals $\bar{d}_\Psi(W)$ by the Følner property. Since $\int \psi \, d\mu = 1$, it follows that the integral is bounded above by $d^*(W)$. \square

4. Large sets in WM groups

This section is dedicated to the proof of Theorem 1.15. We also show that, in general, Theorem 1.15 fails for left substantial sets, even for countable groups and subsets of density 1. We will need some facts about commuting, measure preserving actions of WM groups on probability spaces. Recall that two actions T_1 and T_2 of G are said to

commute if $T_1^g T_2^h = T_2^h T_1^g$ for all g, h in G . Given an action T of G on a probability space (X, \mathcal{B}, μ) , denote by $T \times T$ the induced action of G on the product probability space $(X \times X, \mathcal{B} \otimes \mathcal{B}, \mu \otimes \mu)$.

Definition 4.1. Let T be an action of a locally compact, second countable group G on a probability space (X, \mathcal{B}, μ) . Denote by $\mathcal{I}(T)$ the closed subspace of $L^2(X, \mathcal{B}, \mu)$ consisting of functions invariant under T , and by $\mathcal{K}(T)$ the closed subspace of $L^2(X, \mathcal{B}, \mu)$ spanned by the finite-dimensional, T -invariant subspaces therein.

Lemma 4.2. For any measure-preserving action T of a locally compact, second countable, WM, amenable group G on a probability space (X, \mathcal{B}, μ) we have $\mathcal{I}(T) = \mathcal{K}(T)$ and $\mathcal{I}(T \times T) = \mathcal{I}(T) \otimes \mathcal{I}(T)$.

Proof. The inclusion $\mathcal{I}(T) \subset \mathcal{K}(T)$ is immediate. For the reverse inclusion, consider a finite-dimensional, T -invariant subspace H of $L^2(X, \mathcal{B}, \mu)$. It induces a finite-dimensional representation of G , which must be trivial because G is WM. Therefore $H \subset \mathcal{I}(T)$.

The second assertion follows from the first and the fact that $\mathcal{K}(T \times T) = \mathcal{K}(T) \otimes \mathcal{K}(T)$, which follows from [13]. \square

Definition 4.3. Let T be a measure-preserving action of a locally compact, second countable, amenable group G on a probability space (X, \mathcal{B}, μ) and let \mathcal{D} be a T -invariant sub- σ -algebra of \mathcal{B} . We can think of $L^2(X, \mathcal{B}, \mu)$ as an $L^\infty(X, \mathcal{D}, \mu)$ module. Denote by $\mathcal{K}(T|\mathcal{D})$ the closed subspace of $L^2(X, \mathcal{B}, \mu)$ spanned by the closed, finite-rank, T -invariant $L^\infty(X, \mathcal{D}, \mu)$ sub-modules.

Lemma 4.4. Let T_1 and T_2 be measure-preserving actions of a locally compact, second countable, amenable group G on probability spaces $(X_1, \mathcal{B}_1, \mu_1)$ and $(X_2, \mathcal{B}_2, \mu_2)$ respectively. Let \mathcal{D}_1 and \mathcal{D}_2 be T_1 and T_2 invariant sub- σ -algebras of \mathcal{B}_1 and \mathcal{B}_2 respectively. Then $\mathcal{K}(T_1|\mathcal{D}_1) \otimes \mathcal{K}(T_2|\mathcal{D}_2) = \mathcal{K}(T_1 \times T_2|\mathcal{D}_1 \times \mathcal{D}_2)$.

Proof. The inclusion \subset follows from the definition. For the reverse inclusion, note that if f_1 is orthogonal to $\mathcal{K}(T_1|\mathcal{D}_1)$ or f_2 is orthogonal to $\mathcal{K}(T_2|\mathcal{D}_2)$ then $f_1 \otimes f_2$ is orthogonal to $\mathcal{K}(T_1 \times T_2|\mathcal{D}_1 \otimes \mathcal{D}_2)$ by the characterization (see [28, Theorem 4.7], for example) of the orthogonal complements of these spaces. \square

The next definition is [9, Definition 2.5], which is based on a notion introduced in [23].

Definition 4.5. Let T_1 and T_2 be commuting, measure-preserving actions of a locally compact, second countable, amenable group G on a probability space (X, \mathcal{B}, μ) . We say that (T_1, T_2) is *magic* if $\mathcal{K}(T_1|\mathcal{I}(T_2)) = \mathcal{I}(T_1) \vee \mathcal{I}(T_2)$.

Lemma 4.6. If G is a locally compact, second countable, WM, amenable group and (T_1, T_2) is magic, then $(T_1 \times T_1, T_2 \times T_2)$ is also magic.

Proof. Applying Lemma 4.2 and then Lemma 4.4, we have

$$\mathcal{H}(T_1 \times T_1 | \mathcal{I}(T_2 \times T_2)) = \mathcal{H}(T_1 | \mathcal{I}(T_2)) \otimes \mathcal{H}(T_1 | \mathcal{I}(T_2))$$

so the conclusion follows upon using the fact that (T_1, T_2) is magic, noting that

$$(\mathcal{I}(T_1) \vee \mathcal{I}(T_2)) \otimes (\mathcal{I}(T_1) \vee \mathcal{I}(T_2)) = (\mathcal{I}(T_1) \otimes \mathcal{I}(T_1)) \vee (\mathcal{I}(T_2) \otimes \mathcal{I}(T_2))$$

and further use of Lemma 4.2. \square

Given a left Reiter sequence Φ in $L^1(G, m)$ and a measurable map $u : g \mapsto u_g$ from G to \mathbb{C} , write

$$\text{C-lim}_{g \rightarrow \Phi} u_g = \lim_{N \rightarrow \infty} \int \Phi_N(g) u_g \, dm(g)$$

if this limit exists. Write $\text{UC-lim}_g u_g = u$ if $\text{C-lim}_{g \rightarrow \Phi} u_g = u$ for every left Reiter sequence Φ , write $\text{D-lim}_{g \rightarrow \Phi} u_g = u$ if $\text{C-lim}_{g \rightarrow \Phi} |u_g - u| = 0$, and write $\text{UD-lim}_g u_g = u$ if $\text{UC-lim}_g |u_g - u| = 0$. Write \mathcal{I}_1 for $\mathcal{I}(T_1)$, \mathcal{I}_2 for $\mathcal{I}(T_2)$, and \mathcal{I}_{12} for $\mathcal{I}(T_1 T_2)$.

Theorem 4.7. *If G is a locally compact, second countable, WM, amenable group, (T_1, T_2) is magic, and $(g_1, g_2) \mapsto T_1^{g_1} T_2^{g_2}$ is an ergodic action of $G \times G$, then*

$$\begin{aligned} & \text{UD-lim}_g \int f_0 \cdot T_1^g f_1 \cdot T_1^g T_2^g f_2 \, d\mu \\ &= \int \mathbb{E}(f_0 | \mathcal{I}_1 \vee \mathcal{I}_{12}) \cdot \mathbb{E}(f_1 | \mathcal{I}_1 \vee \mathcal{I}_2) \cdot \mathbb{E}(f_2 | \mathcal{I}_2 \vee \mathcal{I}_{12}) \, d\mu \end{aligned}$$

for any f_0, f_1, f_2 in $L^\infty(X, \mathcal{B}, \mu)$.

Proof. By [9, Lemma 4.4] and our ergodicity assumptions the function

$$g \mapsto \int \mathbb{E}(f_0 | \mathcal{I}_1 \vee \mathcal{I}_{12}) \cdot T_1^g \mathbb{E}(f_1 | \mathcal{I}_1 \vee \mathcal{I}_2) \cdot T_1^g T_2^g \mathbb{E}(f_2 | \mathcal{I}_2 \vee \mathcal{I}_{12}) \, d\mu$$

is almost periodic. Since G is WM, it is in fact constant. Thus it suffices to prove that

$$\text{UD-lim}_g \int f_0 \cdot T_1^g f_1 \cdot T_1^g T_2^g f_2 \, d\mu = 0 \tag{4.8}$$

for any f_0, f_1, f_2 in $L^\infty(X, \mathcal{B}, \mu)$ satisfying one of the following conditions:

1. $f_0 \perp \mathcal{I}_1 \vee \mathcal{I}_{12} \iff f_0 \otimes f_0 \perp \mathcal{I}(T_1 \times T_1) \vee \mathcal{H}(T_1 T_2 \times T_1 T_2)$
2. $f_1 \perp \mathcal{I}_1 \vee \mathcal{I}_2 \iff f_1 \otimes f_1 \perp \mathcal{I}(T_1 \times T_1) \vee \mathcal{I}(T_2 \times T_2)$
3. $f_2 \perp \mathcal{I}_2 \vee \mathcal{I}_{12} \iff f_2 \otimes f_2 \perp \mathcal{I}(T_2 \times T_2) \vee \mathcal{H}(T_1 T_2 \times T_1 T_2)$

where the equivalences all follow from Lemma 4.2. But (4.8) is equivalent to

$$\text{UC-}\lim_g \int (f_0 \otimes f_0)(T_1 \times T_1)^g (f_1 \otimes f_1)(T_1 T_2 \times T_1 T_2)^g (f_2 \otimes f_2) d(\mu \otimes \mu) = 0,$$

which is true under any of the above conditions by Lemma 4.6 and [9, Corollary 3.6]. □

Corollary 4.9. *Let G be a locally compact, second countable, WM, amenable group and let T_1, T_2 be commuting G -actions on a probability space (X, \mathcal{B}, μ) such that the induced $G \times G$ action $(g_1, g_2) \mapsto T_1^{g_1} T_2^{g_2}$ is ergodic. Then*

$$\text{UD-}\lim_g \int f \cdot T_1^g f \cdot T_1^g T_2^g f d\mu \geq \left(\int f d\mu \right)^4$$

for any $0 \leq f \leq 1$ in $L^\infty(X, \mathcal{B}, \mu)$.

Proof. By [9, Corollary 4.6] the system $(X, \mathcal{B}, \mu, T_1, T_2)$ admits an ergodic magic extension which is denoted by the same symbols. Lifting f to this extension, Theorem 4.7 and [8, Lemma 1.6] combined yield

$$\begin{aligned} & \text{UD-}\lim_g \int f \cdot T_1^g f \cdot T_1^g T_2^g f d\mu \\ &= \int \mathbb{E}(f | \mathcal{I}_1 \vee \mathcal{I}_{12}) \mathbb{E}(f | \mathcal{I}_1 \vee \mathcal{I}_2) \mathbb{E}(f | \mathcal{I}_2 \vee \mathcal{I}_{12}) d\mu \\ &\geq \int f \mathbb{E}(f | \mathcal{I}_1 \vee \mathcal{I}_{12}) \mathbb{E}(f | \mathcal{I}_1 \vee \mathcal{I}_2) \mathbb{E}(f | \mathcal{I}_2 \vee \mathcal{I}_{12}) d\mu \geq \left(\int f d\mu \right)^4 \end{aligned}$$

as desired. □

We are now in a position to prove Theorem 1.15.

Proof of Theorem 1.15. Let E be a subset of G that is substantial with respect to some two-sided Reiter sequence Φ . Put $g_0 = \text{id}_G$, $E_0 = E$ and $\Psi_0 = \Phi$. Let also $K_1 \subset K_2 \subset \dots$ be an exhaustion of G by compact sets given by Lemma 2.3. We construct inductively a sequence g_i in G , a sequence E_i of measurable subsets of G with $g_{i+1} \in E_i$, and a sequence Ψ_i of two-sided Reiter sequences in G such that

$$E_{i+1} = g_{i+1}^{-1} E_i \cap E_i \cap E_i g_{i+1}^{-1}, \tag{4.10}$$

the set E_i is substantial with respect to Ψ_i and $g_i \notin K_i$ for all $i \geq 0$. Assume by induction that for some $i \geq 0$, we have g_j, E_j and Ψ_j defined for all $0 \leq j \leq i$ and having the desired properties.

Since E_i is substantial with respect to Ψ_i we can find a symmetric, open neighborhood U of the identity in G and a measurable subset W of G with $\bar{d}_{\Psi_i}(W) > 0$ such that $E_i \supset$

$UUWUU$. Put $S = UWU$. Since S is substantial, Part 2 of [Theorem 3.2](#) yields commuting G actions L and R on a compact, metric probability space (X, \mathcal{B}, ν) , a two-sided Reiter sequence Ψ_{i+1} in G , and a non-negative, continuous function ξ on X such that the $G \times G$ action induced by L and R is ergodic, $\int \xi \, d\nu \geq \bar{d}_{\Psi_i}(W)$ and

$$\bar{d}_{\Psi_{i+1}}(Sg^{-1} \cap S \cap g^{-1}S) \geq \int L^g \xi \cdot \xi \cdot R^{g^{-1}} \xi \, d\nu = \int \xi \cdot R^g \xi \cdot R^g L^g \xi \, d\nu$$

for all g in G .

By [Theorem 4.7](#) with $T_1 = R$, $T_2 = L$ and $f = \xi$ we obtain

$$\text{UD-lim}_g \int \xi \cdot R^g \xi \cdot R^g L^g \xi \, d\nu \geq \bar{d}_{\Psi_i}(W)^4$$

so for any $\varepsilon > 0$ the set

$$F_i = \{g \in G : \bar{d}_{\Psi_{i+1}}(Sg^{-1} \cap S \cap g^{-1}S) \geq \bar{d}_{\Psi_i}(W)^4 - \varepsilon\}$$

has density 1 with respect to any left Reiter sequence. Thus $d_{\Psi_i}(F_i) = 1$ so in particular the intersection $F_i \cap E_i$ is not relatively compact. Choose g_{i+1} in $F_i \cap E_i \setminus K_{i+1}$ and define E_{i+1} by [\(4.10\)](#). To see that E_{i+1} is substantial with respect to Ψ_{i+1} , note that

$$E_i g_{i+1}^{-1} \cap E_i \cap g_{i+1}^{-1} E_i \supset V(Sg_{i+1}^{-1} \cap S \cap g_{i+1}^{-1} S)V$$

where V is the symmetric neighborhood $U \cap g_{i+1} U g_{i+1}^{-1} \cap g_{i+1}^{-1} U g_{i+1}$ of the identity. This concludes the inductive construction.

It remains to prove that $\text{FP}(g_n)$ is contained in E . Note that g_n belongs to E for each n and that [\(4.10\)](#) implies

$$E_i = \cap \{I_\alpha(g_n)^{-1} E D_\beta(g_n)^{-1} : \alpha, \beta \subset \{1, \dots, i\} \text{ and } \alpha \cap \beta = \emptyset\}$$

for each i in \mathbb{N} by induction on i . Thus $I_\alpha(g_n) g_{i+1} D_\beta(g_n)$ belongs to E for any disjoint subsets α, β of $\{1, \dots, i\}$ as desired. \square

We remark that [Theorem 1.15](#) immediately implies the following partition result.

Corollary 4.11. *For any measurable partition $C_1 \cup \dots \cup C_r$ of a locally compact, second countable, WM, amenable group one can find $1 \leq i \leq r$ such that, for any open neighborhood U of the identity, the set $UC_i U$ contains a two-sided finite products set.*

Also, in the proof of [Theorem 1.15](#), there are in fact many choices for each g_i because $F_i \cap E_i$ has positive density with respect to a two-sided Reiter sequence.

The following example shows that [Theorem 1.15](#) does not extend to general left Følner sequences, even in the discrete case.

Example 4.12. Let $A(\mathbb{N})$ be the finite alternating group on \mathbb{N} . As in Example 2.13 we can view $A(\mathbb{N})$ as the increasing union of the alternating groups $A_n = A(\{1, \dots, n\})$. For $n \geq 4$ let $h_n := (1, n)(2, 3) \in A_n$ and $\Phi_n := A_{n-1}h_n$. This is a left Følner sequence, and we see from

$$\Phi_n \subset \{\sigma \in A_n : \sigma(1) = n\}$$

that the sets Φ_n are pairwise disjoint. Let $E := \cup_{n \geq 5} \Phi_n$, so that $d_\Phi(E) = 1$. Suppose that there exists a sequence (g_k) with $\text{FPD}(g_k) \subset E$. We will show the following statement by induction on n .

Claim. *Suppose that for some $\alpha \in \mathcal{F}$ and $n \geq 4$ we have $D_\alpha(g_k) \in \Phi_n$. Then there exists $\beta \in \mathcal{F}$ such that $D_\beta(g_k) \in \Phi_4$.*

Since the assumption is clearly satisfied for some $n \geq 5$ and some $\alpha \in \mathcal{F}$, we obtain a contradiction.

Proof of the claim. For $n = 4$ the conclusion holds with $\beta = \alpha$. Suppose now that the claim is known to hold up to some n and assume $h := D_\alpha(g_k) \in \Phi_{n+1}$. Let $i > \max \alpha$, let j be such that $g_i \in \Phi_j$, and consider $D_{\alpha \cup \{i\}}(g_k) = g_i h$. By the assumption this is an element of E . Consider now the following cases.

$j > n + 1$ We have $g_i h(1) = g_i(n + 1) < j$ and $g_i h(j) = g_i(j) < j$, so that $g_i h \notin E$, contradicting the assumption.

$j \leq n$ The conclusion follows from the inductive assumption.

$j = n + 1$ In this case we have $g_i h(1) = g_i(n + 1) < n + 1$, so that $g_i h \notin \Phi_{n+1}$. Since $g_i h \in E \cap A_{n+1}$, it follows that $g_i h \in \Phi_m$ for some $m \leq n$, and the conclusion again follows from the inductive hypothesis. \square

In view of [4, Theorem 2.4] this implies that the set E in this example has density zero with respect to any right Følner sequence. As discussed in the introduction, this answers a question from [12].

For the proof of Theorem 1.19 we need a tool that allows us to deduce that a set having positive density in a group has positive density in some coset of any cocompact subgroup.

Lemma 4.13. *Let G be a locally compact, second countable, amenable group with a left (respectively two-sided) Følner sequence Φ . Let H be a closed, normal, cocompact subgroup of G . Then Φ has a subsequence, denoted by the same symbol, such that for almost every $z \in G/H$ and any $x \in z$ the sequence $N \mapsto x^{-1}(\Phi_N \cap z)$ is a left (respectively two-sided) Følner sequence in H .*

If E is a measurable subset of G such that $\overline{d}_\Phi(E) > 0$, then for a positive measure set of $z \in G/H$ we have $\overline{d}_{\Phi \cap z}(E) > 0$.

Proof. We will prove the result for two-sided Følner sequences, the proof for left Følner sequences is nearly identical.

Consider for each x in G the measure m_x defined by $m_x(E) = m_H(x^{-1}E)$ for all Borel subsets E of xH . This is a measure on the coset xH and is invariant under left and right translation by H . The measure m_x only depends on the coset of H that x represents. We can therefore define, for each z in G/H , a measure m_z on G by $m_z = m_x$ for any $x \in z$. After suitably normalizing the left Haar measure m_H , we obtain

$$\int f \, dm = \iint f \, dm_z \, d\mu(z),$$

where μ denotes the Haar measure on G/H , for any $f \in C_c(G)$ from [15, Theorem 2.49]. In fact, this holds for any f in $L^1(G, m)$.

We have

$$m_x(\Phi_N \triangle g \Phi_N) \geq |m_x(\Phi_N) - m_{g^{-1}x}(\Phi_N)|$$

for all N in \mathbb{N} and all g, x in G . By the Følner property it follows that

$$\int \frac{|m_z(\Phi_N) - m_{g^{-1}z}(\Phi_N)|}{m(\Phi_N)} \, d\mu(z) \leq \frac{m(\Phi_N \triangle g \Phi_N)}{m(\Phi_N)} \rightarrow 0$$

as $N \rightarrow \infty$. Invariance of μ implies

$$1 = \int \frac{m_z(\Phi_N)}{m(\Phi_N)} \, d\mu(z) = \int \frac{m_{zw}(\Phi_N)}{m(\Phi_N)} \, d\mu(z)$$

for every $w \in G/H$, so

$$\int \left| 1 - \frac{m_w(\Phi_N)}{m(\Phi_N)} \right| \, d\mu(w) \leq \iint \frac{|m_{zw}(\Phi_N) - m_w(\Phi_N)|}{m(\Phi_N)} \, d\mu(z) \, d\mu(w) \rightarrow 0$$

as $N \rightarrow \infty$ by the dominated convergence theorem. Passing to a subsequence, we may assume

$$m_z(\Phi_N)/m(\Phi_N) \rightarrow 1 \text{ for almost every } z \in G/H. \tag{4.14}$$

By the Følner condition we also have $m(\Phi_N \triangle g \Phi_N h)/m(\Phi_N) \rightarrow 0$ locally uniformly for $g, h \in H$. Let K_n be a countable collection of relatively compact open sets that covers $H \times H$. We have

$$\int \int_{K_n} \frac{m_z(\Phi_N \triangle g \Phi_N h)}{m(\Phi_N)} \, d\mu(z) \, d(m_H \times m_H)(g, h) \rightarrow 0$$

for every n . Passing to a subsequence, we may assume that the convergence holds pointwise almost everywhere for every n .

By Fubini’s theorem it follows that for almost every $z \in G/H$ we have

$$\frac{m_z(\Phi_N \triangle g \Phi_N h)}{m(\Phi_N)} \rightarrow 0$$

for almost every pair (g, h) in $H \times H$. Combined with (4.14), this implies

$$\frac{m_z(\Phi_N \triangle g \Phi_N h)}{m_z(\Phi_N)} \rightarrow 0$$

with the same quantifiers. By the remarks at the start of Section 2.2, $\Phi_N \cap z$ is a two-sided Følner sequence in the two-sided H torsor z for almost every z in G/H . It follows that $x^{-1}(\Phi_N \cap z)$ is a two-sided Følner sequence in H for any x in z .

Suppose now that $\bar{d}_\Phi(E) > 0$. Before performing the above construction we may pass to a subsequence of Φ such that $d_\Phi(E) > 0$. This ensures $d_\Phi(E) > 0$ after passing to further subsequences as required by the construction. Now the Fatou lemma justifies

$$\begin{aligned} 0 < \bar{d}_\Phi(E) &= \limsup_{N \rightarrow \infty} \frac{1}{m(\Phi_N)} \int \mathbf{1}_{\Phi_N} \cdot \mathbf{1}_E \, dm_G \\ &= \limsup_{N \rightarrow \infty} \int \frac{1}{m(\Phi_N)} \int \mathbf{1}_{\Phi_N} \cdot \mathbf{1}_E \, dm_z \, d\mu(z) \\ &\leq \int \limsup_{N \rightarrow \infty} \frac{1}{m(\Phi_N)} \int \mathbf{1}_{\Phi_N} \cdot \mathbf{1}_E \, dm_z \, d\mu(z) \\ &= \int \limsup_{N \rightarrow \infty} \frac{1}{m_z(\Phi_N)} \int \mathbf{1}_{\Phi_N} \cdot \mathbf{1}_E \, dm_z \, d\mu(z) = \int \bar{d}_{\Phi \cap z}(E) \, d\mu(z) \end{aligned}$$

so $\bar{d}_{\Phi \cap z}(E) > 0$ for a positive measure set of cosets. \square

We now prove Theorem 1.19.

Proof of Theorem 1.19. Let $H \leq G$ be a closed normal subgroup such that H is WM and G/H is compact. For the second part it suffices to take a nonnull compact subset of G/H that does not contain the equivalence class of id_G and pull it back to G ; the fact that this is not a set of recurrence will be witnessed by the left translation action of G on G/H .

For the first part note that H is unimodular by Lemma 2.1, so that G is also unimodular. Hence, for a given Borel subset $E \subset G$ of positive upper Banach density, the density is realized along some two-sided Følner sequence Φ by Proposition 2.10.

It remains to show that E has positive upper density in one of the cosets of H . By Lemma 4.13 the sequence Φ is a two-sided Følner sequence in almost every coset of H and $\bar{d}_{\Phi \cap z}(E) > 0$ for a positive measure set of cosets. Pick a coset z such that this holds and $\Phi \cap z$ is a two-sided Følner sequence in z . Now if UEU is a substantial set in G and $x \in z$, then $x^{-1}UEU \cap H \supset (x^{-1}Ux \cap H)(x^{-1}E \cap H)(U \cap H)$ is a substantial set in H , and we can apply Theorem 1.15 to it. \square

Proof of Theorem 1.21. As in the proof of Theorem 1.19, shifting S (on the left or on the right) we may assume $S = U^9EU^2$, where $E = UE_0$ has positive upper density with respect to some left Følner sequence Φ in G , $E_0 \subset G_0$ has positive upper density in G_0 with respect to the Følner sequence $N \mapsto \Phi_N \cap G_0$, and U is a symmetric neighborhood of the identity in G .

Put $g_0 = \text{id}_G$, $S_0 = S$, $U_0 = U$, and $\Psi_0 = \Phi$. Let also $K_1 \subset K_2 \subset \dots$ be an exhaustion of G by compact sets given by Lemma 2.3. We construct inductively

1. a nested sequence of subsequences Ψ_i of Φ
2. a decreasing sequence S_i of substantial subsets of G such that $S_i \supseteq U_i^{10}E_iU_i^2$ with $\bar{d}_{\Psi_i \cap G_0}(E_i) > 0$ and symmetric relatively compact neighborhoods of identity U_i in G , and
3. a sequence g_i in G with $g_{i+1} \in S_i \setminus K_i$ such that

$$S_{i+1} = g_{i+1}^{-1}S_i \cap S_i. \tag{4.15}$$

This suffices to conclude $\text{FPI}(g_n) \subset S$.

Assume by induction that for some $i \geq 0$, we have g_j, S_j, U_j , and E_j defined for all $0 \leq j \leq i$ and having the desired properties. Applying the one-sided correspondence principle [4, Theorem 1.1] to $U_i(U_iE_i)$, we obtain a measure-preserving action T of G on a separable probability space (X, \mathcal{B}, μ) and a positive function ξ in $L^\infty(X, \mathcal{B}, \mu)$ such that

$$\bar{d}_\Phi(U_i^2E_i \cap g^{-1}U_i^2E_i) \geq \int \xi \cdot T^g\xi \, d\mu$$

for all g in G . Let $\varepsilon = 10^{-1}|\int \xi \, d\mu|^2$.

Recall that

$$L^2(X, \mathcal{B}, \mu) = \mathcal{K}(T) \oplus \mathcal{W}(T),$$

where $\mathcal{W}(T)$ is the closed subspace consisting of the functions f such that for every $\phi \in L^\infty(X)$ we have

$$\text{UC-}\lim_g \left| \int \phi \cdot T^g f \, d\mu \right| = 0. \tag{4.16}$$

Applying (4.16) with $\phi = \xi$ and $f = \xi - \mathbb{E}(\xi|\mathcal{K}(T))$ we see that

$$\left| \int \xi \cdot T^g(\xi - \mathbb{E}(\xi|\mathcal{K}(T))) \, d\mu \right| < \varepsilon \tag{4.17}$$

for a set of $g \in G$ with density 1. Let $W \subset U_i$ be a neighborhood of identity such that

$$\|\mathbb{E}(\xi|\mathcal{K}(T)) - T^g\mathbb{E}(\xi|\mathcal{K}(T))\| < \varepsilon \tag{4.18}$$

for every $g \in W$. Since almost periodic functions on G are trivial on G_0 , the above inequality holds for all $g \in WE_i$. The latter set has positive upper density with respect to Φ in G , so that there is some $g \in WE_i \setminus K_i \overline{U_i}^2$ for which both inequalities (4.17) and (4.18) hold.

For this g the set $U_i^2 E_i \cap g^{-1} U_i^2 E_i$ has positive upper density in G . Since the latter set is contained in $U_i^2 G_0$, by Lemma 4.13 there exists $h \in U^2$ such that $h^{-1}(U_i^2 E_i \cap g^{-1} U_i^2 E_i) \cap G_0$ has positive upper density in G_0 . It follows that

$$E_{i+1} := U_i^4 E_i \cap g_{i+1}^{-1} U_i^2 E_i \cap G_0$$

has positive upper density in G_0 with $g_{i+1} = gh \in S_i \setminus K_i$. Therefore $S_i \cap g_{i+1}^{-1} S_i$ is a substantial set of the same special form as S_i , and we can continue the induction. \square

5. Large sets in groups with von Neumann kernel not cocompact

In this section we extend Straus’s example to locally compact, second countable, amenable groups whose von Neumann kernel is not cocompact by proving Theorem 1.11. The construction involves the following approximate version of the monotone convergence theorem for the finitely additive measure defined by a Følner sequence.

Lemma 5.1. *Let G be a locally compact, second countable, amenable group and let Φ be a left, right, or two-sided Følner sequence in G . Let $i \mapsto A_i$ be a sequence of Borel subset of G such that $d_\Phi(A_i)$ exists for every i , the sequence $i \mapsto d_\Phi(A_i)$ is summable, and $d_\Phi(A_1 \cup \dots \cup A_n)$ exists for every n . Then there exist cocompact subsets $A'_i \subset A_i$ such that $d_\Phi(C) = \lim d_\Phi(A_1 \cup \dots \cup A_n)$, where $C = \cup\{A'_i : i \in \mathbb{N}\}$.*

Proof. If G is compact, then $d_\Phi = m$, and the conclusion holds with $A'_i = A_i$ by the monotone convergence theorem. Hence we may assume that G is not compact.

For each $i \in \mathbb{N}$ there is some s_i in \mathbb{N} such that

$$\left| \frac{m(A_i \cap \Phi_N)}{m(\Phi_N)} - d_\Phi(A_i) \right| < \frac{1}{2^i}$$

whenever $N \geq s_i$. We may assume that $s_{i+1} > s_i$ for all $i \in \mathbb{N}$. We will show that the conclusion holds for $A'_i = A_i \setminus (\Phi_1 \cup \dots \cup \Phi_{s_i})$. Note that $\frac{m(A'_i \cap \Phi_N)}{m(\Phi_N)} < d_\Phi(A_i) + \frac{1}{2^i}$ for every N .

Define $B_n = A_1 \cup \dots \cup A_n$ and $C_n = A'_1 \cup \dots \cup A'_n$. Let $\alpha = \lim d_\Phi(B_n)$. Since $C_n \subset B_n$ and $B_n \setminus C_n$ is contained in a compact set we have $d_\Phi(C_n) = d_\Phi(B_n)$. From $C_n \subset C_{n+1}$ it follows that $\underline{d}_\Phi(C) \geq \alpha$. Thus it remains to show that $\overline{d}_\Phi(C) \leq \alpha$. For every J and every N we have

$$\begin{aligned} \frac{m(C \cap \Phi_N)}{m(\Phi_N)} &\leq \frac{m(C_J \cap \Phi_N)}{m(\Phi_N)} + \sum_{i=J+1}^{\infty} \frac{m(A'_i \cap \Phi_N)}{m(\Phi_N)} \\ &\leq \frac{m(B_J \cap \Phi_N)}{m(\Phi_N)} + \sum_{i=J+1}^{\infty} d_{\Phi}(A_i) + \frac{1}{2^i}. \end{aligned}$$

Letting $N \rightarrow \infty$ we obtain

$$\bar{d}_{\Phi}(C) \leq d_{\Phi}(B_J) + 2^{-J} + \sum_{i=J+1}^{\infty} d_{\Phi}(A_i).$$

Now, letting $J \rightarrow \infty$, we obtain $\bar{d}_{\Phi}(C) \leq \alpha$ as required. \square

Proof of Theorem 1.11. Fix a left Følner sequence Φ in G and $\varepsilon > 0$. Let $K_1 \subset K_2 \subset \dots$ be an increasing sequence of compact sets whose union is G , as in Lemma 2.3. For every n let f_n be the almost periodic function on G given by Lemma 2.17 with the compact set K_n and $\varepsilon/2^{n+10}$. Let $X_n = \overline{\{l_g f_n : g \in G\}}$, then (X_n, G) is a topologically transitive, equicontinuous topological dynamical system. Therefore (X_n, G) is minimal and uniquely ergodic by Theorem 2.18. Let μ_n be the unique invariant Borel probability measure on (X_n, G) . By minimality μ_n has full support. Let $\varepsilon/2^{n+5} < r_n < \varepsilon/2^{n+4}$ be such that $\mu_n(\partial B_{r_n}(f_n)) = 0$. (Such r_n exist because the boundaries are pairwise disjoint.) Since the action l is isometric on X_n and the range of f_n is $\varepsilon/2^{n+10}$ -dense in $[0, 1]$, there are at least $2^{n+1}/\varepsilon$ disjoint images of $U_n := B_{r_n}(f_n)$ in X_n , so $\mu_n(U_n) \leq \varepsilon/2^{n+1}$. Put $A_n = R_{U_n}(f_n)$, so that $d_{\Phi}(A_n) = \mu_n(U_n) \leq \varepsilon/2^{n+1}$ by Lemma 2.19.

In order to apply Lemma 5.1 we need to prove that

$$B_n = A_1 \cup \dots \cup A_n$$

has density for every n . To do this consider the action L of G on $X_1 \times \dots \times X_n$ induced by applying l in each coordinate. Let Z be the orbit closure of (f_1, \dots, f_n) under this action. By Theorem 2.18 the topological dynamical system (Z, G) is minimal and uniquely ergodic. We have

$$\begin{aligned} B_n &= \cup_{i=1}^n \{x \in G : l_x f_i \in U_i\} \\ &= \{x \in G : L_x(f_1, \dots, f_n) \in \pi_1^{-1}U_1 \cup \dots \cup \pi_n^{-1}U_n\} \end{aligned}$$

so B_n is a set of return times in a uniquely ergodic dynamical system. Let ν be the unique invariant probability measure on (Z, G) . Each of the coordinate projections $\pi_i : Z \rightarrow X_i$ intertwines the actions L and l . Since X_i is uniquely ergodic, this implies that $\pi_i \nu = \mu_i$ for each i . Thus

$$\nu(\partial(\pi_1^{-1}U_1 \cup \dots \cup \pi_n^{-1}U_n)) \leq \mu_1(\partial U_1) + \dots + \mu_n(\partial U_n) = 0$$

and therefore $d_\Phi(B_n)$ exists by Lemma 2.19.

It follows from Lemma 5.1 that there exist cocompact subsets $A'_n \subset A_n$ whose union C has density at most ε . We claim that the set $E := G \setminus C$ satisfies the conclusion of the theorem. Indeed, let K be an arbitrary symmetric compact subset of G . It suffices to show that the complement of KEK is an \mathcal{R}^* set.

By construction we have $K \subset K_n$ for some n . Put $D_n = G \setminus A_n$ and $D'_n = G \setminus A'_n$. We have $E \subset D'_n$ so it suffices to prove that $G \setminus (KD'_nK)$ is \mathcal{R}^* . By Corollary 2.24 this is equivalent to $G \setminus (KD_nK)$ being an \mathcal{R}^* set. Now

$$G \setminus (KD_nK) = \bigcap_{g,h \in K} gA_nh = \bigcap_{g,h \in K} R_{l_g B_{r_n}(f_n)}(l_{h^{-1}}f_n) \supseteq R_{B_{r_n/2}(f_n)}(f_n)$$

since f_n is $\varepsilon/2^{n+10}$ -invariant under K and G acts isometrically on X_n . The latter set is \mathcal{R}^* by Lemma 2.25. \square

In Theorem 1.11 the set E depends on the Følner sequence. It is natural to ask whether it is possible for E to satisfy the conclusion of Theorem 1.11 and have positive density with respect to all left Følner sequences. To see that this is impossible we will show that any set satisfying the conclusion of Theorem 1.11 is not piecewise syndetic. In view of Example 2.22 it suffices to show the following.

Lemma 5.2. *Let G be a locally compact, second countable group that is not compact. Then for every piecewise-syndetic subset P of G there exists $k \in G$ such that for every neighborhood U of id_G the set UkP contains a set of the form $\text{FPI}(g_n)$ with $g_n \rightarrow \infty$.*

Proof. Let P be a piecewise-syndetic subset of G . Fix a compact subset K of G such that $T = KP$ is thick. Let $n \mapsto K_n$ be a sequence of increasing, compact subsets of G that cover G (see Lemma 2.3).

Using Lemma 2.27 with $K = \{\text{id}_G\}$, choose $g_1 \in G \setminus K_1$ such that $g_1 \in T$. Assume by induction that we have found g_1, \dots, g_n in G such that $H = \{I_\alpha(g_n) : \emptyset \neq \alpha \subset \{1, \dots, n\}\}$ is a subset of T and $g_i \notin K_i$ for each $1 \leq i \leq n$. By Lemma 2.27 there is some g_{n+1} in $G \setminus (H \cup \{\text{id}_G\})^{-1}K_{n+1}$ such that $(H \cup \{\text{id}_G\})g_{n+1} \subset T$. It follows that T contains $\text{FPI}(g_n)$ and $I_\alpha(g_n) \rightarrow \infty$ as $\alpha \rightarrow \infty$.

For each finite set $\emptyset \neq \alpha \subset \mathbb{N}$ let $k_\alpha \in K$ be such that $k_\alpha^{-1}I_\alpha(g_n) \in P$. Since G is metrizable, by [18, Theorem 1.3] there is a sub-IP-ring such that $\lim_\alpha k_\alpha = k$ exists. In particular, for every symmetric neighborhood U of id_G we have $k_\alpha \in Uk$ for sufficiently large α . Hence $I_\alpha(g_n) \in k_\alpha P \subset UkP$ as required. \square

Call any subset E of G satisfying the conclusion of Theorem 1.11 a *Straus set*. It follows from Lemma 5.2 that any Straus set E is not piecewise syndetic. In particular, E is not syndetic, and therefore its complement is thick. (This follows from the proof of

Lemma 2.26.) Now, any thick set T has full density with respect to some Følner sequence in G . Indeed, for any Følner sequence Φ in G , one can find for each N some $h_N \in G$ such that $\Phi_N h_N \subset T$, so that T has full density with respect to the Følner sequence $N \mapsto \Phi_N h_N$. Thus it is impossible for a Straus set to have positive upper density with respect to every left Følner sequence.

6. Non-piecewise-syndetic sets with large density

By the remarks at the end of Section 5 we see that Straus sets are not piecewise-syndetic. However, Straus sets only exist in amenable groups whose von Neumann kernel is not cocompact. In this section we prove **Theorem 1.9**, which states that it is possible in any locally compact, second countable, amenable group that is not compact to construct a non-piecewise-syndetic set with positive lower density. The proof of **Theorem 1.9** requires an ample supply of syndetic sets.

Lemma 6.1. *Let G be a group, $S \subset G$ be any set, and $F \subset G$ be a non-empty set. Then there exists a subset $S' \subset S$ such that*

1. $F^{-1}FS' \supseteq S$;
2. $f_1S' \cap f_2S' = \emptyset$ for any $f_1, f_2 \in F, f_1 \neq f_2$.

Proof. The collection

$$\{T \subset S : f_1T \cap f_2T = \emptyset \text{ for any } f_1 \neq f_2 \in F\}$$

is closed under increasing unions, hence it contains a maximal element S' by Zorn's Lemma. Suppose that condition (1) fails for S' , that is, there exists $s \in S \setminus F^{-1}FS'$. We will show that the set $S'' := S' \cup \{s\}$ also satisfies condition (2), thereby contradicting the maximality of S' . To this end let f_1, f_2 be two distinct elements of F . We have

$$f_1S'' \cap f_2S'' = (f_1S' \cap f_2S') \cup (f_1S' \cap f_2\{s\}) \cup (f_1\{s\} \cap f_2S') \cup (f_1\{s\} \cap f_2\{s\}) = \emptyset$$

by the assumptions that S' satisfies (2) and that $s \notin F^{-1}FS$, as required. \square

Lemma 6.2. *Let G be a locally compact, second countable, amenable group that is not compact. Then for every relatively compact neighborhood U of the identity there exists a decreasing sequence $S_1 \supset S_2 \supset \dots$ of discrete, syndetic subsets of G such that $d^*(US_n) \rightarrow 0$ as $n \rightarrow \infty$.*

Proof. Put $S_0 = G$ and let g_1, g_2, \dots be a sequence in G such that the sets g_nU are pairwise disjoint. Suppose that S_n has been constructed for some n . Let S_{n+1} be given by **Lemma 6.1** with $F = g_1U \cup \dots \cup g_{n+1}U$ and $S = S_n$. The set F is relatively compact, so it follows from syndeticity of S_n and Part 1 of **Lemma 6.1** that S_{n+1} is syndetic. By

Part 2 of Lemma 6.1 the sets $g_i U_s$ are pairwise disjoint for $i = 1, \dots, n+1$ and $s \in S_{n+1}$. This implies that S_{n+1} is discrete and $d^*(US_{n+1}) \leq \frac{1}{n+1}$. \square

The final ingredient in the proof of Theorem 1.9 is an appropriate version of Lemma 5.1.

Lemma 6.3. *Let G be a locally compact, second countable, amenable group with a (left, right, or two-sided) Følner sequence Φ and let A_i be a sequence of measurable subsets of G . Then for every $\varepsilon > 0$ there exist cocompact subsets $A'_i \subset A_i$ such that $\bar{d}_\Phi(\cup_i A'_i) \leq \sum_i \bar{d}_\Phi(A_i) + \varepsilon$.*

Proof. By definition of the upper density, for every i there exists s_i such that

$$\frac{m(A_i \cap \Phi_N)}{m(\Phi_N)} < \bar{d}_\Phi(A_i) + \frac{\varepsilon}{2^i}$$

for every $N > s_i$. Consider the cocompact subsets $A'_i := A_i \setminus \cup_{n=1}^{s_i} \Phi_n$. Then for every N we have

$$\begin{aligned} m\left(\bigcup_i A'_i \cap \Phi_N\right) &= m\left(\bigcup_{i:s_i < N} A'_i \cap \Phi_N\right) \leq m\left(\bigcup_{i:s_i < N} (A_i \cap \Phi_N)\right) \leq \sum_{i:s_i < N} m(A_i \cap \Phi_N) \\ &< \sum_{i:s_i < N} (\bar{d}_\Phi(A_i) + \varepsilon/2^i)m(\Phi_N) \leq m(\Phi_N) \left(\sum_i \bar{d}_\Phi(A_i) + \varepsilon\right) \end{aligned}$$

as required. \square

Proof of Theorem 1.9. Let $(g_i)_{i \in \mathbb{N}}$ be a dense subset of G , let U be a symmetric, relatively compact neighborhood of id_G , and let $G \supset S_1 \supset S_2 \supset \dots$ be the decreasing sequence of syndetic sets from Lemma 6.2. Passing to a subsequence we may assume $\sum_i d^*(S_i) < \varepsilon$. By Lemma 6.3 we obtain cocompact subsets $S'_i \subset US_i$ such that $\bar{d}_\Phi(\cup\{g_i S'_i : i \in \mathbb{N}\}) < \varepsilon$. Consider the set

$$Q := G \setminus \cup\{g_i S'_i : i \in \mathbb{N}\}.$$

It follows from the construction that Q has lower density at least $1 - \varepsilon$ with respect to Φ . We claim that Q is not piecewise-syndetic. Suppose Q is piecewise-syndetic, that is, that there exists a compact set $K \subset G$ such that $T := KQ$ is a thick set. By compactness we have $K \subset Ug_1^{-1} \cup \dots \cup Ug_N^{-1}$ for some $N \in \mathbb{N}$. Thus

$$KQ \subseteq Ug_1^{-1}Q \cup \dots \cup Ug_N^{-1}Q \subseteq \cup_{i=1}^N \cup_{u \in U} ug_i^{-1}(G \setminus g_i S'_i)$$

and so

$$G \setminus KQ \supseteq \cap_{i=1}^N \cap_{u \in U} uS'_i \supseteq S_N \setminus \cup_{i=1}^N U(US_i \setminus S'_i).$$

This is a cocompact subset of the syndetic set S_N , hence a syndetic set, contradicting thickness of KQ . Hence Q is not piecewise syndetic. \square

We next show, using the (rather deep) Jewett–Krieger theorem for countable, amenable groups, that if G is a countable, infinite, amenable group then the set E obtained in [Theorem 1.9](#) can be taken to have density.

Theorem 6.4. *For any countably infinite, amenable group G , any left Følner sequence Φ in G , and any $\varepsilon > 0$ there is a subset Q of G with $d_\Phi(Q) > 1 - \varepsilon$ that is not piecewise syndetic.*

Proof. Fix $\varepsilon > 0$. Consider an ergodic action of G on a non-atomic space, for example the action of G on $\{0, 1\}^G$ given by $(g \cdot \mathbf{1}_B)(x) = \mathbf{1}_B(xg)$ equipped with a Bernoulli measure. By Rosenthal’s Jewett–Krieger theorem [\[29\]](#) this action admits a topological model (X, G) that is uniquely ergodic. Let μ be the unique invariant probability measure. Since X is infinite, the measure μ is non-atomic. Fix x in the support of μ . For each $n \in \mathbb{N}$, [Lemma 2.20](#) yields an open ball $A_n \subset X$ centered at x with $\mu(A_n) < \varepsilon/2^n$ and $\mu(\partial A_n) = 0$. Since x is in the support of μ , each A_n has positive measure. Applying [Lemma 2.19](#), we obtain $d_\Phi(R_{A_n}(x)) = \mu(A_n)$ for every left Følner sequence Φ . Thus, for each n , the set $S_n := R_{A_n}(x)$ has positive density with respect to every left Følner sequence. It follows from this that each of the sets S_n is syndetic.

Let $(g_i)_{i \in \mathbb{N}}$ be an enumeration of G . Put $B_i = g_1 S_1 \cup \dots \cup g_i S_i$. We have

$$B_i = \{g \in G : gx \in g_1 A_1 \cup \dots \cup g_i A_i\} = R_{Y_i}(x)$$

where $Y_i := g_1 A_1 \cup \dots \cup g_i A_i$. Since $\mu(\partial Y_i) = 0$ it follows from [Lemma 2.19](#) that $d_\Phi(B_i)$ exists. Applying [Lemma 5.1](#), we obtain cofinite subsets $S'_i \subset S_i$ such that $d_\Phi(\cup\{g_i S'_i : i \in \mathbb{N}\}) < \varepsilon$.

Put $Q = G \setminus \cup\{g_i S'_i : i \in \mathbb{N}\}$. The density of Q exists and is at least $1 - \varepsilon$. Arguing as in the proof of [Theorem 1.9](#) shows that Q is not piecewise syndetic. To this end, note that $S_{i_1} \cap \dots \cap S_{i_k} = S_l$ where $l = \max\{i_1, \dots, i_k\}$. Thus the intersection is syndetic. Hence its cofinite subset $S'_{i_1} \cap \dots \cap S'_{i_k}$ is also syndetic. \square

We will use sets whose existence is ensured by [Theorem 6.4](#) in order to construct non-trivial actions of G . Denote by $\mathbf{1}_\emptyset$ the function $G \rightarrow \{0, 1\}$ mapping every element of G to 0.

Proposition 6.5. *Let G be a countably infinite amenable group and let Φ be a Følner sequence in G . Suppose Q is a subset of G that has positive upper density but is not piecewise-syndetic. Let X be the orbit closure of $\mathbf{1}_Q$ in $\{0, 1\}^G$ under the action of G on $\{0, 1\}^G$ given by $(g \cdot \mathbf{1}_B)(x) = \mathbf{1}_B(xg)$. Then $\{\mathbf{1}_\emptyset\}$ is the only minimal subsystem of X , but there is a non-atomic G -invariant probability measure on X .*

Proof. Let $\mathbf{1}_B$ be any point in X and assume that B is syndetic. Let F be a finite set such that $FB = G$. Let H be any finite subset of G . Since $\mathbf{1}_B$ is in the orbit closure of $\mathbf{1}_Q$ we can find $g \in G$ such that $\mathbf{1}_{Qg}$ and $\mathbf{1}_B$ agree on the finite set $F^{-1}H$. In particular,

$$FQg \supset F(F^{-1}H \cap B) \supset \bigcup_{f \in F} f(f^{-1}H \cap B) = H \cap FB = H,$$

so FQ contains Hg^{-1} . Since H was arbitrary, Q must be piecewise-syndetic, a contradiction. Hence no point of X can correspond to a syndetic set.

Suppose now that Y is a minimal subsystem of X different from $\{\mathbf{1}_\emptyset\}$. Then the set $C = \{\omega \in \{0, 1\}^G : \omega(\text{id}_G) = 1\}$ has non-empty intersection with Y . Since Y is minimal, every point $\mathbf{1}_B \in Y$ visits C syndetically. In particular, every point corresponds to a syndetic set, a contradiction.

For the second assertion, let Ψ be a sub-sequence of Φ such that $d_\Psi(Q) = \underline{d}_\Phi(Q)$ and let μ be any limit point of the sequence

$$\mu_N = \frac{1}{|\Psi_N|} \sum_{g \in \Psi_N} \delta_{g \cdot \mathbf{1}_Q}$$

of probability measures on X . We have $\mu(C) = d_\Psi(Q) > 0$, so $\mu(\{\mathbf{1}_\emptyset\}) < 1$.

Hence there is a G invariant probability measure with no point mass at 0. Suppose now that it has a point mass at some other point $\mathbf{1}_B$. Then the orbit of $\mathbf{1}_B$ is finite, so it is a minimal subsystem of X different from $\{\mathbf{1}_\emptyset\}$, a contradiction. \square

We conclude with an example demonstrating that, in general, the invariant probability measure in Proposition 6.5 will not have full support on X , even in the case of a \mathbb{Z} -action.

Example 6.6. Let $Q \subset \mathbb{Z}$ be a non-piecewise-syndetic set of positive lower density. (One can take, for instance, a Straus set in \mathbb{Z} .) Since Q is not syndetic, its characteristic function $\mathbf{1}_Q$ has two consecutive zeroes. Translating Q if necessary we may assume that $0, 1 \notin Q$. Consider now the set $Q' := 2Q \cup (2Q + 1) \cup \{1\}$. This set still has positive lower density, and is not piecewise-syndetic since it is contained in the union of three translates of the non-piecewise-syndetic set $2Q$. Moreover, 1 is the only member of Q' both of whose neighbors are not in Q' . Let now $X \subset \{0, 1\}^{\mathbb{Z}}$ be the orbit closure of $\mathbf{1}_{Q'}$ and let $C := \{\omega \in \{0, 1\}^{\mathbb{Z}} : \omega(0) = \omega(2) = 0, \omega(1) = 1\}$. Then the cylinder sets $X \cap (C + n) = \{\mathbf{1}_{Q'+n}\}$ are disjoint singletons that are mapped to each other under the action of \mathbb{Z} . Hence any \mathbb{Z} -invariant probability measure assigns zero measure to them. Since they are open, an invariant probability measure cannot have full support.

References

[1] T. Austin, On the norm convergence of non-conventional ergodic averages, Ergodic Theory Dynam. Systems 30 (2) (2010) 321–338, <http://dx.doi.org/10.1017/S014338570900011X>.

- [2] M. Beiglböck, V. Bergelson, N. Hindman, D. Strauss, Multiplicative structures in additively large sets, *J. Combin. Theory Ser. A* 113 (7) (2006) 1219–1242, <http://dx.doi.org/10.1016/j.jcta.2005.11.003>.
- [3] V. Bergelson, T. Downarowicz, Large sets of integers and hierarchy of mixing properties of measure preserving systems, *Colloq. Math.* 110 (1) (2008) 117–150, <http://dx.doi.org/10.4064/cm110-1-4>.
- [4] V. Bergelson, H. Furstenberg, WM groups and Ramsey theory, *Topology Appl.* 156 (16) (2009) 2572–2580, <http://dx.doi.org/10.1016/j.topol.2009.04.007>.
- [5] V. Bergelson, M. Boshernitzan, J. Bourgain, Some results on nonlinear recurrence, *J. Anal. Math.* 62 (1994) 29–46, <http://dx.doi.org/10.1007/BF02835947>.
- [6] V. Bergelson, B. Host, B. Kra, Multiple recurrence and nilsequences, *Invent. Math.* 160 (2) (2005) 261–303, <http://dx.doi.org/10.1007/s00222-004-0428-6>, with an appendix by Imre Ruzsa.
- [7] J.F. Berglund, H.D. Jungheun, P. Milnes, *Compact Right Topological Semigroups and Generalizations of Almost Periodicity*, *Lecture Notes in Math.*, vol. 663, Springer, Berlin, 1978.
- [8] Q. Chu, Multiple recurrence for two commuting transformations, *Ergodic Theory Dynam. Systems* 31 (3) (2011) 771–792, <http://dx.doi.org/10.1017/S0143385710000258>, arXiv:0912.3381.
- [9] Q. Chu, P. Zorin-Kranich, Lower bound in the Roth theorem for amenable groups, *Ergodic Theory Dynam. Systems* 35 (06) (2015) 1746–1766, <http://dx.doi.org/10.1017/etds.2014.13>, arXiv:1309.6095.
- [10] P. Csillag, Über die gleichmässige Verteilung nichtganzer positiver Potenzen mod 1, *Acta Szeged* 5 (1930) 13–18.
- [11] C.W. Curtis, *I. Reiner, Representation Theory of Finite Groups and Associative Algebras*, Wiley Classics Library, John Wiley & Sons, Inc., New York, 1988, reprint of the 1962 original, A Wiley-Interscience Publication.
- [12] M. Di Nasso, I. Goldbring, R. Jin, S. Leth, M. Lupini, K. Mahlborg, Progress on a sumset conjecture of Erdős, preprint, arXiv:1307.0767, 2013.
- [13] H.A. Dye, On the ergodic mixing theorem, *Trans. Amer. Math. Soc.* 118 (1965) 123–130.
- [14] P. Erdős, A survey of problems in combinatorial number theory, in: *Combinatorial Mathematics, Optimal Designs and Their Applications*, Proc. Sympos. Combin. Math. and Optimal Design, in: *Ann. Discrete Math.*, vol. 6, Colorado State Univ., Fort Collins, Colo., 1978, 1980, pp. 89–115.
- [15] G.B. Folland, *A Course in Abstract Harmonic Analysis*, Stud. Adv. Math., CRC Press, Boca Raton, FL, 1995.
- [16] E. Følner, On groups with full Banach mean value, *Math. Scand.* 3 (1955) 243–254.
- [17] H. Furstenberg, *Recurrence in Ergodic Theory and Combinatorial Number Theory*, M.B. Porter Lectures, Princeton University Press, Princeton, NJ, 1981.
- [18] H. Furstenberg, Y. Katznelson, An ergodic Szemerédi theorem for IP-systems and combinatorial theory, *J. Anal. Math.* 45 (1985) 117–168, <http://dx.doi.org/10.1007/BF02792547>.
- [19] J. Gillis, Note on a property of measurable sets, *J. Lond. Math. Soc.* S1-11 (2) (1936) 139–141, <http://dx.doi.org/10.1112/jlms/s1-11.2.139>.
- [20] E. Hewitt, K.A. Ross, *Abstract Harmonic Analysis*, vol. I, 2nd edition, Grundlehren Math. Wiss. (Fundamental Principles of Mathematical Sciences), vol. 115, Springer-Verlag, Berlin, 1979.
- [21] N. Hindman, Finite sums from sequences within cells of a partition of N , *J. Combin. Theory Ser. A* 17 (1974) 1–11.
- [22] N. Hindman, Ultrafilters and combinatorial number theory, in: *Number Theory*, Carbondale, 1979, Proc. Southern Illinois Conf., Southern Illinois Univ., Carbondale, Ill., 1979, in: *Lecture Notes in Math.*, vol. 751, Springer, Berlin, 1979, pp. 119–184.
- [23] B. Host, Ergodic seminorms for commuting transformations and applications, *Studia Math.* 195 (1) (2009) 31–49, <http://dx.doi.org/10.4064/sm195-1-3>, arXiv:0811.3703.
- [24] K. Juschenko, N. Monod, Cantor systems, piecewise translations and simple amenable groups, *Ann. of Math.* (2) 178 (2) (2013) 775–787, <http://dx.doi.org/10.4007/annals.2013.178.2.7>.
- [25] A. Malcev, On isomorphic matrix representations of infinite groups, *Rec. Math. [Mat. Sb.] N.S.* 8 (50) (1940) 405–422.
- [26] D.S. Ornstein, B. Weiss, Entropy and isomorphism theorems for actions of amenable groups, *J. Anal. Math.* 48 (1987) 1–141, <http://dx.doi.org/10.1007/BF02790325>.
- [27] A.L.T. Paterson, *Amenability*, Math. Surveys Monogr., vol. 29, American Mathematical Society, Providence, RI, 1988.
- [28] D. Robertson, Characteristic factors for commuting actions of amenable groups, *J. Analyse Math.* (2015), in press, arXiv:1402.3843.
- [29] A. Rosenthal, Strictly ergodic models and amenable group action, Ph.D. thesis, Paris 6, 1986.

- [30] K. Schmidt, Asymptotic properties of unitary representations and mixing, *Proc. Lond. Math. Soc.* (3) 48 (3) (1984) 445–460, <http://dx.doi.org/10.1112/plms/s3-48.3.445>.
- [31] I.E. Segal, Invariant measures on locally compact spaces, *J. Indian Math. Soc. (N.S.)* 13 (1949) 105–130.
- [32] A.I. Shtern, A version of van der Waerden’s theorem and a proof of Mishchenko’s conjecture for homomorphisms of locally compact groups, *Izv. Ross. Akad. Nauk Ser. Mat.* 72 (1) (2008) 183–224, <http://dx.doi.org/10.1070/IM2008v072n01ABEH002397>.
- [33] A.I. Shtern, The structure of homomorphisms of connected locally compact groups into compact groups, *Izv. Ross. Akad. Nauk Ser. Mat.* 75 (6) (2011) 195–222, <http://dx.doi.org/10.1070/IM2011v075n06ABEH002572>.
- [34] J. von Neumann, Almost periodic functions in a group. I, *Trans. Amer. Math. Soc.* 36 (1934) 445–492, <http://dx.doi.org/10.2307/1989792>.
- [35] J. von Neumann, E.P. Wigner, Minimally almost periodic groups, *Ann. of Math. (2)* 41 (1940) 746–750.
- [36] P. Walters, *An Introduction to Ergodic Theory*, *Grad. Texts in Math.*, vol. 79, Springer-Verlag, New York, 1982.