AFFINE ACTIONS OF A FREE SEMIGROUP ON THE REAL LINE

VITALY BERGELSON, MICHAŁ MISIUREWICZ, AND SAMUEL SENTI

ABSTRACT. We consider actions of the free semigroup with two generators on the real line, where the generators act as affine maps, one contracting and one expanding, with distinct fixed points. Then every orbit is dense in a half-line, which leads to the question whether it is, in some sense, uniformly distributed. We present answers to this question for various interpretations of the phrase "uniformly distributed".

1. INTRODUCTION

Denote the set of all one-sided 0-1 sequences by $\Sigma = \{0,1\}^{\mathbb{N}_0}$, where $\mathbb{N}_0 = \{0,1,2,\dots\}$, and let Σ_n be the set of all 0-1 sequences of length n (Σ_0 consists of one empty sequence). Set

$$G = \bigcup_{n \in \mathbb{N}_0} \Sigma_n.$$

When the semigroup multiplication is the concatenation, then G is a free semigroup with two generators.

Suppose that for some space Y two maps, $T_0, T_1 : Y \to Y$, are given. For $\omega = (\omega_0, \omega_1, \dots, \omega_{n-1}) \in \Sigma_n$ write

$$T_\omega = T_{\omega_{n-1}} \circ \cdots \circ T_{\omega_1} \circ T_{\omega_0}.$$

Then G acts on Y by $\omega \mapsto T_\omega$.

Our starting point is the following result, equivalent to Corollary 3.2 of [MR].

Theorem A. *Let $T_0(x) = x/2$ and $T_1(x) = (3x+1)/2$, $x \in \mathbb{R}_+ = [0, \infty)$. Then for any $x \in \mathbb{R}_+$ the orbit $\{T_\omega(x)\}_{\omega \in G}$ is dense in $\mathbb{R}_+$.*

We now consider a somewhat more general setup, where we have one expanding and one contracting affine transformation of $\mathbb{R}_+$ with distinct fixed points. Any pair of such transformations may be brought into the form $T_0(x) = ax$ and $T_1(x) = bx+1$, where $0 < a < 1 < b$, by conjugating via an affine map. Therefore we will assume from now on that T_0 and T_1 have such form.

Both Theorem 3.1 and Theorem 3.4 of [MR] imply Theorem A. However, the proof of Theorem 3.1 of [MR] cannot be generalized for all T_0 and T_1 as defined in the preceding paragraph, since it requires rational independence of $\log a$ and $\log b$.

Fortunately, one can repeat almost verbatim the proofs of Lemma 3.3 and Theorem 3.4 of [MR], which gives the following result.

Theorem B. *Let $T_0(x) = ax$ and $T_1(x) = bx + 1$, $x \in \mathbb{R}_+ = [0, \infty)$, where $0 < a < 1 < b$. Then for any $x \in \mathbb{R}_+$ the orbit $\{T_\omega(x)\}_{\omega \in G}$ is dense in $\mathbb{R}_+$.*

Theorem B naturally leads to the following question.

Question. *Is it true that under the assumptions of Theorem B every orbit is uniformly distributed in $\mathbb{R}_+$?*

One of the problems with this Question is, of course, the vagueness of the phrase "uniformly distributed". Indeed, while in the classical theory of the uniform distribution of sequences modulo one (see, e.g., [KN]) the sequences are conveniently indexed by the ordered set $\mathbb{N} = \{1, 2, 3, \dots\}$ and belong to the "nice" compact space $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, none of these ingredients is present in our situation.

As we shall see, there are various natural approaches to interpreting and answering our Question, each revealing an interesting facet of the situation at hand.

One possible approach would be to inquire whether one can order the elements of our semigroup G in some way, say $(g_i)_{i \in \mathbb{N}}$, so that for some probability measure[1] μ on $\overline{\mathbb{R}}_+ = [0, \infty]$, one has

$$(1.1) \qquad \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} f(T_{g_i}(x)) = \int_{\overline{\mathbb{R}}_+} f \, d\mu$$

for every $x \in \mathbb{R}_+$ and every $f \in C(\overline{\mathbb{R}}_+)$, where $C(\overline{\mathbb{R}}_+)$ is the space of all continuous real functions on $\overline{\mathbb{R}}_+$. While a priori it is not clear at all whether one can find such ordering of G and such measure μ, the following theorem shows that, actually, a much stronger result holds.

Theorem C. *For any probability measure on $\overline{\mathbb{R}}_+$ there exists a sequence $(g_i)_{i \in \mathbb{N}}$ of elements of G (which can be chosen in such a way that it exhausts G), such that (1.1) holds for any $x \in \mathbb{R}_+$ and any $f \in C(\overline{\mathbb{R}}_+)$.*

We will now describe another approach, which is, in our opinion, much more natural, since it takes into account the structure of the acting semigroup.

By extending T_0 and T_1 to continuous maps of the one-point compactification $\overline{\mathbb{R}}_+$ of $\mathbb{R}_+$ by the rule $T_0(\infty) = T_1(\infty) = \infty$ (and keeping the notation T_0, T_1 for these extensions), we obtain an action of G on a compact space $\overline{\mathbb{R}}_+$. Let $\mathcal{M}$ (respectively $\overline{\mathcal{M}}$) denote the space of all probability measures on $\mathbb{R}_+$ (respectively $\overline{\mathbb{R}}_+$).

A possible approach to answering our Question is to look for a measure $\mu \in \overline{\mathcal{M}}$ such that the natural (from the point of view of ergodic theory) averages

$$\frac{1}{|\Sigma_n|} \sum_{\omega \in \Sigma_n} f(T_\omega(x))$$

converge to $\int_{\overline{\mathbb{R}}_+} f \, d\mu$ for every $f \in C(\overline{\mathbb{R}}_+)$ and every $x \in \mathbb{R}_+$.

A priori, it is not clear whether such measure μ exists. A possible candidate for such μ would be a measure that is g-invariant for any $g \in G$. However, it is easy to see

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[1] By a probability measure we will always understand a Borel probability measure.

that there are no such nontrivial measures in our situation. Indeed, since $T_1(x) > x$ for every $x \in \mathbb{R}_+$, already the only T_1-invariant measure in $\overline{\mathcal{M}}$ is δ_∞. It therefore comes as a pleasant surprise that (under an additional assumption on a and b) such a measure exists and can be described quite explicitly.

Theorem D. *If, in addition to our standard assumption that $0 < a < 1 < b$, the parameters a, b satisfy $ab < 1$, then there exists $\mu \in \mathcal{M}$ such that for any $f \in C(\overline{\mathbb{R}}_+)$ and any $x \in \mathbb{R}_+$ one has*

$$(1.2) \qquad \lim_{n \to \infty} \frac{1}{|\Sigma_n|} \sum_{\omega \in \Sigma_n} f(T_\omega(x)) = \int_{\mathbb{R}_+} f \, d\mu.$$

It turns out that when $ab \geq 1$ then the averages in (1.2) converge to $f(\infty)$. However, as we will see, a natural *weighted* version of Theorem D holds under much more general assumptions. In order to give a precise formulation of this result, we have to introduce additional definitions and notation.

Let $\sigma : \Sigma \to \Sigma$ be the full one-sided shift, i.e. $(\sigma(\bar{\omega}))_i = \omega_{i+1}$ for any $\bar{\omega} = (\omega_i)_{i \in \mathbb{N}_0} \in \Sigma$. Set $X = \Sigma \times \overline{\mathbb{R}}_+$ and denote by π_1 and π_2 the projections of X onto Σ and $\overline{\mathbb{R}}_+$ respectively. Let $\mathcal{N}$ denote the space of all probability measures on Σ. Define $\tau : X \to X$ as the skew product

$$\tau(\bar{\omega}, x) = (\sigma(\bar{\omega}), T_{\omega_0} x).$$

A similar approach to investigation of actions of free semigroups has been used for instance by Sumi [S].

Definition 1.1. *Let ν be a σ-invariant probability measure on Σ. We will say that a measure $\mu \in \mathcal{M}$ is* invariant for (T_0, T_1, ν) *if there exists a τ-invariant probability measure $\varkappa$ on X such that the projection $(\pi_1)_*(\varkappa)$ to $\mathcal{N}$ is ν and the projection $(\pi_2)_*(\varkappa)$ to $\overline{\mathcal{M}}$ is μ.*

Note that by choosing in this definition $\mu \in \mathcal{M}$ (rather than $\mu \in \overline{\mathcal{M}}$) we exclude the case $\mu = \delta_\infty$, although $\varkappa = \nu \times \delta_\infty$ is always τ-invariant and projects to ν and δ_∞.

It makes sense to speak of the Lyapunov exponent of ν, when we look at the derivative of τ in the direction in which it exists, that is, in $\overline{\mathbb{R}}_+$. The derivatives of T_0 and T_1 are constant and equal to a and b respectively. We thus define the *Lyapunov exponent* $\lambda(\nu)$ of an ergodic σ-invariant measure $\nu \in \mathcal{N}$ as the integral of the function which equals to $\ln a$ on C_0 and $\ln b$ on C_1, where $C_i = \{\omega \in \Sigma : \omega_0 = i\}$:

$$\lambda(\nu) = \nu(C_0) \ln a + \nu(C_1) \ln b.$$

Theorem E. *For any ergodic σ-invariant probability measure $\nu \in \mathcal{N}$ there exists a probability measure invariant for (T_0, T_1, ν) if and only if $\lambda(\nu) < 0$. If such a measure μ exists, then it is unique and for every $f \in C(\overline{\mathbb{R}}_+)$ and every $x \in \mathbb{R}_+$ one has*

$$\lim_{n \to \infty} \sum_{\omega \in \Sigma_n} \nu(C_\omega) f(T_\omega(x)) = \int_{\mathbb{R}_+} f \, d\mu,$$

where for $\omega = (\omega_0, \omega_1, \ldots, \omega_{n-1})$, $C_\omega = \{\bar{\omega} \in \Sigma : (\bar{\omega})_i = \omega_i, i = 0, 1, \ldots, n-1\}$.

The issue of what happens (from the point of view of our Question) when $\lambda(\nu) \geq 0$, remains not fully resolved. One can show that the weighted averages which appear in Theorem E converge to $f(\infty)$. This leads to a suspicion that there is an infinite measure on $\mathbb{R}_+$ which is "responsible" for the statistical behavior of G-orbits. We do not address this issue here.

We also have an interesting addition to Theorem E, which deals with, so to say, individual paths of points.

Theorem F. *Assume that $\nu \in \mathcal{N}$ is σ-invariant, ergodic and satisfies $\lambda(\nu) < 0$. Then for ν-almost every $\bar{\omega} = (\omega_0, \omega_1, \dots) \in \Sigma$ and for every $f \in C(\overline{\mathbb{R}}_+)$ and $x \in \mathbb{R}_+$ one has*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f(x_n) = \int_{\mathbb{R}_+} f \, d\mu,$$

where $x_n = T_{(\omega_0, \omega_1, \dots, \omega_{n-1})}(x)$ and μ is the unique measure invariant for (T_0, T_1, ν).

One more way of addressing our Question is to inquire whether for some σ-invariant ergodic measure $\nu \in \mathcal{N}$ the corresponding (T_0, T_1, ν)-invariant measure μ is absolutely continuous with respect to the Lebesgue measure. We prove the following result.

Theorem G. *For every $\gamma > 1$ there exists a σ-invariant ergodic probability measure $\nu_\gamma \in \mathcal{N}$ with negative Lyapunov exponent, for which the corresponding (T_0, T_1, ν_γ)-invariant measure μ_γ is absolutely continuous with respect to the Lebesgue measure and the support of μ_γ is the union of finitely many intervals whose convex hull is $[\frac{\gamma-1}{b}, \frac{\gamma}{a}]$. In particular, $\mu_\gamma \neq \mu_\delta$ for $\gamma \neq \delta$, $\gamma, \delta > 1$.*

The case when ν is $(p, 1-p)$ Bernoulli is of a special interest and needs a separate discussion. If we assume that $a^p b^{1-p} < 1$ then $\lambda(\nu) < 0$, and so the (T_0, T_1, ν)-invariant measure μ exists. If, in addition, G acts effectively (that is, $\omega \neq \omega'$ implies $T_\omega \neq T_{\omega'}$) and $a^p b^{1-p} > p^p(1-p)^{1-p}$, then the formal computation of the Hausdorff dimension of μ, which does not take into account the overlap of the images of $\mathbb{R}_+$ under T_0 and T_1, gives result larger than 1. In such a case it is widely conjectured that μ should be absolutely continuous with respect to the Lebesgue measure (see, e.g., [PSS]). Unfortunately, the existing techniques which could be used to prove this, at least for typical values of a and b (e.g. [SSU], [T], [R]), use strong assumptions which cannot be easily checked in our case. As a matter of fact, we believe that the above conjecture is false in our situation.

On the other hand, even without assuming that G acts effectively, we can prove the following interesting property of μ.

Theorem H. *Assume that ν is $(p, 1 - p)$-Bernoulli with $\lambda(\nu) < 0$. Then the (T_0, T_1, ν)-invariant probability measure μ on $\mathbb{R}_+$ is the image of the Lebesgue measure under an increasing map from $[0, 1)$ onto $\mathbb{R}_+$, which is Hölder continuous with the same exponent on each compact interval.*

A property of this type is ubiquitous in the theory of dynamical systems. For example, any conjugacy between diffeomorphisms on compact hyperbolic sets is Hölder continuous (Theorem 19.1.2 of [KH]). Therefore, the measure with maximal entropy

for any Anosov diffeomorphism on a torus is the image of the Lebesgue measure under a Hölder continuous homeomorphism of the torus.[2]

The paper is organized as follows. We deal with Theorem C in Section 2, with existence of (T_0, T_1, ν)-invariant measures in Section 3, with Theorems E and F in Section 4, with Theorem G in Section 5, and with Theorem H in Section 6. When stating the results in those sections, we use a slightly different language, which employs the notion of the weak-* convergence of measures.

2. If one disregards generators. . .

Let us consider the maps T_0, T_1 and the semigroup G defined in the introduction. It is a well known fact (see for instance Theorem 2.2 of [MR]) that if $a = 1/2$ and $b = 3/2$, then the action of G is effective. In fact, it is easy to see that the set of (a, b) for which this action is effective contains the complement of a union of countably many algebraic curves. Indeed, if the action of G is not effective, then there exist distinct words ω and ω' such that $T_\omega = T_{\omega'}$. In particular $T_\omega(0) = T_{\omega'}(0)$ and this is a polynomial equation in a, b.

Remark 2.1. *While rational independence of the logarithms of a and b seems to play a crucial role for effectiveness of the action of G, it is not sufficient for it. For instance if $a = 1/2$ and $b = 4/3$ then $T_{10001} = T_{01100}$, hence the action is not effective.*

The following theorem (together with Remark 2.5) is equivalent to Theorem C.

Theorem 2.2. *For any $\mu \in \mathcal{M}$ there exists a infinite sequence $(g_i)_{i=1}^\infty$ with $g_i \in G$ such that, for any point $x \in \mathbb{R}_+$ we have*

$$(2.1) \qquad \frac{1}{n} \sum_{i=1}^n \delta_{T_{g_i} x} \xrightarrow{\ *\ } \mu.$$

where $\xrightarrow{\ *\ }$ denotes the weak- convergence in $\overline{\mathbb{R}}_+$.*

For the proof of this theorem we will need to adapt Lemma 3.3 and Theorem 3.4 of [MR] to our more general setup with $T_0(x) = ax$ and $T_1(x) = bx + 1$, with $0 < a < 1 < b$. The proofs are the same as in [MR].

Lemma 2.3. *Let m be the number of 1's in $\omega = (\omega_0, \ldots, \omega_{n-1}) \in G$ of length n. Assume that $x \leq M$ and for every $k < n$ we have $T_{\omega_k} \circ \ldots \circ T_{\omega_0}(x) \leq M$. Then*

$$\frac{1}{a^{n-m}b^m} \geq (m-1)\frac{x}{bM^2}.$$

Theorem 2.4. *Given $x, y > 0$ and $\varepsilon > 0$, there is $\omega = (\omega_0, \ldots, \omega_{n-1}) \in G$ such that $|T_\omega(x) - y| < \varepsilon$. Moreover*

$$\min(x, y - \varepsilon) \leq T_{\omega_k} \circ \ldots \circ T_{\omega_0}(x) \leq K$$

for every $k < n$ where K is a constant depending on x and y.

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[2]The authors thank Anatole Katok for pointing this out.

Proof of Theorem 2.2. For any measure $\mu \in \mathcal{M}$ there exists a sequence $(a_i)_{i=1}^{\infty}$ of points $a_i \in \mathbb{R}_+$ such that the averages of the Dirac delta measures δ_{a_i} at a_i converge to μ:

$$\frac{1}{n} \sum_{i=1}^{n} \delta_{a_i} \xrightarrow{*} \mu.$$

Observe that $\overline{\{T_g(1)\}}_{g \in G} = \mathbb{R}_+$ by Theorem 2.4, and together with Lemma 2.3, this implies that for any i there exists $g_i \in G$ with

$$(2.2) \qquad |T_{g_i}(1) - a_i| < \frac{1}{i} \qquad \text{and} \qquad T'_{g_i} < \frac{1}{i}.$$

Thus, for any $x \in \mathbb{R}_+$ we have

$$|T_{g_i}(x) - a_i| \le |T_{g_i}(x) - T_{g_i}(1)| + |T_{g_i}(1) - a_i| \le \frac{1}{i}|x - 1| + \frac{1}{i} \to 0.$$

as $i \to \infty$. This implies (2.1). $\qquad \square$

Remark 2.5. *Observe that if $\frac{1}{n} \sum_{i=1}^{n} a_i \to a$ for some infinite sequence $(a_i)_{i=1}^{\infty}$ and $\tilde{a}_k$ is a sequence obtained from $(a_i)_{i=1}^{\infty}$ by inserting additional elements from a bounded sequence at sufficiently rarefied times, then $\frac{1}{n} \sum_{i=1}^{n} \tilde{a}_i \to a$. Therefore the sequence g_i in the statement of Theorem 2.2 can be chosen so as to exhaust the semigroup G.*

Note that the proof of Theorem 2.2 relies strongly on the contraction property (2.2) of the action of G on $\mathbb{R}_+$. The following example shows that without this property the situation may be different.

Example 2.6. *Let $T_r(x) = x + r$ and consider the semigroup H of rational translations $\{T_r(x), \ r \in \mathbb{Q}_+\}$ acting on $\overline{\mathbb{R}}_+$ (with $\infty + r = \infty$). For any probability measure μ concentrated on $\mathbb{R}_+$ and any infinite sequence of elements $h_i \in H$ there exists at most one $x \in \mathbb{R}_+$ for which*

$$(2.3) \qquad \frac{1}{n} \sum_{i=1}^{n} \delta_{T_{h_i}(x)} \xrightarrow{*} \mu.$$

Proof. Assume there exist $x < y \in \mathbb{R}_+$ such that

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \delta_{T_{h_i}(x)} = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \delta_{T_{h_i}(y)} = \mu.$$

Since $T_{h_i} y = T_{h_i} x + (y - x)$, the sequence (2.3) with x replaced by y converges to the measure μ shifted by $(y - x)$, so μ is invariant with respect to the shift by $(y - x)$. However, there exists an interval I of positive measure μ and length less than $(y - x)$. Then $\mu(I + k(y - x)) = \mu(I)$ for all $k \ge 0$, so $\mu(\mathbb{R}_+) = \infty$. This is a contradiction, as μ is a probability measure. $\qquad \square$

3. Existence and uniqueness of invariant measures

In Ergodic Theory one usually considers invariant measures. As we explained in the Introduction, requiring the measure to be invariant for each T_ω is too strong. Therefore, we adopt Definition 1.1, which takes into account the particular structure of G as a semigroup with two generators.

Note that in the theory of iterations of random maps (see, e.g., [KL]) one often encounters a definition of invariant measure similar to our Definition 1.1. This is also the situation in the theory of Iterated Function Systems (IFS; see, e.g., [BE], [DF], [NSB] or [PSS]), except that the maps T_{ω_i} are applied in the reverse order. Hence, instead of considering the measure ν, we should, in the IFS case, consider the "dual" measure. By a dual measure we mean the projection to the space of measures on the negative coordinates of the natural extension of ν as in Section 5. When ν is a Bernoulli measure, then it is naturally "isomorphic" to its dual.

Often, in the theory of random iterations and in IFS only Bernoulli measures ν are considered. While this approach can be easily justified, from a purely mathematical point of view all non trivial ergodic invariant probability measures ν are equally interesting (this approach is also adopted for instance in [PSS]). This motivates our Definition 1.1.

We want to show that measures invariant for (T_0, T_1, ν) are unique if they exist. We start by proving that our system is contracting in the sense revealed by the following lemma.

Lemma 3.1. *Fix $x, y \in \mathbb{R}_+$ and $\bar{\omega} = (\omega_0, \omega_1, \dots) \in \Sigma$. Set $x_0 = x$, $y_0 = y$ and by induction $x_{n+1} = T_{\omega_n}(x_n)$ and $y_{n+1} = T_{\omega_n}(y_n)$. Then*

$$(3.1) \qquad \lim_{n \to \infty} \left(\frac{1}{(1 + x_n)} - \frac{1}{(1 + y_n)} \right) = 0.$$

Proof. We consider another system smoothly conjugated to (T_0, T_1). This conjugacy gives us a new metric on the positive real half-line, which enables us to control the estimates related to the contracting properties of the system more efficiently.

The conjugacy is via the logarithmic function: instead of $x \in (0, \infty)$ we consider $t = \ln x \in \mathbb{R}$. Let T_0 and T_1 be conjugate to L_0 and L_1 respectively. Then $L_0(t) = \ln(ae^t) = t + \ln a$ and $L_1(t) = \ln(be^t + 1)$. We get $L_0'(t) = 1$ and $L_1'(t) = be^t/(be^t + 1) < 1$, so both maps are Lipschitz continuous with constant 1. This shows that $|\ln x_{n+1} - \ln y_{n+1}| \leq |\ln x_n - \ln y_n|$ for all n. In particular, if one of the sequences x_n, y_n converges to 0 or ∞, so does the other one.

Assume that x_n and y_n do not converge to zero or infinity. Then there is M such that for infinitely many n's we have $\ln x_n, \ln y_n \leq M$ and $\omega_n = 1$ (we use here the assumption that $a < 1$). For each such n we get

$$|\ln x_{n+1} - \ln y_{n+1}| \leq |\ln x_n - \ln y_n| \cdot \frac{be^M}{be^M + 1}.$$

Since $be^M/(be^M + 1) < 1$, this proves that $\lim_{n \to \infty} |\ln x_n - \ln y_n| = 0$. In order to get $1/(1 + x)$ from $\ln x$ we apply the map $t \mapsto 1/(1 + e^t)$. The derivative of this map is $-e^t/(1 + e^t)^2$. However, since $(1 + e^t)^2 - 4e^t = (1 - e^t)^2 \geq 0$, the absolute value of this derivative is not larger than $1/4$. Therefore $|1/(1 + x_n) - 1/(1 + y_n)| \leq \frac{1}{4}|\ln x_n - \ln y_n|$, so (3.1) holds.

If both x_n and y_n converge to 0 then $1/(1 + x_n)$ and $1/(1 + y_n)$ go to 1, so (3.1) holds. If both x_n and y_n converge to ∞ then $1/(1 + x_n)$ and $1/(1 + y_n)$ go to 0, so (3.1) also holds.

Since we started by taking logarithms of x and y, there remains the case when one of these numbers is 0. However, then either all ω_n's are 0, and both x_n and y_n go to

zero, or there is n such $\omega_n = 1$ and we can apply the main part of the proof to x_{n+1} and y_{n+1} instead of x and y. This completes the proof. □

Now we can prove the following theorem.

Theorem 3.2. *For any ergodic σ-invariant probability measure ν there is at most one τ-invariant probability measure $\varkappa$ on $X = \Sigma \times \overline{\mathbb{R}}_+$, other than $\nu \times \delta_\infty$, whose projection to $\mathcal{N}$ is ν. This measure $\varkappa$, if it exists, is ergodic.*

Proof. If such a measure $\varkappa$ exists and is not ergodic, each of its ergodic components is also τ-invariant and projects to ν. Therefore it is enough to prove that there is at most one ergodic measure $\varkappa$ satisfying the assertions of the theorem.

Suppose that there exist two such ergodic measures, $\varkappa_1$ and $\varkappa_2$. For $i = 1, 2$ and ν-almost every $\bar{\omega} \in \Sigma$, there are generic points of $\varkappa_i$ in $\{\bar{\omega}\} \times \overline{\mathbb{R}}_+$. Take such $\bar{\omega} \in \Sigma$ and $x_1, x_2 \in \mathbb{R}_+$ such that $z_i = (\bar{\omega}, x_i)$ is generic for $\varkappa_i$, $i = 1, 2$ (note that $x_i \neq \infty$, since $\varkappa_i \neq \nu \times \delta_\infty$).

The metric $d(x, y) = |1/(1 + x) - 1/(1 + y)|$ is compatible with the topology of the compactification $\overline{\mathbb{R}}_+$ of $\mathbb{R}_+$, so we may use it for measuring of the distance between two points of X having the same projection to Σ. By Lemma 3.1 we have $\lim_{n\to\infty} d(\tau^n(z_1), \tau^n(z_2)) = 0$ and thus $\lim_{n\to\infty} |\varphi(\tau^n(z_1)) - \varphi(\tau^n(z_2))| = 0$ for any continuous real function φ on X. Since z_i is generic for $\varkappa_i$ $(i = 1, 2)$, this implies that $\int \varphi \, d\varkappa_1 = \int \varphi \, d\varkappa_2$. Since this holds for all φ, we get $\varkappa_1 = \varkappa_2$. □

Corollary 3.3. *For any ergodic σ-invariant probability measure ν, there is at most one probability measure invariant for (T_0, T_1, ν).*

We now investigate when such a τ-invariant measure exists.

Theorem 3.4. *For any ergodic σ-invariant probability measure ν there exists a probability measure invariant for (T_0, T_1, ν) if and only if the Lyapunov exponent $\lambda(\nu)$ is negative. If such a measure μ exists, then it is unique and the sequence of measures*

$$(3.2) \qquad \varkappa_n = \frac{1}{n} \sum_{k=0}^{n-1} \delta_{\tau^k(z)},$$

where $z = (\bar{\omega}, x)$, converges for ν-almost every $\bar{\omega} \in \Sigma$ and every $x \in \mathbb{R}_+$ to a unique τ-invariant measure $\varkappa$ whose projection to $\mathcal{M}$ is μ.

Proof. Suppose first that the Lyapunov exponent $\lambda(\nu)$ of ν is negative, but there does not exist any probability measure invariant for (T_0, T_1, ν). Let $\bar{\omega} \in \Sigma$ be a generic point of ν, and set $z = (\bar{\omega}, 1) \in X$.

Suppose that the sequence of measures $(\varkappa_n)_{n\in\mathbb{N}}$, given by (3.2) with $x = 1$, does not converge to $\nu \times \delta_\infty$ (in the weak-* topology). Then there is a subsequence of the sequence $(\varkappa_n)_{n=1}^\infty$ which converges to some other measure $\varkappa$. By the standard argument (as in the standard proof of the Krylov-Bogolubov Theorem, see, e.g. [W], Theorem 6.9), the measure $\varkappa$ is τ-invariant. Since $\bar{\omega}$ is a generic point of ν, the projection of $\varkappa$ to $\mathcal{N}$ is equal to ν. Note that $\Sigma \times \{\infty\}$ is τ-invariant and $\nu \times \delta_\infty$ is the only invariant measure concentrated on this set and whose projection to $\mathcal{N}$ is ν. Therefore our measure $\varkappa$ has to be an affine combination of $\nu \times \delta_\infty$ and $\varkappa'$, where $\varkappa'$ is an τ-invariant probability measure concentrated on $\Sigma \times \mathbb{R}_+$ and its projection to $\mathcal{N}$ is ν. In this affine combination the coefficient of $\varkappa'$ is non-zero since we assumed

that $\varkappa \neq \nu \times \delta_\infty$. Hence the projection of $\varkappa'$ to $\overline{\mathcal{M}}$ is a probability measure invariant for (T_0, T_1, ν). This contradicts the assumption that such a measure does not exist, thus proving that the sequence $(\varkappa_n)_{n=1}^\infty$ must converge to $\nu \times \delta_\infty$.

Since the Lyapunov exponent of ν is negative, there exists $\eta > 0$ such that $\nu(C_0) \ln a + \nu(C_1) \ln(b + \eta) < 0$. Set $M = \max(1/a, 1/\eta) > 1$. For every positive integer n define

$$
\begin{aligned}
A_n &= \{k \in \{0, 1, \ldots, n-1\} : \pi_2(\tau^k(z)) \geq M \text{ and } \pi_1(\tau^k(z))_0 = 0\}, \\
B_n &= \{k \in \{0, 1, \ldots, n-1\} : \pi_2(\tau^k(z)) \geq M \text{ and } \pi_1(\tau^k(z))_0 = 1\}, \\
D_n &= \{k \in \{0, 1, \ldots, n-1\} : \pi_2(\tau^k(z)) < M\}.
\end{aligned}
$$

These sets form a partition of $\{0, 1, \ldots, n-1\}$. Now fix n and set $\alpha_k = \ln \pi_2(\tau^k(z))$ if $k \in A_n \cup B_n$ and $\alpha_k = 0$ if $k \in D_n$. Let us estimate α_{k+1} from above.

If $k \in A_n$ then either $\alpha_{k+1} = \alpha_k + \ln a$ or $\alpha_{k+1} = 0$. In the second case, since $\alpha_k \geq \ln M$ and $M \geq 1/a$, we get $\alpha_{k+1} \leq \alpha_k + \ln a$; clearly this also holds in the first case.

If $k \in B_n$ then, if $x = \pi_2(\tau^k(z))$, we have $\alpha_k = \ln x$ and $\alpha_{k+1} = \ln(bx + 1) = \ln x + \ln(b + 1/x)$. Since $x \geq M \geq 1/\eta$, we get $\alpha_{k+1}(x) \leq \alpha_k + \ln(b + \eta)$.

If $k \in D_n$ then, with the same notation, $x < M$, so $\alpha_k = 0$ and $\alpha_{k+1} \leq \ln(bx + 1) < \ln(bM + 1)$. Therefore $\alpha_{k+1}(x) \leq \alpha_k + \ln(bM + 1)$.

Summing everything and taking into account that $\alpha_0 = 0$ (because $M > 1$), we get

$$
\sum_{k=0}^{n-1} \alpha_k = \sum_{k=0}^{n-2} \alpha_{k+1} \leq \sum_{k=0}^{n-1} \alpha_k + |A_n| \cdot \ln a + |B_n| \cdot \ln(b + \eta) + |D_n| \cdot \ln(bM + 1).
$$

Therefore

$$
\frac{|A_n|}{n} \ln a + \frac{|B_n|}{n} \ln(b + \eta) + \frac{|D_n|}{n} \ln(bM + 1) \geq 0.
$$

However, as $n \to \infty$, $\varkappa_n$ goes to $\nu \times \delta_\infty$, and hence, $|A_n|/n$ goes to $\nu(C_0)$, $|B_n|/n$ goes to $\nu(C_1)$, and $|D_n|/n$ goes to 0. Therefore we get

$$
\nu(C_0) \ln a + \nu(C_1) \ln(b + \eta) \geq 0,
$$

contrary to our choice of η. This proves that if the Lyapunov exponent of ν is negative then there exists a probability measure μ invariant for (T_0, T_1, ν).

Uniqueness of μ follows from Corollary 3.3 and uniqueness of $\varkappa$ from Theorem 3.2. This shows also that $\varkappa$ is ergodic.

If z is a generic point of $\varkappa$, the measures $\varkappa_n$, given by (3.2), converge to $\varkappa$. By Lemma 3.1, if this holds for some $z = (\bar{\omega}, x)$, it holds also for $z' = (\bar{\omega}, x')$ for every $x' \in \mathbb{R}_+$. Therefore it holds for ν-almost every $\bar{\omega} \in \Sigma$ and every $x \in \mathbb{R}_+$.

Suppose now that the Lyapunov exponent of ν is non-negative, but there exists a probability measure invariant for (T_0, T_1, ν). This measure is the projection to $\overline{\mathcal{M}}$ of some τ-invariant measure $\varkappa$ on X whose projection to $\mathcal{N}$ is ν. In particular, $\varkappa$ is concentrated on $\Sigma \times \mathbb{R}_+$. Let $z' = (\bar{\omega}, y) \in X$ be a generic point of $\varkappa$.

Our assumptions imply $\nu(C_1) > 0$, since $\nu(C_1) = 0$ would imply $\lambda(\nu) = \ln a < 0$ leading to a contradiction. So there exists $M' > 0$ with $\varkappa(\Sigma \times \{M'\}) = 0$ and $\varkappa(C_1 \times [0, M']) > 0$, which means that we can choose an arbitrarily large n such that $\pi_2(\tau^{n-1}(z')) \leq M'$.

Now we make a construction very similar to the one from the proof of the first implication. The definitions of the sets A'_n, B'_n and D'_n and the functions α'_k will be very similar. Namely, fix some M' and n as above and set

$$
\begin{aligned}
A'_n &= \{k \in \{0, 1, \dots, n-1\} : \pi_1(\tau^k(z')) = 0\}, \\
B'_n &= \{k \in \{0, 1, \dots, n-1\} : \pi_2(\tau^k(z')) \le M' \text{ and } \pi_1(\tau^k(z'))_0 = 1\}, \\
D'_n &= \{k \in \{0, 1, \dots, n-1\} : \pi_2(\tau^k(z')) > M' \text{ and } \pi_1(\tau^k(z'))_0 = 1\}.
\end{aligned}
$$

Again, those sets form a partition of $\{0, 1, \dots, n-1\}$.

Set $\alpha'_k = \ln \pi_2(\tau^k(z'))$ for all $k \in \{0, 1, \dots, n-1\}$. Let us estimate α'_{k+1} from below.

If $k \in A'_n$ then $\alpha'_{k+1} = \alpha'_k + \ln a$.

If $k \in B'_n$ then, if $x = \pi_2(\tau^k(z))$, we have $\alpha'_k = \ln x$ and $\alpha'_{k+1} = \ln(bx + 1) = \ln x + \ln(b + 1/x) \ge \alpha'_k + \ln(b + 1/M')$.

If $k \in D'_n$ then, with the same notation, $\alpha'_{k+1} = \ln(bx + 1) > \ln bx = \alpha'_k + \ln b$.

Summing everything we get

$$
\begin{aligned}
\sum_{k=0}^{n-1} \alpha'_k &= \alpha'_0 + \sum_{k=0}^{n-2} \alpha'_{k+1} \\
&\ge \sum_{k=0}^{n-1} \alpha'_k + \alpha'_0 - \alpha'_{n-1} + |A'_n| \cdot \ln a + |B'_n| \cdot \ln(b + 1/M') + |D'_n| \cdot \ln b.
\end{aligned}
$$

Since $\alpha'_0 = \ln y$ and $\alpha'_{n-1} \le \ln M'$ by our choice of n, we get

$$
\frac{|A'_n|}{n} \ln a + \frac{|B'_n|}{n} \ln(b + 1/M') + \frac{|D'_n|}{n} \ln b + \frac{\ln y - \ln M'}{n} \le 0.
$$

As $n \to \infty$ (along a suitable subsequence), $|A'_n|/n$ goes to $\nu(C_0)$, $|B'_n|/n$ goes to $\varkappa(C_1 \times [0, M'])$, $|D'_n|/n$ goes to $\varkappa(C_1 \times (M', \infty))$ and $(\ln y - \ln M')/n$ goes to 0, we get

$$
\nu(C_0) \ln a + \varkappa(C_1 \times [0, M']) \ln(b + 1/M') + \varkappa(C_1 \times (M', \infty)) \ln b \le 0.
$$

By assumption $\lambda(\nu) = \nu(C_0) \ln a + \nu(C_1) \ln b \ge 0$, hence

$$
0 \le \nu(C_0) \ln a + \left(\varkappa(C_1 \times [0, M']) + \varkappa(C_1 \times (M', \infty)) \right) \ln b
$$

and the last two equations yield

$$
\varkappa(C_1 \times [0, M']) \ln(b + 1/M') \le \varkappa(C_1 \times [0, M']) \ln b.
$$

This contradicts our choice of M', thus completing the proof. $\square$

4. Ergodic properties of invariant measures

Recall that the elements of the semigroup G are 0-1 sequences $\omega = (\omega_0, \dots, \omega_{n-1})$, $n \in \mathbb{N}_0$. Denote the cylinder consisting of all elements of Σ whose first n positions coincide with ω by C_ω and write $x_n = x(\omega) = T_{\omega_{n-1}} \circ \cdots \circ T_{\omega_0}(x)$. For $\bar{\omega} \in \Sigma$ denote the cylinder whose first n positions agree with the first n positions of $\bar{\omega}$ by $C_{\bar{\omega}}^n$ and write $x_n = x_n(\bar{\omega}) = T_{\omega_{n-1}} \circ \cdots \circ T_{\omega_0}(x)$. In particular, $0_n(\bar{\omega})$ means $x_n(\bar{\omega})$ with $x = 0$.

In the case of a (semi)group with a single generator T, Birkhoff's Ergodic Theorem can be restated in the following way. If a measure μ is invariant and ergodic, then for μ almost every point x we have

$$\frac{1}{n} \sum_{i=0}^{n-1} T_*^i(\delta_x) \xrightarrow{\ *\ } \mu.$$

We would like to obtain similar results for our semigroup G. What was a single element T^i in the semigroup with one generator now becomes a whole "sphere" Σ_i and thus, before averaging with respect to i, one should also take the average over this sphere. Luckily, it turns out that after averaging over the sphere using appropriate weights, we do not need to average over i.

We start with a simple lemma. For any $x, y \in \mathbb{R}_+$ write

$$d(x, y) = \left| \frac{1}{(1 + x_n)} - \frac{1}{(1 + y_n)} \right|$$

and $\psi_n(\bar{\omega}) = d(x_n(\bar{\omega}), y_n(\bar{\omega}))$.

Lemma 4.1. *For any probability measure ν on Σ, any $\eta > 0$ and any $x, y \in \mathbb{R}_+$ we have*

$$\lim_{n \to \infty} \sum_{\{\bar{\omega}\, :\, \psi_n(\bar{\omega}) \geq \eta\}} \nu(C_{\bar{\omega}}^n) = \lim_{n \to \infty} \nu\big(\{\bar{\omega} \in \Sigma \,:\, \psi_n(\bar{\omega}) \geq \eta\}\big) = 0.$$

Proof. By Lemma 3.1, the sequence of functions ψ_n converges to 0 for any $x, y \in \mathbb{R}_+$, so it converges to 0 in measure. $\qquad\square$

Theorem 4.2. *Assume that ν is an ergodic σ-invariant probability measure with $\lambda(\nu) < 0$. Then for every $x \in \mathbb{R}_+$ the sequence of measures*

$$\mu_{n,x} = \sum_{\omega \in \Sigma_n} \nu(C_\omega) \delta_{T_\omega(x)}$$

converges (in the weak-* topology) to the (T_0, T_1, ν)-invariant measure μ.

Proof. Given $\varepsilon > 0$, we choose M such that $M > x$ and $\mu([0, M]) > 1 - \varepsilon$. By Lemma 4.1, for every $\eta > 0$ there exists N such that if $n \geq N$ then

$$\nu\big(\{\bar{\omega} \in \Sigma \,:\, d\left(0_n(\bar{\omega}), M_n(\bar{\omega})\right) \geq \eta\}\big) < \varepsilon.$$

In other words, if

$$B_n = \bigcup \{C_\omega \colon \omega \in \Sigma_n, d(T_\omega 0, T_\omega M) < \eta\}$$

then $\nu(B_n) > 1 - \varepsilon$ for every $n \geq N$. Therefore,

(4.1) $$\varkappa(B_n \times [0, M]) > 1 - 2\varepsilon.$$

For a continuous function $\varphi : \overline{\mathbb{R}}_+ \to \mathbb{R}$, let $\Phi = \varphi \circ \pi_2$. Then

$$\int \varphi \, d\mu = \int \Phi \, d\varkappa.$$

Set $\varkappa_n = \tau_*^n \left(\varkappa|_{B_n \times [0,M]} \right)$. Since $\varkappa$ is invariant, we have by (4.1)

$$\left| \int \Phi \, d\varkappa_n - \int \Phi \, d\varkappa \right| = \left| \int \Phi \, d\left(\tau_*^n \left(\varkappa|_{(\Sigma \times \mathbb{R}_+) \setminus (B_n \times [0,M])} \right) \right) \right| \leq \|\varphi\| \cdot 2\varepsilon.$$

If we denote by $\sum'$ the sum over those $\omega \in \Sigma_n$ for which $C_\omega \subset B_n$, and by $\sum''$ the sum over the rest of $\omega \in \Sigma_n$, we get

$$\int \Phi \, d\varkappa_n = \int_{B_n \times [0,M]} \Phi \circ \tau^n \, d\varkappa = \sum{}' \int_{C_\omega \times [0,M]} \Phi \circ \tau^n \, d\varkappa$$

and

$$\int \varphi \, d\mu_{n,x} = \sum_{\omega \in \Sigma_n} \nu(C_\omega) \, \varphi(T_\omega x) = \sum{}' \nu(C_\omega) \, \varphi(T_\omega x) + \sum{}'' \nu(C_\omega) \, \varphi(T_\omega x),$$

where

$$\left| \sum{}'' \nu(C_\omega) \, \varphi(T_\omega x) \right| \le \|\varphi\| \cdot \sum{}'' \nu(C_\omega) \le \|\varphi\| \cdot \varepsilon.$$

Therefore

$$(4.2) \quad \left| \int \varphi \, d\mu - \int \varphi \, d\mu_{n,x} \right| = \left| \int \Phi \, d\varkappa - \int \varphi \, d\mu_{n,x} \right|$$

$$\le 3 \, \varepsilon \cdot \|\varphi\| + \sum{}' \left| \int_{C_\omega \times [0,M]} \Phi \circ \tau^n \, d\varkappa - \nu(C_\omega) \, \varphi(T_\omega x) \right|.$$

Since φ is uniformly continuous, for every $\varepsilon > 0$ there exists $\eta > 0$ such that, if $d(x,y) < \eta$ then $|\varphi(x) - \varphi(y)| < \varepsilon$. By the definition of B_n, if $n \ge N$, then for any $\bar{\omega}' \in C_\omega \in B_n$ and any $x, y \in [0,M]$ we have

$$|(\Phi \circ \tau^n)(\bar{\omega}', y) - \varphi(T_\omega x)| = |\varphi(T_\omega y) - \varphi(T_\omega x)| < \varepsilon.$$

We then get for the sum on the right-hand side of (4.2) the following estimate:

$$\sum{}' \left| \int_{C_\omega \times [0,M]} \Phi \circ \tau^n \, d\varkappa - \nu(C_\omega) \, \varphi(T_\omega x) \right| \le \varepsilon \cdot \sum{}' \nu(C_\omega) \le \varepsilon.$$

Thus, for $n \ge N$ we have $\left| \int \varphi \, d\mu - \int \varphi \, d\mu_{n,x} \right| \le \varepsilon(3\|\varphi\| + 1)$. As $\varepsilon > 0$ can be chosen arbitrarily this proves that $\lim_{n \to \infty} \int \varphi \, d\mu_{n,x} = \int \varphi \, d\mu$. As this holds for every φ, the proof of the theorem is complete. $\qquad \square$

One can also mimic Birkhoff's Ergodic Theorem in another way. Namely, choose one sequence $\bar{\omega} \in \Sigma$ and average $\delta_{x_i(\bar{\omega})}$ where i ranges from 0 to $n-1$. Note that this is a similar procedure as in Theorem 2.2, but here we use the structure of G given by its generators. Therefore we get a different result, which is a direct consequence of Theorem 3.4.

Theorem 4.3. *Assume that ν is an ergodic σ-invariant probability measure with $\lambda(\nu) < 0$. Then for ν-almost every $\bar{\omega} \in \Sigma$ and every $x \in \mathbb{R}_+$ the sequence of measures*

$$\frac{1}{n} \sum_{i=0}^{n-1} \delta_{x_n(\bar{\omega})}$$

converges (in the weak-* topology) to the (T_0, T_1, ν)-invariant measure μ.

5. Absolutely continuous invariant measures

In the theory of Dynamical Systems, measures that are absolutely continuous with respect to the Lebesgue measure are of special interest. As we mentioned in the Introduction, we do not know whether certain particular measures are absolutely continuous. Nevertheless, we can show that for some σ-invariant measures ν there exist (T_0, T_1, ν)-invariant measures that are absolutely continuous.

For any $\gamma > 1$, consider that map defined by

$$(5.1) \qquad T(x) = \begin{cases} T_0^{-1}(x) = \frac{x}{a} & \text{if } \frac{\gamma-1}{b} \leq x < \gamma; \\ T_1^{-1}(x) = \frac{x-1}{b} & \text{if } \gamma \leq x \leq \frac{\gamma}{a}; \end{cases}$$

and let $I_0 = [\frac{\gamma-1}{b}, \gamma)$, $I_1 = [\gamma, \frac{\gamma}{a}]$ and $I = I_0 \cup I_1 = [\frac{\gamma-1}{b}, \frac{\gamma}{a}]$. We want to find an absolutely continuous T-invariant measure and investigate its properties. For this, we need the following theorem.

Theorem 5.1. *Let $p < \gamma < q$ be real numbers and let S be a map of $J = [p, q]$ to itself such that both $S|_{[p,\gamma)}$ and $S|_{(\gamma,q]}$ extend to monotone C^2 maps on the closed intervals $[p, \gamma]$ and $[\gamma, q]$ respectively. Assume that there exists n such that S^n is piecewise expanding. Then there exists a unique probability measure μ on J which is invariant for S and absolutely continuous with respect to the Lebesgue measure. The measure μ is ergodic and its support is the union of finitely many intervals, one of which contains γ in its interior.*

Proof. By results of Lasota and Yorke [LaY] and Li and Yorke [LiY] (see also [BG]) there exist absolutely continuous probability measures $\mu_1, \ldots, \mu_k$, $k \geq 1$, invariant for S^n, ergodic, having mutually disjoint supports and such that every absolutely continuous probability measure, invariant for S^n, is a convex combination of $\mu_1, \ldots, \mu_k$. Moreover, the support of each μ_i is the union of finitely many intervals, one of which contains a discontinuity of S^n or of its derivative in its interior. Note that such a discontinuity is a preimage of γ under $S^{j(i)}$ for some $0 \leq j(i) < n$ and the measure $S_*^{j(i)}(\mu_i)$ thus contains γ in the interior of its support.

Since S^n commutes with S, any measure $S_*(\mu_i)$ is an absolutely continuous probability measure, invariant and ergodic for S^n, Therefore S_* maps $\{\mu_1, \ldots, \mu_k\}$ to itself. Since each μ_i is fixed by S_*^n, this map is a bijection. Only one of these measures may contain γ in the interior of its support, hence all $S_*^{j(i)}(\mu_i)$ must be equal. This measure belongs to the S_*-orbit of each μ_i and therefore all μ_i form one orbit of S_*.

Set $\mu = \frac{1}{k} \sum_{i=1}^{k} \mu_i$. All the properties of μ from the statement of the theorem follow directly from the properties of the $\mu_1, \ldots, \mu_k$ described above. $\qquad \square$

We get

Corollary 5.2. *For any $0 < a < 1 < b$ and $\gamma > 1$ there exists an absolutely continuous T-invariant probability measure with the properties listed in Theorem 5.1.*

Proof. From Lemma 2.3 it follows that there exists n such that T^n is piecewise expanding. We then apply Theorem 5.1. $\qquad \square$

Let $\rho : X \to X$ be the skew product map $\rho(\bar{\omega}, x) = (\sigma(\bar{\omega}), T_{\omega_0}^{-1}(x))$.

Lemma 5.3. *Given a T-invariant probability measure μ, there exists $B \subset X$ and a ρ-invariant probability measure $\varkappa$ on B such that $(\pi_2)_*(\varkappa) = \mu$ and $(\rho|_B, \varkappa)$ is isomorphic to (T, μ).*

Proof. For $x \in I$ consider the itinerary map $\varphi : I \to \Sigma$ with $\varphi(x) = (\omega_0, \omega_1, \dots) \in \Sigma$, where $\omega_k = i$ if $T^k(x) \in I_i$ (for $k \geq 0$ and $i \in \{0, 1\}$). This map is a Borel map and its graph $B = \{(\varphi(x), x) : x \in I\}$ is a Borel set in X, and the projection $\pi_2|_B$ is a bijection between B and I. Thus a measure μ can be transported to B by the inverse bijection and we obtain a measure $\varkappa$ on $B \subset X$. By construction $(\pi_2)_*(\varkappa) = \mu$. We have

$$\pi_2(\rho(\varphi(x), x)) = T_{\omega_0}^{-1}(x) = T(\pi_2(\varphi(x), x)),$$

so $(\pi_2)_*$ is an isomorphism between $(B, \rho, \varkappa)$ and (I, T, μ). In particular, $\varkappa$ is ρ-invariant. $\qquad\square$

Denote by Σ_{ts} the space of all infinite two-sided 0-1 sequences

$$\bar{\xi} = (\dots, \xi_{-2}, \xi_{-1}, \xi_0, \xi_1, \xi_2, \dots).$$

Lemma 5.4. *Given an ergodic T-invariant probability measure μ, there exists an ergodic σ-invariant probability measure ν on Σ such that μ is invariant for (T_0, T_1, ν).*

Proof. Let us look at the natural extension (inverse limit) $(\widehat{X}, \widehat{\rho}, \widehat{\varkappa})$ of the system $(X, \rho, \varkappa)$ constructed in Lemma 5.3. The space $\widehat{X}$ consists of all sequences of pairs $(\bar{\omega}^{(i)}, y^{(i)})_{i=-\infty}^0$, for which

$$(5.2) \qquad\qquad \rho(\bar{\omega}^{(i)}, y^{(i)}) = (\bar{\omega}^{(i+1)}, y^{(i+1)})$$

$(i = -1, -2, \dots)$. The map $\widehat{\rho}$ is given by

$$(5.3) \quad \widehat{\rho}(\dots, (\bar{\omega}^{(-2)}, y^{(-2)}), (\bar{\omega}^{(-1)}, y^{(-1)}), (\bar{\omega}^{(0)}, y^{(0)}))$$
$$= (\dots, (\bar{\omega}^{(-1)}, y^{(-1)}), (\bar{\omega}^{(0)}, y^{(0)}), (\sigma(\bar{\omega}^{(0)}), T_{\omega_0^{(0)}}^{-1}(y^{(0)}))).$$

Condition (5.2) is equivalent to the pair of conditions

$$(5.4) \qquad\qquad \sigma(\bar{\omega}^{(i)}) = \bar{\omega}^{(i+1)}$$

and

$$(5.5) \qquad\qquad T_{\omega_0^{(i)}}^{-1}(y^{(i)}) = y^{(i+1)}.$$

Sequences $(\bar{\omega}^{(i)})_{i=-\infty}^0$ of elements of Σ satisfying (5.4) can be identified with doubly infinite 0-1 sequences $\bar{\xi}$ by $\bar{\omega}^{(i)} = (\xi_i, \xi_{i+1}, \dots)$. Then (5.5) becomes

$$T_{\xi_i}^{-1}(y^{(i)}) = y^{(i+1)}.$$

This is equivalent to $T_{\xi_i}(y^{(i+1)}) = y^{(i)}$. Therefore, once $\bar{\xi}$ and $y^{(0)}$ are given, all $y^{(i)}$ are uniquely determined and become redundant. This means that the space $\widehat{X}$ can be written as $\Sigma_{ts} \times \overline{\mathbb{R}}_+$, and then $\widehat{\rho}$ becomes the skew product

$$\widehat{\rho}(\bar{\xi}, y) = (\sigma(\bar{\xi}), T_{\xi_0}^{-1}(y)).$$

Note that although $T_{\xi_0}^{-1}(y)$ is not defined everywhere, it is defined $\widehat{\varkappa}$-almost everywhere, and this is sufficient for our purposes.

The inverse of $\widehat{\rho}$ is given by

$$\widehat{\rho}^{\,-1}(\bar{\xi}, y) = (\sigma^{-1}(\bar{\xi}), T_{\xi_{-1}}(y)).$$

Therefore τ is a factor of $\widehat{\rho}^{\,-1}$ under the projection $\pi : \widehat{X} \to X$, given by

$$\pi(\bar{\xi}, y) = ((\xi_{-1}, \xi_{-2}, \dots), y).$$

Hence, $\widetilde{\varkappa} = \pi_*(\widehat{\varkappa})$ is a τ-invariant probability measure.

By Lemma 5.3, $(\pi_2)_*(\varkappa) = \mu$. Therefore for every measurable set $A \subset \mathbb{R}_+$ we have $\mu(A) = \varkappa(\Sigma \times A)$. Clearly, $\varkappa(\Sigma \times A) = \widehat{\varkappa}(\Sigma_{ts} \times A)$. This proves that the projection of $\widehat{\varkappa}$ to the space of probability measures on the second coordinate of $\widehat{X} = \Sigma_{ts} \times \overline{\mathbb{R}}_+$ is μ. Hence, $(\pi_2)_*(\widetilde{\varkappa}) = \mu$. Therefore, if $\nu = (\pi_1)_*(\widetilde{\varkappa})$, then μ is a (T_0, T_1, ν)-invariant measure.

Since μ is ergodic for T, by Lemma 5.3 $\varkappa$ is ergodic for ρ. Therefore $\widehat{\varkappa}$ is ergodic for $\widehat{\rho}$, so $\widetilde{\varkappa}$ is ergodic for τ, so ν is ergodic for σ. $\qquad\square$

We can now state the main result of this section.

Theorem 5.5. *For every $\gamma > 1$ there exists a σ-invariant ergodic probability measure $\nu_\gamma \in \mathcal{N}$ with negative Lyapunov exponent, for which the corresponding (T_0, T_1, ν_γ)-invariant measure μ_γ is absolutely continuous with respect to the Lebesgue measure and the support of μ_γ is the union of finitely many intervals whose convex hull equals $[\frac{\gamma-1}{b}, \frac{\gamma}{a}]$. In particular, $\mu_\gamma \neq \mu_\delta$ for $\gamma \neq \delta$, $\gamma, \delta > 1$.*

Proof. The existence of ν_γ and μ_γ follows from Corollary 5.2, Lemma 5.4 and Theorem 3.4. Observe that by Theorem 5.1 γ is contained in the interior of the support of μ_γ. Since $T(\gamma) = \frac{\gamma-1}{b}$ and $\lim_{x \to \gamma_-} T(x) = \frac{\gamma}{a}$, the endpoints of I are contained in the support of μ_γ. As the support of μ_γ is contained in I, its convex hull equals I. All other properties of μ_γ follow from Corollary 5.2. $\qquad\square$

6. Invariant measures over Bernoulli systems

Let us see how Definition 1.1 works in the classical case when the system (σ, Σ, ν) is Bernoulli.

Proposition 6.1. *Assume that ν is Bernoulli. If a τ-invariant measure $\varkappa$ projects to ν, then $\varkappa = \nu \times \mu$ for some measure μ.*

Proof. We have $\varkappa = \varkappa_1 + \varkappa_2$ where $\varkappa_1$ is concentrated on $\Sigma \times \mathbb{R}_+$ and $\varkappa_2$ is concentrated on $\Sigma \times \{\infty\}$ (one of these measures may be zero). Since $\Sigma \times \{\infty\}$ is τ-invariant, the measure $\varkappa_2$ is τ-invariant. Together with the ergodicity of ν this implies that $\varkappa_2 = c \cdot \nu \times \delta_\infty$ for some constant c. Therefore it is enough to consider $\varkappa = \varkappa_1$. Thus we assume that $\varkappa$ is concentrated on $\Sigma \times \mathbb{R}_+$.

Observe first that the measure $\tau_*(\nu \times \delta_x)$ is a weighted average of the measures $\nu \times \delta_{T_0 x}$ and $\nu \times \delta_{T_1 x}$ (here we use the assumption that ν is Bernoulli). Therefore, for any measure μ_1 on $\overline{\mathbb{R}}_+$, the measure $\tau_*(\nu \times \mu_1)$ is of the form $\nu \times \mu_2$ for some measure μ_2 on $\overline{\mathbb{R}}_+$.

By Theorem 3.4, for almost every $\bar{\omega} \in \Sigma$, the sequence of measures $\varkappa_n$ given by (3.2) with $x = 1$ converges to $\varkappa$. Integrating those measures with respect to ν and applying the Lebesgue Dominated Convergence Theorem (we can do it since

the weak-* convergence means convergence of integrals of any continuous function $\varphi : \overline{\mathbb{R}}_+ \to \mathbb{R}$ and those integrals are bounded by the supremum of $|\varphi|$) we get that

$$\frac{1}{n} \sum_{i=0}^{n-1} \tau_*^i(\nu \times \delta_1) \overset{*}{\longrightarrow} \varkappa.$$

By the preceding paragraph, all measures $\tau_*^i(\nu \times \delta_1)$ are of the form $\nu \times \mu_i$, so $\varkappa$ is also of the form $\nu \times \mu$. $\qquad\square$

If the system (σ, Σ, ν) is $(p, 1 - p)$ Bernoulli, then the action of G has a simple interpretation. Namely, we apply T_0 or T_1 randomly each time, choosing T_0 with probability p and T_1 with probability $1 - p$.

In this case we can also look at the invariant measures for (T_0, T_1, ν) from another point of view. The maps T_0, T_1 induce operators $(T_0)_*, (T_1)_*$ from $\mathcal{M}$ to itself. Define $\mathcal{T}_p : \mathcal{M} \to \mathcal{M}$ by $\mathcal{T}_p = p(T_0)_* + (1 - p)(T_1)_*$.

Proposition 6.2. *Let ν be the $(p, 1 - p)$ Bernoulli measure on Σ and let $\mu \in \mathcal{M}$. Then the following are equivalent*

(1) *the measure $\nu \times \mu$ is τ-invariant;*
(2) *for any measurable set $B \subset \overline{\mathbb{R}}_+$, we have*

(6.1) $$\mu(B) = p\mu(T_0^{-1}(B)) + (1 - p)\mu(T_1^{-1}(B));$$

(3) *μ is a fixed point of $\mathcal{T}_p$.*

Proof. Denote $C_i = \{\bar{\omega} \in \Sigma : \omega_0 = i\}$ for $i = 0, 1$. We have

$$\tau^{-1}(A \times B) = \left[(\sigma|_{C_0})^{-1}(A) \times T_0^{-1}(B)\right] \cup \left[(\sigma|_{C_1})^{-1}(A) \times T_1^{-1}(B)\right].$$

Since $\nu\big((\sigma|_{C_0})^{-1}(A)\big) = p\nu(A)$ and $\nu\big((\sigma|_{C_1})^{-1}(A)\big) = (1 - p)\nu(A)$, we have

(6.2) $$(\nu \times \mu)(\tau^{-1}(A \times B)) = p\nu(A)\mu(T_0^{-1}(B)) + (1 - p)\nu(A)\mu(T_1^{-1}(B)).$$

If $\nu \times \mu$ is τ-invariant, then by (6.2)

$$\mu(B) = (\nu \times \mu)(\Sigma \times B) = (\nu \times \mu)\tau^{-1}(\Sigma \times B) = p\mu(T_0^{-1}(B)) + (1 - p)\mu(T_1^{-1}(B)),$$

so (2) holds.

On the other hand, multiplying both sides of (6.1) by $\nu(A)$ and using (6.2) one sees that (2) implies (1).

Conditions (2) and (3) are equivalent by the definition of $\mathcal{T}_p$. $\qquad\square$

Note that by the above proposition and Theorem 5.1 of [DF] (cf. [NSB]), our (T_0, T_1, ν)-invariant measure is the same as the invariant (stationary) measure for the corresponding IFS.

As an immediate consequence of Propositions 6.1 and 6.2 we get the following theorem.

Theorem 6.3. *Assume that ν is $(p, 1 - p)$ Bernoulli. A probability measure μ supported by $\mathbb{R}_+$ is (T_0, T_1, ν)-invariant if and only if it is a fixed point of $\mathcal{T}_p$.*

Another immediate consequence is the following lemma, where we use property (2) of Proposition 6.2.

Lemma 6.4. *Assume that ν is $(p, 1-p)$ Bernoulli and μ is the (T_0, T_1, ν)-invariant measure. Then for every interval I and $i = 0, 1$ we have $\mu(T_i(I)) \geq q\mu(I)$, where $q = \min(p, 1-p)$.*

From this we get, in turn, the next lemma (cf. [BE], Theorem 3).

Lemma 6.5. *Assume that ν is $(p, 1-p)$ Bernoulli and μ is the (T_0, T_1, ν)-invariant measure. Then the measure μ of every interval I is positive. In other words, the support of μ is the whole $\overline{\mathbb{R}}_+$.*

Proof. There exists an interval $J \subset \mathbb{R}_+$ with $\mu(J) > 0$. Let x be the center of J. Let $I \subset \mathbb{R}_+$ be an interval with center y. By Lemmas 2.3 and 2.4, we can find $\omega \in G$ such that that $T_\omega(x)$ is so close to y and T'_ω is so small that $T_\omega(J) \subset I$. Then by Lemma 6.4 $\mu(I) > 0$. $\qquad\square$

As we mentioned in the Introduction, we do not know whether the assumption that ν is $(p, 1-p)$ Bernoulli and μ is a (T_0, T_1, ν)-invariant measure implies that μ is absolutely continuous with respect to the Lebesgue measure. The strongest property of the measure μ that we can prove is the following.

Theorem 6.6. *Assume that ν is $(p, 1-p)$-Bernoulli with $\lambda(\nu) < 0$. Then the (T_0, T_1, ν)-invariant measure μ on $\mathbb{R}_+$ is the image of the Lebesgue measure under an increasing map from $[0, 1)$ onto $\mathbb{R}_+$, which is Hölder continuous with the same exponent on each compact interval.*

In order to prove this theorem we need the following lemma. For an interval I we denote its length by $|I|$.

Lemma 6.7. *Under the assumptions of Theorem 6.6, if $I \subset \mathbb{R}_+$ is an interval centered at t, then*

$$\log \mu(I) > c_4(t) + c_5 \log |I| \tag{6.3}$$

holds with $c_4(t)$ depending continuously on t (but independent of I) and

$$c_5 = \frac{-\log q \cdot \log \frac{a+b-1}{a}}{\log b \cdot \log \frac{b}{a+b-1}} > 0, \tag{6.4}$$

where $q = \min(p, 1-p)$.

Proof. For an interval $I \subset \mathbb{R}_+$, we need to estimate $\mu(I)$ from below. If we find a sequence ω of length n such that $T_\omega([0, 1/a]) \subset I$, then by Lemma 6.4, $\mu(I) \geq q^n \mu([0, 1/a])$. Note that by Lemma 6.5, $\mu([0, 1/a]) > 0$. Thus, we will look for preimages of I along a specific branch.

We start by choosing the smallest k such that for the midpoint t of I we have $T_1^{-k}(t) \leq 1/a$. Since $T_1(0) = 1 < 1/a$, we have then $T_1^{-k}(t) \in [0, 1/a]$. Moreover, if $t > 1/a$ then $1/a < T_1^{-k+1}(t) < t/b^{k-1}$, so

$$k < \frac{\log^+ ta}{\log b} + 1, \tag{6.5}$$

where we use the notation $\log^+ x = \max(0, \log x)$.

Consider the map $\varphi : [0, 1/a) \to [0, 1/a)$ given by

$$(6.6) \qquad \varphi(x) = \begin{cases} \frac{x}{a} & \text{if } x < 1, \\ \frac{x-1}{b} & \text{if } x \geq 1. \end{cases}$$

(cf. formula (5.1)). Starting at the point $T_1^{-k}(t)$ and applying φ amounts to choosing further preimages of t along some branch. Our aim is to find m such that $(\varphi^m)'(T_1^{-k}(t))$ is larger than $1/a$ divided by the length of the half of the interval $T_1^{-k}(I)$. Then the preimage of I under the corresponding branch (of length $k + m$) will contain $[0, 1/a)$.

Since we do not know the location of $T_1^{-k}(t)$, we have to estimate from below $(\varphi^n)'(x)$ for all $x \in [0, 1/a)$. We can treat φ as a map of the circle with one discontinuity at which the left-hand limit is larger than the right-hand limit (so its lifting is an *old heavy* map,[3] see [M]). We will compare it with the map $\psi : [0, 1/a) \to [0, 1/a)$ given by

$$\psi(x) = \begin{cases} \frac{1-a}{ab} + \frac{a+b-1}{ab} \, x & \text{if } x < 1, \\ \frac{x-1}{b} & \text{if } x \geq 1. \end{cases}$$

This map can be treated as an orientation preserving homeomorphism of the circle. Let us find its rotation number. We have

$$\frac{1-a}{ab} + \frac{a+b-1}{ab} \left(y - \frac{1}{b-1} \right) = \frac{a+b-1}{ab} y - \frac{1}{b-1}$$

and

$$\frac{\left(y - \frac{1}{b-1} \right) - 1}{b} = \frac{y}{b} - \frac{1}{b-1} \; .$$

This means that the map

$$\xi : \left[0, \frac{1}{a} \right) \to \left[\log \frac{1}{b-1}, \log \left(\frac{1}{a} + \frac{1}{b-1} \right) \right),$$

given by the formula

$$\xi(x) = \log \left(x + \frac{1}{b-1} \right),$$

conjugates ψ with the map

$$\zeta : \left[\log \frac{1}{b-1}, \log \left(\frac{1}{a} + \frac{1}{b-1} \right) \right) \to \left[\log \frac{1}{b-1}, \log \left(\frac{1}{a} + \frac{1}{b-1} \right) \right),$$

given by the formula

$$\zeta(x) = \begin{cases} x + \log \frac{a+b-1}{ab} & \text{if } \log \frac{1}{b-1} \leq x < \log \frac{b}{b-1}, \\ x + \log \frac{1}{b} & \text{if } \log \frac{b}{b-1} \leq x < \log \left(\frac{1}{a} + \frac{1}{b-1} \right). \end{cases}$$

The map ζ, treated as a circle homeomorphism, is just a rotation. Its rotation number is

$$\rho(\zeta) = \frac{\log \frac{a+b-1}{ab}}{\log \frac{a+b-1}{a}} \; .$$

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[3] A map $\Phi : \mathbb{R} \to \mathbb{R}$ is *old* if $\Phi(x + 1) = \Phi(x) + 1$ for all $x \in \mathbb{R}$; it is *heavy* if $\lim_{y \to x^-} \Phi(y) \geq \Phi(x) \geq \lim_{y \to x^+} \Phi(y)$ for all $x \in \mathbb{R}$.

The lifting of ζ, with the length of the circle normalized to 1, is the translation by $\rho(\zeta)$. The n-th iterate of this lifting sends a point $y \in [0,1)$ to $y + n\rho(\zeta) \in [p, p+1)$, where p equals $\lfloor n\rho(\zeta) \rfloor$ or $\lfloor n\rho(\zeta) + 1 \rfloor$. Thus, the same is true for ψ replacing ζ.

Let Ψ be the lifting of ψ and Φ the lifting of φ, agreeing with Ψ on the right lap. Then $\Psi \geq \Phi$. Moreover, Ψ is increasing, and thus $\Psi^n \geq \Phi^n$ for all n (see [ALM]). Thus if $y \in [0,1)$ then $\Phi^n(y) \in [p', p'+1)$, where $p' \leq \lfloor n\rho(\zeta) + 1 \rfloor$. This means that if we iterate φ n times, its right lap is used at most p' times. Therefore the left lap is used at least $n - p' \geq n - 2 - n\rho(\zeta)$ times. From (6.6) we read off the derivatives of both laps of φ, and we get that

$$(\varphi^n)'(x) \geq \left(\frac{1}{a}\right)^{n-2-n\rho(\zeta)} \cdot \left(\frac{1}{b}\right)^{2+n\rho(\zeta)} = \left(\frac{a}{b}\right)^2 \left[\left(\frac{1}{a}\right) \cdot \left(\frac{a}{b}\right)^{\rho(\zeta)}\right]^n.$$

Substituting the value of $\rho(\zeta)$, we get after a short algebraic manipulation on the formulas

(6.7)
$$\log(\varphi^n)'(x) \geq 2 \log \frac{a}{b} + \frac{\log b \cdot \log \frac{b}{a+b-1}}{\log \frac{a+b-1}{a}} \, n.$$

We want to find m such that

$$(\varphi^m)'(x) \geq \frac{1}{a} \cdot \frac{2}{|T_1^{-k}(I)|} = \frac{2b^k}{a|I|} \, .$$

According to (6.5) and (6.7), this inequality will be satisfied if

$$2 \log \frac{a}{b} + \frac{\log b \cdot \log \frac{b}{a+b-1}}{\log \frac{a+b-1}{a}} \, m \geq \log \frac{2b^{(\log^+ ta)/\log b + 1}}{a|I|} \, ,$$

that is,

$$m \geq \frac{\log \frac{2b^{(\log^+ ta)/\log b + 3}}{a^3|I|} \cdot \log \frac{a+b-1}{a}}{\log b \cdot \log \frac{b}{a+b-1}} \, .$$

Therefore we can find required integer m satisfying

$$m < \frac{\log \frac{2b^{(\log^+ ta)/\log b + 3}}{a^3|I|} \cdot \log \frac{a+b-1}{a}}{\log b \cdot \log \frac{b}{a+b-1}} + 1.$$

We can rewrite this inequality as

(6.8)
$$m < c_1(t) - c_2 \log |I|,$$

where

(6.9)
$$c_1(t) = \frac{\log \frac{2b^{(\log^+ ta)/\log b + 3}}{a^3} \cdot \log \frac{a+b-1}{a}}{\log b \cdot \log \frac{b}{a+b-1}} + 1$$

and

(6.10)
$$c_2 = \frac{\log \frac{a+b-1}{a}}{\log b \cdot \log \frac{b}{a+b-1}}.$$

Note that all logarithms appearing in the formulas for $c_1(t)$ and c_2 are positive, so $c_1(t)$ and c_2 are positive.

In such a way we found a 0-1 sequence ω of length $k + m$, such that $T_\omega([0, 1/a]) \subset I$, and hence $\mu(I) \geq q^{(k+m)}\mu([0, 1/a])$. Moreover, by (6.5) and (6.8), we have the following estimate for $k + m$:

$$(6.11) \qquad k + m < \frac{\log^+ ta}{\log b} + 1 + c_1(t) - c_2 \log |I|.$$

We can rewrite it as

$$k + m < c_3(t) - c_2 \log |I|.$$

According to formulas (6.8)-(6.11), the dependence of $c_3(t)$ on t is continuous, and c_2 is independent of t. We get

$$\log \mu(I) > (c_3(t) - c_2 \log |I|) \log q + \log \mu([0, 1/a]).$$

Therefore (6.3) holds with c_5 given by (6.4) and $c_4(t)$ depending continuously on t. $\qquad \square$

Proof of Theorem 6.6. Consider the map $H : [0, 1) \to \mathbb{R}_+$, such that $H(x) = y$ if and only if $\mu([0, y]) = x$. By Lemma 6.7, and since $\mu(\mathbb{R}_+) = 1$, this map is well defined. By the definition, the measure μ is the image of the Lebesgue measure under H. In particular, for an interval $J \subset [0, 1)$ we have $|J| = \mu(H(J))$, so (6.3) gives us

$$|J| > \exp(c_4(t)) \cdot |H(J)|^{c_5},$$

where t is a certain point of $H(J)$. This shows that H is Hölder continuous with exponent $1/c_5$ on each compact interval. $\qquad \square$

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DEPARTMENT OF MATHEMATICS, OHIO STATE UNIVERSITY, COLUMBUS, OH 43210, USA
E-mail address: vitaly@math.ohio-state.edu

DEPARTMENT OF MATHEMATICAL SCIENCES, INDIANA UNIVERSITY PURDUE UNIVERSITY INDIANAPOLIS, INDIANAPOLIS, IN 46202-3216, USA
E-mail address: mmisiure@math.iupui.edu

INSTITUTO DE MATEMATICA PURA E APLICADA, RIO DE JANEIRO 22460-320, BRAZIL
E-mail address: senti@impa.br