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Polynomials at Iterated Spectra Near Zero

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Abstract

Central sets in $\mathbb{N}$ are sets known to have substantial combinatorial structure. Given $x \in \mathbb{R}$, let $w(x) = x - \lfloor x + \frac{1}{2} \rfloor$. Kronecker's Theorem [19] says that if $1, \alpha_1, \alpha_2, \ldots, \alpha_v$ are linearly independent over $\mathbb{Q}$ and U is an open subset of $(-\frac{1}{2}, \frac{1}{2})^v$, then $\{x \in \mathbb{N} : (w(\alpha_1 x), \ldots, w(\alpha_v x)) \in U\}$ is nonempty and Weyl [22] showed that this set has positive density. In a previous paper we showed that if $\overline{0}$ is in the closure of U, then this set is central. More generally, let $P_1, P_2, \ldots, P_v$ be real polynomials with zero constant term. We showed that

$$\{x \in \mathbb{N} : (w(P_1(x)), \ldots, w(P_v(x))) \in U\}$$

is non empty for every open U with $\overline{0} \in c\ell U$ if and only if it is central for every

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such U and we obtained a simple necessary and sufficient condition for these to occur.

In this paper we show that the same conclusion applies to compositions of polynomials with functions of the form $n \mapsto \lfloor \alpha n + \gamma \rfloor$ where α is a positive real and $0 < \gamma < 1$. (The ranges of such functions are called *nonhomogeneous spectra* and by extension we refer to the functions as spectra.) We characterize precisely when we can compose with a single function of the form $n \mapsto \lfloor \alpha n \rfloor$ or $n \mapsto \lfloor \alpha n + 1 \rfloor$. With the stronger assumption that U is a neighborhood of $\overline{0}$, we show when we can allow the composition with two such spectra and investigate some related questions.

Key words: central set, IP set, Stone-Čech compactification, spectra of numbers, iterated spectra
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1. Introduction

Let $v \in \mathbb{N}$, let $P_1, P_2, \ldots, P_v$ be real polynomials with zero constant term, and let U be an open subset of $(-\frac{1}{2}, \frac{1}{2})^v$ with $\overline{0} \in c\ell U$. Let $\nu(x) = \lfloor x + \frac{1}{2} \rfloor$, the nearest integer to x, and let $w(x)$ be defined as in the abstract, so that $w(x) = x - \nu(x)$. In [8] we showed that if

$$\{x \in \mathbb{N} : \big(w\big(P_1(x)\big), \ldots, w\big(P_v(x)\big)\big) \in U\} \neq \emptyset$$

for every such U, then it is large; it must be *central*, in fact *strongly central*. (We will present the definitions of these terms shortly.) Related results from [12] and [10] deal with the function $||x|| = |w(x)|$. When one is dealing with neighborhoods of $\overline{0}$, both notations are equally convenient. But the use of $w(x)$ allows us to distinguish between points which are close to zero from the right and points which are close to zero from the left.

We take $\mathbb{N}$ to be the set of positive integers and $\omega = \mathbb{N} \cup \{0\}$.

Definition 1.1. Let α be a positive real and let $0 \leq \gamma \leq 1$. The function $g_{\alpha,\gamma} : \mathbb{N} \to \mathbb{N}$ is defined by $g_{\alpha,\gamma}(n) = \lfloor \alpha n + \gamma \rfloor$ for each $n \in \mathbb{N}$.

In terminology introduced by Graham, Lin, and Lin in [13], $g_{\alpha,\gamma}[\mathbb{N}]$ is called the γ-non homogeneous spectrum of α. See [6] for a discussion of the history of the study of such spectra.

We shall be concerned in this paper primarily with determining conditions under which one may conclude that $\{x \in \mathbb{N} : \big(w\big(P_1(x)\big), \ldots, w\big(P_v(x)\big)\big) \in U\}$ is central or strongly central where U is only assumed to have $\overline{0}$ in its closure and each P_i is a polynomial composed with one or more functions of the form $g_{\alpha,\gamma}$.

In order to discuss the notions of largeness with which we are dealing, we need to briefly discuss the algebraic structure of the Stone-Čech compactification of a discrete semigroup. For an elementary introduction to this structure and any unfamiliar facts mentioned here, see [18]. Or see the papers [1], [2], or [3], with the caution that there βS is taken to be left topological rather than right topological.

Let $(S,+)$ be a discrete semigroup. (We shall be primarily concerned with subsemigroups of $(\mathbb{R},+)$ so we shall denote the operation by $+$. However, we are not assuming that S is commutative. And even if S is commutative, it is very unlikely that $(\beta S,+)$ is commutative. In particular, if S is a subsemigroup of $(\mathbb{R},+)$, then $(\beta S,+)$ is not commutative.) The points of βS are the ultrafilters on S, the principal ultrafilters being identified with the points of S, allowing us to pretend that $S \subseteq \beta S$. There is a unique extension of the operation to βS making $(\beta S,+)$ a right topological semigroup (meaning that for each $p \in \beta S$, the function ρ_p is continuous where $\rho_p(q) = q+p$) with the additional properth that for each $x \in S$, the function λ_x is continuous where $\lambda_x(q) = x+q$. Given $A \subseteq S$, $\overline{A} = \{p \in \beta S : A \in p\}$ and $\{\overline{A} : A \subseteq S\}$ is a basis for the open sets (as well as a basis for the closed sets) of βS. Given $p,q \in \beta S$ and $A \subseteq S$, $A \in p+q$ if and only if $\{x \in S : -x+A \in q\} \in p$ where $-x+A = \{y \in S : x+y \in A\}$. (There no requirement that S has an identity, nor, even if it does, that it is a group. However, if S is a group, then $-x+A = \{-x+y : y \in A\}$.)

Any compact Hausdorff right topological semigroup $(T,+)$ has idempotents [11, Lemma 1]. Let $E(T)$ be the set of idempotents in T. A set $A \subseteq S$ is said to be an *IP set* if and only if A is a member of an idempotent in βS. Equivalently, A is an IP set if and only if there is a sequence $\langle x_n \rangle_{n=1}^{\infty}$ such that $FS(\langle x_n \rangle_{n=1}^{\infty}) \subseteq A$, where $FS(\langle x_n \rangle_{n=1}^{\infty}) = \{\sum_{t \in F} x_t : F \in \mathcal{P}_f(\mathbb{N})\}$ and for any set X, $\mathcal{P}_f(F)$ is the set of finite nonempty subsets of X. The sums in $\sum_{t \in F} x_t$ are taken in increasing order of indices. A set $A \subseteq S$ is an *IP* set* if and only if it has nontrivial intersection with every IP set. Equivalently A is an IP* set if and only if it is a member of every idempotent in βS.

Any compact Hausdorff right topological semigroup $(T,+)$ has a smallest two sided ideal $K(T)$, which is the union of all minimal left ideals of T and is also the union of all minimal right ideals of T. The intersection of any minimal left ideal and any minimal right ideal is a group, and any two such groups are isomorphic. An idempotent is said to be *minimal* if and only if it is a member of $K(T)$.

If S is a discrete space and C is a compact Hausdorff space, then any mapping $f : S \to C$ has a continuous extension $\widetilde{f} : \beta S \to C$.

Central subsets of $\mathbb{N}$ were introduced by Furstenberg in [12], defined in terms of the notions *proximal* and *uniformly recurrent* of topological dynamics. The property of being central was shown in [5] (with help from B. Weiss) to be equivalent to being a member of a minimal idempotent. (Later Shi and Yang [21] showed that the natural extension of Furstenberg's definition to an arbitrary semigroup S is equivalent to membership in a minimal idempotent of βS.) We take this to be the definition of central. That is, $A \subseteq S$ is *central* if and only if A is a member of some minimal idempotent. And A is *central** if and only if it is a member of every minimal idempotent, equivalently it has nontrivial intersection with every central set. From the above description, one easily sees that A is central if and only if there is a minimal left ideal L of βS such that A is a member of some idempotent in L. We say that A is *strongly central* if and only if for every minimal left ideal L of βS, there is some idempotent $p \in L$ such that $A \in p$.

Central sets are guaranteed to contain substantial combinatorial structure. (See [12, Chapter 8] and [18, Chapters 14 through 16] for examples of much of this structure.) Further, from the definition, it is easy to see that the notions of central and IP are partition regular. That is, if A is central (respectively IP), and A is divided into finitely many sets, then one of those sets is central (respectively IP).

If A is a subset of $\mathbb{N}$ which contains arbitrarily long intervals, then $\overline{A}$ contains a left ideal of $\beta\mathbb{N}$ by [7, Theorem 2.9]. Therefore if both A and its complement contain arbitrarily long intervals, then A is central but not strongly central. For example A could consist of those integers that have even maximum of their binary supports. We will show that several of the sets with which we are interested in this paper are strongly central if and only if they are central.

In Section 2 we shall show that the characterization given in [8] describing when $\{x \in \mathbb{N} : (w(P_1(x)), \ldots, w(P_v(x))) \in U\}$ is central extends to polynomials composed with finitely many functions of the form $g_{\alpha,\gamma}$ with $0 < \gamma < 1$. For example, suppose $P_1(x) = x^2$ and $P_2(x) = 2x^3 - x$, α and δ are positve reals, and γ and μ are elements of the interval $(0, 1)$. Then, with

$$Q_1(x) = P_1 \circ g_{\alpha,\gamma} \circ g_{\delta,\mu}(x) = \big\lfloor \alpha \lfloor \delta x + \mu \rfloor + \gamma \big\rfloor^2 \text{ and}$$
$$Q_2(x) = P_2 \circ g_{\alpha,\gamma} \circ g_{\delta,\mu}(x) = 2\big\lfloor \alpha \lfloor \delta x + \mu \rfloor + \gamma \big\rfloor^3 - \big\lfloor \alpha \lfloor \delta x + \mu \rfloor + \gamma \big\rfloor\,,$$

and with U any open subset of $(-\frac{1}{2}, \frac{1}{2})^2$ with $\overline{0} \in c\ell U$, one is guaranteed that $\{x \in \mathbb{N} : \big(w\big(Q_1(x)\big), w\big(Q_2(x)\big)\big) \in U\}$ is large exactly when $\{x \in \mathbb{N} : \big(w\big(P_1(x)\big), w\big(P_2(x)\big)\big) \in U\}$ is large.

The functions $g_{\alpha,0}$ and $g_{\alpha,1}$ are not nearly so nice. We will characterize precisely when we can guarantee that

$$\{x \in \mathbb{N} : \big(w\big(Q_1(x)\big), w\big(Q_2(x)\big), \ldots, w\big(Q_v(x)\big)\big) \in U\}$$

is large where $Q_u(x) = P_u \circ g_{\alpha,i}$ and $i \in \{0, 1\}$.

In Section 3 we address the issues of when we can guarantee that

$$\{x \in \mathbb{N} : \big(w\big(Q_1(x)\big), w\big(Q_2(x)\big), \ldots, w\big(Q_v(x)\big)\big) \in U\}$$

is large where $Q_u(x) = P_u \circ g_{\alpha,i} \circ g_{\delta,j}$ and $i, j \in \{0, 1\}$. We are only able to do this under the much stronger assumption that U is a neighborhood of $\overline{0}$. And, surprisingly, the answer depends on whether or not $i = j$.

In Section 4 we turn our attention to the *generalized polynomials* studied in [14], [4], [9], [10], [15], and [17]. Generalized polynomials allow expressions involving the greatest integer function such as $2x^2\big\lfloor x + 3x^5\lfloor 2x^2 - 3x\rfloor\big\rfloor$. The examples of $Q_1 = P_1 \circ g_{\alpha,\gamma} \circ g_{\delta,\mu}$ and $Q_2 = P_2 \circ g_{\alpha,\gamma} \circ g_{\delta,\mu}$ given three paragraphs above are also generalized polynomials. We obtain a characterization of those generalized polynomials P that have the property that $\{x \in \mathbb{R} : w\big(\alpha P(x)\big) \in U\}$ is IP* whenever $\alpha \in \mathbb{R}$ and U is a neighborhood of 0.

2. Evaluating Polynomials at Iterated Spectra

In [6] it was shown that if $0 < \gamma < 1$, then $g_{\alpha,\gamma}$ (see Definition 1.1) preserves much of the largeness structure of $\mathbb{N}$. In particular, it takes central sets to central sets. A key to this was the fact that, while $g_{\alpha,\gamma}$ is not a homomorphism, its continuous extension to $\beta\mathbb{N}$ is a homomorphism on a natural subset of $\beta\mathbb{N}$ containing all of the idempotents.

We regard the circle group $\mathbb{T}$ as being $\mathbb{R}/\mathbb{Z}$, and we shall denote it additively. We shall use real numbers in $[-\frac{1}{2},\frac{1}{2})$ to denote the corresponding elements of $\mathbb{T}$. Then w can be regarded as a mapping from $\mathbb{R}$ to $\mathbb{T}$, and is then, of course, the canonical homomorphism from $\mathbb{R}$ onto $\mathbb{R}/\mathbb{Z}$. It follows that, if $\mathbb{R}_d$ denotes $\mathbb{R}$ with the discrete topology, then $\widetilde{w} : \beta\mathbb{R}_d \to \mathbb{T}$ is also a homomomorphism (by [18, Corollary 4.22], due originally to P. Milnes in [20]). Recall that $\omega = \mathbb{N}\cup\{0\}$.

Definition 2.1. Let α be a positive real, let $h_\alpha = g_{\alpha,1/2}$, and define $f_\alpha : \mathbb{N} \to \mathbb{T}$ by $f_\alpha(x) = w(\alpha x)$. Let $\widetilde{h}_\alpha : \beta\mathbb{N} \to \beta\omega$ and $\widetilde{f}_\alpha : \beta\mathbb{N} \to \mathbb{T}$ be their continuous extensions. Then

$$\begin{array}{rcl} Z_\alpha & = & \{p \in \beta\mathbb{N} : \widetilde{f}_\alpha(p) = 0\}\,, \\ X_\alpha & = & \{p \in \beta\mathbb{N} : \{x \in \mathbb{N} : 0 < f_\alpha(x) < \frac{1}{2}\} \in p\}, \text{ and} \\ Y_\alpha & = & \{p \in \beta\mathbb{N} : \{x \in \mathbb{N} : -\frac{1}{2} < f_\alpha(x) < 0\} \in p\}\,. \end{array}$$

Note that for $x \in \mathbb{N}$, $f_\alpha(x) = \alpha x - h_\alpha(x)$.

Theorem 2.2. *Let α be a positive real. Then Z_α is a compact subsemigroup of $\beta\mathbb{N}$ containing the idempotents and $\widetilde{h}_\alpha$ is an isomorphism and a homeomorphism from Z_α onto $Z_{1/\alpha}$. If α is irrational, then X_α and Y_α are subsemigroups of $\beta\mathbb{N}$, $Z_\alpha = X_\alpha \cup Y_\alpha$, and $\widetilde{h}_\alpha$ takes X_α onto $Y_{1/\alpha}$ and takes Y_α onto $X_{1/\alpha}$. If α is irrational, $0 < \gamma < 1$, and $\widetilde{g}_{\alpha,\gamma} : \beta\mathbb{N} \to \beta\omega$ is the continuous extension of $g_{\alpha,\gamma}$, then for all $p \in Z_\alpha$, $\widetilde{g}_{\alpha,\gamma}(p) = \widetilde{h}_\alpha(p)$.*

Proof. [6, Lemma 5.7 and Theorems 5.8 and 5.10]. □

The following lemma is Lemma 5.12(a) of [6], except that we had an unnecessary additional hypothesis there.

Lemma 2.3. *Let $(S,+)$ be a compact Hausdorff right topological semigroup and let T be a compact subsemigroup of S. If M is a minimal left ideal of S and $M \cap T \neq \emptyset$, then $M \cap T$ is a minimal left ideal of T.*

Proof. Trivially $M \cap T$ is a left ideal of T, so pick a minimal left ideal L of T such that $L \subseteq M \cap T$. To see that $M \cap T \subseteq L$ let $x \in M \cap T$. Pick an idempotent $e \in L$. Then $e \in M$ and $M + e$ is a left ideal of S contained in M so $M = M + e$. Therefore $x \in M + e$ and so $x = x + e$ so $x \in T + e$. Since $T + e$ is a left ideal of T contained in L, $L = T + e$, so $x \in L$. □

Lemma 2.4. *Let $(S,+)$ be a compact Hausdorff right topological semigroup and let T be a compact subsemigroup of S. If $K(S) \cap T \neq \emptyset$ and L is a minimal left ideal of T, then there is a minimal left ideal M of S such that $M \cap T = L$.*

Proof. Pick $x \in L$. Then $x \in K(T) \subseteq K(S)$ so $S+x$ is a minimal left ideal of S. By Lemma 2.3, $(S+x) \cap T$ is a minimal left ideal of T and $L = T+x \subseteq (S+x) \cap T$ so $L = (S+x) \cap T$. $\square$

We now show that we can preserve the characterization given by [8, Theorem 2.8] when the polynomials are replaced by the composition of polynomials with finitely many iterated spectra.

Theorem 2.5. *Let $v, m \in \mathbb{N}$ and for $u \in \{1, 2, \ldots, v\}$, let P_u be a polynomial with real coefficients and zero constant term. For $t \in \{1, 2, \ldots, m\}$, let α_t be a positive real and let $0 < \gamma_t < 1$. For $u \in \{1, 2, \ldots, v\}$ let $Q_u = P_u \circ g_{\alpha_1,\gamma_1} \circ \cdots \circ g_{\alpha_m,\gamma_m}$. The following statements are equivalent.*

(a) *Whenever U is an open subset of $(-\frac{1}{2}, \frac{1}{2})^v$ with $\overline{0} \in c\ell U$, $\{x \in \mathbb{N} : \big(w\big(P_1(x)\big), w\big(P_2(x)\big), \ldots, w\big(P_v(x)\big)\big) \in U\}$ is strongly central.*

(b) *Whenever U is an open subset of $(-\frac{1}{2}, \frac{1}{2})^v$ with $\overline{0} \in c\ell U$, $\{x \in \mathbb{N} : \big(w\big(P_1(x)\big), w\big(P_2(x)\big), \ldots, w\big(P_v(x)\big)\big) \in U\}$ is central.*

(c) *Whenever U is an open subset of $(-\frac{1}{2}, \frac{1}{2})^v$ with $\overline{0} \in c\ell U$, $\{x \in \mathbb{N} : \big(w\big(P_1(x)\big), w\big(P_2(x)\big), \ldots, w\big(P_v(x)\big)\big) \in U\} \neq \emptyset$.*

(d) *Any nontrivial linear combination of $\big\{P_u : u \in \{1, 2, \ldots, v\}\big\}$ over $\mathbb{Q}$ has at least one irrational coefficient.*

(e) *Whenever U is an open subset of $(-\frac{1}{2}, \frac{1}{2})^v$ with $\overline{0} \in c\ell U$, $\{x \in \mathbb{N} : \big(w\big(Q_1(x)\big), w\big(Q_2(x)\big), \ldots, w\big(Q_v(x)\big)\big) \in U\}$ is strongly central.*

(f) *Whenever U is an open subset of $(-\frac{1}{2}, \frac{1}{2})^v$ with $\overline{0} \in c\ell U$, $\{x \in \mathbb{N} : \big(w\big(Q_1(x)\big), w\big(Q_2(x)\big), \ldots, w\big(Q_v(x)\big)\big) \in U\}$ is central.*

(g) *Whenever U is an open subset of $(-\frac{1}{2}, \frac{1}{2})^v$ with $\overline{0} \in c\ell U$, $\{x \in \mathbb{N} : \big(w\big(Q_1(x)\big), w\big(Q_2(x)\big), \ldots, w\big(Q_v(x)\big)\big) \in U\} \neq \emptyset$.*

Proof. Statements (a) through (d) are equivalent by [8, Theorem 2.8] and statement (g) trivially implies statement (c). So it suffices to show that statement (a) implies statement (e). The proof is by induction on m. Let $m \in \mathbb{N}$ and assume the conclusion holds for $m-1$. (In the case that $m = 1$, we are simply assuming that statement (a) holds.)

Let U be an open subset of $(-\frac{1}{2}, \frac{1}{2})^v$ with $\overline{0} \in c\ell U$ and let

$$B = \{x \in \mathbb{N} : \big(w\big(Q_1(x)\big), w\big(Q_2(x)\big), \ldots, w\big(Q_v(x)\big)\big) \in U\}.$$

For $u \in \{0, 1, \ldots, m-1\}$, let $R_u = P_u \circ g_{\alpha_1,\gamma_1} \circ \cdots \circ g_{\alpha_{m-1},\gamma_{m-1}}$ and let

$$C = \{x \in \mathbb{N} : \big(w\big(R_1(x)\big), w\big(R_2(x)\big), \ldots, w\big(R_v(x)\big)\big) \in U\}.$$

Let L be a minimal left ideal of $(\beta\mathbb{N}, +)$. By the induction hypothesis, we know there is an idempotent $r \in L \cap \overline{C}$. Consequently, we have by Theorem 2.2 that

$L \cap Z_{\alpha_m} \neq \emptyset$ and therefore is a minimal left ideal of Z_{α_m} by Lemma 2.3. Let k be the restriction of $\widetilde{g}_{\alpha_m,\gamma_m}$ to Z_{α_m}. Then by Theorem 2.2, k is a homeomorphism and an isomorphism onto Z_{1/α_m} and so $M = k[L \cap Z_{\alpha_m}]$ is a minimal left ideal of Z_{1/α_m} and is therefore the intersection of a minimal left ideal M' of $\beta\mathbb{N}$ with Z_{1/α_m}. Pick an idempotent $p \in M' \cap \overline{C}$. Then $p \in Z_{1/\alpha_m}$ so $k^{-1}(p)$ is an idempotent in L. Since $C \in p$, $g^{-1}_{\alpha_m,\gamma_m}[C] \in k^{-1}(p)$. And $g^{-1}_{\alpha_m,\gamma_m}[C] \subseteq B$. $\square$

As was shown in [6, Section 6], if α is irrational, then the functions $g_{\alpha,0}$ and $g_{\alpha,1}$ have much weaker properties than $g_{\alpha,\gamma}$ for $0 < \gamma < 1$. They take central* sets to central sets, but they do not even take all central sets to IP sets. We characterize now when we can add one function of the form $g_{\alpha,0}$ or $g_{\alpha,1}$ where we could add finitely many of the form $g_{\alpha,\gamma}$ with $0 < \gamma < 1$.

Theorem 2.6. *Let $v \in \mathbb{N}$ and for $u \in \{1, 2, \ldots, v\}$, let P_u be a polynomial with real coefficients and zero constant term. Let α be a positive irrational. The following statements are equivalent.*

(a) *For every minimal left ideal L of $\beta\mathbb{N}$ and every open $U \subseteq (-\frac{1}{2}, \frac{1}{2})^v$ with $\overline{0} \in c\ell U$, there is an idempotent $q \in L \cap X_\alpha$ such that* $\{x \in \mathbb{N} : \big(w\big(P_1(g_{\alpha,0}(x))\big), w\big(P_2(g_{\alpha,0}(x))\big), \ldots, w\big(P_v(g_{\alpha,0}(x))\big)\big) \in U\} \in q$.

(a′) *For every minimal left ideal L of $\beta\mathbb{N}$ and every open $U \subseteq (-\frac{1}{2}, \frac{1}{2})^v$ with $\overline{0} \in c\ell U$, there is an idempotent $q \in L \cap Y_\alpha$ such that* $\{x \in \mathbb{N} : \big(w\big(P_1(g_{\alpha,1}(x))\big), w\big(P_2(g_{\alpha,1}(x))\big), \ldots, w\big(P_v(g_{\alpha,1}(x))\big)\big) \in U\} \in q$.

(b) *For every open $U \subseteq (-\frac{1}{2}, \frac{1}{2})^v$ with $\overline{0} \in c\ell U$,* $\{x \in \mathbb{N} : \big(w\big(P_1(g_{\alpha,0}(x))\big), w\big(P_2(g_{\alpha,0}(x))\big), \ldots, w\big(P_v(g_{\alpha,0}(x))\big)\big) \in U\}$ *is strongly central.*

(b′) *For every open $U \subseteq (-\frac{1}{2}, \frac{1}{2})^v$ with $\overline{0} \in c\ell U$,* $\{x \in \mathbb{N} : \big(w\big(P_1(g_{\alpha,1}(x))\big), w\big(P_2(g_{\alpha,1}(x))\big), \ldots, w\big(P_v(g_{\alpha,1}(x))\big)\big) \in U\}$ *is strongly central.*

(c) *For every open $U \subseteq (-\frac{1}{2}, \frac{1}{2})^v$ with $\overline{0} \in c\ell U$,* $\{x \in \mathbb{N} : \big(w\big(P_1(g_{\alpha,0}(x))\big), w\big(P_2(g_{\alpha,0}(x))\big), \ldots, w\big(P_v(g_{\alpha,0}(x))\big)\big) \in U\}$ *is central.*

(c′) *For every open $U \subseteq (-\frac{1}{2}, \frac{1}{2})^v$ with $\overline{0} \in c\ell U$,* $\{x \in \mathbb{N} : \big(w\big(P_1(g_{\alpha,1}(x))\big), w\big(P_2(g_{\alpha,1}(x))\big), \ldots, w\big(P_v(g_{\alpha,1}(x))\big)\big) \in U\}$ *is central.*

(d) *For every open $U \subseteq (-\frac{1}{2}, \frac{1}{2})^v$ with $\overline{0} \in c\ell U$,* $\{x \in \mathbb{N} : \big(w\big(P_1(g_{\alpha,0}(x))\big), w\big(P_2(g_{\alpha,0}(x))\big), \ldots, w\big(P_v(g_{\alpha,0}(x))\big)\big) \in U\}$ *is an IP set.*

(d′) *For every open $U \subseteq (-\frac{1}{2}, \frac{1}{2})^v$ with $\overline{0} \in c\ell U$,* $\{x \in \mathbb{N} : \big(w\big(P_1(g_{\alpha,1}(x))\big), w\big(P_2(g_{\alpha,1}(x))\big), \ldots, w\big(P_v(g_{\alpha,1}(x))\big)\big) \in U\}$ *is an IP set.*

(e) *For every open* $U \subseteq (-\frac{1}{2}, \frac{1}{2})^v$ *with* $\overline{0} \in c\ell U$*, there exists* $q \in Z_\alpha$ *such that* $\{x \in \mathbb{N} : \big(w\big(P_1(g_{\alpha,0}(x))\big), w\big(P_2(g_{\alpha,0}(x))\big), \ldots, w\big(P_v(g_{\alpha,0}(x))\big)\big) \in U\} \in q.$

(e′) *For every open* $U \subseteq (-\frac{1}{2}, \frac{1}{2})^v$ *with* $\overline{0} \in c\ell U$*, there exists* $q \in Z_\alpha$ *such that* $\{x \in \mathbb{N} : \big(w\big(P_1(g_{\alpha,1}(x))\big), w\big(P_2(g_{\alpha,1}(x))\big), \ldots, w\big(P_v(g_{\alpha,1}(x))\big)\big) \in U\} \in q.$

(f) *If* $P_0(x) = \frac{1}{\alpha}x$*, and* $(c_0, c_1, \ldots, c_v) \in \mathbb{Q}^{v+1} \setminus \{\overline{0}\}$*, then* $\sum_{u=0}^{v} c_u P_u$ *has at least one irrational coefficient.*

Proof. Let $k = \max\big\{\deg(P_u) : u \in \{1, 2, \ldots, v\}\big\}$ and pick $\langle a_{u,s}\rangle_{s=1}^{k}$ in $\mathbb{R}$ for each $u \in \{1, 2, \ldots, v\}$ such that for $x \in \mathbb{R}$, $P_u(x) = \sum_{s=1}^{k} a_{u,s}x^s$.

It suffices to show that (e) implies (f), (e') implies (f), (f) implies (a), and (f) implies (a').

To see that (e) implies (f), assume that (e) holds and that (f) does not. Then there are integers, $c_0, c_1, c_2, \cdots, c_v$, not all zero, for which $\sum_{u=0}^{v} c_u P_u$ has integer coefficients. So $w\big(\sum_{u=0}^{v} c_u P_u(n)\big) = 0$ for every $n \in \mathbb{N}$. If $c_0 = 0$, then by Theorem 2.5, there is some open $U \subseteq (-\frac{1}{2}, \frac{1}{2})^v$ with $\overline{0} \in c\ell U$ such that

$$\{x \in \mathbb{N} : \big(w\big(P_1(x)\big), w\big(P_2(x)\big), \ldots, w\big(P_v(x)\big)\big) \in U\} = \emptyset\,.$$

So we may assume that $c_0 > 0$.

Let $\tau : \mathbb{T}^v \to \mathbb{T}$ be defined by $\tau(t_1, t_2, \ldots, t_v) = \sum_{u=1}^{v} c_u t_u$. Now choose $d \in (0, \frac{1}{2})$ for which $w(\frac{c_0}{\alpha}) \notin (-d, d)$ and let $U = \tau^{-1}[(-d, 0)] \cap (-\frac{1}{2}, \frac{1}{2})^v$. Let $A = \{n \in \mathbb{N} : \big(w\big(P_1(g_{\alpha,0}(n))\big), w\big(P_2(g_{\alpha,0}(n))\big), \ldots, w\big(P_v(g_{\alpha,0}(n))\big)\big) \in U\}$ and pick $q \in Z_\alpha$ such that $A \in q$.

First suppose that $q \in X_\alpha$ and let $B = \{n \in \mathbb{N} : \lfloor n\alpha\rfloor < n\alpha < \lfloor n\alpha\rfloor + \frac{d\alpha}{c_0}\}$. Then $B \in q$ so pick $n \in A \cap B$ and let $m = \lfloor n\alpha\rfloor$. Then $c_0 n > \frac{c_0 m}{\alpha} > c_0 n - d$ and $\frac{c_0 m}{\alpha} = c_0 P_0\big(g_{\alpha,0}(n)\big)$ so $-d < w\big(c_0 P_0(g_{\alpha,0}(n))\big) < 0$. Since

$$\tau\Big(w\big(P_1(g_{\alpha,0}(n))\big), w\big(P_2(g_{\alpha,0}(n))\big), \ldots, w\big(P_v(g_{\alpha,0}(n))\big)\Big) = -w\big(c_0 P_0(g_{\alpha,0}(n))\big)$$

we have a contradiction to the fact that $n \in A$.

Now suppose that $q \in Y_\alpha$ and let

$$B = \{n \in \mathbb{N} : \lfloor n\alpha\rfloor + 1 - \tfrac{d\alpha}{c_0} < n\alpha < \lfloor n\alpha\rfloor + 1\}\,.$$

Then $B \in q$ so pick $n \in A \cap B$ and let $m = \lfloor n\alpha\rfloor$. Then $c_0 n < \frac{c_0}{\alpha}(m+1) < c_0 n + d$ so $0 < w\big(\frac{c_0}{\alpha}(m+1)\big) < d$. Also

$$\tau\Big(w\big(P_1(g_{\alpha,0}(n))\big), w\big(P_2(g_{\alpha,0}(n))\big), \ldots, w\big(P_v(g_{\alpha,0}(n))\big)\Big) = -w\big(c_0 P_0(g_{\alpha,0}(n))\big)\,,$$

$\frac{c_0 m}{\alpha} = c_0 P_0\big(g_{\alpha,0}(n)\big)$, and $n \in A$ so $0 < w(\frac{c_0 m}{\alpha}) < d$. We thus have integers k and l such that $k < \frac{c_0}{\alpha}(m+1) < k + d$ and $l < \frac{c_0 m}{\alpha} < l + d$ from which it follows that $w(\frac{c_0}{\alpha}) \in (-d, d)$, a contradiction.

The fact that (e') implies (f) can be shown in a similar way. We omit the details.

To see that (f) implies (a), let

$$B = \{x \in \mathbb{N} : \big(w\big(P_1(g_{\alpha,0}(x))\big), w\big(P_2(g_{\alpha,0}(x))\big), \ldots, w\big(P_v(g_{\alpha,0}(x))\big)\big) \in U\}\,.$$

Note that by [6, Theorem 5.5], there is a right ideal U_α of $\beta\mathbb{N}$ such that $X_\alpha = U_\alpha \cap Z_\alpha$, so there is an idempotent in $L \cap U_\alpha$ and thus $L \cap X_\alpha \neq \emptyset$. By Lemma 2.3, $L \cap X_\alpha$ is a minimal left ideal of X_α. By [6, Lemma 5.9 and Theorem 5.10], $\widetilde{g}_{\alpha,0}$ is an isomorphism and a homeomorphism from X_α onto $Y_{1/\alpha}$ so $\widetilde{g}_{\alpha,0}[L \cap X_\alpha]$ is a minimal left ideal of $Y_{1/\alpha}$. Pick by Lemma 2.4 a minimal left ideal M of $\beta\mathbb{N}$ such that $M \cap Y_{1/\alpha} = \widetilde{g}_{\alpha,0}[L \cap X_\alpha]$. Let $\epsilon = \min\{\frac{1}{2}, \frac{1}{\alpha}\}$. Let $C = \{y \in \mathbb{N} : \big(w\big(P_0(y)\big), w\big(P_1(y)\big), \ldots, w\big(P_v(y)\big)\big) \in (-\epsilon, 0) \times U\}$. By Theorem 2.5, pick an idempotent $p \in M \cap \overline{C}$. Since p is an idempotent, $p \in Z_{1/\alpha}$. Since $C \subseteq \{y \in \mathbb{N} : w(\frac{1}{\alpha}y) \in (-\epsilon, 0)\}$, $p \in Y_{1/\alpha}$. Thus there is an idempotent $q \in L \cap X_\alpha$ such that $p = \widetilde{g}_{\alpha,0}(q)$. By [6, Lemma 5.9 and Theorem 5.10], $\widetilde{g}_{1/\alpha,1}(p) = q$. Since $C \in p$ it suffices to show that $g_{1/\alpha,1}[C] \subseteq B$. So let $y \in C$ and let $x = g_{1/\alpha,1}(y)$. Then since $w(\frac{1}{\alpha}y) \in (-\epsilon, 0)$, $x - \epsilon < \frac{1}{\alpha}y < x$ and so $y < \alpha x < y + \epsilon\alpha \leq y + 1$ and thus $y = g_{\alpha,0}(x)$ and so $x \in B$.

The proof that (f) implies (a') has only obvious changes from the proof that (f) implies (a). $\square$

3. Results assuming that U is a neighborhood of $\overline{0}$

In this section we establish some results utilizing a much stronger assumption. That is, we assume that U is a neighborhood of $\overline{0}$, rather than just an open set with $\overline{0}$ in its closure. (Of course, we would rather use the weaker assumption if we could.)

To set the stage, we note the following consequence of Theorem 4.2, showing that the stronger assumption can produce stronger conclusions. (The set B in the following theorem could not be an IP* set if both U and its complement had $\overline{0}$ in their closures.)

Theorem 3.1. *Let $v, m \in \mathbb{N}$ and for $u \in \{1, 2, \ldots, v\}$, let P_u be a polynomial with real coefficients and zero constant term. For $t \in \{1, 2, \ldots, m\}$, let α_t be a positive real and let $0 < \gamma_t < 1$. For $u \in \{1, 2, \ldots, v\}$ let*

$$Q_u = P_u \circ g_{\alpha_1,\gamma_1} \circ \cdots \circ g_{\alpha_m,\gamma_m}$$

and let U be a neighborhood of $\overline{0}$ in $(-\frac{1}{2}, \frac{1}{2})^v$. Then

$$B = \{x \in \mathbb{N} : \big(w\big(Q_1(x)\big), w\big(Q_2(x)\big), \ldots, w\big(Q_v(x)\big)\big) \in U\}$$

is an IP set in $\mathbb{N}$.*

Proof. Let p be an idempotent in $\beta\mathbb{N}$, let $k = g_{\alpha_1,\gamma_1} \circ \cdots \circ g_{\alpha_m,\gamma_m}$, and let $q = \widetilde{k}(p)$. Then by Theorem 2.2 applied m times, q is an idempotent so by Theorem 4.2, if $C = \{x \in \mathbb{N} : \big(w\big(P_1(x)\big), w\big(P_2(x)\big), \ldots, w\big(P_v(x)\big)\big) \in U\}$, then $C \in q$. Consequently $k^{-1}[C] \in p$ and $k^{-1}[C] \subseteq B$. $\square$

We now turn our attention to seeing to what extent we can let the γ's in Theorem 3.1 be 0 or 1. Restricting our attention to $g_{\alpha,\gamma} \circ g_{\delta,\tau}$ we obtain simple necessary and sufficient conditions for the existence of one (and thus many) idempotents p with B as a member in Theorem 3.9 in the case $\gamma = \tau$ and Theorem 3.11 in the case $\gamma \neq \tau$.

By $\mathbb{R}_d$ we mean the set $\mathbb{R}$ with the discrete topology.

Lemma 3.2. *Let p be an (additive) idempotent in $\beta\mathbb{R}_d$ and let $\alpha \in \mathbb{R} \setminus \{0\}$.*

(a) Let αp be the product in $(\beta\mathbb{R}_d, \cdot)$. Then αp is an additive idempotent in $\beta\mathbb{R}_d$.

(b) For every $\epsilon > 0$, $\{x \in \mathbb{R} : h_\alpha(x) - \epsilon < \alpha x < h_\alpha(x) + \epsilon\} \in p$.

Proof. (a). If $l_\alpha : \mathbb{R} \to \mathbb{R}$ is defined by $l_\alpha(x) = \alpha x$, then $\alpha p = \widetilde{l}_\alpha(p)$ where $\widetilde{l}_\alpha : \beta\mathbb{R}_d \to \beta\mathbb{R}_d$ is the continuous extension of l_α. Since l_α is a homomorphism on $(\mathbb{R}, +)$, $\widetilde{l}_\alpha$ is a homomorphism on $(\beta\mathbb{R}_d, +)$ by [18, Corollary 4.22] (due originally to P. Milnes in [20]), so $\widetilde{l}_\alpha(p)$ is an idempotent.

(b) Since $w : \mathbb{R} \to \mathbb{T}$ is a homomorphism, so is $\widetilde{w} : \beta\mathbb{R}_d \to \mathbb{T}$. Since αp is an idempotent, $\widetilde{w}(\alpha p) = 0$ so $\{y \in \mathbb{T} : -\epsilon < y < \epsilon\}$ is a neighborhood of $\widetilde{w}(\alpha p) = \widetilde{w} \circ \widetilde{l}_\alpha(p)$ so there exists $A \in p$ such that $\widetilde{w} \circ \widetilde{l}_\alpha[\overline{A}] \subseteq \{y \in \mathbb{T} : -\epsilon < y < \epsilon\}$. Then

$$A \subseteq \{x \in \mathbb{R} : h_\alpha(x) - \epsilon < \alpha x < h_\alpha(x) + \epsilon\}\,.$$

□

Lemma 3.3. *Let P be a polynomial with real coefficients and zero constant term and let $\widetilde{w \circ P} : \beta\mathbb{R}_d \to \mathbb{T}$ be the continuous extension of $w \circ P$. Then for every idempotent $p \in \beta\mathbb{R}_d$ and every $q \in \beta\mathbb{R}_d$, $\widetilde{w \circ P}(p) = 0$ and $\widetilde{w \circ P}(q+p) = \widetilde{w \circ P}(q)$.*

Proof. [8, Lemma 2.1]. □

Lemma 3.4. *Let α and δ be positive irrationals and let p be an idempotent in $\beta\mathbb{N}$. Then $\{n \in \mathbb{N} : h_\alpha\big(h_\delta(n)\big) = h_{\alpha\delta}(n)\} \in p$. In particular, $\widetilde{h}_\alpha\big(\widetilde{h}_\delta(p)\big) = \widetilde{h}_{\alpha\delta}(p)$.*

Proof. Let $\epsilon = \min\{\frac{1}{4}, \frac{1}{4\alpha}\}$. Let $A = \{n \in \mathbb{N} : h_\delta(n) - \epsilon < \delta n < h_\delta(n) + \epsilon\}$ and let $B = \{k \in \mathbb{N} : h_\alpha(k) - \epsilon < \alpha k < h_\alpha(k) + \epsilon\}$. Let $P(x) = \delta x$ and $Q(x) = \alpha x$. By Lemma 3.3, $\widetilde{w \circ P}(p) = 0$ so $A \in p$. By Theorem 2.2, $\widetilde{h}_\delta(p)$ is an idempotent so by Lemma 3.3, $\widetilde{w \circ Q}\big(\widetilde{h}_\delta(p)\big) = 0$ and consequently $B \in \widetilde{h}_\delta(p)$ and therefore $h_\delta^{-1}[B] \in p$. We shall show that $A \cap h_\delta^{-1}[B] \subseteq \{n \in \mathbb{N} : h_\alpha\big(h_\delta(n)\big) = h_{\alpha\delta}(n)\}$.

To this end, let $n \in A \cap h_\delta^{-1}[B]$, let $k = h_\delta(n)$ and let $m = h_\alpha(k)$. Then

$$k - \epsilon < \delta n < k + \epsilon \text{ so } \alpha k - \alpha\epsilon < \alpha\delta n < \alpha k + \alpha\epsilon\,.$$

Also $m - \epsilon < \alpha k < m + \epsilon$. Therefore,

$$m - \epsilon < \alpha k < \alpha\delta n + \alpha\epsilon < \alpha k + 2\alpha\epsilon < m + 2\alpha\epsilon + \epsilon$$

and consequently $m - \frac{1}{2} \leq m - \epsilon - \alpha\epsilon < \alpha\delta n < m + \epsilon + \alpha\epsilon \leq m + \frac{1}{2}$. Therefore $h_{\alpha\delta}(n) = m$ as required. □

The following lemma can be derived from [6, Lemmas 6.5 and 6.6]. We present the simple self contained proof of part I, leaving part II to the reader.

Lemma 3.5. *Let α and δ be positive irrationals.*

(I) The following statements are equivalent.

- *(a) $X_\alpha \cap Y_\delta \neq \emptyset$.*
- *(b) $X_\delta \cap Y_\alpha \neq \emptyset$.*
- *(c) There do not exist $m, r \in \mathbb{N}$ such that $m\alpha - r\delta \in \mathbb{Z}$.*

(II) The following statements are equivalent.

- *(a) $X_\alpha \cap X_\delta \neq \emptyset$.*
- *(b) $Y_\delta \cap Y_\alpha \neq \emptyset$.*
- *(c) There do not exist $m, r \in \mathbb{N}$ such that $m\alpha + r\delta \in \mathbb{Z}$.*

Proof. (I). We prove that (a) and (c) are equivalent. The equivalence of (b) and (c) then follows by interchanging α and δ.

To see that (a) implies (c), pick $p \in X_\alpha \cap Y_\delta$ and suppose that we have $m, r \in \mathbb{N}$ and $k \in \mathbb{Z}$ such that $m\alpha - r\delta = k$. Let $\epsilon = \frac{1}{m+r}$. Then

$$\{n \in \mathbb{N} : h_\alpha(n) < \alpha n < h_\alpha(n) + \epsilon \text{ and } h_\delta(n) - \epsilon < \delta n < h_\delta(n)\} \in p\,,$$

so pick $n \in \mathbb{N}$ such that $h_\alpha(n) < \alpha n < h_\alpha(n) + \epsilon$ and $h_\delta(n) - \epsilon < \delta n < h_\delta(n)$. Let $l = h_\alpha(n)$ and $s = h_\delta(n)$. Then

$$lm < m\alpha n < lm + \epsilon m \text{ and } -rs < -r\delta n < -rs + \epsilon r$$

so $lm - rs < kn < lm - rs + \epsilon(m + r) = lm - rs + 1$, a contradiction since kn is an integer.

To see that (c) implies (a), we show that for every $\epsilon > 0$,

$$A_\epsilon = \{n \in \mathbb{N} : h_\alpha(n) < \alpha n < h_\alpha(n) + \epsilon \text{ and } h_\delta(n) - \epsilon < \delta n < h_\delta(n)\} \neq \emptyset\,,$$

so let $\epsilon > 0$. If 1, α, and δ are linearly independent over $\mathbb{Q}$, then one may apply Kronecker's Theorem directly to produce $n \in A_\epsilon$. So assume that we have some $m, r, s \in \mathbb{Q}$ such that $m\alpha + r\delta = s$. By multiplying by a multiple of the denominators, we may presume that m, r, and s are integers. If $m = 0$ we get that $\delta \in \mathbb{Q}$, so we may assume without loss of generality that $m > 0$. If $r = 0$ we get that $\alpha \in \mathbb{Q}$, so by (c) we must have that $r \in \mathbb{N}$ and therefore $s \in \mathbb{N}$. Pick by Kronecker's Theorem some $n \in \mathbb{N}$ such that $h_\alpha(n) < \alpha n < h_\alpha(n) + \frac{\epsilon}{m+r}$ and let $k = h_\alpha(n)$. Then $kr < \alpha nr < kr + \frac{\epsilon r}{m+r} < kr + \epsilon$. Also

$$km < \alpha nm = sn - \delta nr < km + \tfrac{\epsilon m}{m+r} < km + \epsilon$$

so $ns - km - \epsilon < \delta nr < ns - km$. Therefore $nr \in A_\epsilon$. □

The following technical lemma is used in several parts of the subsequent proofs.

Lemma 3.6. *Let α and δ be positive rationals, let $\epsilon > 0$, let $n \in \mathbb{N}$, let $k = h_\delta(n)$, and let $m = h_{\alpha\delta}(n)$. If $k - \epsilon < \delta n < k + \epsilon$ and $m - \epsilon < \alpha\delta n < m + \epsilon$, then*

$$m - \epsilon - \alpha\epsilon < \alpha k < m + \epsilon + \alpha\epsilon\,.$$

Proof. We have $\alpha k - \alpha\epsilon < \alpha\delta n < \alpha k + \alpha\epsilon$ so

$$m - \epsilon < \alpha\delta n < \alpha k + \alpha\epsilon < \alpha\delta n + 2\alpha\epsilon < m + \epsilon + 2\alpha\epsilon$$

and therefore $m - \epsilon - \alpha\epsilon < \alpha k < m + \epsilon + \alpha\epsilon$. $\square$

Lemma 3.7. *Let α and δ be positive irrationals, let $r = \lceil \alpha \rceil$, let*

$$\gamma = \tfrac{1}{2} \min\left\{|w(\tfrac{t}{\alpha})| : t \in \{1, 2, \ldots, r\}\right\},$$

let p be an idempotent in $\beta\mathbb{N}$, let $Q : \mathbb{N} \to \mathbb{R}$, let $s \in \{-r, -r+1, \ldots, r-1, r\}$, and assume that $\{n \in \mathbb{N} : w\big(Q(n)\big) \in (-\gamma, \gamma)$ and $Q(n) = \frac{1}{\alpha} h_{\alpha\delta}(n) + \frac{s}{\alpha}\} \in p$. Then $s = 0$.

Proof. Let $A = \{n \in \mathbb{N} : w\big(Q(n)\big) \in (-\gamma, \gamma)$ and $Q(n) = \frac{1}{\alpha} h_{\alpha\delta}(n) + \frac{s}{\alpha}\}$ and let $B = \{n \in \mathbb{N} : w\big(\frac{1}{\alpha} h_{\alpha\delta}(n)\big) \in (-\gamma, \gamma)\}$. We are given that $A \in p$. Now $\widetilde{h}_{\alpha\delta}(p)$ is an idempotent by Theorem 2.2, so by Lemma 3.2, $\frac{1}{\alpha}\widetilde{h}_{\alpha\delta}(p)$ is an idempotent and so $\widetilde{w}\big(\frac{1}{\alpha}\widetilde{h}_{\alpha\delta}(p)\big) = 0$ and therefore $B \in p$. Pick $n \in A \cap B$. Then

$$w\big(\tfrac{1}{\alpha} h_{\alpha\delta}(n)\big) \in (-\gamma, \gamma) \text{ and } w\big(Q(n)\big) = w\big(\tfrac{1}{\alpha} h_{\alpha\delta}(n) + \tfrac{s}{\alpha}\big) \in (-\gamma, \gamma)\,.$$

Then $|w(\frac{s}{\alpha})| < 2\gamma$ so $s = 0$ as required. $\square$

Lemma 3.8. *Let α and δ be positive irrationals, let p be an idempotent in $\beta\mathbb{N}$, and let $r = \lceil \alpha \rceil$.*

(a) If $\widetilde{h}_\delta(p) \notin X_\alpha \cap Y_{1/\delta}$, then there exists $s \in \{1, 2, \ldots, r\}$ such that $\{n \in \mathbb{N} : g_{\alpha,0} \circ g_{\delta,0}(n) = h_{\alpha\delta}(n) - s\} \in p$.

(b) If $\widetilde{h}_\delta(p) \notin Y_\alpha \cap X_{1/\delta}$, then there exists $s \in \{1, 2, \ldots, r\}$ such that $\{n \in \mathbb{N} : g_{\alpha,1} \circ g_{\delta,1}(n) = h_{\alpha\delta}(n) + s\} \in p$.

(c) If $\alpha > 1$ and $\widetilde{h}_\delta(p) \notin X_\alpha \cap X_{1/\delta}$, then there exists $s \in \{1, 2, \ldots, r-1\} \cup \{-1\}$ such that $\{n \in \mathbb{N} : g_{\alpha,0} \circ g_{\delta,1}(n) = h_{\alpha\delta}(n) + s\} \in p$.

(d) If $\alpha > 1$ and $\widetilde{h}_\delta(p) \notin Y_\alpha \cap Y_{1/\delta}$, then there exists $s \in \{1, 2, \ldots, r-1\} \cup \{-1\}$ such that $\{n \in \mathbb{N} : g_{\alpha,1} \circ g_{\delta,0}(n) = h_{\alpha\delta}(n) - s\} \in p$.

Proof. (a). Case 1. $\widetilde{h}_\delta(p) \notin Y_{1/\delta}$. Then by Theorem 2.2, $p \in Y_\delta$. Let $\epsilon = \min\{\frac{\alpha}{1+\alpha}, \frac{r-\alpha}{1+\alpha}\}$. Let $A = \{n \in \mathbb{N} : h_\delta(n) - \frac{1}{2} < \delta n < h_\delta(n)\}$. Since $p \in Y_\delta$, $A \in p$. Let

$$B = \{n \in \mathbb{N} : h_\delta(n) - \epsilon < \delta n < h_\delta(n) + \epsilon \text{ and } h_{\alpha\delta}(n) - \epsilon < \alpha\delta n < h_{\alpha\delta}(n) + \epsilon\}\,.$$

By Lemma 3.2, $B \in p$. We shall show that

$$A \cap B \subseteq \bigcup_{s=1}^{r} \{n \in \mathbb{N} : g_{\alpha,0} \circ g_{\delta,0}(n) = h_{\alpha\delta}(n) - s\}\,.$$

So let $n \in A \cap B$, let $k = h_\delta(n)$, and let $m = h_{\alpha\delta}(n)$. Since $n \in A$, $g_{\delta,0}(n) = k-1$ so $g_{\alpha,0} \circ g_{\delta,0}(n) = \lfloor \alpha k - \alpha \rfloor$. By Lemma 3.6,

$$m - \epsilon - \alpha\epsilon - \alpha < \alpha k - \alpha < m + \epsilon + \alpha\epsilon - \alpha\,.$$

Since $\epsilon \leq \frac{\alpha}{1+\alpha}$, $\epsilon + \alpha\epsilon - \alpha \leq 0$. Since $\epsilon \leq \frac{r-\alpha}{1+\alpha}$, $-\epsilon - \alpha\epsilon - \alpha \geq -r$. Therefore $m - r < \alpha k - \alpha < m$ so $\lfloor \alpha k - \alpha \rfloor = m - s$ for some $s \in \{1, 2, \ldots, r\}$.

Case 2. $\widetilde{h}_\delta(p) \in Y_{1/\delta}$, in which case $p \in X_\delta$ and $\widetilde{h}_\delta(p) \in Y_\alpha$. Let

$$\begin{array}{rcl} A & = & \{n \in \mathbb{N} : h_\delta(n) < \delta n < h_\delta(n) + \frac{1}{2}\}, \text{ let} \\ B & = & \{k \in \mathbb{N} : h_\alpha(k) - \frac{1}{2} < \alpha k < h_\alpha(k)\}, \text{ and let} \\ C & = & \{n \in \mathbb{N} : h_\alpha\big(h_\delta(n)\big) = h_{\alpha\delta}(n)\}\,. \end{array}$$

Then $A \in p$ and $B \in \widetilde{h}_\delta(p)$ so $h_\delta^{-1}[B] \in p$. By Lemma 3.4, $C \in p$. We shall show that $A \cap h_\delta^{-1}[B] \cap C \subseteq \{n \in \mathbb{N} : g_{\alpha,0} \circ g_{\delta,0}(n) = h_{\alpha\delta}(n) - 1\}$. So let $n \in A \cap h_\delta^{-1}[B] \cap C$, let $k = h_\delta(n)$, and let $m = h_\alpha(k)$. Since $n \in C$, $m = h_{\alpha\delta}(n)$.

Since $n \in A$, $k = g_{\delta,0}(n)$ and so $g_{\alpha,0}\big(g_{\delta,0}(n)\big) = \lfloor \alpha k \rfloor$. Since $k \in B$, $\lfloor \alpha k \rfloor = m - 1$.

(b). This proof involves only obvious changes from the proof of (a). If $\widetilde{h}_\delta(p) \notin X_{1/\delta}$, one shows that $\bigcup_{s=1}^{r} \{n \in \mathbb{N} : g_{\alpha,1} \circ g_{\delta,1}(n) = h_{\alpha\delta}(n) + s\} \in p$. If $\widetilde{h}_\delta(p) \in X_{1/\delta}$, one shows that $\{n \in \mathbb{N} : g_{\alpha,1} \circ g_{\delta,1}(n) = h_{\alpha\delta}(n) + 1\} \in p$.

(c). Case 1. $\widetilde{h}_\delta(p) \notin X_{1/\delta}$. Then by Theorem 2.2, $p \in X_\delta$. Let $\epsilon = \min\{\frac{\alpha-1}{1+\alpha}, \frac{r-\alpha}{1+\alpha}\}$. Let $A = \{n \in \mathbb{N} : h_\delta(n) < \delta n < h_\delta(n) + \epsilon\}$. Since $p \in X_\delta$, $A \in p$. Let

$$B = \{n \in \mathbb{N} : h_\delta(n) - \epsilon < \delta n < h_\delta(n) + \epsilon \text{ and } h_{\alpha\delta}(n) - \epsilon < \alpha\delta n < h_{\alpha\delta}(n) + \epsilon\}\,.$$

By Lemma 3.2, $B \in p$. We shall show that

$$A \cap B \subseteq \bigcup_{s=1}^{r-1} \{n \in \mathbb{N} : g_{\alpha,0} \circ g_{\delta,1}(n) = h_{\alpha\delta}(n) + s\}\,.$$

So let $n \in A \cap B$, let $k = h_\delta(n)$, and let $m = h_{\alpha\delta}(n)$. Since $n \in A$, $g_{\delta,1}(n) = k+1$ so $g_{\alpha,0} \circ g_{\delta,1}(n) = \lfloor \alpha k + \alpha \rfloor$. By Lemma 3.6,

$$m - \epsilon - \alpha\epsilon + \alpha < \alpha k + \alpha < m + \epsilon + \alpha\epsilon + \alpha\,.$$

Since $\epsilon \leq \frac{\alpha-1}{1+\alpha}$, $m - \epsilon - \alpha\epsilon + \alpha \geq m + 1$. Since $\epsilon \leq \frac{r-\alpha}{1+\alpha}$,

$$m + \epsilon + \alpha\epsilon + \alpha \leq m + r\,.$$

Therefore $m + 1 < \alpha k - \alpha < m + r$ so $\lfloor \alpha k + \alpha \rfloor = m + s$ for some $s \in \{1, 2, \ldots, r-1\}$.

Case 2. $\widetilde{h}_\delta(p) \in X_{1/\delta}$, in which case $p \in Y_\delta$ and $h_\delta(p) \in Y_\alpha$. Let

$$\begin{array}{rcl} A & = & \{n \in \mathbb{N} : h_\delta(n) - \frac{1}{2} < \delta n < h_\delta(n)\}, \text{ let} \\ B & = & \{k \in \mathbb{N} : h_\alpha(k) - \frac{1}{2} < \alpha k < h_\alpha(k)\}, \text{ and let} \\ C & = & \{n \in \mathbb{N} : h_\alpha\big(h_\delta(n)\big) = h_{\alpha\delta}(n)\}\,. \end{array}$$

Then $A \in p$ and $B \in \widetilde{h}_\delta(p)$ so $h_\delta^{-1}[B] \in p$. By Lemma 3.4, $C \in p$. We shall show that $A \cap h_\delta^{-1}[B] \cap C \subseteq \{n \in \mathbb{N} : g_{\alpha,0} \circ g_{\delta,1}(n) = h_{\alpha\delta}(n) - 1\}$. So let $n \in A \cap h_\delta^{-1}[B] \cap C$, let $k = h_\delta(n)$, and let $m = h_\alpha(k)$. Since $n \in C$, $m = h_{\alpha\delta}(n)$.

Since $n \in A$, $g_{\delta,1}(n) = k$ and so $g_{\alpha,0}\big(g_{\delta,1}(n)\big) = \lfloor \alpha k \rfloor$. Since $k \in B$, $\lfloor \alpha k \rfloor = m - 1$.

(d). This proof involves only obvious changes from the proof of (c). $\square$

Theorem 3.9. *Let α and δ be positive irrationals. The following statements are equivalent.*

(a) *Whenever $v \in \mathbb{N}$, $P_1, P_2, \ldots, P_v$ are real polynomials with zero constant term, for each $u \in \{1, 2, \ldots, v\}$, $Q_u = P_u \circ g_{\alpha,0} \circ g_{\delta,0}$, and U is a neighborhood of $\overline{0}$ in $(-\frac{1}{2}, \frac{1}{2})^v$, $\{n \in \mathbb{N} : \big(w\big(Q_1(n)\big), w\big(Q_2(n)\big), \ldots, w\big(Q_v(n)\big)\big) \in U\}$ is strongly central.*

(a′) *Whenever $v \in \mathbb{N}$, $P_1, P_2, \ldots, P_v$ are real polynomials with zero constant term, for each $u \in \{1, 2, \ldots, v\}$, $Q_u = P_u \circ g_{\alpha,1} \circ g_{\delta,1}$, and U is a neighborhood of $\overline{0}$ in $(-\frac{1}{2}, \frac{1}{2})^v$, $\{n \in \mathbb{N} : \big(w\big(Q_1(n)\big), w\big(Q_2(n)\big), \ldots, w\big(Q_v(n)\big)\big) \in U\}$ is strongly central.*

(b) *Whenever $v \in \mathbb{N}$, $P_1, P_2, \ldots, P_v$ are real polynomials with zero constant term, for each $u \in \{1, 2, \ldots, v\}$, $Q_u = P_u \circ g_{\alpha,0} \circ g_{\delta,0}$, and U is a neighborhood of $\overline{0}$ in $(-\frac{1}{2}, \frac{1}{2})^v$, $\{n \in \mathbb{N} : \big(w\big(Q_1(n)\big), w\big(Q_2(n)\big), \ldots, w\big(Q_v(n)\big)\big) \in U\}$ is central.*

(b′) *Whenever $v \in \mathbb{N}$, $P_1, P_2, \ldots, P_v$ are real polynomials with zero constant term, for each $u \in \{1, 2, \ldots, v\}$, $Q_u = P_u \circ g_{\alpha,1} \circ g_{\delta,1}$, and U is a neighborhood of $\overline{0}$ in $(-\frac{1}{2}, \frac{1}{2})^v$, $\{n \in \mathbb{N} : \big(w\big(Q_1(n)\big), w\big(Q_2(n)\big), \ldots, w\big(Q_v(n)\big)\big) \in U\}$ is central.*

(c) *Whenever $v \in \mathbb{N}$, $P_1, P_2, \ldots, P_v$ are real polynomials with zero constant term, for each $u \in \{1, 2, \ldots, v\}$, $Q_u = P_u \circ g_{\alpha,0} \circ g_{\delta,0}$, and U is a neighborhood of $\overline{0}$ in $(-\frac{1}{2}, \frac{1}{2})^v$, $\{n \in \mathbb{N} : \big(w\big(Q_1(n)\big), w\big(Q_2(n)\big), \ldots, w\big(Q_v(n)\big)\big) \in U\}$ is an IP set.*

(c′) *Whenever $v \in \mathbb{N}$, $P_1, P_2, \ldots, P_v$ are real polynomials with zero constant term, for each $u \in \{1, 2, \ldots, v\}$, $Q_u = P_u \circ g_{\alpha,1} \circ g_{\delta,1}$, and U is a neighborhood of $\overline{0}$ in $(-\frac{1}{2}, \frac{1}{2})^v$, $\{n \in \mathbb{N} : \big(w\big(Q_1(n)\big), w\big(Q_2(n)\big), \ldots, w\big(Q_v(n)\big)\big) \in U\}$ is an IP set.*

(d) *For each polynomial P with real coefficients and zero constant term and each neighborhood U of 0 in $(-\frac{1}{2}, \frac{1}{2})$, if $Q = P \circ g_{\alpha,0} \circ g_{\delta,0}$, then $\{n \in \mathbb{N} : w\big(Q(n)\big) \in U\}$ is an IP set.*

(d') *For each polynomial P with real coefficients and zero constant term and each neighborhood U of 0 in $(-\frac{1}{2},\frac{1}{2})$, if $Q = P \circ g_{\alpha,1} \circ g_{\delta,1}$, then $\{n \in \mathbb{N} : w\big(Q(n)\big) \in U\}$ is an IP set.*

(e) $X_\alpha \cap Y_{1/\delta} \neq \emptyset$.

(e') $Y_\alpha \cap X_{1/\delta} \neq \emptyset$.

(f) *There do not exist $m, r \in \mathbb{N}$ such that $m\alpha - \frac{r}{\delta} \in \mathbb{Z}$.*

Proof. That (a) implies (b), (b) implies (c), and (c) implies (d) is trivial, as is the fact that (a') implies (b'), (b') implies (c'), and (c') implies (d'). The fact that (e), (e'), and (f) are equivalent is Lemma 3.5(I).

To see that (e) implies (a), assume that $X_\alpha \cap Y_{1/\delta} \neq \emptyset$. Then by [6, Theorem 5.5], $X_\alpha \cap Y_{1/\delta}$ is a right ideal of $Z_{1/\delta}$. By Lemma 2.3, $L \cap Z_\delta$ is a minimal left ideal of Z_δ so by Theorem 2.2, $\widetilde{h}_\delta[L \cap Z_\delta]$ is a minimal left ideal of $Z_{1/\delta}$. Pick an idempotent $q \in \widetilde{h}_\delta[L \cap Z_\delta] \cap X_\alpha \cap Y_{1/\delta}$ and let $p = \widetilde{h}_{1/\delta}(q)$. Then p is an idempotent in $L \cap X_\delta$ and $\widetilde{h}_\delta(q) = p$.

Let

$$\begin{aligned} A &= \{n \in \mathbb{N} : h_\alpha\big(h_\delta(n)\big) = h_{\alpha\delta}(n)\}\,, \\ B &= \{n \in \mathbb{N} : h_\delta(n) < \delta n < h_\delta(n) + \tfrac{1}{2}\}, \text{ and} \\ C &= \{k \in \mathbb{N} : h_\alpha(k) < \alpha k < h_\alpha(k) + \tfrac{1}{2}\}\,. \end{aligned}$$

By Lemma 3.4, $A \in p$. Since $p \in X_\delta$, $B \in p$. Since $q \in X_\alpha$, $C \in q$ and so $h_\delta^{-1}[C] \in p$. We claim that

$$A \cap B \cap h_\delta^{-1}[C] \subseteq \{n \in \mathbb{N} : g_{\alpha,0} \circ g_{\delta,0}(n) = h_{\alpha\delta}(n)\}\,,$$

so let $n \in A \cap B \cap h_\delta^{-1}[C]$. Let $k = h_\delta(n)$ and let $m = h_\alpha(k)$. Since $n \in A$, $m = h_{\alpha\delta}(n)$. Since $n \in B$, $k = g_{\delta,0}(n)$ and so $g_{\alpha,0} \circ g_{\delta,0}(n) = \lfloor \alpha k \rfloor$. Since $k \in C$, $\lfloor \alpha k \rfloor = m$ as required.

Now let $D = \{n \in \mathbb{N} : \big(w\big(P_1(n)\big), w\big(P_2(n)\big), \ldots, w\big(P_v(n)\big)\big) \in U\}$. Then by Theorem 4.2, $D \in \widetilde{h}_{\alpha\delta}(p)$ so $h_{\alpha\delta}^{-1}[D] \in p$. Then

$$A \cap B \cap h_\delta^{-1}[C] \cap h_{\alpha\delta}^{-1}[D] \subseteq \{n \in \mathbb{N} : \big(w\big(Q_1(n)\big), w\big(Q_2(n)\big), \ldots, w\big(Q_v(n)\big)\big) \in U\}\,.$$

The proof that (e') implies (a') is very similar and we omit it.

To see that (d) implies (e), let $P(x) = \frac{1}{\alpha}x$ and let $Q = P \circ g_{\alpha,0} \circ g_{\delta,0}$. Let $\gamma = \frac{1}{2}\min\big\{|w(\frac{t}{\alpha})| : t \in \{1, 2, \ldots, \lceil \alpha \rceil\}\big\}$ and let $U = (-\gamma, \gamma)$. Pick an idempotent $p \in \beta\mathbb{N}$ such that $\{n \in \mathbb{N} : w\big(Q(n)\big) \in U\} \in p$. We claim that $\widetilde{h}_\delta(p) \in X_\alpha \cap Y_{1/\delta}$. Suppose instead $\widetilde{h}_\delta(p) \notin X_\alpha \cap Y_{1/\delta}$. Then by Lemma 3.8(a), pick $s \in \{1, 2, \ldots, \lceil \alpha \rceil\}$ such that $\{n \in \mathbb{N} : g_{\alpha,0} \circ g_{\delta,0}(n) = h_{\alpha\delta}(n) - s\} \in p$. That is, $\{n \in \mathbb{N} : Q(n) = \frac{1}{\alpha}h_{\alpha\delta}(n) - \frac{s}{\alpha}\} \in p$, contradicting Lemma 3.7.

The proof that (d') implies (e') is nearly identical, using Lemma 3.8(b). $\square$

The situation with respect to $g_{\alpha,0} \circ g_{\delta,1}$ and $g_{\alpha,1} \circ g_{\delta,0}$ is significantly different. In these cases it matters whether or not $\alpha > 1$. (If $\alpha < 1$, then the major conclusions are simply true.)

We need one more preliminary lemma.

Lemma 3.10. *Let α and δ be positive irrationals with $\alpha < 1$ and let p be an idempotent in $\beta\mathbb{N}$.*

(a) If $p \in X_\delta$, then $\{n \in \mathbb{N} : g_{\alpha,0} \circ g_{\delta,1}(n) = h_{\alpha\delta}(n)\} \in p$.

(b) If $p \in Y_\delta$, then $\{n \in \mathbb{N} : g_{\alpha,1} \circ g_{\delta,0}(n) = h_{\alpha\delta}(n)\} \in p$.

Proof. Let $\epsilon = \min\{\frac{\alpha}{1+\alpha}, \frac{1-\alpha}{1+\alpha}\}$. Let

$A = \{n \in \mathbb{N} : h_\delta(n) - \epsilon < \delta n < h_\delta(n) + \epsilon$ and $h_{\alpha\delta}(n) - \epsilon < \alpha\delta n < h_{\alpha\delta}(n) + \epsilon\}$.

By Lemma 3.2, $A \in p$.

Assume first that $p \in X_\delta$ and let $B = \{n \in \mathbb{N} : h_\delta(n) < \delta n < h_\delta(n) + \frac{1}{2}\}$. We claim that $A \cap B \subseteq \{n \in \mathbb{N} : g_{\alpha,0} \circ g_{\delta,1}(n) = h_{\alpha\delta}(n)\}$, so let $n \in A \cap B$, let $k = h_\delta(n)$, and let $m = h_{\alpha\delta}(n)$. Since $n \in B$, $g_{\delta,1}(n) = k + 1$ so $g_{\alpha,0} \circ g_{\delta,1}(n) = \lfloor \alpha k + \alpha \rfloor$. By Lemma 3.6, $m - \epsilon - \alpha\epsilon + \alpha < \alpha k + \alpha < m + \epsilon + \alpha\epsilon + \alpha$. Since $\epsilon \leq \frac{\alpha}{1+\alpha}$, $m - \epsilon - \alpha\epsilon + \alpha \geq m$. Since $\epsilon \leq \frac{1-\alpha}{1+\alpha}$, $m + \epsilon + \alpha\epsilon + \alpha \leq m + 1$. Thus $\lfloor \alpha k + \alpha \rfloor = m$.

Now assume that $p \in Y_\delta$ and let $B = \{n \in \mathbb{N} : h_\delta(n) - \frac{1}{2} < \delta n < h_\delta(n)\}$. Then as above, one shows that $A \cap B \subseteq \{n \in \mathbb{N} : g_{\alpha,1} \circ g_{\delta,0}(n) = h_{\alpha\delta}(n)\}$. □

Theorem 3.11. *Let α and δ be positive irrationals. Statements (e), (e′), and (f) are equivalent and imply the other statements. If $\alpha < 1$, then each of statements (a), (a′), (b), (b′), (c), (c′), (d), and (d′) are true. If $\alpha > 1$, then all of the following statements are equivalent.*

(a) *Whenever $v \in \mathbb{N}$, $P_1, P_2, \ldots, P_v$ are real polynomials with zero constant term, for each $u \in \{1, 2, \ldots, v\}$, $Q_u = P_u \circ g_{\alpha,0} \circ g_{\delta,1}$, and U is a neighborhood of $\overline{0}$ in $(-\frac{1}{2}, \frac{1}{2})^v$,*
$\{n \in \mathbb{N} : \big(w(Q_1(n)), w(Q_2(n)), \ldots, w(Q_v(n))\big) \in U\}$ is strongly central.

(a′) *Whenever $v \in \mathbb{N}$, $P_1, P_2, \ldots, P_v$ are real polynomials with zero constant term, for each $u \in \{1, 2, \ldots, v\}$, $Q_u = P_u \circ g_{\alpha,1} \circ g_{\delta,0}$, and U is a neighborhood of $\overline{0}$ in $(-\frac{1}{2}, \frac{1}{2})^v$,*
$\{n \in \mathbb{N} : \big(w(Q_1(n)), w(Q_2(n)), \ldots, w(Q_v(n))\big) \in U\}$ is strongly central.

(b) *Whenever $v \in \mathbb{N}$, $P_1, P_2, \ldots, P_v$ are real polynomials with zero constant term, for each $u \in \{1, 2, \ldots, v\}$, $Q_u = P_u \circ g_{\alpha,0} \circ g_{\delta,1}$, and U is a neighborhood of $\overline{0}$ in $(-\frac{1}{2}, \frac{1}{2})^v$,*
$\{n \in \mathbb{N} : \big(w(Q_1(n)), w(Q_2(n)), \ldots, w(Q_v(n))\big) \in U\}$ is central.

(b′) *Whenever $v \in \mathbb{N}$, $P_1, P_2, \ldots, P_v$ are real polynomials with zero constant term, for each $u \in \{1, 2, \ldots, v\}$, $Q_u = P_u \circ g_{\alpha,1} \circ g_{\delta,0}$, and U is a neighborhood of $\overline{0}$ in $(-\frac{1}{2}, \frac{1}{2})^v$,*
$\{n \in \mathbb{N} : \big(w(Q_1(n)), w(Q_2(n)), \ldots, w(Q_v(n))\big) \in U\}$ is central.

(c) *Whenever $v \in \mathbb{N}$, $P_1, P_2, \ldots, P_v$ are real polynomials with zero constant term, for each $u \in \{1, 2, \ldots, v\}$, $Q_u = P_u \circ g_{\alpha,0} \circ g_{\delta,1}$, and U is a neighborhood of $\overline{0}$ in $(-\frac{1}{2}, \frac{1}{2})^v$,*
$\{n \in \mathbb{N} : \big(w(Q_1(n)), w(Q_2(n)), \ldots, w(Q_v(n))\big) \in U\}$ is an IP set.

(c') *Whenever* $v \in \mathbb{N}$, $P_1, P_2, \ldots, P_v$ *are real polynomials with zero constant term, for each* $u \in \{1, 2, \ldots, v\}$, $Q_u = P_u \circ g_{\alpha,1} \circ g_{\delta,0}$, *and* U *is a neighborhood of* $\overline{0}$ *in* $(-\frac{1}{2}, \frac{1}{2})^v$,
$\{n \in \mathbb{N} : \big(w\big(Q_1(n)\big), w\big(Q_2(n)\big), \ldots, w\big(Q_v(n)\big)\big) \in U\}$ *is an IP set.*

(d) *For each polynomial* P *with real coefficients and zero constant term and each neighborhood* U *of* 0 *in* $(-\frac{1}{2}, \frac{1}{2})$, *if* $Q = P \circ g_{\alpha,0} \circ g_{\delta,1}$, *then* $\{n \in \mathbb{N} : w\big(Q(n)\big) \in U\}$ *is an IP set.*

(d') *For each polynomial* P *with real coefficients and zero constant term and each neighborhood* U *of* 0 *in* $(-\frac{1}{2}, \frac{1}{2})$, *if* $Q = P \circ g_{\alpha,1} \circ g_{\delta,0}$, *then* $\{n \in \mathbb{N} : w\big(Q(n)\big) \in U\}$ *is an IP set.*

(e) $X_\alpha \cap X_{1/\delta} \neq \emptyset$.

(e') $Y_\alpha \cap Y_{1/\delta} \neq \emptyset$.

(f) *There do not exist* $m, r \in \mathbb{N}$ *such that* $m\alpha + r\frac{1}{\delta} \in \mathbb{Z}$.

Proof. That (a) implies (b), (b) implies (c), and (c) implies (d) is trivial, as is the fact that (a') implies (b'), (b') implies (c'), and (c') implies (d'). The fact that (e), (e'), and (f) are equivalent is Lemma 3.5(II).

Assume first that $\alpha < 1$. It suffices to show that statements (a) and (a') hold. To this end, let L be a minimal left ideal of $\beta\mathbb{N}$ and let $P_1, P_2, \ldots, P_v$ and U be as in statements (a) and (a'). For $u \in \{1, 2, \ldots, v\}$, let $Q_u = P_u \circ g_{\alpha,0} \circ g_{\delta,1}$ and let $Q'_u = P_u \circ g_{\alpha,1} \circ g_{\delta,0}$. By [6, Theorem 5.5], X_δ and Y_δ are right ideals of Z_δ. Further, by Theorem 2.2, $L \cap Z_\delta \neq \emptyset$ and therefore $L \cap Z_\delta$ is a left ideal of Z_δ. Pick idempotents $p \in L \cap X_\delta$ and $q \in L \cap Y_\delta$. Let

$$\begin{array}{rcl} A & = & \{n \in \mathbb{N} : g_{\alpha,0} \circ g_{\delta,1}(n) = h_{\alpha\delta}(n)\} \text{ and} \\ B & = & \{n \in \mathbb{N} : g_{\alpha,1} \circ g_{\delta,0}(n) = h_{\alpha\delta}(n)\}\,. \end{array}$$

By Lemma 3.10, $A \in p$ and $B \in q$.

Now let $C = \{n \in \mathbb{N} : \big(w\big(P_1(n)\big), w\big(P_2(n)\big), \ldots, w\big(P_v(n)\big)\big) \in U\}$. Then by Theorem 4.2, $C \in \widetilde{h}_{\alpha\delta}(p) \cap \widetilde{h}_{\alpha\delta}(q)$ so $h_{\alpha\delta}^{-1}[C] \in p \cap q$. Then

$$\begin{array}{rcl} A \cap h_{\alpha\delta}^{-1}[C] & \subseteq & \{n \in \mathbb{N} : \big(w\big(Q_1(n)\big), w\big(Q_2(n)\big), \ldots, w\big(Q_v(n)\big)\big) \in U\} \text{ and} \\ B \cap h_{\alpha\delta}^{-1}[C] & \subseteq & \{n \in \mathbb{N} : \big(w\big(Q'_1(n)\big), w\big(Q'_2(n)\big), \ldots, w\big(Q'_v(n)\big)\big) \in U\}\,. \end{array}$$

Now assume that $\alpha > 1$. The proofs that (e) implies (a) and (e') implies (a') are very similar to the corresponding parts of Theorem 3.9. The proofs that (d) implies (e) and (d') implies (e') are also similar to the corresponding parts of Theorem 3.9, using Lemma 3.8(c) and Lemma 3.8(d) respectively. $\square$

4. Generalized Polynomials

As we have mentioned in the introduction, generalized polynomials have been extensively studied. Loosely speaking, they are algebraic expressions which allow applications of the greatest integer function as often as one wishes. The functions of the form $P_u \circ g_{\alpha_1,\gamma_1} \circ \ldots \circ g_{\alpha_m,\gamma_m}$ with which we have been dealing are all examples of generalized polynomials. In [10], generalized polynomials were formalized as follows.

Definition 4.1. Let GP_0 be the set of polynomials with real coefficients and for $n \in \mathbb{N}$, define

$$\begin{aligned} GP_n &= GP_{n-1} \cup \{P+Q : P, Q \in GP_{n-1}\} \cup \\ &\quad \{P \cdot Q : P, Q \in GP_{n-1}\} \cup \{\lfloor P \rfloor : P \in GP_{n-1}\}\,. \end{aligned}$$

Let $\mathcal{GP} = \bigcup_{n=0}^{\infty} GP_n$. Then P is a *generalized polynomial* if and only if $P \in \mathcal{GP}$.

Theorem 4.2. *Let $P_1, P_2, \ldots, P_v$ be generalized polynomials such that all polynomials occurring in their representations have zero constant term and let U be a neighborhood of $\overline{0}$ in $(-\frac{1}{2}, \frac{1}{2})^v$. Let*

$$A = \Big\{ x \in \mathbb{N} : \big(w\big(P_1(x)\big), w\big(P_2(x)\big) \ldots, w\big(P_v(x)\big) \big) \in U \Big\}\,.$$

Then A is an IP set.*

Proof. [10, Theorem D]. $\square$

Recall that $\nu(x) = \lfloor x + \frac{1}{2} \rfloor = h_1(x)$ and denotes the integer closest to x. In this section we provide in Corollary 4.9 a characterization of those generalized polynomials P that have the property that $\{x \in \mathbb{R} : w\big(\alpha P(x)\big) \in U\}$ is IP* whenever $\alpha \in \mathbb{R}$ and U is a neighborhood of 0. This characterization is in terms of a set $\mathcal{D}$ of generalized polynomials which we define now.

Definition 4.3. Let $D_0 = \{0\}$ and let D_1 be the set of linear polynomials with zero constant term. Let $\phi(0) = 0$ and let $\phi(P) = 1$ for $P \in D_1 \setminus \{0\}$. Given $n > 1$, assume that D_{n-1} has been defined and $\phi(P)$ has been defined for $P \in D_{n-1}$. Let

$$\begin{aligned} D_n &= \{P+Q : P, Q \in D_{n-1}\} \cup \\ &\quad \{PQ : P, Q \in D_{n-1} \text{ and } \phi(P) + \phi(Q) \le n\} \cup \\ &\quad \{\alpha \cdot (\nu \circ P) : \alpha \in \mathbb{R} \text{ and } P \in D_{n-1}\}\,. \end{aligned}$$

For $P \in D_n \setminus D_{n-1}$, define $\phi(P) = n$. Let $\mathcal{D} = \bigcup_{n=0}^{\infty} D_n$.

For each $n \in \omega$, let $E_n = \{\sum_{i=1}^{r} P_i : r \in \mathbb{N} \text{ and each } P_i \in D_n\}$.

Notice that for each $n \in \mathbb{N}$, $D_{n-1} \subseteq D_n$ and if $\alpha \in \mathbb{R}$ and $P \in D_n$, then $\alpha P \in D_n$. Notice also that $\mathcal{D}$ includes all real polynomials with zero constant term. In fact $\mathcal{D}$ is the smallest algebra of real functions which includes the real polynomials with zero constant term and has the property that $\nu \circ P \in \mathcal{D}$ whenever $P \in \mathcal{D}$.

Given an ultrafilter p on X and a function $f : X \to Y$ where Y is a topological space, $p\text{-}\lim_{x\in X} f(x) = y$ if and only if for every neighborhood U of y, $\{x \in X : f(x) \in U\} \in p$. This notion is very well behaved. (See [18, Section 3.5].) In particular, if g is a continuous function from the space Y to the space Z, then $p\text{-}\lim_{x\in X} g\big(f(x)\big) = g\big(p\text{-}\lim_{x\in X} f(x)\big)$.

Lemma 4.4. *For $n \in \mathbb{N}$, let $C_n = \{p \in \beta\mathbb{R}_d : \widetilde{w \circ P}(p) = 0$ for every $P \in D_n\}$.*

(1) For every $n \in \mathbb{N}$ and every $P \in D_n$ there exists $X \subseteq \mathbb{R}$ such that $C \cup E(\beta\mathbb{R}_d) \subseteq cl_{\beta\mathbb{R}_d}(X)$ and, for each $x \in X$, there exists $R \in E_{n-1}$ such that $C \cup E(\beta\mathbb{R}_d) \subseteq cl_{\beta\mathbb{R}_d}\big(\{y \in \mathbb{R} : P(x+y) = P(x) + P(y) + R(y)\}\big)$.

(2) For every $n \in \mathbb{N}$, $E(\beta\mathbb{R}_d) \subseteq C_n$.

Proof. We proceed by induction.

Let $n = 1$. Then (1) clearly holds. Let $P \in D_1$. Since $P : \mathbb{R} \to \mathbb{R}$ is a homomorphism, $\widetilde{P} : \beta\mathbb{R}_d \to \beta\mathbb{R}_d$ is a homomorphism, by [18, Theorem 4.8]. So $\widetilde{w \circ P} : \beta\mathbb{R}_d \to \mathbb{T}$ is a homomorphism and therefore (2) holds.

Now let $n > 1$ and assume that our lemma is true for all smaller positive integers. It follows, in particular, that $\widetilde{w \circ R}(p) = 0$ for every $p \in E(\beta\mathbb{R}_d)$ and every $R \in E_{n-1}$. We claim that (1) then implies (2). To see this, let $P \in D_n$ and assume that (1) holds for P. Let $x \in X$ be given and pick $R \in E_{n-1}$ as guaranteed for x. Let $p \in E(\beta\mathbb{R}_d)$ and let

$$Y = \{y \in \mathbb{R} : P(x+y) = P(x) + P(y) + R(y)\}\,.$$

Then $Y \in p$ so

$$\begin{aligned}
\widetilde{w \circ P}(x+p) &= p\text{-}\lim_{y\in Y} w \circ P(x+y) \\
&= w \circ P(x) + p\text{-}\lim_{y\in Y} w \circ P(y) + p\text{-}\lim_{y\in Y} w \circ R(y) \\
&= w \circ P(x) + \widetilde{w \circ P}(p) + \widetilde{w \circ R}(p) \\
&= w \circ P(x) + \widetilde{w \circ P}(p)
\end{aligned}$$

where the second equality holds because addition in $\mathbb{T}$ is jointly continuous. Therefore

$$\begin{aligned}
\widetilde{w \circ P}(p) &= \widetilde{w \circ P}(p+p) \\
&= p\text{-}\lim_{x\in X} \widetilde{w \circ P}(x+p) \\
&= p\text{-}\lim_{x\in X} w \circ P(x) + \widetilde{w \circ P}(p) \\
&= \widetilde{w \circ P}(p) + \widetilde{w \circ P}(p)
\end{aligned}$$

and therefore $\widetilde{w \circ P}(p) = 0$.

We now show that (1) holds, so let $P \in D_n$. We consider three cases.

(i) There exist U and V in D_{n-1} such that $P = U + V$.

(ii) There exist U and V in D_{n-1} such that $\phi(U) + \phi(V) \leq n$ and $P = UV$.

(iii) There exist $U \in D_{n-1}$ and $\alpha \in \mathbb{R}$ and $P = \alpha(\nu \circ U)$.

Case (i). This is obvious.

Case (ii). If $U = 0$ or $V = 0$ the conclusion is trivial. So we may assume that $\phi(U) < n$ and $\phi(V) < n$. Let $k = \phi(U)$ and let $l = \phi(V)$. Pick subsets X and Z of $\mathbb{R}$ such that

(a) $C \cup E(\beta\mathbb{R}_d) \subseteq c\ell_{\beta\mathbb{R}_d} X$;

(b) $C \cup E(\beta\mathbb{R}_d) \subseteq c\ell_{\beta\mathbb{R}_d} Z$;

(c) for each $x \in X$ there exists $R \in E_{k-1}$ such that $C \cup E(\beta\mathbb{R}_d) \subseteq c\ell_{\beta\mathbb{R}_d}\{y \in \mathbb{R} : U(x+y) = U(x) + U(y) + R(y)\}$; and

(d) for each $x \in Z$ there exists $S \in E_{l-1}$ such that $C \cup E(\beta\mathbb{R}_d) \subseteq c\ell_{\beta\mathbb{R}_d}\{y \in \mathbb{R} : V(x+y) = V(x) + V(y) + S(y)\}$.

Then $C \cup E(\beta\mathbb{R}_d) \subseteq c\ell_{\beta\mathbb{R}_d}(X \cap Z)$. Let $x \in X \cap Z$ and pick $R \in E_{k-1}$ and $S \in E_{l-1}$ as guaranteed by (c) and (d). Define $T : \mathbb{R} \to \mathbb{R}$ by, for $y \in \mathbb{R}$, $T(y) = U(x)V(y) + U(x)S(y) + U(y)V(x) + U(y)S(y) + R(y)V(x) + R(y)V(y) + R(y)S(y)$. Then $T \in E_{n-1}$ and

$$\begin{aligned}&\{y \in \mathbb{R} : U(x+y) = U(x) + U(y) + R(y)\} \cap \\ &\{y \in \mathbb{R} : V(x+y) = V(x) + V(y) + S(y)\} \subseteq \\ &\{y \in \mathbb{R} : P(x+y) = P(x) + P(y) + T(y)\}\,.\end{aligned}$$

Case (iii). Pick $X \subseteq \mathbb{R}$ such that $C \cup E(\beta\mathbb{R}_d) \subseteq c\ell_{\beta\mathbb{R}_d} X$ and for each $x \in X$ there exists $R \in E_{n-2}$ such that

$$C \cup E(\beta\mathbb{R}_d) \subseteq c\ell_{\beta\mathbb{R}_d}\{y \in \mathbb{R} : U(x+y) = U(x) + U(y) + R(y)\}\,.$$

Let $Z = \{x \in \mathbb{R} : w\big(U(x)\big) \in (-\frac{1}{6}, \frac{1}{6})\}$. Then by (2) at $n-1$, $C \cup E(\beta\mathbb{R}_d) \subseteq c\ell_{\beta\mathbb{R}_d} Z$. Let $x \in X \cap Z$ and pick $R \in E_{n-2}$ such that

$$C \cup E(\beta\mathbb{R}_d) \subseteq c\ell_{\beta\mathbb{R}_d}\{y \in \mathbb{R} : U(x+y) = U(x) + U(y) + R(y)\}\,.$$

Let $V = \{y \in \mathbb{R} : w\big(U(y)\big) \in (-\frac{1}{6}, \frac{1}{6})$ and $w\big(R(y)\big) \in (-\frac{1}{6}, \frac{1}{6})\}$. Then $C \cup E(\beta\mathbb{R}_d) \subseteq c\ell_{\beta\mathbb{R}_d} V$ and for $y \in V$,

$$\nu\big(U(x) + U(y) + R(y)\big) = \nu\big(U(x)\big) + \nu\big(U(y)\big) + \nu\big(R(y)\big)\,,$$

so $V \cap \{y \in \mathbb{R} : U(x+y) = U(x) + U(y) + R(y)\} \subseteq \big\{y \in \mathbb{R} : P(x+y) = P(x) + P(y) + \alpha\nu\big(R(y)\big)\big\}$. $\square$

Lemma 4.5. *Let $P \in \mathcal{D}$. If $\alpha \in \mathbb{R}$ and $\epsilon > 0$, then*

$$\{x \in \mathbb{R} : w\big(\alpha P(x)\big) \in (-\epsilon, \epsilon)\}$$

is an IP set in $\mathbb{R}_d$.*

Proof. This is an immediate consequence of Lemma 4.4(2). $\square$

Theorem 4.6. *For every $P \in \mathcal{GP}$ there exist $r \in \mathbb{N}$, a partition*

$$\{X_i : i \in \{1, 2, \ldots, r\}\} \text{ of } \mathbb{R},$$

members $Q_1, Q_2, \ldots, Q_r$ of $\mathcal{D}$, and $\{\alpha_1, \alpha_2, \ldots, \alpha_r\} \subseteq \mathbb{R}$ such that for each $i \in \{1, 2, \ldots, r\}$ and each $x \in X_i$, $P(x) = Q_i(x) + \alpha_i$.

Proof. The set of functions $\mathcal{GP}$ for which this statement holds contains all real polynomials and it is easy to see that it is closed under addition and multiplication. So it is sufficient to prove that, if this statement holds for $P \in \mathcal{GP}$, then it also holds for $\lfloor P \rfloor$. So pick $r \in \mathbb{N}$, a partition

$$\{X_i : i \in \{1, 2, \ldots, r\}\} \text{ of } \mathbb{R},$$

members $Q_1, Q_2, \ldots, Q_r$ of $\mathcal{D}$, and $\{\alpha_1, \alpha_2, \ldots, \alpha_r\} \subseteq \mathbb{R}$ such that for each $i \in \{1, 2, \ldots, r\}$ and each $x \in X_i$, $P(x) = Q_i(x) + \alpha_i$.

Now, given $i \in \{1, 2, \ldots, r\}$ and $x \in X_i$, $\lfloor P(x) \rfloor = \nu\big(Q_i(x)\big) + \nu(\alpha_i) - k$ for some $k \in \{0, 1, 2\}$. So if $X_{i,k} = \{x \in X_i : \lfloor P(x) \rfloor = \nu\big(Q_i(x)\big) + \nu(\alpha_i) - k\}$, Then $\big\{X_{i,k} : i \in \{1, 2, \ldots, r\} \text{ and } k \in \{0, 1, 2\}\big\}$ is the required partition of $\mathbb{R}$.$\square$

We omit the easy proof of the following corollary.

Corollary 4.7. *Let $P \in \mathcal{GP}$ and let r, $\big\{X_i : i \in \{1, 2, \ldots, r\}\big\}$, $Q_1, Q_2, \ldots, Q_r$, and $\{\alpha_1, \alpha_2, \ldots, \alpha_r\}$ be as in the statement of Theorem 4.6. Then $\widetilde{w \circ P}(p) = 0$ for every $p \in E(\beta\mathbb{R}_d)$ if and only if $\alpha_i \in \mathbb{Z}$ for every $i \in \{1, 2, \ldots, r\}$ for which X_i is an IP set.*

Definition 4.8. $\mathcal{H}$ will denote the set of functions $P \in \mathcal{GP}$ for which there exist an IP* subset X of $\mathbb{R}$, $r \in \mathbb{N}$, a partition $\big\{X_i : i \in \{1, 2, \ldots, r\}\big\}$ of X and members $Q_1, Q_2, \ldots, Q_r$ of $\mathcal{D}$, such that, for every $i \in \{1, 2, \ldots, r\}$ and every $x \in X_i$, $P(x) = Q_i(x)$.

Corollary 4.9. *Let $P \in \mathcal{GP}$. The following statements are equivalent.*

(a) $P \in \mathcal{H}$.

(b) For every $\alpha \in \mathbb{R}$ and every $p \in E(\beta\mathbb{R}_d)$, $\widetilde{w \circ \alpha P}(p) = 0$.

(c) For every $\alpha \in \mathbb{R}$ and every $\epsilon > 0$, $\{x \in \mathbb{R} : w\big(\alpha P(x)\big) \in (-\epsilon, \epsilon)\}$ is an IP set in $\mathbb{R}$.*

Proof. To see that (a) implies (b), pick X, r, $\big\{X_i : i \in \{1, 2, \ldots, r\}\big\}$, and $Q_1, Q_2, \ldots, Q_r$ as guaranteed by the definition of $\mathcal{H}$ for P. Let $\alpha \in \mathbb{R}$ and let $p \in E(\beta\mathbb{R}_d)$. Pick $i \in \{1, 2, \ldots, r\}$ such that $X_i \in p$. By Lemma 4.4(2), $\widetilde{w \circ \alpha P}(p) = \widetilde{w \circ \alpha Q_i}(p) = 0$.

That (b) implies (c) is trivial. To see that (c) implies (a), by Theorem 4.6 pick $r \in \mathbb{N}$, a partition $\big\{X_i : i \in \{1, 2, \ldots, r\}\big\}$ of $\mathbb{R}$, members $Q_1, Q_2, \ldots, Q_r$ of

$\mathcal{D}$, and $\{\delta_1, \delta_2, \ldots, \delta_r\} \subseteq \mathbb{R}$ such that for each $i \in \{1, 2, \ldots, r\}$ and each $x \in X_i$, $P(x) = Q_i(x) + \delta_i$. By reordering, we may assume that we have $s \in \{1, 2, \ldots, r\}$ such that for all $i \in \{1, 2, \ldots, s\}$, X_i is an IP set and for all $i \in \{s+1, s+2, \ldots, r\}$, if any, X_i is not an IP set. Let $X = \bigcup_{i=1}^{s} X_i$. Then X is an IP* set.

We claim that for $i \in \{1, 2, \ldots, s\}$, $\delta_i = 0$. So let $i \in \{1, 2, \ldots, s\}$ and pick an idempotent p such that $X_i \in p$. Suppose that $\delta_i \neq 0$ and let $\alpha = \frac{1}{4|\delta_i|}$. Then $\{x \in \mathbb{R} : w\big(\alpha P(x)\big) \in (-\frac{1}{8}, \frac{1}{8})\} \in p$ by assumption and, since $\alpha Q_i \in \mathcal{D}$, $\{x \in \mathbb{R} : w\big(\alpha Q_i(x)\big) \in (-\frac{1}{8}, \frac{1}{8})\} \in p$ by Lemma 4.5. Pick $x \in X_i$ such that $w\big(\alpha P(x)\big) \in (-\frac{1}{8}, \frac{1}{8})$ and $w\big(\alpha Q_i(x)\big) \in (-\frac{1}{8}, \frac{1}{8})$. Then $w(\alpha\delta_i) \in (-\frac{1}{4}, \frac{1}{4})$, which is a contradiction since $|\alpha\delta_i| = \frac{1}{4}$. □

Corollary 4.10. *$\mathcal{H}$ is an ideal of $\mathcal{GP}$.*

Proof. Let $P \in \mathcal{H}$ and $Q \in \mathcal{GP}$. It is sufficient to show that $PQ \in \mathcal{H}$ because $\mathcal{H}$ is clearly an algebra.

We can choose an IP$*$ subset X of $\mathbb{R}_d$, $r \in \mathbb{N}$, a partition $\{X_1, X_2, \ldots, X_r\}$ of X and functions $Q_1, Q_2, \ldots, Q_r \in \mathcal{D}$ such that, for every $i \in \{1, 2, \ldots, r\}$ and every $x \in X_i$, $P(x) = Q_i(x)$. We can also choose $s \in \mathbb{N}$, a partition $\{Y_1, Y_2, \ldots, Y_s\}$ of $\mathbb{R}$, numbers $\alpha_1, \alpha_2, \ldots, \alpha_s \in \mathbb{R}$ and functions $R_1, R_2, \ldots, R_s$ in $\mathcal{D}$ such that, for every $j \in \{1, 2, \ldots, s\}$ and every $x \in Y_j$, $Q(x) = R_j(x) + \alpha_j$. Then, for each $i \in \{1, 2, \ldots, r\}$ and each $j \in \{1, 2, \ldots, s\}$, $Q_i(R_j + \alpha_j) \in \mathcal{D}$. Since $P(x)Q(x) = Q_i(x)(R_j(x) + \alpha_j)$ if $x \in X_i \cap Y_j$ and since

$$\big\{X_i \cap Y_j : (i, j) \in \{1, 2, \ldots, r\} \times \{1, 2, \ldots, j\}\big\}$$

is a partition of X, it follows that $PQ \in \mathcal{H}$. □

The next result deals with *admissible generalized polynomials*. We now define a subset $\mathcal{G}_{\mathbb{Z}}$ of $\mathcal{G}$ with the property that, for each $P \in \mathcal{G}_{\mathbb{Z}}$, $P[\mathcal{G}_{\mathbb{Z}}] \subseteq \mathbb{Z}$.This is defined to be the smallest set of real functions which contains the polynomials with integer coefficients, is closed under sums and products, and whenever $v \in \mathbb{N}$, $c_1, c_2, \ldots, c_v \in \mathbb{R}$, and $p_1, p_2, \ldots, p_v \in \mathcal{G}_{\mathbb{Z}}$, then the function $n \mapsto \lfloor \sum_{i=1}^{v} c_i p_i(n) \rfloor$ is in $\mathcal{G}_{\mathbb{Z}}$. (In [9], the set $\mathcal{G}_{\mathbb{Z}}$, as defined here, is called the set of generalized polynomials. However, this usage differs from the definition in [10], presented as Definition 4.1 above, which includes all real polynomials in the set of generalized polynomials.)

Definition 4.11. The class $\mathcal{G}_a$ of *admissible generalized polynomials* is defined to be the smallest subset of $\mathcal{G}_{\mathbb{Z}}$ which includes the identity function and has the following properties:

(1) $P - Q \in \mathcal{G}_a$ whenever $P, Q \in \mathcal{G}_a$;

(2) $PQ \in \mathcal{G}_a$ whenever $P \in \mathcal{G}_a$ and $Q \in \mathcal{G}_{\mathbb{Z}}$; and

(3) $\lfloor c_1P_1 + c_2P_2 + \ldots + c_vP_v + \gamma \rfloor \in \mathcal{G}_a$ whenever $v \in \mathbb{N}$, $P_1, P_2, \ldots, P_v \in \mathcal{G}_a$, $c_1, c_2, \ldots, c_v \in \mathbb{R}$, and $0 < \gamma < 1$.

Lemma 4.12. $\mathcal{G}_a \subseteq \mathcal{H}$.

Proof. It is clear that $\mathcal{H}$ contains the identity function and has the property that $\mathcal{H} - \mathcal{H} \subseteq \mathcal{H}$. By Corollary 4.10, $\mathcal{H}\mathcal{G}_{\mathbb{Z}} \subseteq \mathcal{H}$. So it is sufficient to show that, for every $P_1, P_2, \ldots, P_v \in \mathcal{H}$, every $c_1, c_2, \ldots, c_v \in \mathbb{R}$ and every $\gamma \in (0,1)$, the function $Q = \lfloor c_1P_1 + c_2P_2 + \ldots + c_vP_v + \gamma \rfloor \in \mathcal{H}$. To see this, let $P = c_1P_1 + c_2P_2 + \ldots + c_vP_v \in \mathcal{H}$. Let $X = \{x \in \mathbb{R} : -\gamma < w(P(x)) < 1 - \gamma\}$. Then X is an IP* set by Corollary 4.9 and, for every $x \in X$, $Q(x) = \nu(P(x))$. It is now routine to check that $Q \in \mathcal{H}$. $\square$

Theorem 4.13. *Let P be an admissible generalized polynomial, let $\alpha \in \mathbb{R}$, and let $\epsilon > 0$. Then $\{x \in \mathbb{R} : w\big(\alpha P(x)\big) \in (-\epsilon, \epsilon)\}$ is an IP* set in $\mathbb{R}_d$.*

Proof. Lemma 4.12 and Corollary 4.9. $\square$

It is an immediate consequence of Theorem 4.13 that if P is an admissible generalized polynomial, $\alpha \in \mathbb{R}$, and $\epsilon > 0$, then $\{x \in \mathbb{N} : w\big(\alpha P(x)\big) \in (-\epsilon, \epsilon)\}$ is an IP* set in $\mathbb{N}$. This fact is also a consequence of [9, Theorem A].

It is very easy to give examples of functions $P \in \mathcal{D}$ and open subsets U of $\mathbb{T}$ for which $\{x \in \mathbb{R} : w\big(P(x)\big) \in U\} = \emptyset$. For example, if $P(x) = \nu(x)$, $(w \circ P)[\mathbb{R}] = \{0\}$. Just slightly less trivially, if $P(x) = (x - \nu(x))^2$, $(w \circ P)[\mathbb{R}] = [0, \frac{1}{4}]$. However, we now see that, if $\{x \in \mathbb{R} : w\big(P(x)\big) \in U\}$ is an IP set in $\mathbb{R}_d$, then it has a very rich structure.

Corollary 4.14. *Let $P \in \mathcal{D}$ and let S be a subsemigroup of $\mathbb{R}_d$. Let U be an open subset of $\mathbb{T}$ and let $A = \{x \in S : w\big(P(x)\big) \in U\}$. If A is an IP set in S, then A is a strongly central subset and an IP$^*_+$ subset of S.*

Proof. Let $p \in \overline{A} \cap E(\beta S)$ and let L be a minimal left ideal of βS. By Lemma 4.4, there is an IP* subset X of $\beta\mathbb{R}_d$ such that for every $x \in X$ there exists $R \in \mathcal{D}$ such that $Y = \{y \in \mathbb{R} : P(x+y) = P(x) + P(y) + R(y)\}$ is IP* in $\mathbb{R}_d$. Since $p + \beta S$ is a right ideal of βS, there is an idempotent $q \in L \cap (p + \beta S)$. We observe that $p + q = q$. Let $B = A \cap X \cap \{x \in \mathbb{R} : w\big(P(x)\big) \in (-\frac{1}{4}, \frac{1}{4})\}$ and let $x \in B$. Since $w\big(P(x)\big) \in U$ and since $\widetilde{w \circ P}(q) = \widetilde{w \circ R}(q) = 0$, it follows that $-x + A = \{y \in S : w\big(P(x+y)\big) \in U\} \in q$. Now $\{x + y : x \in B \text{ and } y \in -x + A\} \in p + q = q$. Since this set is contained in A, $A \in q$. Thus A is strongly central in S.

A similar argument, using the fact that Y is IP* in $\mathbb{R}_d$, shows that, for each $x \in B$, $-x + A$ is IP* in S. So A is an IP$^*_+$ set in S. $\square$

Corollary 4.15. *Let $P \in \mathcal{D}$ and let $0 < \epsilon < \frac{1}{2}$. Then at least one of the following statements must hold:*

(a) $\{x \in \mathbb{R} : w(P(x)) = 0\}$ *is IP* in $\mathbb{R}_d$;*

(b) $\{x \in \mathbb{R} : w(P(x)) \in (0, \epsilon)\}$ *is strongly central and IP$^*_+$ in $\beta\mathbb{R}_d$.*

(c) $\{x \in \mathbb{R} : w(P(x)) \in (-\epsilon, 0)\}$ *is strongly central and IP$^*_+$ in $\beta\mathbb{R}_d$.*

Proof. Lemma 4.5 and Corollary 4.14. $\square$

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