POLYNOMIAL EXTENSIONS OF THE MILLIKEN-TAYLOR THEOREM

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Abstract. *Milliken-Taylor systems* are some of the most general infinitary configurations that are known to be partition regular. These are sets of the form $MT(\langle a_i\rangle_{i=1}^m, \langle x_n\rangle_{n=1}^\infty) = \{\sum_{i=1}^m a_i \sum_{t\in F_i} x_t : F_1, F_2, \ldots, F_m$ are increasing finite nonempty subsets of $\mathbb{N}\}$, where $a_1, a_2, \ldots, a_m \in \mathbb{Z}$ with $a_m > 0$ and $\langle x_n\rangle_{n=1}^\infty$ is a sequence in $\mathbb{N}$. That is, if $p(y_1, y_2, \ldots, y_m) = \sum_{i=1}^m a_i y_i$ is a given linear polynomial and a finite coloring of $\mathbb{N}$ is given, one gets a sequence $\langle x_n\rangle_{n=1}^\infty$ such that all sums of the form $p(\sum_{t\in F_1} x_t, \ldots, \sum_{t\in F_m} x_t)$ are monochromatic. In this paper we extend these systems to images of very general *extended polynomials*. We work with the Stone-Čech compactification $\beta\mathcal{F}$ of the discrete space $\mathcal{F}$ of finite subsets of $\mathbb{N}$, whose points we take to be the ultrafilters on $\mathcal{F}$. We utilize a simply stated result about the tensor products of ultrafilters and the algebraic structure of $\beta\mathcal{F}$.

1. Introduction

Given a sequence $\langle x_n\rangle_{n=1}^\infty$ in the set $\mathbb{N}$ of positive integers, let $FS(\langle x_n\rangle_{n=1}^\infty) = \{\sum_{n\in F} x_n : F \in \mathcal{P}_f(\mathbb{N})\}$, where, for any set X, $\mathcal{P}_f(X)$ is the set of finite nonempty subsets of X. Similarly, given a sequence $\langle H_n\rangle_{n=1}^\infty$ in $\mathcal{P}_f(\mathbb{N})$, $FU(\langle H_n\rangle_{n=1}^\infty) = \{\bigcup_{n\in F} H_n : F \in \mathcal{P}_f(\mathbb{N})\}$. (Here FS and FU stand for "finite sums" and "finite unions".) In 1972 the following theorems were proved. Before they were proved, the statements were known to be equivalent via a consideration of the binary expansions of positive integers. (If $H_n \cap H_m = \emptyset$, then $\sum_{t\in H_n} 2^{t-1} + \sum_{t\in H_m} 2^{t-1} = \sum_{t\in H_n\cup H_m} 2^{t-1}$ so Theorem 1.2 trivially implies Theorem 1.1. To see that Theorem 1.1 implies Theorem 1.2, one first shows that the sequence $\langle x_n\rangle_{n=1}^\infty$ can be chosen so that, if $x_n = \sum_{t\in H_n} 2^{t-1}$, then $\max H_n < \min H_{n+1}$.)

Theorem 1.1 (Finite Sums Theorem). *Let $r \in \mathbb{N}$ and let $\mathbb{N} = \bigcup_{i=1}^r C_i$. There exist $i \in \{1, 2, \ldots, r\}$ and a sequence $\langle x_n\rangle_{n=1}^\infty$ such that $FS(\langle x_n\rangle_{n=1}^\infty) \subseteq C_i$.*

Proof. [17, Theorem 3.1]. □

Theorem 1.2 (Finite Unions Theorem). *Let $r \in \mathbb{N}$ and let $\mathcal{P}_f(\mathbb{N}) = \bigcup_{i=1}^r \mathcal{C}_i$. There exist $i \in \{1, 2, \ldots, r\}$ and a sequence $\langle H_n\rangle_{n=1}^\infty$ such that $\max F_n < \min F_{n+1}$ for each $n \in \mathbb{N}$ and $FU(\langle H_n\rangle_{n=1}^\infty) \subseteq \mathcal{C}_i$.*

Proof. [17, Corollary 3.3]. □

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Before the actual publication of [17], while both were still graduate students, Keith Milliken and Alan Taylor independently used Theorem 1.1 to prove what has come to be known as the Milliken-Taylor Theorem, a result which provides a simultaneous generalization of the Finite Sums Theorem and Ramsey's Theorem and which has been often utilized in the literature, including various powerful generalizations of Szemerédi's Theorem on arithmetic progressions. (See for example [1], [3], [10], and [11].) To state the Milliken-Taylor Theorem, we need to introduce some notation. Given $F, G \in \mathcal{P}_f(\mathbb{N})$, we write $F < G$ to mean that $\max F < \min G$. Further, when we write $F < G$ we intend to implicitly include the assertion that $F, G \in \mathcal{P}_f(\mathbb{N})$. When we say that a sequence $\langle H_n \rangle_{n=1}^\infty$ in $\mathcal{P}_f(\mathbb{N})$ is an *increasing* sequence we mean that for each n, $\max H_n < \min H_{n+1}$. In a semigroup $(S, \cdot)$, analogous to the notation $FS(\langle x_n \rangle_{n=1}^\infty)$, we have $FP(\langle x_n \rangle_{n=1}^\infty) = \{\prod_{t \in F} x_t : F \in \mathcal{P}_f(\mathbb{N})\}$, where $\prod_{t \in F} x_t$ is taken in increasing order of indices.

The notions defined in (5) and (6) below are special cases of (4). We present the different terminology because these special cases arise frequently. In each of these, the object defined depends not only on the set $FP(\langle y_n \rangle_{n=1}^\infty)$ or $FS(\langle y_n \rangle_{n=1}^\infty)$ or $FU(\langle F_n \rangle_{n=1}^\infty)$, but on the sequence which generates the set.

Definition 1.3. Let $k \in \mathbb{N}$.

(1) For any set X, $[X]^k = \{A \subseteq X : |A| = k\}$.

(2) For a sequence $\langle x_n \rangle_{n=1}^\infty$ in $\mathbb{N}$,

$$[FS(\langle x_n \rangle_{n=1}^\infty)]^k_< = \big\{ \{\textstyle\sum_{n \in F_1} x_n, \sum_{n \in F_2} x_n, \ldots, \sum_{n \in F_k} x_n\} : F_1 < F_2 < \ldots < F_k \big\}.$$

(3) For a sequence $\langle H_n \rangle_{n=1}^\infty$ in $\mathcal{P}_f(\mathbb{N})$

$$[FU(\langle H_n \rangle_{n=1}^\infty)]^k_< = \big\{ \{\textstyle\bigcup_{n \in F_1} H_n, \bigcup_{n \in F_2} H_n, \ldots, \bigcup_{n \in F_k} H_n\} : F_1 < F_2 < \ldots < F_k \big\}.$$

(4) In a semigroup $(S, \cdot)$, $FP(\langle y_n \rangle_{n=1}^\infty)$ is a *product subsystem* of $FP(\langle x_n \rangle_{n=1}^\infty)$ if and only if there exists an increasing sequence $\langle H_n \rangle_{n=1}^\infty$ in $\mathcal{P}_f(\mathbb{N})$ such that for each $n \in \mathbb{N}$, $y_n = \prod_{t \in H_n} x_t$, where the products $\prod_{t \in H_n} x_t$ are computed in increasing order of indices.

(5) In a semigroup $(S, +)$, $FS(\langle y_n \rangle_{n=1}^\infty)$ is a *sum subsystem* of $FS(\langle x_n \rangle_{n=1}^\infty)$ if and only if there exists an increasing sequence $\langle H_n \rangle_{n=1}^\infty$ in $\mathcal{P}_f(\mathbb{N})$ such that for each $n \in \mathbb{N}$, $y_n = \sum_{t \in H_n} x_t$.

(6) In the semigroup $(\mathcal{P}_f(\mathbb{N}), \cup)$, $FU(\langle K_n \rangle_{n=1}^\infty)$ is a *union subsystem* of $FU(\langle F_n \rangle_{n=1}^\infty)$ if and only if there exists an increasing sequence $\langle H_n \rangle_{n=1}^\infty$ in $\mathcal{P}_f(\mathbb{N})$ such that for each $n \in \mathbb{N}$, $K_n = \bigcup_{t \in H_n} F_t$.

The subsystems defined in (4) are called *IP-subsystems* in [14, Chapter 8].

Version (1) of the following theorem is due to K. Milliken and version (2) is due to A. Taylor. The fact that they are equivalent is established similarly to the way Theorems 1.1 and 1.2 are standardly shown to be equivalent.

Theorem 1.4 (Milliken-Taylor Theorem)**.** *Let $m, r \in \mathbb{N}$.*

(1) *Let $[\mathbb{N}]^m = \bigcup_{i=1}^r C_i$, and let $\langle x_n \rangle_{n=1}^\infty$ be a sequence in $\mathbb{N}$. There exist $i \in \{1, 2, \ldots, r\}$ and a sum subsystem $FS(\langle y_n \rangle_{n=1}^\infty)$ of $FS(\langle x_n \rangle_{n=1}^\infty)$ such that*
$[FS(\langle y_n \rangle_{n=1}^\infty)]^m_< \subseteq C_i$.

(2) *Let* $[\mathcal{P}_f(\mathbb{N})]^m = \bigcup_{i=1}^r C_i$. *There exist* $i \in \{1,2,\ldots,r\}$ *and an increasing sequence* $\langle H_n\rangle_{n=1}^\infty$ *in* $\mathcal{P}_f(\mathbb{N})$ *such that* $[FU(\langle H_n\rangle_{n=1}^\infty)]^m_< \subseteq C_i$.

Proof. (1) [21, Theorem 2.2], or see [20, Theorem 18.7].
(2) [24, Lemma 2.2], or see [20, Corollary 18.8]. □

The case $m = 1$ of Theorem 1.4(1) is an apparent strengthening of the Finite Sums Theorem. That is, not only is one guaranteed a sequence with its finite sums in one color class, but one may get such a sequence $\langle x_n\rangle_{n=1}^\infty$ with $FS(\langle x_n\rangle_{n=1}^\infty)$ as a sum subsystem of any specified sequence. (This strengthening is also easily derivable from the Finite Sums Theorem itself. See the sequence of exercises at the end of [20, Section 5.2].)

A goal of this paper is to establish a polynomial version of the Milliken-Taylor Theorem, thereby adding to the circle of results represented by the Polynomial van der Waerden Theorem and the Polynomial Hales-Jewett Theorem. (See [6] and [7].) To start explaining our approach, we introduce the notion of *Milliken-Taylor System*.

Definition 1.5. Let $m \in \mathbb{N}$ and let $\langle a_j\rangle_{j=1}^m$ and $\langle x_n\rangle_{n=1}^\infty$ be sequences in $\mathbb{N}$. The *Milliken-Taylor System determined by* $\langle a_j\rangle_{j=1}^m$ *and* $\langle x_n\rangle_{n=1}^\infty$ is

$$MT(\langle a_j\rangle_{j=1}^m, \langle x_n\rangle_{n=1}^\infty) = \{\textstyle\sum_{j=1}^m a_j \sum_{t\in F_j} x_t : F_1 < F_2 < \ldots < F_m\}\,.$$

Milliken-Taylor systems are partition regular. That is, there is the following result which is well known among the experts.

Theorem 1.6. *Let* $m, r \in \mathbb{N}$, *let* $\langle a_j\rangle_{j=1}^m$ *and* $\langle x_n\rangle_{n=1}^\infty$ *be sequences in* $\mathbb{N}$, *and let* $\mathbb{N} = \bigcup_{i=1}^r C_i$. *There exist* $i \in \{1,2,\ldots,r\}$ *and a sum subsystem* $FS(\langle y_n\rangle_{n=1}^\infty)$ *of* $FS(\langle x_n\rangle_{n=1}^\infty)$ *such that* $MT(\langle a_j\rangle_{j=1}^m, \langle y_n\rangle_{n=1}^\infty) \subseteq C_i$.

One may extend Theorem 1.6 by producing sum subsystems of the set of finite sums of several different sequences, and allowing them to occur in arbitrary order. For example with $m = 3$ and $k = 2$ in the following theorem, one can be asking for partition regularity of sums of the form $a_1 \sum_{t\in F_1} y_{1,t} + a_2 \sum_{t\in F_1} y_{2,t} + a_3 \sum_{t\in F_3} y_{1,t}$. See the explanation following the statement of Theorem 1.13, regarding the difficulties introduced by possible repetitions of sequences.

Theorem 1.7. *Let* $k, m, r \in \mathbb{N}$, *let* $\langle a_j\rangle_{j=1}^m$ *be a sequence in* $\mathbb{N}$ *and for each* $j \in \{1,2,\ldots,k\}$, *let* $\langle x_{j,n}\rangle_{n=1}^\infty$ *be a sequence in* $\mathbb{N}$. *Let* $f : \{1,2,\ldots,m\} \to \{1,2,\ldots,k\}$ *and let* $\mathbb{N} = \bigcup_{i=1}^r C_i$. *There exists* $i \in \{1,2,\ldots,r\}$ *and for each* $j \in \{1,2,\ldots,k\}$, *there exists a sum subsystem* $FS(\langle y_{j,n}\rangle_{n=1}^\infty)$ *of* $FS(\langle x_{j,n}\rangle_{n=1}^\infty)$ *such that* $\{\sum_{j=1}^m a_j \sum_{t\in F_j} y_{f(j),t} : F_1 < F_2 < \ldots < F_m\} \subseteq C_i$.

It is quite easy to see that Theorem 1.7 implies Theorem 1.6. The converse is not so obvious. In fact, we shall show in the appendix that Theorems 1.6, 1.7, and the following theorem are all equivalent to the Milliken-Taylor Theorem (Theorem 1.4) (in the informal sense that each is easily derivable from the other).

Theorem 1.8. *Let* $m, k, r \in \mathbb{N}$, *and let* $f : \{1,2,\ldots,m\} \to \{1,2,\ldots,k\}$. *For each* $j \in \{1,2,\ldots,k\}$, *let* $\langle H_{j,n}\rangle_{n=1}^\infty$ *be an increasing sequence in* $\mathcal{P}_f(\mathbb{N})$, *and let* $\mathcal{P}_f(\mathbb{N})^m = \bigcup_{i=1}^r C_i$. *There exists* $i \in \{1,2,\ldots,r\}$ *and for each* $j \in \{1,2,\ldots,k\}$ *there exists a union subsystem* $FU(\langle K_{j,n}\rangle_{n=1}^\infty)$ *of* $FU(\langle H_{j,n}\rangle_{n=1}^\infty)$ *such that*

$$\{(\textstyle\bigcup_{t\in F_1} K_{f(1),t}, \bigcup_{t\in F_2} K_{f(2),t}, \ldots, \bigcup_{t\in F_m} K_{f(m),t}) : F_1 < F_2 < \ldots < F_m\} \subseteq C_i\,.$$

Before the proof of the Finite Sums Theorem, Fred Galvin knew that this theorem would be an easy consequence of the existence of what he called an *almost translation invariant* ultrafilter on $\mathbb{N}$, namely an ultrafilter p with the property that whenever $A \in p$, $\{x \in \mathbb{N} : -x + A \in p\} \in p$, where $-x + A = \{y \in \mathbb{N} : x + y \in A\}$. (We are here viewing an ultrafilter as a maximal filter on $\mathbb{N}$, that is a set of subsets of $\mathbb{N}$ which is maximal with respect to the finite intersection property. The "almost translation invariant" terminology reflects the fact that an ultrafilter can be thought of as a finitely additive, $\{0, 1\}$-valued, measure on $\mathcal{P}(\mathbb{N})$, wherein the assertion that $A \in p$ is the same as saying that p assigns measure 1 to A. Thus when one says that p is almost translation invariant, one is saying that a p-large set p-almost always translates to a p-large set.) In 1975, Galvin met Steven Glazer who knew that an almost translation invariant ultrafilter was simply an idempotent in the compact right topological semigroup $(\beta\mathbb{N}, +)$ and that any compact Hausdorff right topological semigroup has idempotents [13, Lemma 1]. Consequently, a very easy proof of the Finite Sums Theorem became available. (See for example [20, Theorem 5.8]. And the process of exploiting the algebraic structure of the Stone-Čech compactification for combinatorial applications began.

We pause now to briefly introduce the algebra of the Stone-Čech compactification of a discrete semigroup. Given a discrete semigroup $(S, \cdot)$, we take the points of βS to be the ultrafilters on S, identifying the points of S with the principal ultrafilters. (The principal ultrafilter associated with the point $x \in S$ is $\{A \subseteq S : x \in S\}$. If one is thinking of an ultrafilter as a measure, this is the point mass measure which has $\mu(\{x\}) = 1$.) The operation on S extends to an associative operation on βS, customarily denoted by the same symbol. (In particular, if the operation on S is denoted by $+$, so is the operation on βS. But the reader should be warned that $(\beta S, +)$ is very unlikely to be commutative. In fact, the centers of $(\beta\mathbb{N}, +)$ and $(\beta\mathbb{N}, \cdot)$ are both equal to $\mathbb{N}$ [20, Theorem 6.10].) Given $p \in \beta S$ and $x \in S$, the functions ρ_p and λ_x from βS to itself are continuous, where, for $q \in \beta S$, $\rho_p(q) = q \cdot p$ and $\lambda_x(q) = x \cdot q$. Given $A \subseteq S$ and $p, q \in \beta S$, $A \in p \cdot q$ if and only if

$$\{x \in S : x^{-1}A \in q\} \in p\,,$$

where $x^{-1}A = \{y \in S : xy \in A\}$. In particular, since we are identifying the points of S with the principal ultrafilters, if $a \in S$, $p \in \beta S$, and $A \subseteq S$, then $A \in ap$ if and only if $a^{-1}A \in p$. A great deal is known about the algebraic structure of βS and its combinatorial consequences. (See [20] for much of this information as well as an elementary introduction to the subject.)

In this paper, we will be primarily concerned with applications of the basic fact cited above that any compact Hausdorff right topological semigroup has idempotents and the relationship with what are known as *IP-sets*. In a semigroup $(S, \cdot)$, a set A is an IP-set if and only if there exists a sequence $\langle x_n \rangle_{n=1}^{\infty}$ in S such that $FP(\langle x_n \rangle_{n=1}^{\infty}) \subseteq A$.

Lemma 1.9. *Let $(S, \cdot)$ be a semigroup and let $\langle x_n \rangle_{n=1}^{\infty}$ be a sequence in S. Then $T = \bigcap_{m=1}^{\infty} \overline{FP(\langle x_n \rangle_{n=m}^{\infty})}$ is a compact subsemigroup of βS, and thus there is an idempotent in T.*

Proof. [20, Lemma 5.11]. □

Theorem 1.10. *Let $(S, \cdot)$ be a semigroup and let $A \subseteq S$. Then A is an IP-set if and only if there is an idempotent $p \in \beta S$ such that $A \in p$.*

Proof. The sufficiency is the Galvin-Glazer proof referred to above. The necessity is Lemma 1.9. □

Milliken-Taylor systems are images of infinite matrices under matrix multiplication and provide most of the known examples of infinite image partition regular matrices. (See [18].) Classifying infinite image partition regular matrices is a major unsolved problem. (By way of contrast, the finite image partition regular matrices are completely characterized in terms of *first entries matrices* [19], which are essentially the same thing as Deuber's (m, p, c)-sets [12], or see [16].)

The following relationship between a Milliken-Taylor system and a linear form in one variable evaluated at an ultrafilter is the starting point for our current investigation. The proof follows from [20, Theorems 17.31 and 17.32]. (It is also a special case of Corollary 3.5.) Note that, if $p \in \beta\mathbb{N}$ and $a \in \mathbb{N}$, then ap is the product in $\beta\mathbb{N}$ and not the sum of p with itself a times. It is not true in general that $a_1p + a_2p = (a_1 + a_2)p$.

Theorem 1.11. *Let $k \in \mathbb{N}$, let $\langle a_j\rangle_{j=1}^k$ and $\langle x_n\rangle_{n=1}^\infty$ be sequences in $\mathbb{N}$, let $g(z) = \sum_{j=1}^k a_j z$, and let $A \subseteq \mathbb{N}$. The following statements are equivalent.*

(a) *There is an idempotent $p \in \bigcap_{m=1}^\infty \overline{FS(\langle x_n\rangle_{n=m}^\infty)}$ such that $A \in g(p)$.*
(b) *There is a sum subsystem $FS(\langle y_n\rangle_{n=1}^\infty)$ of $FS(\langle x_n\rangle_{n=1}^\infty)$ such that $MT(\langle a_j\rangle_{j=1}^k, \langle y_n\rangle_{n=1}^\infty) \subseteq A$.*

One can also simply extend Theorem 1.11 to apply to linear expressions with multiple variables. The following result is also a special case of Corollary 3.5.

Theorem 1.12. *Let $k \in \mathbb{N}$, let $\langle a_j\rangle_{j=1}^k$ be a sequence in $\mathbb{N}$ and for each $j \in \{1, 2, \ldots, k\}$, let $\langle x_{j,n}\rangle_{n=1}^\infty$ be a sequence in $\mathbb{N}$. Let $g(z_1, z_2, \ldots, z_k) = \sum_{j=1}^k a_j z_j$, and let $A \subseteq \mathbb{N}$. The following statements are equivalent.*

(a) *For each $j \in \{1, 2, \ldots, k\}$, there is an idempotent $p_j \in \bigcap_{m=1}^\infty \overline{FS(\langle x_{j,n}\rangle_{n=m}^\infty)}$ such that $A \in g(p_1, p_2, \ldots, p_k)$.*
(b) *For each $j \in \{1, 2, \ldots, k\}$, there is a sum subsystem $FS(\langle y_{j,n}\rangle_{n=1}^\infty)$ of $FS(\langle x_{j,n}\rangle_{n=1}^\infty)$ such that $\{\sum_{j=1}^k a_j \sum_{t\in F_j} y_{j,t} : F_1 < F_2 < \ldots < F_k\} \subseteq A$.*

Since $\beta\mathbb{N}$ is a semigroup under both addition and multiplication, it makes sense to talk about polynomials in multiple variables evaluated at members of $\beta\mathbb{N}$. Consider, for example the polynomial $h(z_1, z_2) = -3z_1 + 2z_2z_1$. The following theorem is a special case of [25, Theorem 2.3], a result which is in turn a special case of Corollary 3.5 of the current paper.

Theorem 1.13. *Let $h(z_1, z_2) = -3z_1 + 2z_2z_1$, $A \subseteq \mathbb{N}$, and for $i \in \{1, 2\}$, let $\langle x_{i,t}\rangle_{t=1}^\infty$ be a sequence in $\mathbb{N}$. The following are equivalent:*

(a) *For each $i \in \{1, 2\}$, there exists an idempotent $p_i \in \bigcap_{m=1}^\infty \overline{FS(\langle x_{i,t}\rangle_{t=m}^\infty)}$ such that $A \in h(p_1, p_2)$.*
(b) *There exist sum subsystems $FS(\langle y_{i,t}\rangle_{t=1}^\infty)$ of $FS(\langle x_{i,t}\rangle_{t=1}^\infty)$ for each $i \in \{1, 2\}$ such that $\{-3\sum_{t\in F_1} y_{1,t} + 2\sum_{t\in F_2} y_{2,t} \sum_{t\in F_3} y_{1,t} : F_1 < F_2 < F_3\} \subseteq A$.*

If, instead of the polynomial $h(z_1, z_2) = -3z_1 + 2z_2z_1$, one were dealing with the polynomial $g(z_1, z_2, z_3) = -3z_1 + 2z_2z_3$, the directly analogous result with three idempotents p_1, p_2, and p_3 and three given sequences would hold. But then it would be a routine computation. The reason involves the continuity of the operations in $(\beta\mathbb{Z},+)$ and $(\beta\mathbb{Z},\cdot)$. Given $p,q \in \beta\mathbb{Z}$, let $\rho_p^+(q) = q+p$, $\rho_p^\bullet(q) = q\cdot p$, $\lambda_p^+(q) = p+q$, and $\lambda_p^\bullet(q) = p\cdot q$. Then both ρ_p^+ and $\rho_p^\bullet$ are continuous, and if $p \in \mathbb{Z}$, then λ_p^+ and $\lambda_p^\bullet$ are continuous. Let $p_1,p_2,p_3 \in \beta\mathbb{N}$ and let U be an open neighborhood of $g(p_1,p_2,p_3)$. Since $g(p_1,p_2,p_3) = \rho^+_{2p_2p_3}\big(\lambda^\bullet_{-3}(p_1)\big)$, one gets a neighborhood V_1 of p_1 such that $-3V_1+2p_2p_3 \subseteq U$. Given any $z_1 \in V_1$, $-3z_1+2p_2p_3 = \lambda^+_{-3z_1}\Big(\rho^\bullet_{p_3}\big(\lambda^\bullet_2(p_2)\big)\Big)$ so one gets a neighborhood V_2 of p_2 such that $-3z_1+2V_2p_3 \subseteq U$. Given any $z_2 \in V_2$, $-3z_1+2z_2p_3 = \lambda^+_{-3z_1}\big(\lambda^\bullet_{2z_2}(p_3)\big)$ so one gets a neighborhood V_3 of p_3 such that $-3z_1+2z_2V_3 \subseteq U$ and given any $z_3 \in V_3$ one has $-3z_1+2z_2z_3 \in U$. In the event that $p_3 = p_1$, this routine argument does not allow one to choose $z_3 = z_1$. That is, the challenge in the case of h is to choose the sum subsystem to simultaneously satisfy the requirements on F_1 and F_3.

In recent years, some of the classical results of Ramsey Theory have been "polynomialized", beginning with [6] where the following extension of Szemerédi's Theorem [22] was established: If A is a subset of $\mathbb{N}$ with positive upper density, $k \in \mathbb{N}$, and $P_1,P_2,\ldots,P_k$ are polynomials taking integer values at the integers and having zero constant term, then there exist a and m such that $\{a+P_1(m), a+P_2(m),\ldots,a+P_k(m)\} \subseteq A$. More recently [23] the same result was established where A is only assumed to have positive relative density in the set of prime numbers. Other polynomializations of versions of Szemerédi's Theorem can be found in [9] and [10].

The works cited in the paragraph above all used ordinary polynomials. In [7], *set polynomials* were used to obtain a generalization of the Hales-Jewett Theorem [15]. In [8], [4], [5], [11], *generalized polynomials* were studied and applied. These are functions with values in $\mathbb{R}$ or $\mathbb{R}^d$ that are built up using addition, multiplication, and applications of the greatest integer function. In the current paper, we introduce *extended polynomials* in which we allow arbitrarily many associative operations.

In our applications of extended polynomials, we deal with idempotents with respect to any of the operations. The following theorem, dealing with the same polynomial h defined by $h(z_1,z_2) = -3z_1+2z_2z_1$, is a special case of Corollary 3.5 applied to this polynomial, wherein p_1 is a multiplicative idempotent and p_2 is an additive idempotent.

Theorem 1.14. *Let $h(z_1,z_2) = -3z_1+2z_2z_1$, $A \subseteq \mathbb{N}$, and for $i \in \{1,2\}$, let $\langle x_{i,t}\rangle_{t=1}^\infty$ be a sequence in $\mathbb{N}$. The following are equivalent:*

(a) *There exist $p_1 = p_1\cdot p_1 \in \bigcap_{m=1}^\infty \overline{FP(\langle x_{1,t}\rangle_{t=m}^\infty)}$ and $p_2 = p_2+p_2 \in \bigcap_{m=1}^\infty \overline{FS(\langle x_{2,t}\rangle_{t=m}^\infty)}$ such that $A \in h(p_1,p_2)$.*

(b) *There exist a product subsystem $FP(\langle y_{1,t}\rangle_{t=1}^\infty)$ of $FP(\langle x_{1,t}\rangle_{t=1}^\infty)$ and a sum subsytem $FS(\langle y_{2,t}\rangle_{t=1}^\infty)$ of $FS(\langle x_{i,t}\rangle_{t=1}^\infty)$ such that*
$\{-3\prod_{t\in F_1} y_{1,t} + 2(\sum_{t\in F_2} y_{2,t})(\prod_{t\in F_3} y_{1,t}) : F_1 < F_2 < F_3\} \subseteq A.$

Our main tool is the notion of tensor products of ultrafilters, which we introduce now.

Definition 1.15. Let $k \in \mathbb{N}$ and for $i \in \{1,2,\ldots,k\}$, let S_i be a semigroup and let $p_i \in \beta S_i$. We define $\bigotimes_{i=1}^k p_i \in \beta(\times_{i=1}^k S_i)$ inductively as follows.

(1) $\bigotimes_{i=1}^1 p_i = p_1$.

(2) Given $k \in \mathbb{N}$ and $A \subseteq \times_{i=1}^{k+1} S_i$, $A \in \bigotimes_{i=1}^{k+1} p_i$ if and only if
$\{(x_1,x_2,\ldots,x_k) \in \times_{i=1}^k S_i : \{x_{k+1} \in S_{k+1} :$
$(x_1,x_2,\ldots,x_{k+1}) \in A\} \in p_{k+1}\} \in \bigotimes_{i=1}^k p_i$.

It is routine but mildly tedious to verify that $\bigotimes_{i=1}^k p_i$ is an ultrafilter on $\times_{i=1}^k S_i$. (We shall present the details in the appendix.)

One can establish the following two generalizations of Theorem 1.10. We will also provide proofs of these in the appendix.

Theorem 1.16. *Let $m \in \mathbb{N}$ and for each $i \in \{1, 2, \ldots, m\}$, let S_i be a semigroup and let $A \subseteq \times_{i=1}^m S_i$. The following statements are equivalent.*

(a) *For each $i \in \{1, 2, \ldots, m\}$, there is a sequence $\langle x_{i,n}\rangle_{n=1}^\infty$ in S_i such that*
$$\{(\textstyle\prod_{t\in F_1} x_{1,t}, \prod_{t\in F_2} x_{2,t}, \ldots, \prod_{t\in F_m} x_{m,t}) : F_1 < F_2 < \ldots < F_m\} \subseteq A\,.$$

(b) *For each $i \in \{1, 2, \ldots, m\}$, there is an idempotent $p_i \in \beta S_i$ such that $A \in \bigotimes_{i=1}^m p_i$.*

Theorem 1.17. *Let S be a semigroup, let $m \in \mathbb{N}$, and let $A \subseteq \times_{i=1}^m S$. The following statements are equivalent.*

(a) *There is a sequence $\langle x_n\rangle_{n=1}^\infty$ in S such that*
$$\{(\textstyle\prod_{t\in F_1} x_t, \prod_{t\in F_2} x_t, \ldots, \prod_{t\in F_m} x_t) : F_1 < F_2 < \ldots < F_m\} \subseteq A\,.$$

(b) *There is an idempotent $p \in \beta S$ such that $A \in \bigotimes_{i=1}^m p$.*

The reader will see that the proof of Theorem 1.16 is much simpler than the proof of Theorem 1.17, again because in the latter case one needs to be concerned with an appearence of a given x_t at each position in the expression
$$(\textstyle\prod_{t\in F_1} x_t, \prod_{t\in F_2} x_t, \ldots, \prod_{t\in F_m} x_t)\,.$$

We should point out that we do not see a way to show that the Milliken-Taylor Theorem implies either Theorem 1.16 or Theorem 1.17 (beyond the formal fact that these theorems are true.)

The basic facts which we need about the tensor products of ultrafilters are presented in Section 2 in the context of the semigroup $\mathcal{F}$ of finite nonempty subsets of $\mathbb{N}$ under the operation of union. As we have noted, we customarily use the same notation to denote the extension of an operation to βS as used for the operation in S. However, for $p, q \in \beta\mathcal{F}$, $p \cup q$ already means something, so we denote the extended operation by $*$. Thus, for $A \subseteq \mathcal{F}$, $A \in p * q$ if and only if $\{F \in \mathcal{F} : \{G \in \mathcal{F} : F \cup G \in A\} \in q\} \in p$.

Definition 1.18. $\delta\mathcal{F} = \{p \in \beta\mathcal{F} : (\forall n \in \mathbb{N})(\{F \in \mathcal{F} : \min F > n\} \in p)\}$.

Equivalently, given $p \in \beta\mathcal{F}$, $p \in \delta\mathcal{F}$ if and only if for each $G \in \mathcal{F}$, $\{F \in \mathcal{F} : F \cap G = \emptyset\} \in p$. By [2, Proposition 2.6], $\delta\mathcal{F}$ is a subsemigroup of $\beta\mathcal{F}$. One deals with $\delta\mathcal{F}$ rather than $\beta\mathcal{F}$ because the operation is better behaved there. For example, the function $\varphi : \mathcal{F} \to \mathbb{N}$ defined by $\varphi(F) = \sum_{t\in F} 2^t$ extends to a continuous function $\widetilde{\varphi} : \beta\mathcal{F} \to \beta\mathbb{N}$. This extension is not a homomorphism on $\beta\mathcal{F}$, but its restriction to $\delta\mathcal{F}$ is a homomorphism.

The principal result of Section 2 is the following generalization of the Milliken-Taylor Theorem.

Theorem 2.6. *Let $m, k \in \mathbb{N}$ and let $f : \{1, 2, \ldots, m\} \to \{1, 2, \ldots, k\}$. For each $i \in \{1, 2, \ldots, k\}$, let p_i be an idempotent in $\delta\mathcal{F}$, and let $A \in \bigotimes_{i=1}^m p_{f(i)}$. Then for each $i \in \{1, 2, \ldots, k\}$ there exists an increasing sequence $\langle H_{i,n}\rangle_{n=1}^\infty$ in $\mathcal{F}$ such that*
$$\{(\textstyle\bigcup_{t\in F_1} H_{f(1),t}, \bigcup_{t\in F_2} H_{f(2),t}, \ldots, \bigcup_{t\in F_m} H_{f(m),t}) : F_1 < F_2 < \ldots < F_m\} \subseteq A\,.$$

In Section 3 we will introduce the notion of an *extended polynomial* over a set S, where S is a semigroup with respect to each of a set $\mathfrak{S}$ of operations on S. In the special case that $S=\mathbb{N}$ these include the ordinary polynomials. We will then define the set $\mathfrak{P}_m$ to be the set of extended polynomials with m variables. For example, if $S=\mathbb{N}$, $\{+,\cdot,\vee\}\subseteq\mathfrak{S}$, and $g(x_1,x_2,x_3,x_4)=\Big(\big((2+x_1)\vee 3\big)\cdot x_2\Big)+\big((3\cdot x_3)+x_4\big)$, then $g\in\mathfrak{P}_4$. We will then prove the following theorem, which is the fundamental result that allows us to prove quite intricate results about extended polynomials based on the relatively simple Theorem 2.6. One of these results is Corollary 3.5 and as we have already mentioned, several results presented in this introduction are consequences of Corollary 3.5.

Theorem 3.2. *Let S be a nonempty set, let $\mathfrak{S}$ be a nonempty set of associative operations on S, let $g\in\mathfrak{P}_m$ and let $p_1,p_2,\ldots,p_m\in\beta S$. Let $\widetilde{g}:\beta(S^m)\to\beta S$ be the continuous extension of g. Then $\widetilde{g}(\bigotimes_{j=1}^m p_j)=g(p_1,p_2,\ldots,p_m)$.*

2. Tensor products of idempotents

In this section we will primarily be concerned with the proof of Theorem 2.6 as stated in the introduction. We begin with the following two lemmas, whose routine proofs will be presented in the appendix.

Lemma 2.1. *Let $k,l\in\mathbb{N}$. For $i\in\{1,2,\ldots,k+l\}$, let S_i be a semigroup and let $p_i\in\beta S_i$. Let $A\subseteq\times_{i=1}^{k+l}S_i$. Then $A\in\bigotimes_{i=1}^{k+l}p_i$ if and only if*

$$\{(x_1,x_2,\ldots,x_k)\in\times_{i=1}^{k}S_i:\{(x_{k+1},x_{k+2},\ldots,x_{k+l})\in\times_{i=k+1}^{k+l}S_i:$$
$$(x_1,x_2,\ldots,x_{k+l})\in A\}\in\bigotimes_{i=k+1}^{k+l}p_i\}\in\bigotimes_{i=1}^{k}p_i\,.$$

Lemma 2.2. *Let $m\in\mathbb{N}$ and for $j\in\{1,2,\ldots,m\}$, let $p_j\in\delta\mathcal{F}$. Then $\{(F_1,F_2,\ldots,F_m)\in\times_{j=1}^{m}\mathcal{F}:F_1<F_2<\ldots<F_m\}\in\bigotimes_{j=1}^m p_j$.*

Lemma 2.3. *Let S be a semigroup, let $\langle x_n\rangle_{n=1}^\infty$ be a sequence in S, and let p be an idempotent in $\bigcap_{m=1}^\infty\overline{FP(\langle x_n\rangle_{n=m}^\infty)}$. Define $\psi:\mathcal{F}\to S$ by $\psi(F)=\prod_{n\in F}x_n$ and let $\widetilde{\psi}:\beta\mathcal{F}\to\beta S$ be its continuous extension. There exists an idempotent $q\in\delta\mathcal{F}$ such that $\widetilde{\psi}(q)=p$.*

Proof. It suffices to show that $\delta\mathcal{F}\cap\widetilde{\psi}^{-1}[\{p\}]$ is a subsemigroup of $\delta\mathcal{F}$, which therefore contains an idempotent. For this it in turn suffices to show that $\delta\mathcal{F}\cap\widetilde{\psi}^{-1}[\{p\}]\neq\emptyset$ and that the restriction of $\widetilde{\psi}$ to $\delta\mathcal{F}$ is a homomorphism.

We show first that $\delta\mathcal{F}\cap\widetilde{\psi}^{-1}[\{p\}]\neq\emptyset$. Given $A\in p$ and $m\in\mathbb{N}$, let $B(A,m)=\{F\in\mathcal{F}:\psi(F)\in A\text{ and }\min F\geq m\}$. Given $C_1,C_2\in p$ and $m_1,m_2\in\mathbb{N}$, we have that $B(C_1\cap C_2,\max\{m_1,m_2\})\subseteq B(C_1,m_1)\cap B(C_2,m_2)$ so to see that $\{B(A,m):A\in p\text{ and }m\in\mathbb{N}\}$ has the finite intersection property, it suffices to show that each $B(A,m)\neq\emptyset$, so let $A\in p$ and $m\in\mathbb{N}$. Then $A\cap FP(\langle x_n\rangle_{n=m}^\infty)\in p$ so pick $F\in\mathcal{F}$ such that $\min F\geq m$ and $\prod_{n\in F}x_n\in A$. Then $F\in B(A,m)$.

To see that the restriction of $\widetilde{\psi}$ to $\delta\mathcal{F}$ is a homomorphism, it suffices by [20, Theorem 4.21] to observe that if $F,G\in\mathcal{F}$ and $F<G$, then $\psi(F\cup G)=\psi(F)\cdot\psi(G)$. $\square$

Lemma 2.4. *Let $m\in\mathbb{N}$. For $j\in\{1,2,\ldots,m\}$. let T_j and S_j be discrete topological spaces, let $\psi_j:T_j\to S_j$, and let $\widetilde{\psi}_j:\beta T_j\to\beta S_j$ be the continuous extension of ψ_j. Define $\varphi_m:\times_{j=1}^m T_j\to\times_{j=1}^m S_j$ by $\varphi_m(x_1,x_2,\ldots,x_m)=$*

$\big(\psi_1(x_1),\psi_2(x_2),\ldots,\psi_m(x_m)\big)$ and let $\widetilde{\varphi}_m:\beta(\times_{j=1}^m T_j)\to\beta(\times_{j=1}^m S_j)$ be its continuous extension. For each $j\in\{1,2,\ldots,m\}$ let $q_j\in\beta T_j$ and let $p_j=\widetilde{\psi}_j(q_j)$. Then $\widetilde{\varphi}_m(\bigotimes_{j=1}^m q_j)=\bigotimes_{j=1}^m p_j$.

Proof. We proceed by induction on m. Since $\varphi_1=\psi_1$, the case $m=1$ is trivial. So let $m\in\mathbb{N}$ and assume the conclusion holds for m. It suffices to let $A\in\bigotimes_{j=1}^{m+1} q_j$ and show that $\varphi_{m+1}[A]\in\bigotimes_{j=1}^{m+1} p_j$. Let

$$\begin{aligned}B=&\{(x_1,x_2,\ldots,x_m)\in\times_{j=1}^m T_j:\\&\{x_{m+1}\in T_{m+1}:(x_1,x_2,\ldots,x_{m+1})\in A\}\in q_{m+1}\}\,.\end{aligned}$$

Then $B\in\bigotimes_{j=1}^m q_j$ so by the induction hypothesis, $\varphi_m[B]\in\bigotimes_{j=1}^m p_j$.

We claim that

$$\begin{aligned}\varphi_m[B]\subseteq&\{(y_1,y_2,\ldots,y_m)\in\times_{j=1}^m S_j:\\&\{y_{m+1}\in S_{m+1}:(y_1,y_2,\ldots,y_{m+1})\in\varphi_{m+1}[A]\}\in p_{m+1}\}\end{aligned}$$

so that $\varphi_{m+1}[A]\in\bigotimes_{t=1}^{m+1} p_t$ as required. So let $(x_1,x_2,\ldots,x_m)\in B$. We need to show that

$$\{y_{m+1}\in S_{m+1}:\big(\psi_1(x_1),\psi_2(x_2),\ldots,\psi_m(x_m),y_{m+1}\big)\in\varphi_{m+1}[A]\}\in p_{m+1}\,.$$

Now $(x_1,x_2,\ldots,x_m)\in B$ so $C=\{x_{m+1}\in T_{m+1}:(x_1,x_2,\ldots,x_{m+1})\in A\}\in q_{m+1}$ and thus $\psi_{m+1}[C]\in p_{m+1}$. Further, if $x_{m+1}\in C$, then

$$\big(\psi_1(x_1),\psi_2(x_2),\ldots,\psi_{m+1}(x_{m+1})\big)\in\varphi_{m+1}[A]$$

so $\varphi_{m+1}[C]\subseteq\{y_{m+1}\in S_{m+1}:\big(\psi_1(x_1),\psi_2(x_2),\ldots,\psi_m(x_m),y_{m+1}\big)\in\varphi_{m+1}[A]\}$ as required. □

In the following lemma, we take $\times_{l=1}^0 S_{f(l)}=\{\emptyset\}$. And similarly, we take $D_1(w_1,w_2,\ldots,w_0)=D_1(\emptyset)$.

Lemma 2.5. *Let $m,k\in\mathbb{N}$, let $f:\{1,2,\ldots,m\}\to\{1,2,\ldots,k\}$, and for $i\in\{1,2,\ldots,k\}$ let S_i be a semigroup and let $p_i\in\beta S_i$. Let $A\in\bigotimes_{j=1}^m p_{f(j)}$. Then for $j\in\{1,2,\ldots,m\}$ there exists $D_j:\times_{l=1}^{j-1}S_{f(l)}\to\mathcal{P}(S_{f(j)})$ such that*

1. *for $j\in\{1,2,\ldots,m\}$, if for each $t\in\{1,2,\ldots,j-1\}$, $w_t\in D_t(w_1,w_2,\ldots,w_{t-1})$, then $D_j(w_1,w_2,\ldots,w_{j-1})\in p_{f(j)}$; and*
2. *if for each $t\in\{1,2,\ldots,m\}$, $w_t\in D_t(w_1,w_2,\ldots,w_{t-1})$, then $(w_1,w_2,\ldots,w_m)\in A$.*

Proof. We proceed by induction on m. If $m=1$. we let $D_1(\emptyset)=A$. Both conclusions hold. Now assume that $m>1$ and the lemma holds for $m-1$. Let

$$\begin{aligned}B=&\{(x_1,x_2,\ldots,x_{m-1})\in\times_{j=1}^{m-1}S_{f(j)}:\\&\{x_m\in S_{f(m)}:(x_1,x_2,\ldots,x_m)\in A\}\in p_{f(m)}\}\,.\end{aligned}$$

Then $B\in\bigotimes_{j=1}^{m-1} p_{f(j)}$ so by the induction hypothesis, pick for each $j\in\{1,2,\ldots,m-1\}$ some $D_j:\times_{l=1}^{j-1}S_{f(l)}\to\mathcal{P}(S_{f(j)})$ such that

1. for $j\in\{1,2,\ldots,m-1\}$, if for each $t\in\{1,2,\ldots,j-1\}$, $w_t\in D_t(w_1,w_2,\ldots,w_{t-1})$, then $D_j(w_1,w_2,\ldots,w_{j-1})\in p_{f(j)}$; and
2. if for each $t\in\{1,2,\ldots,m-1\}$, $w_t\in D_t(w_1,w_2,\ldots,w_{t-1})$, then $(w_1,w_2,\ldots,w_{m-1})\in B$.

To define $D_m : \times_{l=1}^{m-1} S_{f(l)} \to \mathcal{P}(S_{f(m)})$, let $(w_1, w_2, \ldots, w_{m-1}) \in \times_{l=1}^{m-1} S_{f(l)}$. If there is some $t \in \{1, 2, \ldots, m-1\}$ such that $w_t \notin D_t(w_1, w_2, \ldots, w_{t-1})$, let $D_m(w_1, w_2, \ldots, w_{m-1}) = S_{f(m)}$ (or any other subset of $S_{f(m)}$). If for each $t \in \{1, 2, \ldots, m-1\}$ we have $w_t \in D_t(w_1, w_2, \ldots, w_{t-1})$, then we have $(w_1, w_2, \ldots, w_{m-1}) \in B$ so let $D_m(w_1, w_2, \ldots, w_{m-1}) = \{w_m \in S_{f(m)} : (w_1, w_2, \ldots, w_m) \in A\}$ and note that $D_m(w_1, w_2, \ldots, w_{m-1}) \in p_{f(m)}$.

To verify conclusion (1), we have by assumption that it holds for $j \in \{1, 2, \ldots, m-1\}$. Assume that for each $t \in \{1, 2, \ldots, m-1\}$, $w_t \in D_t(w_1, w_2, \ldots, w_{t-1})$. We have just seen that then $D_m(w_1, w_2, \ldots, w_{m-1}) \in p_{f(m)}$. To verify conclusion (2), assume that for each $t \in \{1, 2, \ldots, m\}$, $w_t \in D_t(w_1, w_2, \ldots, w_{t-1})$. Then by the definition of $D_m(w_1, w_2, \ldots, w_{m-1})$, we have that $(w_1, w_2, \ldots, w_m) \in A$. □

If p is an idempotent in a semigroup S and $A \in p$, then the set $A^\star(p) = \{x \in A : x^{-1}A \in p\}$. By [20, Lemma 4.14], if $x \in A^\star(p)$, then $x^{-1}A^\star(p) \in p$. In case the semigroup is $(\mathcal{F}, \cup)$ and $p \in \delta\mathcal{F}$, if $D \in p$, then $D^\star(p) = \{H \in D : H^{-1}D \in p\}$ where $H^{-1}D = \{F \in \mathcal{F} : H \cup F \in D\}$.

The reader who wishes to follow the proof of the following theorem may wish to read first the proof in the appendix that (b) implies (a) in Theorem 1.17.

Theorem 2.6. *Let $m, k \in \mathbb{N}$ and let $f : \{1, 2, \ldots, m\} \to \{1, 2, \ldots, k\}$. For each $i \in \{1, 2, \ldots, k\}$, let p_i be an idempotent in $\delta\mathcal{F}$, and let $A \in \bigotimes_{i=1}^m p_{f(i)}$. Then for each $i \in \{1, 2, \ldots, k\}$ there exists an increasing sequence $\langle H_{i,n}\rangle_{n=1}^\infty$ in $\mathcal{F}$ such that*

$$\{(\textstyle\bigcup_{t\in F_1} H_{f(1),t}, \bigcup_{t\in F_2} H_{f(2),t}, \ldots, \bigcup_{t\in F_m} H_{f(m),t}) : F_1 < F_2 < \ldots < F_m\} \subseteq A\,.$$

Proof. By Lemma 2.5 for each $j \in \{1, 2, \ldots, m\}$ pick $D_j : \mathcal{F}^{j-1} \to \mathcal{P}(\mathcal{F})$ such that

(1) for $j \in \{1, 2, \ldots, m\}$, if for each $s \in \{1, 2, \ldots, j-1\}$, $F_s \in D_t(F_1, F_2, \ldots, F_{s-1})$, then $D_j(F_1, F_2, \ldots, F_{j-1}) \in p_{f(j)}$; and
(2) if for each $j \in \{1, 2, \ldots, m\}$, $F_j \in D_j(F_1, F_2, \ldots, F_{j-1})$, then $(F_1, F_2, \ldots, F_m) \in A$.

For $n \in \{1, 2, \ldots, m\}$ and $i \in \{1, 2, \ldots, k\}$, let $B_{i,n} = \{j \in \{1, 2, \ldots, n\} : f(j) = i\}$. For $n > m$, let $B_{i,n} = B_{i,m} = \{j \in \{1, 2, \ldots, m\} : f(j) = i\}$. Now $D_1(\emptyset) \in p_{f(1)}$. Pick $H_{f(1),1} \in D_1(\emptyset)^\star(p_{f(1)})$. Then $D_2(H_{f(1),1}) \in p_{f(2)}$ and $H_{f(1),1}^{-1} D_1(\emptyset)^\star(p_{f(1)}) \in p_{f(1)}$. For $i \in \{1, 2, \ldots, k\} \setminus \{f(1)\}$, let $H_{i,1} = \{1\}$.

Inductively, let $n \in \mathbb{N}$ and assume that for $i \in \{1, 2, \ldots, k\}$, we have chosen $\langle H_{i,t}\rangle_{t=1}^n$ in $\mathcal{F}$ such that

(i) for $i \in \{1, 2, \ldots, k\}$ and $t \in \{1, 2, \ldots, n-1\}$, $H_{i,t} < H_{i,t+1}$ and
(ii) for $j \in \{1, 2, \ldots, m\}$, if $F_1, F_2, \ldots, F_j \in \mathcal{P}_f(\{1, 2, \ldots, n\})$ and $F_1 < F_2 < \ldots < F_j$, then

$$\textstyle\bigcup_{t\in F_j} H_{f(j),t} \in D_j(\bigcup_{t\in F_1} H_{f(1),t}, \bigcup_{t\in F_2} H_{f(2),t}, \ldots, \bigcup_{t\in F_{j-1}} H_{f(j-1),t})^\star(p_{f(j)})\,.$$

At $n = 1$, hypothesis (i) is vacuous. To verify (ii), let $j \in \{1, 2, \ldots, m\}$ and assume that $F_1, F_2, \ldots, F_j \in \mathcal{P}_f(\{1\})$ and $F_1 < F_2 < \ldots < F_j$. Then $j = 1$ and $F_1 = \{1\}$ so the conclusion says that $H_{f(1),1} \in D_1(\emptyset)^\star(p_{f(1)})$, which is true.

For $i \in \{1, 2, \ldots, k\}$, let $r_{i,n} = \max H_{i,n} + 1$. If $B_{i,n+1} = \emptyset$, let $H_{i,n+1} = \{r_{i,n}\}$. Now assume that $B_{i,n+1} \neq \emptyset$. For $j \in B_{i,n+1}$, let

$$\begin{aligned} G_j = \textstyle\bigcap\big\{ &(\textstyle\bigcup_{t\in F_j} H_{f(j),t})^{-1} D_j(\bigcup_{t\in F_1} H_{f(1),t}, \ldots, \bigcup_{t\in F_{j-1}} H_{f(j-1),t})^\star(p_{f(j)}) : \\ &F_1 < F_2 < \ldots < F_j < \{n+1\}\big\}\,. \end{aligned}$$

and let

$$C_j = \bigcap\big\{D_j(\bigcup_{t\in F_1} H_{f(1),t},\ldots,\bigcup_{t\in F_{j-1}} H_{f(j-1),t})^\star(p_{f(j)}) :\\ F_1 < F_2 < \ldots < F_{j-1} < \{n+1\}\big\}\,.$$

Let $E = \{H \in \mathcal{F} : \min H \geq r_{i,n}\} \cap \bigcap_{j\in B_{i,n+1}} (G_j \cap C_j)$. Note that for all $j \in B_{i,n+1}$, $G_j \in p_{f(j)} = p_i$ by induction hypothesis (ii). Next we claim that for all $j \in B_{i,n+1}$, $C_j \in p_i$, so let $j \in B_{i,n+1}$ and let

$$F_1 < F_2 < \ldots < F_j < \{n+1\}\,.$$

For $s \in \{1,2,\ldots,j-1\}$ we have by hypothesis (ii) that

$$\bigcup_{t\in F_s} H_{f(s),t} \in D_s(\bigcup_{t\in F_1} H_{f(1),t},\ldots,\bigcup_{t\in F_{s-1}} H_{f(s-1),t})$$

so the hypothesis of (1) holds so $D_j(\bigcup_{t\in F_1} H_{f(1),t},\ldots,\bigcup_{t\in F_{j-1}} H_{f(j-1),t}) \in p_{f(j)} = p_i$. Thus $C_j \in p_i$ as claimed. Since $p_i \in \delta\mathcal{F}$, $\{H \in \mathcal{F} : \min H \geq r_{i,n}\} \in p_i$ and thus $E \in p_i$. Pick $H_{i,n+1} \in E$.

Hypothesis (i) holds by construction. To verify hypothesis (ii), let $j \in \{1,2,\ldots, m\}$ and let $F_1, F_2, \ldots, F_j \in \mathcal{P}_f(\{1,2,\ldots,n+1\})$ with $F_1 < F_2 < \ldots < F_j$. If $n+1 \notin F_j$ then the conclusion holds by the fact that (ii) holds for n. So assume that $n+1 \in F_j$. Assume first that $F_j = \{n+1\}$. Let $i = f(j)$. Then $j \in B_{i,n+1}$ so $H_{i,n+1} \in C_j \subseteq D_j(\bigcup_{t\in F_1} H_{f(1),t},\ldots,\bigcup_{t\in F_{j-1}} H_{f(j-1),t})^\star(p_i)$.

Now assume that $F_j \neq \{n+1\}$ and let $F'_j = F_j \setminus \{n+1\}$. Let $i = f(j)$. Then $j \in B_{i,n+1}$ and $F_1 < F_2 < \ldots < F_{j-1} < F'_j$ so

$$H_{i,n+1} \in G_j \subseteq (\bigcup_{t\in F'_j} H_{f(j),t})^{-1} D_j(\bigcup_{t\in F_1} H_{f(1),t},\ldots,\bigcup_{t\in F_{j-1}} H_{f(j-1),t})^\star(p_{f(j)})$$

so $\bigcup_{t\in F_j} H_{f(j),t} \in D_j(\bigcup_{t\in F_1} H_{f(1),t},\ldots,\bigcup_{t\in F_{j-1}} H_{f(j-1),t})^\star(p_{f(j)})$.

The inductive construction being complete, let $F_1 < F_2 < \ldots < F_m$ be given. Then for each $j \in \{1,2,\ldots,m\}$ we have that

$$\bigcup_{t\in F_j} H_{f(j),t} \in D_j(\bigcup_{t\in F_1} H_{f(1),t},\ldots,\bigcup_{t\in F_{j-1}} H_{f(j-1),t})^\star(p_{f(j)})$$

so by (2) we have that $(\bigcup_{t\in F_1} H_{f(1),t},\ldots,\bigcup_{t\in F_m} H_{f(m),t}) \in A$. □

We see easily that Theorem 2.6 generalizes the Milliken-Taylor Theorem.

Corollary 2.7 (Milliken-Taylor Theorem)**.** *Let $m, r \in \mathbb{N}$. Let $[\mathcal{F}]^m = \bigcup_{i=1}^r C_i$. There exist $i \in \{1,2,\ldots,r\}$ and an increasing sequence $\langle H_n\rangle_{n=1}^\infty$ in $\mathcal{F}$ such that $[FU(\langle H_n\rangle_{n=1}^\infty)]^m_< \subseteq C_i$.*

Proof. For $i \in \{1,2,\ldots,r\}$, let

$$B_i = \{(F_1,F_2,\ldots,F_m) \in \mathcal{F}^m : F_1 < F_2 < \ldots < F_m \text{ and } \{F_1,F_2,\ldots,F_m\} \in C_i\}\,.$$

Let $B_{r+1} = \mathcal{F}^m \setminus \bigcup_{i=1}^r B_i$. Pick an idempotent $p \in \delta\mathcal{F}$ and pick $i \in \{1,2,\ldots,r+1\}$ such that $B_i \in \bigotimes_{j=1}^m p$. By Lemma 2.2, $i \neq r+1$. By Theorem 2.6 pick an increasing sequence $\langle H_n\rangle_{n=1}^\infty$ in $\mathcal{F}$ such that

$$\{(\bigcup_{t\in F_1} H_t, \bigcup_{t\in F_2} H_t,\ldots,\bigcup_{t\in F_m} H_t) : F_1 < F_2 < \ldots < F_m\} \subseteq B_i\,.$$

Then $[FU(\langle H_n\rangle_{n=1}^\infty)]^m_< \subseteq C_i$. □

Corollary 2.8. *Let $m,k\in\mathbb{N}$. For each $i\in\{1,2,\ldots,k\}$, let S_i be a semigroup, let $\langle x_{i,n}\rangle_{n=1}^\infty$ be a sequence in S_i, and let p_i be an idempotent in $\bigcap_{r=1}^\infty \overline{FP(\langle x_{i,n}\rangle_{n=r}^\infty)}$. Let $f:\{1,2,\ldots,m\}\to\{1,2,\ldots,k\}$ and let $A\in\bigotimes_{j=1}^m p_{f(j)}$. Then for each $i\in\{1,2,\ldots,k\}$ there is a product subsystem $FP(\langle y_{i,n}\rangle_{n=1}^\infty)$ of $FP(\langle x_{i,n}\rangle_{n=1}^\infty)$ such that*

$$\{\textstyle\prod_{t\in F_1} y_{f(1),t},\prod_{t\in F_2} y_{f(2),t},\ldots,\prod_{t\in F_m} y_{f(m),t}) : F_1<F_2<\ldots<F_m\}\subseteq A\,.$$

Proof. For each $i\in\{1,2,\ldots,k\}$, define $\psi_i:\mathcal{F}\to FP(\langle x_{i,n}\rangle_{n=1}^\infty)\subseteq S_i$ by $\psi_i(F)=\prod_{t\in F} x_{i,t}$ and let $\widetilde{\psi}_i:\beta\mathcal{F}\to\beta S_i$ be the continuous extension of ψ_i. By Lemma 2.3 pick an idempotent $q_i\in\delta\mathcal{F}$ such that $\widetilde{\psi}(q_i)=p_i$.

Define $\varphi:\mathcal{F}^m\to \times_{j=1}^m S_{f(j)}$ by

$$\varphi(F_1,F_2,\ldots,F_m)=\big(\psi_{f(1)}(F_1),\psi_{f(2)}(F_2),\ldots,\psi_{f(m)}(F_m)\big)$$

and let $\widetilde{\varphi}:\beta(\mathcal{F}^m)\to\beta(\times_{j=1}^m S_{f(j)})$ be its continuous extension. By Lemma 2.4, we have that $\widetilde{\varphi}(\bigotimes_{j=1}^m q_{f(j)})=\bigotimes_{j=1}^m p_{f(j)}$. Pick $B\in\bigotimes_{j=1}^m q_{f(j)}$ such that $\widetilde{\varphi}[\,\overline{B}\,]\subseteq\overline{A}$.

By Theorem 2.6 pick for each $i\in\{1,2,\ldots,k\}$, an increasing sequence $\langle H_{i,n}\rangle_{n=1}^\infty$ in $\mathcal{F}$ such that

$$\{(\textstyle\bigcup_{t\in F_1} H_{f(1),t},\bigcup_{t\in F_2} H_{f(2),t},\ldots,\bigcup_{t\in F_m} H_{f(m),t}) : F_1<F_2<\ldots<F_m\}\subseteq B\,.$$

For $i\in\{1,2,\ldots,k\}$ and $n\in\mathbb{N}$, let $y_{i,n}=\prod_{t\in H_{i,n}} x_{i,t}$. Then

$$\begin{aligned}&(\textstyle\prod_{t\in F_1} y_{f(1),t},\prod_{t\in F_2} y_{f(2),t},\ldots,\prod_{t\in F_m} y_{f(m),t})\\ &=\big(\psi_{f(1)}(\textstyle\bigcup_{t\in F_1} H_{f(1),t}),\psi_{f(2)}(\bigcup_{t\in F_2} H_{f(2),t}),\ldots,\psi_{f(m)}(\bigcup_{t\in F_m} H_{f(m),t})\big)\\ &=\varphi(\textstyle\bigcup_{t\in F_1} H_{f(1),t},\bigcup_{t\in F_2} H_{f(2),t},\ldots,\bigcup_{t\in F_m} H_{f(m),t})\\ &\in\varphi[B]\subseteq A\,.\end{aligned}$$

□

We shall conclude this section by showing that we have a characterization of members of tensor products of idempotents. For this we shall need the following preliminary result. We shall need this result again in the next section.

Lemma 2.9. *Let $m,k\in\mathbb{N}$. For each $i\in\{1,2,\ldots,k\}$, let S_i be a semigroup, let $\langle x_{i,n}\rangle_{n=1}^\infty$ be a sequence in S_i, and let $p_i\in\bigcap_{r=1}^\infty\overline{FP(\langle x_{i,n}\rangle_{n=r}^\infty)}$. Let $f:\{1,2,\ldots,m\}\to\{1,2,\ldots,k\}$. Then*

$$\big\{\big(\textstyle\prod_{t\in F_1} x_{f(1),t},\ldots,\prod_{t\in F_m} x_{f(m),t}\big) : F_1<\ldots<F_m\big\}\in\bigotimes_{j=1}^m p_{f(j)}\,.$$

Proof. We proceed by induction on m. If $m=1$, we have that

$$\big\{\textstyle\prod_{t\in F_1} x_{f(1),t} : F_1\in\mathcal{P}_f(\mathbb{N})\big\}=FP(\langle x_{f(1),n}\rangle_{n=1}^\infty)\in p_{f(1)}\,.$$

Now let $m\in\mathbb{N}$ and assume the lemma is valid for m. Let

$$A=\big\{\big(\textstyle\prod_{t\in F_1} x_{f(1),t},\ldots,\prod_{t\in F_{m+1}} x_{f(m+1),t}\big) : F_1<\ldots<F_{m+1}\big\}$$

and let $B=\big\{\big(\prod_{t\in F_1} x_{f(1),t},\ldots,\prod_{t\in F_m} x_{f(m),t}\big) : F_1<\ldots<F_m\big\}$. By assumption, $B\in\bigotimes_{j=1}^m p_{f(j)}$. We show that $A\in\bigotimes_{j=1}^{m+1} p_{f(j)}$, for which it suffices that $B\subseteq\{(z_1,z_2,\ldots,z_m)\in\times_{i=1}^m S_{f(i)} : \{z_{m+1}\in S_{f(m+1)} : (z_1,z_2,\ldots,z_{m+1})\in A\}\in p_{f(m+1)}\}$. So let $F_1<F_2<\ldots<F_m$ and let

$$(z_1,z_2,\ldots,z_m)=\big(\textstyle\prod_{t\in F_1} x_{f(1),t},\prod_{t\in F_2} x_{f(2),t},\ldots,\prod_{t\in F_m} x_{f(m),t}\big)\,.$$

Let $r = \max F_m + 1$ and let $C = \{\prod_{t \in F_{m+1}} x_{f(m+1),t} : F_m < F_{m+1}\}$. Then $C = FP(\langle x_{f(m+1),n}\rangle_{n=r}^{\infty}) \in p_{f(m+1)}$ and $C \subseteq \{z_{m+1} \in S_{f(m+1)} : (z_1, z_2, \ldots, z_{m+1}) \in A\}$ so $\{z_{m+1} \in S : (z_1, z_2, \ldots, z_{m+1}) \in A\} \in p_{f(m+1)}$ as required. $\square$

Theorem 2.10. *Let $m, k \in \mathbb{N}$, let $f : \{1, 2, \ldots, m\} \to \{1, 2, \ldots, k\}$, and let $A \subseteq \mathcal{F}^m$. The following statements are equivalent.*

(a) *For each $i \in \{1, 2, \ldots, k\}$ there exists an idempotent p_i in $\delta\mathcal{F}$ such that $A \in \bigotimes_{i=1}^{m} p_{f(i)}$.*

(b) *For each $i \in \{1, 2, \ldots, k\}$ there exists an increasing sequence $\langle H_{i,n}\rangle_{n=1}^{\infty}$ in $\mathcal{F}$ such that*

$$\{(\textstyle\bigcup_{t\in F_1} H_{f(1),t}, \bigcup_{t\in F_2} H_{f(2),t}, \ldots, \bigcup_{t\in F_m} H_{f(m),t}) : F_1 < F_2 < \ldots < F_m\} \subseteq A\,.$$

Proof. That (a) implies (b) is Theorem 2.6.

To see that (b) implies (a), assume that (b) holds. For each $i \in \{1, 2, \ldots, k\}$, we have by [20, Theorem 4.20] that $T_i = \bigcap_{m=1}^{\infty} \overline{FU(\langle H_{i,n}\rangle_{n=m}^{\infty})}$ is a subsemigroup of $\delta\mathcal{F}$, so pick an idempotent $p_i \in T_i$. Now we apply Lemma 2.9 with $S_i = \mathcal{F}$, for each $i \in \{1, 2, \ldots, k\}$ and for $n \in \mathbb{N}$, $x_{i,n} = H_{i,n}$. Then

$$\{(\textstyle\bigcup_{t\in F_1} H_{f(1),t}, \ldots, \bigcup_{t\in F_m} H_{f(m),t}) : F_1 < F_2 < \ldots < F_m\} \in \bigotimes_{j=1}^{m} p_{f(j)}$$

and so $A \in \bigotimes_{j=1}^{m} p_{f(j)}$. $\square$

3. Extended polynomials

We introduce in this section a very general variety of polynomials, extending the notion of ordinary polynomials. We then characterize the members of these polynomials when they are evaluated at certain idempotents in βS.

Definition 3.1. Let S be a nonempty set and let $\mathfrak{S}$ be a nonempty (finite or infinite) set of associative operations on S. Define a set $\mathfrak{P}$ of "polynomials" on S as follows.

(1) If $g(x_1) = x_1$, then $g \in \mathfrak{P}_1$.
(2) If $a \in S$, $* \in \mathfrak{S}$, and $g(x_1) = a * x_1$ or $g(x_1) = x_1 * a$, then $g \in \mathfrak{P}_1$.
(3) If $a, b \in S$, $*, \diamond \in \mathfrak{S}$, and $g(x_1) = (a * x_1) \diamond b$ or $g(x_1) = a * (x_1 \diamond b)$, then $g \in \mathfrak{P}_1$.
(4) If $k, l \in \mathbb{N}$, $g \in \mathfrak{P}_k$, $h \in \mathfrak{P}_l$, $* \in \mathfrak{S}$, and $r(x_1, x_2, \ldots, x_{k+l}) = \big(g(x_1, x_2, \ldots, x_k)\big) * \big(h(x_{k+1}, x_{k+2}, \ldots, x_{k+l})\big)$, then $r \in \mathfrak{P}_{k+l}$.
(5) $\mathfrak{P} = \bigcup_{k=1}^{\infty} \mathfrak{P}_k$.

As we mentioned in the introduction, if $S = \mathbb{N}$, $\{+, \cdot, \vee\} \subseteq \mathfrak{S}$, and

$$g(x_1, x_2, x_3, x_4) = \Big(\big((2 + x_1) \vee 3\big) \cdot x_2\Big) + \big((3 \cdot x_3) + x_4\big)\,,$$

then $g \in \mathfrak{P}_4$.

Notice that the variables in members of $\mathfrak{P}_k$ occur in increasing order from left to right. This simplifies the proofs immensely, but places no real restriction on the kind of polynomials we deal with. For example, suppose that g is as in the paragraph above and

$$h(x_1, x_2) = \Big(\big((2 + x_2) \vee 3\big) \cdot x_2\Big) + \big((3 \cdot x_1) + x_2\big)\,.$$

Then given any p and q in $\beta\mathbb{N}$, $h(p, q) = g(q, q, p, q)$.

Theorem 3.2. *Let S be a nonempty set, let $\mathfrak{S}$ be a nonempty set of associative operations on S, let $g \in \mathfrak{P}_m$ and let $p_1, p_2, \ldots, p_m \in \beta S$. Let $\widetilde{g} : \beta(S^m) \to \beta S$ be the continuous extension of g. Then $\widetilde{g}(\bigotimes_{j=1}^m p_j) = g(p_1, p_2, \ldots, p_m)$.*

Proof. Since both $\widetilde{g}(\bigotimes_{j=1}^m p_j)$ and $g(p_1, p_2, \ldots, p_m)$ are ultrafilters, it suffices in each case to show that $\widetilde{g}(\bigotimes_{j=1}^m p_j) \subseteq g(p_1, p_2, \ldots, p_m)$.

We proceed by induction on m, so assume that $g \in \mathfrak{P}_1$ and assume first that $g(x_1) = x_1$. Let $A \in \widetilde{g}(p_1)$ and pick $B \in p_1$ such that $\widetilde{g}\,[\,\overline{B}\,] \subseteq \overline{A}$. Then $B = g[B] \subseteq A$ so $A \in p_1 = g(p_1)$.

Now, if $a, b \in S$, $B \in p_1$, and $*, \diamond \in \mathfrak{S}$, then $a * B \in a * p_1$, $B * a \in p_1 * a$, $(a * B) \diamond b \in (a * p_1) \diamond b$, and $a * (B \diamond b) \in a * (p_1 \diamond b)$ so in any event we see as above that for any $g \in \mathfrak{P}_1$, $\widetilde{g}(p_1) = g(p_1)$.

Now let $k, l \in \mathbb{N}$, let $g \in \mathfrak{P}_k$, let $h \in \mathfrak{P}_l$, let $* \in \mathfrak{S}$, and define

$$r(x_1, x_2, \ldots, x_{k+l}) = \big(g(x_1, x_2, \ldots, x_k)\big) * \big(h(x_{k+1}, x_{k+2}, \ldots, x_{k+l})\big)\,.$$

Let $p_1, p_2, \ldots, p_{k+l} \in \beta S$, and assume that $\widetilde{g}(\bigotimes_{j=1}^k p_j) = g(p_1, p_2, \ldots, p_k)$ and $\widetilde{h}(\bigotimes_{j=k+1}^{k+l} p_j) = h(p_{k+1}, p_{k+2}, \ldots, p_{k+l})$. Let $A \in \widetilde{r}(\bigotimes_{j=1}^{k+l} p_j)$ and pick $B \in \bigotimes_{j=1}^{k+l} p_j$ such that $\widetilde{r}\,[\,\overline{B}\,] \subseteq \overline{A}$. Let

$$\begin{aligned} C = &\{(x_1, x_2, \ldots, x_k) \in S^k : \\ &\quad \{(x_{k+1}, x_{k+2}, \ldots, x_{k+l}) \in S^l : (x_1, x_2, \ldots, x_{k+l}) \in B\} \in \textstyle\bigotimes_{j=k+1}^{k+l} p_j\}\,. \end{aligned}$$

Then by Lemma 2.1, $C \in \bigotimes_{j=1}^k p_j$. Thus by [20, Lemma 3.30] $g[C] \in \widetilde{g}\,[\bigotimes_{j=1}^k p_j] = g(p_1, p_2, \ldots, p_k)$.

To see that $A \in r(p_1, p_2, \ldots, p_{k+l})$ it suffices to show that

$$g[C] \subseteq \{y \in S : y^{-1}A \in h(p_{k+1}, p_{k+2}, \ldots, p_{k+l})\}$$

where $y^{-1}A = \{z \in S : y * z \in A\}$. So let $(x_1, x_2, \ldots, x_k) \in C$ and let $y = g(x_1, x_2, \ldots, x_k)$. Let $D = \{(x_{k+1}, x_{k+2}, \ldots, x_{k+l}) \in S^l : (x_1, x_2, \ldots, x_{k+l}) \in B\}$. Then $D \in \bigotimes_{j=k+1}^{k+l} p_j$ and so $h[D] \in \widetilde{h}(\bigotimes_{j=k+1}^{k+l} p_j) = h(p_{k+1}, p_{k+2}, \ldots, p_{k+l})$. We claim that $h[D] \subseteq y^{-1}A$ so that $y^{-1}A \in h(p_{k+1}, p_{k+2}, \ldots, p_{k+l})$ as required. So let $(x_{k+1}, x_{k+2}, \ldots, x_{k+l}) \in D$ and let $z = h(x_{k+1}, x_{k+2}, \ldots, x_{k+l})$. Now

$$(x_1, x_2, \ldots, x_{k+l}) \in B \text{ so } r(x_1, x_2, \ldots, x_{k+l}) \in A\,.$$

Since $r(x_1, x_2, \ldots, x_{k+l}) = y * z$, we have that $z \in y^{-1}A$. □

If we have several, possibly different, operations on S denoted by $*_i$, we write $\overset{i}{\prod}_{n \in F} x_n$ for the product (in increasing order of indices) with respect to the operation $*_i$ and let $FP_i(\langle x_n \rangle_{n=1}^\infty) = \{\overset{i}{\prod}_{n \in F} x_n : F \in \mathcal{P}_f(\mathbb{N})\}$.

Notice that in the next result we do not demand that each p_i be an idempotent.

Theorem 3.3. *Let S be a nonempty set, let $\mathfrak{S}$ be a nonempty set of associative operations on S, let $\mathfrak{P}$ be as in Definition 3.1, and let $m, k \in \mathbb{N}$. For each $i \in \{1, 2, \ldots, k\}$, let $\langle x_{i,n} \rangle_{n=1}^\infty$ be a sequence in S, let $*_i \in \mathfrak{S}$, and let $p_i \in \bigcap_{r=1}^\infty \overline{FP_i(\langle x_{i,n} \rangle_{n=r}^\infty)}$. Let $g \in \mathfrak{P}_m$ and let $f : \{1, 2, \ldots, m\} \to \{1, 2, \ldots, k\}$. Then*

$$\big\{g\big(\overset{f(1)}{\prod}_{t \in F_1} x_{f(1),t}, \ldots, \overset{f(m)}{\prod}_{t \in F_m} x_{f(m),t}\big) : F_1 < \ldots < F_m\big\} \in g(p_{f(1)}, \ldots, p_{f(m)})\,.$$

Proof. By Theorem 3.2 $\widetilde{g}(\bigotimes_{j=1}^{m} p_{f(j)}) = g(p_{f(1)}, p_{f(2)}, \ldots, p_{f(m)})$. Let

$$A = \big\{ \big(\prod_{t\in F_1}^{f(1)} x_{f(1),t}, \ldots, \prod_{t\in F_m}^{f(m)} x_{f(m),t}\big) : F_1 < \ldots < F_m \big\}\,.$$

Then by Lemma 2.9, $A \in \bigotimes_{j=1}^{m} p_{f(j)}$ so by [20, Lemma 3.30] $g[A] \in \widetilde{g}(\bigotimes_{j=1}^{m} p_{f(j)}) = g(p_{f(1)}, p_{f(2)}, \ldots, p_{f(m)})$. □

Theorem 3.4. *Let S be a nonempty set, let $\mathfrak{S}$ be a nonempty set of associative operations on S, let $\mathfrak{P}$ be as in Definition 3.1, and let $m, k \in \mathbb{N}$. For each $i \in \{1, 2, \ldots, k\}$, let $\langle x_{i,n}\rangle_{n=1}^{\infty}$ be a sequence in S, let $*_i \in \mathfrak{S}$, and let $p_i *_i p_i = p_i \in \bigcap_{r=1}^{\infty} \overline{FP_i(\langle x_{i,n}\rangle_{n=r}^{\infty})}$. Let $f : \{1, 2, \ldots, m\} \to \{1, 2, \ldots, k\}$, let $g \in \mathfrak{P}_m$, and let $A \in g(p_{f(1)}, p_{f(2)}, \ldots, p_{f(m)})$. Then for each $i \in \{1, 2, \ldots, k\}$ there is a $*_i$-product subsystem $FP_i(\langle y_{i,n}\rangle_{n=1}^{\infty})$ of $FP_i(\langle x_{i,n}\rangle_{n=1}^{\infty})$ such that*

$$\big\{g\big(\prod_{t\in F_1}^{f(1)} y_{f(1),t}, \prod_{t\in F_2}^{f(2)} y_{f(2),t}, \ldots, \prod_{t\in F_m}^{f(m)} y_{f(m),t}\big) : F_1 < F_2 < \ldots < F_m\big\} \subseteq A\,.$$

Proof. By Theorem 3.2 $g(p_{f(1)}, p_{f(2)}, \ldots, p_{f(m)}) = \widetilde{g}(\bigotimes_{j=1}^{m} p_{f(j)})$ so pick $B \in \bigotimes_{j=1}^{m} p_{f(j)}$ such that $\widetilde{g}[\,\overline{B}\,] \subseteq \overline{A}$. By Corollary 2.8, for each $i \in \{1, 2, \ldots, k\}$ pick a $*_i$-product subsystem $FP_i(\langle y_{i,n}\rangle_{n=1}^{\infty})$ of $FP_i(\langle x_{i,n}\rangle_{n=1}^{\infty})$ such that

$$\big\{\big(\prod_{t\in F_1}^{f(1)} y_{f(1),t}, \prod_{t\in F_2}^{f(2)} y_{f(2),t}, \ldots, \prod_{t\in F_m}^{f(m)} y_{f(m),t}\big) : F_1 < F_2 < \ldots < F_m\big\} \subseteq B\,.$$

Then $\big\{g\big(\prod_{t\in F_1}^{f(1)} y_{f(1),t}, \prod_{t\in F_2}^{f(2)} y_{f(2),t}, \ldots, \prod_{t\in F_m}^{f(m)} y_{f(m),t}\big) : F_1 < F_2 < \ldots < F_m\big\} \subseteq A$ □

We thus have the following characterization of members of idempotents evaluated at idempotents.

Corollary 3.5. *Let S be a nonempty set, let $\mathfrak{S}$ be a nonempty set of associative operations on S, let $\mathfrak{P}$ be as in Definition 3.1, let $m, k \in \mathbb{N}$, let $f : \{1, 2, \ldots, m\} \to \{1, 2, \ldots, k\}$, let $g \in \mathfrak{P}_m$, and let $A \subseteq S$. For each $i \in \{1, 2, \ldots, k\}$, let $\langle x_{i,n}\rangle_{n=1}^{\infty}$ be a sequence in S and let $*_i \in \mathfrak{S}$. The following statements are equivalent.*

(a) *For each $i \in \{1, 2, \ldots, k\}$, there exists $p_i = p_i *_i p_i \in \bigcap_{r=1}^{\infty} \overline{FP_i(\langle x_{i,n}\rangle_{n=r}^{\infty})}$ such that $A \in g(p_{f(1)}, p_{f(2)}, \ldots, p_{f(m)})$.*

(b) *For each $i \in \{1, 2, \ldots, k\}$, there is a $*_i$-product subsystem $FP_i(\langle y_{i,n}\rangle_{n=1}^{\infty})$ of $FP_i(\langle x_{i,n}\rangle_{n=1}^{\infty})$ such that*

$$\big\{g\big(\prod_{t\in F_1}^{f(1)} y_{f(1),t}, \prod_{t\in F_2}^{f(2)} y_{f(2),t}, \ldots, \prod_{t\in F_m}^{f(m)} y_{f(m),t}\big) : F_1 < F_2 < \ldots < F_m\big\} \subseteq A\,.$$

Proof. $(a) \Rightarrow (b)$. Theorem 3.4.

$(b) \Rightarrow (a)$. For each $i \in \{1, 2, \ldots, k\}$, by [20, Lemma 5.11] pick $p_i = p_i *_i p_i \in \bigcap_{r=1}^{\infty} \overline{FP_i(\langle y_{i,n}\rangle_{n=r}^{\infty})}$. Since for each $i \in \{1, 2, \ldots, k\}$, $\bigcap_{r=1}^{\infty} \overline{FP_i(\langle y_{i,n}\rangle_{n=r}^{\infty})} \subseteq \bigcap_{r=1}^{\infty} \overline{FP_i(\langle x_{i,n}\rangle_{n=r}^{\infty})}$, we have for each $i \in \{1, 2, \ldots, k\}$, $p_i \in \bigcap_{r=1}^{\infty} \overline{FP_i(\langle x_{i,n}\rangle_{n=r}^{\infty})}$. By Theorem 3.3

$$\big\{g\big(\prod_{t\in F_1}^{f(1)} y_{f(1),t}, \prod_{t\in F_2}^{f(2)} y_{f(2),t}, \ldots, \prod_{t\in F_m}^{f(m)} y_{f(m),t}\big) : F_1 < F_2 < \ldots < F_m\big\} \in g(p_{f(1)}, p_{f(2)}, \ldots, p_{f(m)})$$

so $A \in g(p_{f(1)}, p_{f(2)}, \ldots, p_{f(m)})$. □

4. Appendix

In this section we provide for the convenience of the reader elementary proofs of some results that were mentioned in the introduction, as well as some proofs of results that were omitted earlier.

We show first that Theorems 1.6, 1.7, and 1.8 are equivalent to the Milliken-Taylor Theorem (in the informal sense that each is easily derivable from the others).

Theorem 4.1. *The following statements are equivalent.*

(a) *(Theorem 1.4). Let $m, r \in \mathbb{N}$. Let $[\mathcal{P}_f(\mathbb{N})]^m = \bigcup_{i=1}^r C_i$. There exist $i \in \{1, 2, \ldots, r\}$ and an increasing sequence $\langle H_n \rangle_{n=1}^\infty$ in $\mathcal{P}_f(\mathbb{N})$ such that $[FU(\langle H_n \rangle_{n=1}^\infty)]_<^m \subseteq C_i$.*

(b) *(Theorem 1.7). Let $k, m, r \in \mathbb{N}$, let $\langle a_j \rangle_{j=1}^m$ be a sequence in $\mathbb{N}$ and for each $j \in \{1, 2, \ldots, k\}$, let $\langle x_{j,n} \rangle_{n=1}^\infty$ be a sequence in $\mathbb{N}$. Let $f : \{1, 2, \ldots, m\} \to \{1, 2, \ldots, k\}$ and let $\mathbb{N} = \bigcup_{i=1}^r C_i$. There exists $i \in \{1, 2, \ldots, r\}$ and for each $j \in \{1, 2, \ldots, k\}$, there exists a sum subsystem $FS(\langle y_{j,n} \rangle_{n=1}^\infty)$ of $FS(\langle x_{j,n} \rangle_{n=1}^\infty)$ such that $\{\sum_{j=1}^m a_j \sum_{t \in F_j} y_{f(j),t} : F_1 < F_2 < \ldots < F_m\} \subseteq C_i$.*

(c) *(Theorem 1.6). Let $m, r \in \mathbb{N}$, let $\langle a_j \rangle_{j=1}^m$ and $\langle x_n \rangle_{n=1}^\infty$ be sequences in $\mathbb{N}$, and let $\mathbb{N} = \bigcup_{i=1}^r C_i$. There exist $i \in \{1, 2, \ldots, r\}$ and a sum subsystem $FS(\langle y_n \rangle_{n=1}^\infty)$ of $FS(\langle x_n \rangle_{n=1}^\infty)$ such that $MT(\langle a_j \rangle_{j=1}^m, \langle y_n \rangle_{n=1}^\infty) \subseteq C_i$.*

(d) *(Theorem 1.8). Let $m, k, r \in \mathbb{N}$, and let $f : \{1, 2, \ldots, m\} \to \{1, 2, \ldots, k\}$. For each $j \in \{1, 2, \ldots, k\}$, let $\langle H_{j,n} \rangle_{n=1}^\infty$ be an increasing sequence in $\mathcal{P}_f(\mathbb{N})$, and let $\mathcal{P}_f(\mathbb{N})^m = \bigcup_{i=1}^r C_i$. There exists $i \in \{1, 2, \ldots, r\}$ and for each $j \in \{1, 2, \ldots, k\}$ there exists a union subsystem $FU(\langle K_{j,n} \rangle_{n=1}^\infty)$ of $FU(\langle H_{j,n} \rangle_{n=1}^\infty)$ such that*

$$\{(\bigcup_{t \in F_1} K_{f(1),t}, \bigcup_{t \in F_2} K_{f(2),t}, \ldots, \bigcup_{t \in F_m} K_{f(m),t}) : F_1 < F_2 < \ldots < F_m\} \subseteq C_i \,.$$

Proof. (a) implies (b). Let $k, m, r \in \mathbb{N}$, let $\langle a_j \rangle_{j=1}^m$ be a sequence in $\mathbb{N}$ and for each $j \in \{1, 2, \ldots, k\}$, let $\langle x_{j,n} \rangle_{n=1}^\infty$ be a sequence in $\mathbb{N}$. Let $f : \{1, 2, \ldots, m\} \to \{1, 2, \ldots, k\}$ and let $\mathbb{N} = \bigcup_{i=1}^r C_i$. For $i \in \{1, 2, \ldots, r\}$, let

$$D_i = \big\{\{F_1, F_2, \ldots, F_m\} : F_1 < F_2 < \ldots < F_m \text{ and } \textstyle\sum_{j=1}^m a_j \sum_{t \in F_j} x_{f(j),t} \in C_i\big\}$$

and let $D_0 = [\mathcal{P}_f(\mathbb{N})]^m \setminus \bigcup_{i=1}^r D_i$. Pick $i \in \{0, 1, \ldots, r\}$ and $\langle H_n \rangle_{n=1}^\infty$ as guaranteed by (a), and note that $i \neq 0$ since $\{F_1, F_2, \ldots, F_m\} \in D_0$ if and only if the F_j's are all distinct and it is not the case that $F_1 < F_2 < \ldots < F_m$. For $n \in \mathbb{N}$ and $j \in \{1, 2, \ldots, k\}$ let $y_{j,n} = \sum_{t \in H_n} x_{j,t}$.

(b) implies (c). Let $m, r \in \mathbb{N}$, let $\langle a_j \rangle_{j=1}^m$ and $\langle x_{1,n} \rangle_{n=1}^\infty$ be sequences in $\mathbb{N}$, and let $\mathbb{N} = \bigcup_{i=1}^r C_i$. Let $k = 1$ and for $j \in \{1, 2, \ldots, m\}$ let $f(j) = 1$.

(c) implies (a). Let $m, r \in \mathbb{N}$ and let $[\mathcal{P}_f(\mathbb{N})]^m = \bigcup_{i=1}^r C_i$. For $j \in \{1, 2, \ldots, m\}$ let $a_j = 2^{j-1}$ and for $n \in \mathbb{N}$, let $x_n = 2^{mn}$. Let for each $i \in \{1, 2, \ldots, r\}$,

$$D_i = \{\textstyle\sum_{j=1}^m a_j \sum_{t \in F_j} x_t : F_1 < F_2 < \ldots < F_m \text{ and } \{F_1, F_2, \ldots, F_m\} \in C_i\} \,.$$

Let $D_0 = \mathbb{N} \setminus \bigcup_{i=1}^r D_i$. Pick $i \in \{0, 1, \ldots, r\}$ and a sum subsystem $FS(\langle y_n \rangle_{n=1}^\infty)$ of $FS(\langle x_n \rangle_{n=1}^\infty)$ such that $MT(\langle a_j \rangle_{j=1}^m, \langle y_n \rangle_{n=1}^\infty) \subseteq D_i$ and note that $i \neq 0$. For each $n \in \mathbb{N}$ let H_n be the unique subset of $\mathbb{N}$ such that $y_n = \sum_{t \in H_n} x_t$. Then $[FU(\langle H_n \rangle_{n=1}^\infty)]_<^m \subseteq C_i$.

(a) implies (d). This proof is essentially identical to the proof that (a) implies (b).

(d) implies (a). Let $m, r \in \mathbb{N}$ and let $[\mathcal{P}_f(\mathbb{N})]^m = \bigcup_{i=1}^r C_i$. Let $k = 1$ and for each $n \in \mathbb{N}$, let $H_{1,n} = \{n\}$. For $i \in \{1, 2, \ldots, r\}$, let $D_i = \{(F_1, F_2, \ldots, F_m) : \{F_1, F_2, \ldots, F_m\} \in C_i\}$. Let $D_0 = \{(F_1, F_2, \ldots, F_m) : |\{F_1, F_2, \ldots, F_m\}| < m\}$. Pick $i \in \{0, 1, \ldots, r\}$ and a union subsystem $FU(\langle K_{1,n}\rangle_{n=1}^\infty)$ of $FU(\langle H_{1,n}\rangle_{n=1}^\infty)$ such that

$$\{(\textstyle\bigcup_{t\in F_1} K_{1,t}, \bigcup_{t\in F_2} K_{1,t}, \ldots, \bigcup_{t\in F_m} K_{1,t}) : F_1 < F_2 < \ldots < F_m\} \subseteq D_i$$

and note that $i \neq 0$. Then $[FU(\langle K_{1,n}\rangle_{n=1}^\infty)]_<^m \subseteq C_i$. □

Lemma 4.2. *Let $k \in \mathbb{N}$ and for $i \in \{1, 2, \ldots, k\}$, let S_i be a semigroup and let $p_i \in \beta S_i$. Then $\bigotimes_{i=1}^k p_i$ is an ultrafilter on $\times_{i=1}^k S_i$.*

Proof. We proceed by induction on k, the case $k = 1$ being trivial. So let $k \in \mathbb{N}$ and assume that $\bigotimes_{i=1}^k p_i$ is an ultrafilter on $\times_{i=1}^k S_i$. It is immediate that $\emptyset \notin \bigotimes_{i=1}^{k+1} p_i$ and $\bigotimes_{i=1}^{k+1} p_i$ is closed under passage to supersets.

For $A \subseteq \times_{i=1}^{k+1} S_i$, let

$$\psi(A) = \{(x_1, x_2, \ldots, x_k) \in \times_{i=1}^k S_i : \{x_{k+1} \in S_{k+1} : (x_1, x_2, \ldots, x_{k+1}) \in A\} \in p_{k+1}\}\,.$$

Let $A, B \in \bigotimes_{i=1}^{k+1} p_i$. Then $\psi(A) \in \bigotimes_{i=1}^k p_i$ and $\psi(B) \in \bigotimes_{i=1}^k p_i$ so $\psi(A) \cap \psi(B) \in \bigotimes_{i=1}^k p_i$. Since $\psi(A) \cap \psi(B) \subseteq \psi(A \cap B)$, $\psi(A \cap B) \in \bigotimes_{i=1}^k p_i$ so $A \cap B \in \bigotimes_{i=1}^{k+1} p_i$.

Now let $A \subseteq \times_{i=1}^{k+1} S_i$ and assume that $A \notin \bigotimes_{i=1}^{k+1} p_i$. Then $\psi(A) \notin \bigotimes_{i=1}^k p_i$ so $\times_{i=1}^k S_i \setminus \psi(A) \in \bigotimes_{i=1}^k p_i$. Also $\times_{i=1}^k S_i \setminus \psi(A) \subseteq \psi(\times_{i=1}^{k+1} S_i \setminus A)$ so $\psi(\times_{i=1}^{k+1} S_i \setminus A) \in \bigotimes_{i=1}^k p_i$ and thus $\times_{i=1}^{k+1} S_i \setminus A \in \bigotimes_{i=1}^{k+1} p_i$. □

Recall that if p is an idempotent in S and $A \in p$, then $A^\star = \{x \in A : x^{-1}A \in p\}$ and if $x \in A^\star$, then $x^{-1}A^\star \in p$.

Theorem 1.16. *Let $m \in \mathbb{N}$ and for each $i \in \{1, 2, \ldots, m\}$, let S_i be a semigroup and let $A \subseteq \times_{i=1}^m S_i$. The following statements are equivalent.*

(a) *For each $i \in \{1, 2, \ldots, m\}$, there is a sequence $\langle x_{i,n}\rangle_{n=1}^\infty$ in S_i such that*

$$\{(\textstyle\prod_{t\in F_1} x_{1,t}, \prod_{t\in F_2} x_{2,t}, \ldots, \prod_{t\in F_m} x_{m,t}) : F_1 < F_2 < \ldots < F_m\} \subseteq A\,.$$

(b) *For each $i \in \{1, 2, \ldots, m\}$, there is an idempotent $p_i \in \beta S_i$ such that $A \in \bigotimes_{i=1}^m p_i$.*

Proof. (a) implies (b). For each $i \in \{1, 2, \ldots, m\}$, pick by Lemma 1.9 an idempotent $p_i \in \bigcap_{r=1}^\infty \overline{FP(\langle x_{i,n}\rangle_{n=r}^\infty)}$. Let $k = m$ and let $f : \{1, 2, \ldots, m\} \to \{1, 2, \ldots, k\}$ be the identity function. Apply Lemma 2.9.

(b) implies (a). We proceed by induction on m, the case $m = 1$ being Theorem 1.10. Let $m \in \mathbb{N}$ and let $A \in \bigotimes_{i=1}^{m+1} p_i$. Let $B = \{(x_1, x_2, \ldots, x_m) \in \times_{i=1}^m S_i : \{x_{m+1} \in S_{m+1} : (x_1, x_2, \ldots, x_{m+1}) \in A\} \in p_{m+1}\}$. For each $i \in \{1, 2, \ldots, m\}$, pick a sequence $\langle x_{i,n}\rangle_{n=1}^\infty$ in S_i such that

$$\{(\textstyle\prod_{t\in F_1} x_{1,t}, \prod_{t\in F_2} x_{2,t}, \ldots, \prod_{t\in F_m} x_{m,t}) : F_1 < F_2 < \ldots < F_m\} \subseteq B\,.$$

For $t \in \{1, 2, \ldots, m\}$, let $x_{m+1,t}$ be any member of S_{m+1}. (These terms are not involved in the conclusion.) For $n > m$, let

$$C_n = \bigcap\big\{\{z \in S_{m+1} : (\textstyle\prod_{t\in F_1} x_{1,t}, \prod_{t\in F_2} x_{2,t}, \ldots, \prod_{t\in F_m} x_{m,t}, z) \in A\} : F_1 < \ldots < F_m < \{n\}\big\}\,.$$

Then each $C_n \in p_{m+1}$ and $C_{n+1} \subseteq C_n$. Pick $x_{m+1,m+1} \in C^\star_{m+1}$. Let $r \geq m+1$ and assume that for each $t \in \{m+1, m+2, \ldots, r\}$, $x_{m+1,t}$ has been chosen such that for each $k \in \{m+1, m+2, \ldots, r\}$ and each nonempty $G \subseteq \{k, k+1, \ldots, r\}$, $\prod_{t \in G} x_{m+1,t} \in C^\star_k$. Pick

$$x_{m+1,r+1} \in C^\star_{r+1} \cap \bigcap_{k=m+1}^{r} \left\{ (\prod_{t \in G} x_{m+1,t})^{-1} C^\star_k : \emptyset \neq G \subseteq \{k, k+1, \ldots, r\} \right\}.$$

□

In order to prove Theorem 1.17 we shall need the following lemma.

Lemma 4.3. *Let $(S, \cdot)$ be a semigroup, let $m \in \mathbb{N}$, and let $p \in \beta S$. Let $A \in \bigotimes_{j=1}^m p$. Then for $j \in \{1, 2, \ldots, m\}$ there exists $D_j : S^{j-1} \to \mathcal{P}(S)$ such that*

1. *for $j \in \{1, 2, \ldots, m\}$, if for each $s \in \{1, 2, \ldots, j-1\}$, $w_s \in D_s(w_1, w_2, \ldots, w_{s-1})$, then $D_j(w_1, w_2, \ldots, w_{j-1}) \in p$; and*
2. *if for each $s \in \{1, 2, \ldots, m\}$, $w_s \in D_s(w_1, w_2, \ldots, w_{s-1})$, then $(w_1, w_2, \ldots, w_m) \in A$.*

Proof. This is the special case of Lemma 2.5 in which $k = 1$, so necessarily the function f is constant. □

Theorem 1.17. *Let S be a semigroup, let $m \in \mathbb{N}$, and let $A \subseteq \times_{i=1}^m S$. The following statements are equivalent.*

(a) *There is a sequence $\langle x_n \rangle_{n=1}^\infty$ in S such that*
$$\{(\prod_{t \in F_1} x_t, \prod_{t \in F_2} x_t, \ldots, \prod_{t \in F_m} x_t) : F_1 < F_2 < \ldots < F_m\} \subseteq A.$$

(b) *There is an idempotent $p \in \beta S$ such that $A \in \bigotimes_{i=1}^m p$.*

Proof. (a) implies (b). Pick by Lemma 1.9 an idempotent $p \in \bigcap_{r=1}^\infty \overline{FP(\langle x_n \rangle_{n=r}^\infty)}$. By Lemma 2.9 with $k = 1$, $A \in \bigotimes_{i=1}^m p$.

(b) implies (a). Pick an idempotent $p \in \beta S$ such that $A \in \bigotimes_{i=1}^m p$. By Lemma 4.3, pick for each $j \in \{1, 2, \ldots, m\}$, some $D_j : S^{j-1} \to \mathcal{P}(S)$ such that

1. for $j \in \{1, 2, \ldots, m\}$, if for each $s \in \{1, 2, \ldots, j-1\}$, $w_s \in D_s(w_1, w_2, \ldots, w_{s-1})$, then $D_j(w_1, w_2, \ldots, w_{j-1}) \in p$; and
2. if for each $s \in \{1, 2, \ldots, m\}$, $w_s \in D_s(w_1, w_2, \ldots, w_{s-1})$, then $(w_1, w_2, \ldots, w_m) \in A$.

Recall that if $x \in B^\star$, then $x^{-1} B^\star \in p$.

Choose $x_1 \in D_1(\emptyset)^\star$. Let $n \in \mathbb{N}$ and assume that we have chosen $\langle x_t \rangle_{t=1}^n$ such that if $j \in \{1, 2, \ldots, m\}$ and $F_1 < F_2 < \ldots < F_j < \{n+1\}$, then $\prod_{t \in F_j} x_t \in D_j(\prod_{t \in F_1} x_t, \prod_{t \in F_2} x_t, \ldots, \prod_{t \in F_{j-1}} x_t)^\star$.

Let

$$G = \bigcap \big\{ (\prod_{t \in F_j} x_t)^{-1} D_j(\prod_{t \in F_1} x_t, \prod_{t \in F_2} x_t, \ldots, \prod_{t \in F_{j-1}} x_t)^\star : \\ j \in \{1, 2, \ldots, m\} \text{ and } F_1 < F_2 < \ldots < F_j < \{n+1\} \big\}$$

and let

$$E = \bigcap \big\{ D_j(\prod_{t \in F_1} x_t, \prod_{t \in F_2} x_t, \ldots, \prod_{t \in F_{j-1}} x_t)^\star : \\ j \in \{1, 2, \ldots, m\} \text{ and } F_1 < F_2 < \ldots < F_{j-1} < \{n+1\} \big\}.$$

By the induction hypothesis we have directly that $G \in p$. By the induction hypothesis and condition (1), we have that $E \in p$. Pick $x_{n+1} \in D \cap E$.

To verify the induction hypothesis at $n+1$, let $j \in \{1,2,\ldots,m\}$ and let $F_1 < F_2 < \ldots < F_j < \{n+2\}$. If $n+1 \notin F_j$, the conclusion holds by assumption so assume tha $n+1 \in F_j$. If $F_j = \{n+1\}$, the conclusion holds since $x_{n+1} \in E$. So assume $F_j \neq \{n+1\}$ and let $F'_j = F_j \setminus \{n+1\}$. Then

$$x_{n+1} \in (\prod_{t\in F'_j} x_t)^{-1} D_j(\prod_{t\in F_1} x_t, \prod_{t\in F_2} x_t, \ldots, \prod_{t\in F_{j-1}} x_t)^\star$$

so $\prod_{t\in F_j} x_t \in D_j(\prod_{t\in F_1} x_t, \prod_{t\in F_2} x_t, \ldots, \prod_{t\in F_{j-1}} x_t)^\star$.

The induction being complete, we have by condition (2) that

$$\{(\prod_{t\in F_1} x_t, \prod_{t\in F_2} x_t, \ldots, \prod_{t\in F_m} x_t) : F_1 < F_2 < \ldots < F_m\} \subseteq A\,.$$

$\square$

Lemma 2.1. *Let $k, l \in \mathbb{N}$. For $i \in \{1,2,\ldots,k+l\}$, let S_i be a semigroup and let $p_i \in \beta S_i$. Let $A \subseteq \times_{i=1}^{k+l} S_i$. Then $A \in \bigotimes_{i=1}^{k+l} p_i$ if and only if*

$$\{(x_1,x_2,\ldots,x_k) \in \times_{i=1}^{k} S_i : \{(x_{k+1},x_{k+2},\ldots,x_{k+l}) \in \times_{i=k+1}^{k+l} S_i : (x_1,x_2,\ldots,x_{k+l}) \in A\} \in \bigotimes_{i=k+1}^{k+l} p_i\} \in \bigotimes_{i=1}^{k} p_i\,.$$

Proof. We proceed by induction on l, the case $l=1$ being the definition of $\bigotimes_{i=1}^{k+1} p_i$. So let $l \in \mathbb{N}$, and assume the statement is true for l.

Sufficiency. Let

$$B = \{(x_1,x_2,\ldots,x_k) \in \times_{i=1}^{k} S_i : \{(x_{k+1},x_{k+2},\ldots,x_{k+l+1}) \in \times_{i=k+1}^{k+l+1} S_i : (x_1,x_2,\ldots,x_{k+l+1}) \in A\} \in \bigotimes_{i=k+1}^{k+l+1} p_i\}$$

and assume that $B \in \bigotimes_{i=1}^{k} p_i$. To see that $A \in \bigotimes_{i=1}^{k+l+1} p_i$, we let

$$C = \{(x_1,x_2,\ldots,x_{k+l}) \in \times_{i=1}^{k+l} S_i : \{x_{k+l+1} \in S_{k+l+1} : (x_1,x_2,\ldots,x_{k+l+1}) \in A\} \in p_{k+l+1}\}$$

and show that $C \in \bigotimes_{i=1}^{k+l} p_i$.

Let $D = \{(x_1,x_2,\ldots,x_k) \in \times_{i=1}^{k} S_i : \{(x_{k+1},x_{k+2},\ldots,x_{k+l}) \in \times_{i=k+1}^{k+l} S_i : (x_1,x_2,\ldots,x_{k+l}) \in C\} \in \bigotimes_{i=k+1}^{k+l} p_i\}$. To see that $C \in \bigotimes_{i=1}^{k+l} p_i$, it suffices to show that $D \in \bigotimes_{i=1}^{k} p_i$, for which it in turn suffices that $B \subseteq D$. So let $(x_1,x_2,\ldots,x_k) \in B$ and let $E = \{(x_{k+1},x_{k+2},\ldots,x_{k+l+1}) \in \times_{i=k+1}^{k+l+1} S_i : (x_1,x_2,\ldots,x_{k+l+1}) \in A\}$. Then $E \in \bigotimes_{i=k+1}^{k+l+1} p_i$. Let

$$F = \{(x_{k+1},x_{k+2},\ldots,x_{k+l}) \in \times_{i=k+1}^{k+l} S_i : \{x_{k+l+1} \in S_{k+l+1} : (x_{k+1},x_{k+2},\ldots,x_{k+l+1}) \in E\} \in p_{k+l+1}\}\,.$$

Then $F \in \bigotimes_{i=k+1}^{k+l} p_i$. To see that $(x_1,x_2,\ldots,x_k) \in D$, it suffices to show that

$$F \subseteq \{(x_{k+1},x_{k+2},\ldots,x_{k+l}) \in \times_{i=k+1}^{k+l} S_i : (x_1,x_2,\ldots,x_{k+l}) \in C\}\,,$$

so let $(x_{k+1},x_{k+2},\ldots,x_{k+l}) \in F$. Let

$$G = \{x_{k+l+1} \in S_{k+l+1} : (x_{k+1},x_{k+2},\ldots,x_{k+l+1}) \in E\}\,.$$

Then $G \in p_{k+l+1}$ so to see that $(x_1,x_2,\ldots,x_{k+l}) \in C$, it suffices that $G \subseteq \{x_{k+l+1} \in S_{k+l+1} : (x_1,x_2,\ldots,x_{k+l+1}) \in A\}$. Let $x_{k+l+1} \in G$. Then

$$(x_{k+1},x_{k+2},\ldots,x_{k+l+1}) \in E$$

so $(x_1, x_2, \ldots, x_{k+l+1}) \in A$ as required.

Necessity. Assume that $A \in \bigotimes_{i=1}^{k+l+1} p_i$ and suppose that

$$\{(x_1, x_2, \ldots, x_k) \in \times_{i=1}^{k} S_i : \{(x_{k+1}, x_{k+2}, \ldots, x_{k+l+1}) \in \times_{i=k+1}^{k+l+1} S_i : (x_1, x_2, \ldots, x_{k+l+1}) \in A\} \in \bigotimes_{i=k+1}^{k+l+1} p_i\} \notin \bigotimes_{i=1}^{k} p_i \,.$$

Then

$$\{(x_1, x_2, \ldots, x_k) \in \times_{i=1}^{k} S_i : \{(x_{k+1}, x_{k+1}, \ldots, x_{k+l+1}) \in \times_{i=k+1}^{k+l+1} S_i : (x_1, x_2, \ldots, x_{k+l+1}) \in \times_{i=1}^{k+l+1} S_i \setminus A\} \in \bigotimes_{i=k+1}^{k+l+1} p_i\} \in \bigotimes_{i=1}^{k} p_i \,.$$

So by the just established sufficiency, $\times_{i=1}^{k+l+1} S_i \setminus A \in \bigotimes_{i=1}^{k+l+1} p_i$, a contradiction. □

Lemma 2.2. *Let $m \in \mathbb{N}$ and for $j \in \{1, 2, \ldots, m\}$, let $p_j \in \delta\mathcal{F}$. Then $\{(F_1, F_2, \ldots, F_m) \in \times_{j=1}^{m} \mathcal{F} : F_1 < F_2 < \ldots < F_m\} \in \bigotimes_{j=1}^{m} p_j$.*

Proof. We proceed by induction on m, the case $m = 1$ being trivial. So let $m \in \mathbb{N}$, let $p_j \in \delta\mathcal{F}$ for $j \in \{1, 2, \ldots, m+1\}$, and assume that

$$A = \{(F_1, F_2, \ldots, F_m) \in \times_{j=1}^{m} \mathcal{F} : F_1 < F_2 < \ldots < F_m\} \in \bigotimes_{j=1}^{m} p_j \,.$$

Let $B = \{(F_1, F_2, \ldots, F_{m+1}) \in \times_{j=1}^{m+1} \mathcal{F} : F_1 < F_2 < \ldots < F_{m+1}\}$. We show that $A \subseteq \{(F_1, F_2, \ldots, F_m) \in \times_{j=1}^{m} \mathcal{F} : \{F_{m+1} \in \mathcal{F} : (F_1, F_2, \ldots, F_{m+1}) \in B\}\} \in p_{m+1}$ so that $B \in \bigotimes_{j=1}^{m+1} p_j$ as required. So let $(F_1, F_2, \ldots, F_m) \in A$ and let $r = \max F_m$. Then $\{F_{m+1} \in \mathcal{F} : (F_1, F_2, \ldots, F_{m+1}) \in B\} = \{F \in \mathcal{F} : \min F > r\}$, which is in p_{m+1} because $p_{m+1} \in \delta\mathcal{F}$. □

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