Simultaneous Diophantine Approximation and VIP Sets

1. Our starting point is the following celebrated theorem of H. Weyl ([W1], [W2]).

Theorem W. Let $f(t) \in \mathbf{R}[t]$ be a polynomial having at least one coefficient other than the constant term irrational. Then for any $0 \leq \alpha < \beta \leq 1$ there exists an integer x such that $\alpha < f(x) \mathrm{mod}\ 1 < \beta$.

A simple consequence of Theorem W is that the set $W_{\alpha,\beta}(f) = \{x \in \mathbf{Z} : \alpha < f(x) \mathrm{mod}\ 1 < \beta\}$ is not merely non-empty, but infinite. As a matter of fact, Weyl obtained Theorem W as a corollary to a limiting theorem which says that the sequence $\big(f(n)\big)_{n=1}^{\infty}$ is equidistributed mod 1, which in particular implies that the density of $W_{\alpha,\beta}(f)$, defined to be $d = \lim_{N\to\infty} \frac{|\{x: -N \leq x < N, f(x) \in (\alpha,\beta)\}|}{2N+1}$, exists and equals $\beta - \alpha$. (Replacing the lim in the definition of density by lim sup or lim inf, one obtains the notions of *upper density* and *lower density*, respectively. Note that the family of sets having positive lower density is closed under supersets, which is a desired feature of any notion of "largeness." Indeed, positive lower density is the first of several progressively stronger "largeness" properties that we shall be concerned with in this paper.)

A set S in $\mathbf{Z}^d$ is *syndetic* if the union of finitely many of its additive shifts is all of $\mathbf{Z}^d$. Alternatively, S is syndetic if it intersects non-trivially any large enough d-dimensional cube; namely, if there exists k such that for all choices of $M_1, \ldots, M_d$, $S \cap \prod_{i=1}^{d} [M_i, M_i + k] \neq \emptyset$. In $\mathbf{Z}$, then, S is syndetic if it intersects non-trivially any large enough interval, i.e. has *bounded gaps.*) Syndeticity is a property that is strictly stronger than that of positive lower density and is the second notion of largeness of interest to us.

Van der Corput provided the following impressive generalization of Theorem W in [VdC].

Theorem VdC. Let $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n$ be real numbers and let $m \in \mathbf{N}$. For each $k = 1, \ldots, n$, let f_k be a polynomial of $m + k - 1$ unknowns. If the system

$$\begin{aligned}
&\alpha_1 < f_1(x_1, \ldots, x_m) - y_1 < \beta_1 \\
&\alpha_2 < f_2(x_1, \ldots, x_m, y_1) - y_2 < \beta_2 \\
&\alpha_3 < f_3(x_1, \ldots, x_m, y_1, y_2) - y_3 < \beta_3 \\
&\vdots \\
&\alpha_n < f_n(x_1, \ldots, x_m, y_1, \ldots, y_n - 1) - y_n < \beta_n
\end{aligned} \tag{1.1}$$

has at least one integer-valued solution then it has infinitely many integer-valued solutions. Moreover, the set of $(x_1, \ldots, x_m) \in \mathbf{Z}^m$ for which there is some $y = (y_1, \ldots, y_n)$, $y_i \in \mathbf{Z}$, so that (x, y) satisfies the system (1.1) is syndetic.

Taking $n = m = 1$ in Theorem VdC, one obtains that $W_{\alpha,\beta}(f)$ is syndetic.

Note: Syndeticity of $W_{\alpha,\beta}$ of follows from *well distribution* of the sequence $\big(f(x)\big)_{x\in\mathbf{N}}$, a concept introduced by E. Hlawka [Hl] and G. Petersen [P] in the mid-fifties. (See also [F], where well-distribution of the sequence $\big((f(x)\big)_{x\in\mathbf{N}}$ is established via ergodic considerations.)

The goal of this paper is to strengthen Theorem VdC in two respects. First we shall show that, in the case that the set of solutions of system (1.1) is non-empty, then it is large in a more powerful sense than mere syndeticity. In doing so, we shall be at the same time extending a result of Furstenberg and Weiss (see [FW]) having a similar flavor. (We mention that while neither [FW] nor Theorem VdC contain the other, our result will contain both.) Second, we shall show that our generalization holds for a wide class of *generalized polynomials*, namely mappings $\mathbf{R}^n \to \mathbf{R}$ one constructs from the constants and coordinate maps $(x_1, \ldots, x_n) \to x_i$ using not only the conventional arithmetic operations of addition and multiplication (as in conventional polynomials), but also the operation of taking the integer part.

We will presently introduce the notions of largeness germain to our paper. First, however, we note that a natural way of defining a notion of largeness, say in $\mathbf{Z}^n$, is to introduce a family $\mathcal{S}$ of subsets of $\mathbf{Z}^n$ and declare a set $E \subset \mathbf{Z}$ to be an $\mathcal{S}^*$ set if for every $S \in \mathcal{S}$, $E \cap S \neq \emptyset$. For example, if $\mathcal{S}$ consists of the sets S in $\mathbf{Z}$ having upper density 1 then $\mathcal{S}^*$ sets are precisely those of positive lower density. If $\mathcal{T}$ is the family of subsets of $\mathbf{Z}^n$ containing arbitrarily large n-dimensional cubes (so-called *thick* sets), then the $\mathcal{T}^*$ sets are precisely those that are syndetic. For more discussion and examples of this type, the reader is referred to [F, Section 9.1].

A set $S \subset \mathbf{Z}^n$ is called an *IP set* (or *finite sums* set) if it contains the set of finite sums, without repetition, of a fixed sequence. (By "without repetition" we mean here repetition of the indices, not the elements appearing in the sequence. If an element appears multiple times in the sequence, it may appear an equal number of times in any finite sum. In particular, any set containing 0 is an IP set by default. This is in contrast the situation in the semigroup $\mathbf{N}$, where all IP sets have infinite cardinality.)

Let us call a set $E \subset \mathbf{Z}^n$ an IP* set if E intersects every IP set nontrivially. It is not hard to see that any IP* set is syndetic, as any set containing arbitarily large n-dimensional cubes may easily be shown to contain an IP set. On the other hand, it is easy to see that not every syndetic set is IP* (consider for example the set of odd integers in $\mathbf{Z}$). Therefore the IP* property is strictly stronger that that of syndeticity.

However, the real strength of the IP* property is that it is preserved under finite intersections; if $S_1, \ldots, S_k$ are IP* sets then $\bigcap_{i=1}^k S_i$ is IP*. This non-trivial fact follows from the following theorem due to Hindman ([Hi]) which plays a prominent role in our paper. (Later, we shall give a different version of Hindman's theorem.)

Theorem H1. Let R be an IP set, let $k \in \mathbf{N}$ and suppose $R = R_1 \cup R_2 \cup \cdots \cup R_k$. Then some R_i, $1 \leq i \leq k$, is an IP set.

We now show via Theorem H1 that the IP* property is passed on to finite intersections. First we note that it suffices to establish this for intersections of two sets. So let A and B be IP* sets and suppose $A \cap B$ is not IP*. Then there exists an IP set R in the complement of $A \cap B$, whereby $R = (R \setminus A) \cup (R \setminus B)$. It follows by Theorem H that either $(R \setminus A)$ or $(R \setminus B)$ is an IP set. In either case this is a contradiction as both A and B are IP*.

We shall not, however, content ourselves with the IP* property. An even stronger notion of largeness may be obtained by considering *VIP sets*–variants of IP sets having a "polynomial" nature (see [BFM]).

Let $\mathcal{F}$ denote the family of finite, non-empty subsets of $\mathbf{N}$. In a commutative group $(G,+)$, a sequence indexed by $\mathcal{F}$, say $(v_\alpha)_{\alpha\in\mathcal{F}}$, is called a *VIP system* if there exists some non-negative integer d such that for every pairwise disjoint $\alpha_0,\alpha_1,\cdots,\alpha_d\in\mathcal{F}$ we have

$$\sum_{\substack{\{\beta_1,\cdots,\beta_t\}\subset\{\alpha_0,\cdots,\alpha_d\}\\ \beta_i\neq\beta_j,\ 1\le i<j\le t}} (-1)^t v_{\beta_1\cup\cdots\cup\beta_t}=0. \tag{1.2}$$

If $(v_\alpha)_{\alpha\in\mathcal{F}}$ is a VIP system then the least non-negative d for which (1.2) holds is called the *degree* of the system. VIP systems of degree 1 are called IP systems, and one may easily show that a set R is an IP set if and only if there exists an IP system $\mathcal{F}\to R$. Similarly we say that R is a VIP set if there exists a VIP system $\mathcal{F}\to R$. The distinction between sets and systems here is very simple. IP sets and VIP sets are sets; IP systems and VIP systems are functions from $\mathcal{F}$ to some group.

A different, though equivalent, characterization of VIP systems is often useful. For $d\in\mathbf{N}$, let $\mathcal{F}_d$ denote the family of non-empty subsets of $\mathbf{N}$ having cardinality at most d.

Proposition 1.1 ([M, Proposition 2.5]). Let G be an additive abelian group and let $d\in\mathbf{N}$. A sequence indexed by $\mathcal{F}$, $(v_\alpha)_{\alpha\in\mathcal{F}}$, in G is a VIP system of degree at most d if and only if there exists a function from $\mathcal{F}_d$ to G, written $\gamma\to n_\gamma$, $\gamma\in\mathcal{F}_d$, such that

$$v_\alpha=\sum_{\gamma\subset\alpha,\gamma\in\mathcal{F}_d} n_\gamma$$

for all $\alpha\in\mathcal{F}$.

We shall prove the following theorem, which contains Theorem VdC as a special case:

Theorem A. Let $\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n$ be real numbers and let $m\in\mathbf{N}$. For each $k=1,\ldots,n$, let p_k be a polynomial of $m+k-1$ unknowns. If the system

$$\begin{aligned}
&\alpha_1<p_1(x_1,\ldots,x_m)-x_{m+1}<\beta_1\\
&\alpha_2<p_2(x_1,\ldots,x_m,x_{m+1})-x_{m+2}<\beta_2\\
&\alpha_3<p_3(x_1,\ldots,x_m,x_{m+1},x_{m+2})-x_{m+3}<\beta_3\\
&\vdots\\
&\alpha_n<p_n(x_1,\ldots,x_m,x_{m+1},\ldots,x_{m+n-1})-x_{m+n}<\beta_n
\end{aligned}\tag{1.3}$$

has an integer valued solution $(a_1,\ldots,a_{m+n})$ then the following set is VIP*:

$$\big\{(s_1,\ldots,s_m):\exists \text{ a solution } (s_1+a_1,s_2+a_2,\ldots,s_m+a_m,s_{m+1},\ldots,s_{m+n})\big\}. \tag{1.4}$$

To formulate the aforementioned result [FW], one considers the special case of Theorem A where $m=1$ and $\alpha_i<0<\beta_i$ for all $i=1,2,\ldots,n$ (so that, in particular, the system (1.3) always has at least one solution, namely the zero solution) and replaces "VIP*" by "IP*."

As noted, our Theorem A contains both Theorem VdC and the results of [FW] as special cases. Indeed, our proof is much different (and shorter), owing to a fortuitous usage of the language of VIP-systems. Still, we feel that van der Corput's original proof has quite a few interesting and neglected facets which deserve to be more widely known. Since [VdC] was written in German and the proof therein is long and somewhat cumbersome, we take the liberty of summarizing this proof in an Appendix (section 3 below). In particular, we would like to attract the reader's attention to what van der Corput calls "rythmic" systems of functions–a notion that we feel could perhaps find new applications.

2. Proof of the Main Theorem.

Before proving Theorem A, we introduce some notation as well as a few lemmas. We denote by $\mathcal{F}$ the family of all finite subsets of $\mathbf{N}$. Note that $(\mathcal{F}, \cup)$ is a semigroup. For $\alpha, \beta \in \mathcal{F}$, we write $\alpha < \beta$ if $i < j$ for every $i \in \alpha$ and every $j \in \beta$. If $(\alpha_i)_{i=1}^\infty \subset \mathcal{F}$ with $\alpha_1 < \alpha_2 < \cdots$, then the sub-family

$$\mathcal{F}^{(1)} = \Big\{ \bigcup_{i \in \beta} \alpha_i : \beta \in \mathcal{F} \Big\} = FU\big((\alpha_i)_{i=1}^\infty\big)$$

is called an *IP-ring*. Notice that $(\mathcal{F}^{(1)}, \cup)$ is isomorphic as a semigroup to $(\mathcal{F}, \cup)$ under the bijection $\pi(\beta) = \bigcup_{i \in \beta} \alpha_i$, and it is often useful to think of them interchangably. For example, if $\mathcal{F}^{(1)}$ is an IP-ring and $(x_\alpha)_{\alpha \in \mathcal{F}^{(1)}}$ are members of a group then we say $(x_\alpha)_{\alpha \in \mathcal{F}^{(1)}}$ is a VIP system if $(x_{\pi(\beta)})_{\beta \in \mathcal{F}}$ is a VIP system.

Here now, as promised, is the second formulation of Hindman's theorem.

Theorem H2. Let $\mathcal{F}^{(1)}$ be an IP-ring. For any finite coloring of $\mathcal{F}^{(1)}$, there exists a monochromatic IP-ring $\mathcal{F}^{(2)} \subset \mathcal{F}^{(1)}$.

Hindman's theorem has important ramifications for a certain mode of convergence along $\mathcal{F}$ we shall define presently. Suppose that $\{x_\alpha\}_{\alpha \in \mathcal{F}}$ is an $\mathcal{F}$-sequence in a topological space and $\mathcal{F}^{(1)}$ is an IP-ring. We write

$$\operatorname*{IP-lim}_{\alpha \in \mathcal{F}^{(1)}} x_\alpha = z$$

if for every neighborhood U of z there exists $\beta \in \mathcal{F}$ having the property that for every $\alpha \in \mathcal{F}^{(1)}$ with $\alpha > \beta$, $x_\alpha \in U$.

The following lemma is a simple consequence of Hindman's theorem.

Lemma 2.1. Suppose that X is a compact metric space and $\{x_\alpha\}_{\alpha \in \mathcal{F}}$ is an $\mathcal{F}$-sequence in X. Then for any IP-ring $\mathcal{F}^{(1)}$, there exists an IP-ring $\mathcal{F}^{(2)} \subset \mathcal{F}^{(1)}$ such that

$$\operatorname*{IP-lim}_{\alpha \in \mathcal{F}^{(2)}} x_\alpha = x$$

exists.

Proof. Using total boundedness of X and Hindman's theorem, for any IP-ring $\mathcal{F}^{(1)}$ there exists an IP-ring $\mathcal{G} \subset \mathcal{F}^{(1)}$ having the property that the the diameter of $\{x_\alpha : \alpha \in \mathcal{G}\}$ is at most ϵ. Therefore, given $\mathcal{F}^{(1)}$ we may let $\mathcal{F}^{(1)} \supset \mathcal{G}^{(1)} \supset \mathcal{G}^{(2)} \supset \mathcal{G}^{(3)} \supset \cdots$ be a descending sequence of IP-rings such that the diameter of $\{x_\alpha : \alpha \in \mathcal{G}^{(n)}\}$ is at most $\frac{1}{n}$ for all $n \in \mathbf{N}$. Let now $\alpha_1 < \alpha_2 < \cdots$ be an increasing sequence in $\mathcal{F}$ with $\alpha_i \in \mathcal{G}^{(i)}$, $i \in \mathbf{N}$, and let $\mathcal{F}^{(2)}$ consist of the finite unions of this seqence.

Lemma 2.2. Let $(v_\alpha^{(i)})_{\alpha\in\mathcal{F}}$ be VIP systems in $\mathbf{R}$, $i \in \mathbf{N}$. For any IP-ring $\mathcal{F}^{(1)}$ there exists an IP-ring $\mathcal{F}^{(2)}$ such that

$$\operatorname*{IP-lim}_{\alpha\in\mathcal{F}^{(2)}} \ \langle v_i(\alpha)\rangle = 0$$

for all $i \in \mathbf{N}$.

Proof. We prove the result for a single VIP system $\big(v(\alpha)\big)_{\alpha\in\mathcal{F}}$, whereupon the general result follows by a standard diagonal argument. $\big(v(\alpha) - [v(\alpha)]\big)_{\alpha\in\mathcal{F}^{(1)}}$ is a VIP system on the torus $[0,1)$ under addition modulo 1. Choose $\mathcal{F}^{(2)} \subset \mathcal{F}^{(1)}$ such that

$$\operatorname*{IP-lim}_{\alpha\in\mathcal{F}^{(2)}} \ \big(v(\alpha) - [v(\alpha)]\big) = v_0$$

exists. Letting all of the α_i's go to ∞ in (1.2), we have $-v_0 = 0$. This follows from the simple fact that any finite set has one more non-empty subset of odd cardinality than it has subsets of even cardinality. At any rate, we are done. $\square$

Lemma 2.3. Let $\big(v_i(\alpha)\big)_{\alpha\in\mathcal{F}}$ be VIP systems in $\mathbf{R}$, $i = 1, 2$, and let c_1, c_2 and $k \in \mathbf{R}$ with $0 < k < 1$. Then there exists an IP-ring $\mathcal{F}^{(1)}$ such that the following are VIP systems:

(a) $\big(c_1 v_1(\alpha) + c_2 v_2(\alpha)\big)_{\alpha\in\mathcal{F}}$.
(b) $\big(v_1(\alpha)v_2(\alpha)\big)_{\alpha\in\mathcal{F}}$.
(c) $\big([v_1(\alpha) + k]\big)_{\alpha\in\mathcal{F}^{(1)}}$.

Proof. Write $v_i(\alpha) = \sum_{\gamma\subset\alpha, |\gamma|\leq d} f_i(\gamma)$, $i = 1, 2$. Then $c_1 v_1(\alpha) + c_2 v_2(\alpha) = \sum_{\gamma\subset\alpha,|\gamma|\leq d} \big(c_1 f_1(\gamma) + c_2 f_2(\gamma)\big)$ and $v_1(\alpha)v_2(\gamma) = \sum_{\gamma\subset\alpha,|\gamma|\leq 2d} \Big(\sum_{\gamma_1\cup\gamma_x=\gamma} f_1(\gamma_1)f_2(\gamma_2)\Big)$. This proves (a) and (b).

For (c), by Lemma 2.1 there exists an IP-ring $\mathcal{F}^{(1)}$ (arising from a sequence $\alpha_1 < \alpha_2 < \cdots$) such that for all $\alpha \in \mathcal{F}^{(1)}$, $\langle v_1(\alpha)\rangle < \frac{\min\{k,1-k\}}{2^{d+1}}$, where d is the degree of the system. Then for any $\gamma \in \mathcal{F}$ with $|\gamma| = d+1$,

$$\begin{aligned}
&\Big|\sum_{\emptyset\neq\beta\subset\gamma} (-1)^{|\gamma\setminus\beta|}\big[v_1\big(\bigcup_{i\in\beta}\alpha_i\big) + k\big]\Big| \\
=&\Big|\sum_{\emptyset\neq\beta\subset\gamma} (-1)^{|\gamma\setminus\beta|}\Big(\big[v_1\big(\bigcup_{i\in\beta}\alpha_i\big) + k\big] - v_1\big(\bigcup_{i\in\beta}\alpha_i\big)\Big)\Big| \\
\leq& \sum_{\emptyset\neq\beta\subset\gamma} (-1)^{|\gamma\setminus\beta|}\big\langle v_1\big(\bigcup_{i\in\beta}\alpha_i\big)\big\rangle < 1.
\end{aligned}$$

Therefore, since this quantity must be an integer, it is zero. This establishes that $\big([v_1(\alpha) + k]\big)_{\alpha\in\mathcal{F}^{(1)}}$ is a VIP system of degree d. $\square$

This immediately gives:

Lemma 2.4. (a) Let $p(x)$ be a polynomial mapping $\mathbf{R}^l \to \mathbf{R}^t$ with $p(0) = 0$ and suppose $\big(x(\alpha)\big)_{\alpha\in\mathcal{F}}$ is a VIP system in $\mathbf{R}^l$. Then $\big(p(x(\alpha))\big)_{\alpha\in\mathcal{F}}$ is a VIP system in $\mathbf{R}^t$. In particular, if $p(x)$ is an arbitrary polynomial mapping $\mathbf{R}^l \to \mathbf{R}^t$ and $c \in \mathbf{R}^l$ is constant then $\big(p(x(\alpha)+c) - p(c)\big)_{\alpha\in\mathcal{F}}$ is a VIP system in $\mathbf{R}^t$.

Proof of Theorem A. Let $(a_1, \dots, a_{m+n})$ be any integer valued solution to (1.3) and let $\big(x(\alpha)\big)_{\alpha\in\mathcal{F}}$ be any VIP system in $\mathbf{Z}^m$. Put

$$\begin{aligned}
&v_1(\alpha) = x(\alpha) + (a_1, \dots, a_m)\\
&u_2(\alpha) = p_1\big(v_1(\alpha)\big) - p_1(a_1, \dots, a_m)\\
&v_2(\alpha) = \big(v_1(\alpha), [u_2(\alpha) + \frac{1}{2}] + a_{m+1}\big)\\
&\vdots\\
&u_{n+1}(\alpha) = p_n\big(v_n(\alpha)\big) - p_n(a_1, \dots, a_{m+n-1})\\
&v_{n+1}(\alpha) = \big(v_n(\alpha), [u_{n+1}(\alpha) + \frac{1}{2}] + a_{m+n}\big).
\end{aligned}$$

Then by iterated use of Lemmas 1.6 (c) and 1.7, there exists an IP-ring $\mathcal{F}^{(1)}$ such that each $\big(v_k(\alpha)\big)_{\alpha\in\mathcal{F}^{(1)}}$ is a shift of a VIP system in $\mathbf{Z}^{m+k-1}$ by $(a_1, \dots, a_{m+k-1})$ and each $\big(u_k(\alpha)\big)_{\alpha\in\mathcal{F}^{(1)}}$ is a VIP system in $\mathbf{R}$.

Choose $\epsilon > 0$ so small that $\alpha_{s+1} < p_{s+1}(a_1, \dots, a_{m+s}) - a_{m+s+1} < \beta_{s+1} - \epsilon$, $s = 0, 1, \dots, n-1$. By Lemma 2.1 there exists $\alpha \in \mathcal{F}^{(1)}$ such that $\big\langle u_i(\alpha)\big\rangle < \epsilon$, $2 \le i \le n+1$. Let $(x_1, \dots, x_{m+n}) = v_{n+1}(\alpha)$. Then for $k = 1, 2, \dots, n+1$,

$$\begin{aligned}
p_k(x_1, \dots x_{m+k-1}) - x_{m+k} =& p_k\big(v_k(\alpha)\big) - x_{m+k}\\
=& u_{k+1}(\alpha) + p_k(a_1, \dots, a_{m+k-1}) - [u_{k+1}(\alpha) + \frac{1}{2}] - a_{m+k}\\
\sim& p_k(a_1, \dots, a_{m+k-1}) - a_{m+k},
\end{aligned}$$

where we write $t \sim s$ for $|t - s| < \epsilon$. This is enough to show that $(x_1, \dots, x_{m+n})$ is a solution. Moreover, $(x_1, \dots, x_m) = x(\alpha) + (a_1, \dots, a_m)$, and $\big(x(\alpha)\big)_{\alpha\in\mathcal{F}}$ was an arbitrary VIP system. $\square$

Given $l \in \mathbf{N}$, let $\mathcal{G}_l$ be the smallest set of mappings $\mathbf{Z}^l \to \mathbf{Z}$ that includes, for $1 \le i \le l$, $(n_1, \dots, n_l) \to n_i$, is closed under sums and products, and has the property that for all $m \in \mathbf{N}$, $c_1, \dots, c_m \in \mathbf{R}$, $p_1, \dots, p_m \in \mathcal{G}_l$ and $0 < k < 1$, the mapping $n \to [\sum_{i=1}^m c_i p_i(n) + k]$ is in $\mathcal{G}_l$.

Members of $\mathcal{G}_l$ will be called *admissible generalized polynomials,* and any map $p : \mathbf{Z}^l \to \mathbf{Z}^t$ will be called an admissible generalized polynomial if its coordinate functions are such. Any map that differs by a constant from an admissible generalized polynomial will be called a *shifted admissible generalized polynomial.* Polynomial mappings $p : \mathbf{Z}^l \to \mathbf{Z}$ are shifted admissible generalized polynomials, and they are admissible if and only if $p(0) = 0$. There are of course many other examples, such as

$$p(n_1, n_2) = [\sqrt{3}[\sqrt{2}n_1^2 n_2 + \frac{\pi}{4}]n_2^5 + \sqrt{17}n_1^3 + \frac{1}{2}][\sqrt{5}n_2 + .17].$$

The following more general version of Lemma 2.4 may easily be obtained from Lemma 2.3.

Lemma 2.5. (a) Let $p(x)$ be an admissible generalized polynomial mapping $\mathbf{R}^l \to \mathbf{R}^t$ and suppose $\big(x(\alpha)\big)_{\alpha\in\mathcal{F}}$ is a VIP system in $\mathbf{R}^l$. Then there exists an IP-ring $\mathcal{F}^{(1)}$ such that $\big(p(x(\alpha))\big)_{\alpha\in\mathcal{F}^{(1)}}$ is a VIP system in $\mathbf{R}^t$. In particular, if $p(x)$ is a shifted admissible generalized polynomial mapping $\mathbf{R}^l \to \mathbf{R}^t$ and $c \in \mathbf{R}^l$ is constant then there exists an IP-ring $\mathcal{F}^{(1)}$ such that $\big(p(x(\alpha)+c) - p(c)\big)_{\alpha\in\mathcal{F}^{(1)}}$ is a VIP system in $\mathbf{R}^t$.

We remark that by using Lemma 2.5 in place of Lemmas 2.3 (c) and 2.4, the following more general form of Theorem A can be obtained by the same method.

Theorem 2.6. Let $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n$ be real numbers and let $m \in \mathbf{N}$. For each $k = 1, \ldots, n$, let p_k be a shifted admissible generalized polynomial of $m+k-1$ unknowns. If the system

$$\begin{aligned}
&\alpha_1 < p_1(x_1, \ldots, x_m) - x_{m+1} < \beta_1\\
&\alpha_2 < p_2(x_1, \ldots, x_m, x_{m+1}) - x_{m+2} < \beta_2\\
&\alpha_3 < p_3(x_1, \ldots, x_m, x_{m+1}, x_{m+2}) - x_{m+3} < \beta_3\\
&\vdots\\
&\alpha_n < p_n(x_1, \ldots, x_m, x_{m+1}, \ldots, x_{m+n-1}) - x_{m+n} < \beta_n
\end{aligned}$$

has an integer valued solution $(a_1, \ldots, a_{m+n})$ then the following set is VIP*:

$$\big\{(s_1, \ldots, s_m) : \text{ there exists a solution } (s_1 + a_1, s_2 + a_2, \ldots, s_m + a_m, s_{m+1}, \ldots, s_{m+n})\big\}.$$

3. Appendix: Van der Corput's proof.

The purpose of this final section is to give a sketch of van der Corput's original proof of Theorem VdC, which we presently restate (in its original formulation) as Theorem 3.1.

Theorem 3.1 ([VdC, Satz 1, p. 253]). Let $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n$ be real numbers and let $m \in \mathbf{N}$. For each $k = 1, \ldots, n$, let f_k be a polynomial of $m+k-1$ unknowns. If the system

$$\begin{aligned}
&\alpha_1 < f_1(x_1, \ldots, x_m) - y_1 < \beta_1\\
&\alpha_2 < f_2(x_1, \ldots, x_m, y_1) - y_2 < \beta_2\\
&\alpha_3 < f_3(x_1, \ldots, x_m, y_1, y_2) - y_3 < \beta_3\\
&\alpha_n < f_n(x_1, \ldots, x_m, y_1, \ldots, y_{n-1}) - y_n < \beta_n
\end{aligned} \tag{3.1}$$

has at least one integer valued solution then it has infinitely many integer-valued solutions. Furthermore, there exists L so that in any m-dimensional cube K_L which has edges of length L and parallel to the coordinate axes, there is at least one point $x = (x_1, \ldots, x_m) \in K_L$, $x_i \in \mathbf{Z}$, for which there is some $y = (y_1, \ldots, y_n)$, $y_i \in \mathbf{Z}$, so that (x, y) satisfies the system (3.1).

Van der Corput's proof proceeds by way of establishing that the system (3.1) has the same solutions as a system involving functions (generalized polynomials, in fact) that are members of a class he calls *rythmic* (see Definition 3.3 below). He then establishes a general result of this type for rythmic systems (see Theorem 3.4 below).

Definition 3.2 ([VdC, p. 229]). Let (f_ν) be a system of real-valued functions defined on $\mathbf{Z}^m$, let $\varepsilon > 0$ and suppose $x = (x_1, \ldots, x_m) \in \mathbf{Z}^m$. An element $\tau = (\tau_1, \ldots, \tau_m) \in \mathbf{Z}^m$ is called a *translation vector* (Verschiebungspunkt) for (f_ν) corresponding to ε, x, written $\tau = \tau(\varepsilon, x, (f_\nu))$ if for each $h = (h_1, \ldots, h_m) \in \mathbf{Z}^m$ with $|h_\mu| \leq \frac{1}{\varepsilon}$, $\mu = 1, 2, \ldots, m$), the n inequalities

$$-\varepsilon < f_\nu(x + \tau + h) - f_\nu(x + h) < \varepsilon \ (\mathrm{mod}\ 1), \tag{3.2}$$

$\nu = 1, 2, \ldots, n$, are satisfied.

Definition 3.3. A system (f_ν) of real valued functions on $\mathbf{Z}^m$, $\nu = 1, 2, \ldots, n$, is called *rythmic* (rhytmisch) if for each $\varepsilon > 0$ there exists $L = L(\varepsilon, (f_\nu))$ such that for any $x \in \mathbf{Z}^m$ there exists a translation vector $\tau = \tau(\varepsilon, x, (f_\nu))$ in any m-dimensional cube with edges of length L.

Note that in Definition 3.3, L is not allowed to depend on x. This requirement is actually illusory, as van der Corput proves in [VdC, Satz 6, p. 233] that if for every $\varepsilon > 0$ and every $x \in Z^m$ there is some L having the property that a translation vector $\tau(\varepsilon, x, (f_n u))$ may be found in every cube having sides of length L, then in fact some L may be shown to work for all x, meaning that (f_ν) is a rythmic system.

Notice that it follows from Definition 3.3 that any subsystem of a rythmic system is again rythmic. If the system consisting of f alone is rythmic, f is called a *rythmic function.*

Examples.

1. Polynomials. It is easy to see that the constant functions and linear polynomials are rythmic. In fact, by repeated use of Theorem 3.7 below it follows that any set of polynomials is a system of rythmic functions. As an example, however, let us now see directly from Definition 3.3 that the function $f(x) = \sqrt{2}x^2$ is rythmic.

Let $x \in \mathbf{Z}$ and $\varepsilon > 0$ be given. Then

$$f(x + h + \tau) - f(x + h) = 2\sqrt{2}(x + h)\tau + \sqrt{2}\tau^2.$$

Since the double sequence $(\sqrt{2}n, \sqrt{2}n^2)$, $n = 1, 2, \ldots$, is well distributed mod 1, there exists a syndetic set of τ satisfying $\{2\sqrt{2}\tau\} < \frac{\varepsilon}{2(x+\frac{1}{\varepsilon})}$ and $\{\sqrt{2}\tau^2\} < \varepsilon/2$. For such τ and for any $h \in \mathbf{Z}$ with $|h| < 1/\varepsilon$ we have

$$-\varepsilon/2 = -\frac{\varepsilon(x + 1/\varepsilon)}{2(x + 1/\varepsilon)} + 0 < 2\sqrt{2}\tau(x + h) + \sqrt{2}\tau^2 < \frac{\varepsilon(x + 1/\varepsilon)}{2(x + 1/\varepsilon)} + \varepsilon/2 = \varepsilon,$$

which shows that $f(x)$ is rythmic.

2. Generalized polynomials. We show here that $f(x) = \frac{1}{2}[\sqrt{2}x]$ is rythmic. Let x and $\varepsilon > 0$ be given. Let $a = \max\{\{\sqrt{2}(x+h)\} \mid h \in \mathbf{Z}, |h| < 1/\varepsilon\}$ and let $\tau \in \mathbf{N}$ with $0 < \{\frac{\sqrt{2}}{2}\tau\} < \frac{1}{2}(1-a)$, which implies $\{\sqrt{2}\tau\} < 1-a < 1-\{\sqrt{2}(x+h)\}$ for all $h \in \mathbf{Z}$ with $|h| < 1/\varepsilon$ and that $[\sqrt{2}\tau] = 2[\frac{\sqrt{2}}{2}\tau]$. Hence,

$$\frac{1}{2}[\sqrt{2}(x+h+\tau)] - \frac{1}{2}[\sqrt{2}(x+h)] = \frac{1}{2}[\sqrt{2}\tau] = [\frac{\sqrt{2}}{2}\tau] \equiv 0 \pmod 1,$$

which shows that $f(x)$ is rythmic. In the same way one can show that $g(x) = \frac{1}{2}[-\sqrt{2}x]$ is rythmic. Note however, that (f,g) is not a rythmic system. For let $x = h = 0$. Then $f(x+h+\tau) - f(x+h) = \frac{1}{2}[\sqrt{2}\tau]$ and $g(x+h+\tau) - g(x+h) = \frac{1}{2}[-\sqrt{2}\tau] = -\frac{1}{2}([\sqrt{2}\tau]+1)$ so that both cannot be zero for the same τ. This problem at zero disappears if we instead look at, for example, $f_1(x) = \frac{1}{2}[\sqrt{2}x + \frac{1}{2}]$ and $g_1(x) = \frac{1}{2}[-\sqrt{2}x - \frac{1}{2}]$. It turns out that (f_1, g_1) is a rythmic system.

As van der Corput remarks ([VdC, p. 215-216]), the definition of rythmicity is similar to that of Bohr almost periodicity. A function $f(x)$ is *(Bohr) almost periodic* if for any $\varepsilon > 0$ there exists a syndetic set of τ such that

$$|f(x+\tau) - f(x)| < \varepsilon \tag{3.3}$$

for all x. Nevertheless, there is an important distinction between these two notions: for rythmic functions, the translation vector τ depends on x, while for almost periodic functions, τ is independent of x. Van der Corput calls a function $f(x)$ *almost periodic mod 1* if for every ε there is a syndetic set T such that for all $\tau \in T$ one has $-\varepsilon < f(x+\tau) - f(x) < \varepsilon \pmod 1$ for all x. If a system (f_ν) consists of almost periodic mod 1 functions then for a syndetic set of τ, (3.2) would hold for *any* $h \in \mathbf{Z}^m$, not just for h with $|h_\nu| < \frac{1}{\varepsilon}$. The reader may wish to check that our first example $f(x) = \sqrt{2}x^2$ is not almost periodic mod 1.

The importance of rythmic systems for the theory of diophantine inequalities is shown by the following theorem.

Theorem 3.4 ([VdC, Satz 1, p. 230]). If (f_ν) is a rythmic system and if the system of inequalities

$$\alpha_\nu < f_\nu(x) < \beta_\nu \pmod 1, \qquad \nu = 1, 2, \ldots, n \tag{3.4}$$

has at least one integer solution $\overline{x} = (\overline{x}_1, \ldots, \overline{x}_m)$, then there exists $L = L((\alpha_\nu), (\beta_\nu), (f_\nu))$ such that any m-dimensional cube which has edges of length L and is parallel to the coordinate axes contains an integer solution.

Proof. By assumption $\alpha_\nu < f_\nu(\overline{x}) < \beta_\nu$, $\nu = 1, 2, \ldots, n$, hence there exists $\varepsilon > 0$ with $\alpha_\nu + \varepsilon < f_\nu(x) < \beta_\nu - \varepsilon$, $\nu = 1, 2, \ldots, n$. Since (f_ν), is rythmic, there exists $L = L(\varepsilon, (f_\nu))$ such that any cube with edge L contains $x \in \mathbf{Z}^m$ with $-\varepsilon < f_\nu(x) - f_\nu(\overline{x}) < \varepsilon \pmod 1$, $\nu = 1, 2, \ldots, n$. Hence, $\alpha_\nu < f_\nu(x) < \beta_\nu \pmod 1$, $\nu = 1, 2, \ldots, n$. □

Example. Van der Corput illustrates how Theorem 3.1 may be obtained as a consequence of Theorem 3.4 by considering the following system ([VdC, p. 224]):

$$\varepsilon < \sqrt{2}x^2 - y < \varepsilon, \qquad \varepsilon < \sqrt{3}y^2 - z < \varepsilon. \tag{3.5}$$

Without loss of generality $\varepsilon < \frac{1}{2}$, so that

$$0 < \frac{1}{2} - \varepsilon < (\sqrt{2}x^2 + \frac{1}{2}) - y < \frac{1}{2} + \varepsilon < 1$$

and therefore $y = [\sqrt{2}x^2 + \frac{1}{2}]$. It follows that (3.5) is equivalent to the system

$$-\varepsilon < \sqrt{2}x^2 < \varepsilon \pmod 1, \qquad -\varepsilon < \sqrt{3}[\sqrt{2}x^2 + \frac{1}{2}]^2 < \varepsilon \pmod 1. \tag{3.6}$$

Since it turns out that $(\sqrt{2}x^2, \sqrt{3}[\sqrt{2}x^2 + \frac{1}{2}]^2)$ is a rythmic system (see Theorem 3.12 at the end of this section) and (3.6) has a solution, namely $x = 0$, it has infinitely many solutions by Theorem 3.4.

Van der Corput points out that rythmic systems are especially well suited for obtaining results in the theory of diophantine inequalities, in part because they are invariant under the following operations ([VdC, p. 212-214]).

(i) Addition. If (f, g) is a rythmic system, then $(f, g, f + g)$ is also a rythmic system ([VdC, Satz 4, p. 231]).

(ii) Composition with continuous functions. If (f_ν) is rythmic and ψ is periodic mod 1 and continuous then $(f_\nu, \psi(f_1, \ldots, f_n))$ is rythmic; see Theorem 3.10 below. (In particular (cf_ν) is rythmic for any $c \in \mathbf{Z}$, however note that this does *not* imply the same conclusion for non-integer constants c.)

(iii) Formation of "summation"-functions, namely passing to $f(x)$ from the difference $\Delta f(x) = f(x+1) - f(x)$. Van der Corput proves that (f_ν) is rythmic if and only if (Δf_ν) is rythmic ([VdC, Satz 9, p. 247]; see also Theorem 3.7 below).

These facts are important tools for showing that, for example, the generalized polynomial $\sqrt{3}[\sqrt{2}x^2 + \frac{1}{2}]^2$ mentioned above is rythmic.

Note that functions (of the discrete variable $x \in \mathbf{Z}^m$) that are uniformly distributed modulo 1 are not necessarily invariant under any of these operations, and that almost periodic functions are not invariant under the third operation. For example, $f(x) = \sqrt{2}x^2$ is not almost periodic mod 1 but $\Delta f(x)$ is.

In order to show that a certain system of generalized polynomials coming from the polynomials in (3.1) is rythmic, van der Corput constructs a ring $\mathcal{R}$ of functions (see Definition 3.11 below) which contains these generalized polynomials and such that any finite subset of $\mathcal{R}$ is a rythmic system. Note that the set of rythmic functions itself does not form a ring–indeed, is not closed under addition. (The union of rythmic systems need not be rythmic in general; hence (i) above need not apply to rythmic f, g when (f, g) is not a rythmic system. For let $f(x) = \frac{1}{2}[\sqrt{2}x]$ and $g(x) = \frac{1}{2}[-\sqrt{2}x]$. Then $f(x)$ and $g(x)$ are rythmic but their sum $f(x) + g(x)$, which is 0 at $x = 0$ and $-\frac{1}{2}$ otherwise, is not.)

In light of the limitations of the class of rythmic functions, van der Corput considers the following subclass.

Definition 3.5 ([VdC, p. 248]). A function $f(x) = f(x_1, \ldots, x_m)$ is called *absolutely rythmic* (absolut rhytmisch) if for any rythmic system (g_ρ) containing r functions

$$g_\rho(x) = g_\rho(x_1, \ldots, x_m), \qquad \rho = 1, 2, \ldots, r,$$

the system (g_ρ, f) containing the $r+1$ functions $g_1(x), \ldots, g_r(x), f(x)$, is rythmic.

It follows from Theorem 3.12 below that polynomials are absolutely rythmic.

The difference operator Δ_μ, which we now define, plays a crucial role in building up the ring $\mathcal{R}$.

Definition 3.6 ([VdC, p. 237]). Let $f(x)$ be a function on $\mathbf{Z}^m$. Put

$$\Delta_\mu f(x) = f(x_1, \ldots, x_{\mu-1}, x_\mu + 1, x_{\mu+1}, \ldots, x_m) - f(x_1, \ldots, x_m), \qquad \mu = 1, 2, \ldots, m.$$

Notation: If (f_ν), $\nu = 1, 2, \ldots, n$ is a system of functions on $\mathbf{Z}^m$ then $(\Delta_\mu f_\nu)$ is the system consisting of the mn functions $\Delta_\mu f_\nu$, $\mu = 1, 2, \ldots, m$, $\nu = 1, 2, \ldots n$.

Theorem 3.7 ([VdC, Satz 9, p. 247 and Satz 7, p. 251]).
(a) If (f_ν) is a rythmic system then $(f_\nu, \Delta_\mu f_\nu)$ is a rythmic system.
(b) If $(\Delta_\mu f_\nu)$ is a rythmic system then (f_ν) is rythmic.
(c) A function f is absolutely rythmic if and only if the functions $\Delta_\mu f(x)$, $\mu = 1, 2, \ldots, m$, are absolutely rythmic.

Van der Corput's proof of (a) is straightforward and quite short. For part (b), he employs a number of preliminary lemmas. The set $M_x((f_\nu))$ defined presently plays an important role.

Definition 3.8 ([VdC, Definition 4, p. 236]). For fixed $x \in \mathbf{Z}^m$ let $M_x((f_\nu))$ be the set of all $u \in \mathbf{R}^n$ having the property that for every $\varepsilon > 0$ there exists $\tau \in \mathbf{Z}^m$ such that for each $h \in \mathbf{Z}^m$ with $|h_\nu| \leq \frac{1}{\varepsilon}$, $\nu = 1, 2, \ldots, m$, one has

$$-\varepsilon < f_\nu(x + \tau + h) - f_\nu(x + h) - u_\nu < \varepsilon \pmod 1, \qquad \nu = 1, 2, \ldots, n.$$

Van der Corput proves that $M_x((f_\nu))$ is closed, periodic mod 1 and is a module, i.e. is non-empty and has the property that when $u, v \in M_x((f_\nu))$ then $\{u+v, u-v\} \subset M_x((f_\nu))$. He then establishes the following.

Lemma 3.9 ([VdC, Hilfssatz 7, p. 244]). Let (Δf_ν) be rythmic. For each $\varepsilon > 0$ and each $x \in \mathbf{Z}^m$ there exists $\Lambda = \Lambda(\varepsilon, x)$ having the property that for every $u \in M_x((f_\nu))$ there exists a lattice point τ in any cube with edges of length Λ such that for each $h \in \mathbf{Z}^m$ with $|h_\mu| \leq \frac{1}{\varepsilon}$, $\mu = 1, 2, \ldots, m$,

$$-\varepsilon < f_\nu(x + \tau + h) - f_\nu(x + h) - u_\nu < \varepsilon \pmod 1, \qquad \nu = 1, 2, \ldots, n.$$

To complete the proof of Theorem 3.7 (b), suppose (Δf_ν) is rythmic. Since $M_x((f_\nu))$ is a module, it contains 0, hence we can let $u = 0$ in Lemma 3.9. It now follows from [VdC, Satz 6, p. 233] (see the discussion following Definition 3.3 above) that (f_ν) is rythmic.

Finally, in order to prove (c), let (g_ρ) be an arbitrary rythmic system. If f is absolutely rythmic then (g_ρ, f) is rythmic, so that by (a) $(g_\rho, f, \Delta_\mu g_\rho, \Delta_\mu f)$ is rythmic. In particular the subsystem $(g_\rho, \Delta_\mu f)$ is rythmic. Since (g_ρ) was arbirary, $(\Delta_\mu f)$ is absolutely rythmic.

Conversely, suppose $(\Delta_\mu f)$ is absolutely rythmic. By (a), $(\Delta_\mu g_\rho)$ is rythmic, hence $\Delta_\mu f, \Delta_\mu g_\rho)$ is rythmic and so by part (b), (f, g_ρ) is rythmic. Again, since (g_ρ) is arbitrary, f is absolutely rythmic. $\square$

If $f(x)$ is an absolutely rythmic function and $f(x) \notin \mathbf{Z}$ for all $x \in \mathbf{Z}^m$, by the following theorem $[f(x)]$ is absolutely rythmic.

Theorem 3.10 ([VdC, Satz 5 and remark, p. 250]). Suppose that the functions $f_\nu(x) = f_\nu(x_1, \ldots, x_m)$, $\nu = 1, \ldots, n$, are absolutely rythmic and $\psi(v_1, \ldots, v_m)$ is a periodic mod 1 function which is continuous at all the points $(f_1(x), \ldots, f_n(x))$, $x \in \mathbf{Z}^m$. Then $\psi(f_1(x), \ldots, f_n(x))$ is absolutely rythmic.

Note that the set of absolutely rythmic functions is not closed under multiplication. For let $f(x) \equiv \frac{1}{2}$ and $\phi(0) = 1$ and $\phi(x) = 0$, $x \neq 0$. Then $f(x)$ and $\phi(x)$ are absolutely rythmic but $f(x)\phi(x)$ is not rythmic, hence not absolutely rythmic. However, van der Corput demonstrates how to construct a ring $\mathcal{R}$ *consisting* of absolutely rythmic functions.

Definition 3.11 ([VdC, p. 254]). Let $\mathcal{R}$ be the smallest space of real-valued functions on $\mathbf{Z}^m$ having the following properties:

1. $\mathcal{R}$ contains the zero function, $f(x) \equiv 0$.
2. If $f, \phi \in \mathcal{R}$ then $f + \phi \in \mathcal{R}$.
3. If $g(x)$ is integer-valued and bounded on $\mathbf{Z}^m$ and $cg(x)$ is absolutely rythmic for any constant c, then $f(x)g(x) \in \mathcal{R}$ for any $f(x) \in \mathcal{R}$.
4. If $f(x)$ is a function so that the m differences $\Delta_\mu f(x)$, $\mu = 1, 2, \ldots, m$ are in $\mathcal{R}$, then $f(x) \in \mathcal{R}$.

Then we have:

Theorem 3.12 ([VdC, p. 255]). $\mathcal{R}$ has the following 4 properties:

1. $\mathcal{R}$ is a ring.
2. Any function in $\mathcal{R}$ is absolutely rythmic.
3. $\mathcal{R}$ contains all polynomials in m variables.
4. If $f(x) \in \mathcal{R}$ such that for all $x \in \mathbf{Z}^m$, $f(x) \notin \mathbf{Z}$, then $[f(x)] \in \mathcal{R}$.

The proof uses the following property of $\mathcal{R}$. For each $f(x) \in \mathcal{R}$, $f \not\equiv 0$, one can find functions $F_1(x), \ldots, F_l(x)$ such that $F_1 \equiv 0$ and so that, putting $F_{l+1}(x) = f(x)$, for each λ, $\lambda = 1, 2, \ldots, l$, at least one of the following three properties holds:

1. There are two functions $g(x)$ and $h(x)$ from the sequence $F_1, \ldots, F_\lambda$ so that $F_{\lambda+1}(x) = g(x) + h(x)$.
2. There exists a function $h(x)$ in the sequence $F_1, \ldots, F_\lambda$ and an integer valued, bounded function $g(x)$, $g(x)$ having the property that $cg(x)$ is absolutely rythmic for all $c \in \mathbf{R}$, with $F_{\lambda+1}(x) = h(x)g(x)$.

3. For all $\mu = 1, 2, \ldots, m$, $\Delta_\mu F_{\lambda+1}(x)$ appears in the sequence $F_1, \ldots, F_\lambda$.

The smallest such l is called the *index* of $f(x)$. To establish properties of $\mathcal{R}$ one can then use induction on the index.

$\mathcal{R}$ contains all generalized polynomials $f(x)$ which have the following property: If $[g(x)]$ appears in the representation of $f(x)$ then for all $x \in \mathbf{Z}^m$, $g(x) \notin \mathbf{Z}$. Since all finite systems composed of function in $\mathcal{R}$ are rythmic, Theorem 3.4 applies to these and Theorem 3.1 follows via the general scheme sketched earlier in the text.

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