Derivation of the Ten Einstein Field Equations from the Semiclassical Approximation to Quantum Geometrodynamics

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All ten Einstein field equations are derived from (a) a single equation, the Einstein-Hamilton-Jacobi equation for general relativity, and (b) the principle of constructive interference of de Broglie waves in superspace. In this derivation one obtains by introducing Tomonaga's many-time parametrization the manifestly covariant Hamiltonian equations of general relativity.

I. INTRODUCTION

THIS work deals with the Hamilton-Jacobi equation for Einstein's theory of gravity. The aim is to show that all ten Einstein field equations are a direct consequence of the principle of constructive interference of wave fronts. The propagation of the wave fronts themselves is determined by the Einstein-Hamilton-Jacobi equation, an equation which marks in one formulation (that of Hamilton and Jacobi) perhaps the furthest step to date in formulating general relativity in quantum language.

The efforts exerted in trying to put general relativity within the framework of a quantum theory and thus obtain answers to a number of problems[1] (gravitational collapse, fluctuations in the geometry of space, the relation between elementary particles and geometrodynamical excitations, etc.) inherent in geometrodynamics have been frustrated repeatedly; nevertheless, a great deal has been learned about the structure of the field equations.[2] The present state of geometrodynamics reminds one of the times when Bohr was trying to understand why an electron does not collapse into the nucleus, and when Planck was arguing for the zero-point fluctuations[3] in an ensemble of simple harmonic oscillators at zero temperature. Subsequently the *ad hoc* assumptions underlying their explanations were given a physical basis by associating with a particle de Broglie waves that are capable of interference. In the semiclassical limit, these de Broglie waves are describable in terms of waves and wave fronts. By writing down his wave equation, Schrödinger gave mathematical rigor to these and additional geometrical aspects of motion, such as phase, wavelength, and frequency. The short-wavelength limit of wave mechanics is classical mechanics. An approximation that stands in between the two extreme descriptions (wave and ray) is the semiclassical approximation. Its mathematical basis is the phase functions S, the solutions of the Hamilton-Jacobi (HJ) equation, together with the principle of *constructive interference* of waves,

$$\Psi \sim e^{iS/\hbar}.$$

Once one has a classical theory, it usually is easier to go to the semiclassical approximation than it is to go to the wave equation or its equivalent. A necessary condition that the semiclassical approximation be correct is that one must be able to deduce the classical equations of motion from it. This is what we aim to do for the case of the Einstein field equations. We deal with a scalar functional S defined on the superspace of three-geometries; that is, the space in which each "point" represents one three-geometry ($S[{}^{(3)}\mathcal{G}]$).

The proposition that we shall prove is the following: Given:

(a) The Einstein-Hamilton-Jacobi (EHJ) equation for general relativlty,[4]

$$0={}^{(3)}R+g^{-1}(\tfrac{1}{2}g_{ij}g_{kl}-g_{ik}g_{jl})(\delta S/\delta g_{ij})(\delta S/\delta g_{kl}). \quad (1)$$

Here g_{ij} denotes the metric of the spatial hypersurface. The EHJ equation is defined for each point of this surface. The curvature invariant on this surface is ${}^{(3)}R$ and $g\equiv \det g_{ij}$. The equation must be solved for S ("HJ function," "Hamilton's principal function," "action," $\hbar$ times the phase of the "Schrödinger" function in the semiclassical limit). The functional derivative of S with respect to $g_{ij}(x)$ is defined by

$$\delta S=\int \frac{\delta S}{\delta g_{ij}(x)}\delta g_{ij}(x)d^3x,$$

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[1] Although there does not yet exist a detailed quantum theory of geometrodynamics, we already today perceive in broad outline the qualitative character of quantum geometrodynamics and a number of its problems. See J. A. Wheeler, in *Battelle Rencontres: 1967 Lectures in Mathematics and Physics*, edited by J. A. Wheeler and C. De Witt (W. A. Benjamin, Inc., New York, 1968); J. A. Wheeler, *Einstein's Vision* (Julius Springer-Verlag, Berlin, 1968); J. A. Wheeler, in *Relativity, Groups, and Topology*, edited by C. De Witt and B. De Witt (Gordon and Breach Science Publishers, Inc., New York, 1964), p. 507.

[2] P. G. Bergmann, Phys. Rev. **75**, 680 (1949); P. A. M. Dirac, Can. J. Math. **2**, 129 (1950); Proc. Roy. Soc. (London) **A246**, 326 (1958); *Lectures on Quantum Mechanics* (Academic Press Inc., New York, 1966); Proc. Roy. Soc. (London) **A246**, 333 (1958); R. Arnowitt, S. Deser, and C. W. Misner, Phys. Rev. **116**, 1322 (1959); in *The Dynamics of General Relativity*, edited by L. Witten (John Wiley & Sons, Inc., New York, 1962); A. Peres, Bull. Res. Counc. (Israel) **8F**, 179 (1959); Nuovo Cimento **26**, 53 (1962).

[3] M. Planck, *The Theory of Heat Radiation* (P. Blakiston's Son and Company, Philadelphia, 1914), pp. 142, 164.

[4] The EHJ equation was first written down by A. Peres, Nuovo Cimento **26**, 53 (1962).

where the integration is performed over the whole spatial hypersurface.

(b) The functional S is a function of the three-geometry[5] only:

$$S = S[{}^{(3)}\mathcal{G}], \tag{2}$$

i.e., S is coordinate-independent.

(c) The principle of constructive interference.[6]

(d) The boundary condition that the spatial hypersurface either (i) be finite and have no boundary[7] or (ii) be asymptotically flat.

Conclusion:

(a) There exist four functions[8] N, N_i $(i=1, 2, 3)$ which together with g_{ij} give a space-time metric

$$\begin{aligned} ds^2 &= g_{ij}(N^i dx^0 + dx^i)(N^j dx^0 + dx^j) - N^2(dx^0)^2 \\ &= g_{ij}dx^i dx^j + 2N_i dx^i dx^0 + (N_j N^j - N^2)(dx^0)^2 \end{aligned} \tag{3}$$

that satisfies the Einstein field equations.

(b) The manifestly covariant equations of geometrodynamics,

$$\frac{\delta g_{ij}(x)}{\delta\sigma} = \frac{\delta H}{\delta\pi^{ij}(x)}, \quad \frac{\delta\pi^{ij}(x)}{\delta\sigma} = -\frac{\delta H}{\delta g_{ij}(x)},$$

are a consequence of the semiclassical approximation to quantum geometrodynamics. Here H is a functional of g_{ij} and $\pi^{ij} \equiv \delta S/\delta g_{ij}$. The Tomonaga-Schwinger[9] many-time parameter is denoted by σ.

In light of these conclusions, one should emphasize the utility of the EHJ equation. Its solution is the fountainhead from which one can, with the help of the principle of constructive interference, obtain any of the histories satisfying the ten Einstein vacuum field equations.[10]

Before we address ourselves to proving the above proposition in geometrodynamics, we recall in Sec. II how all the dynamics of a free particle arise from its HJ equation and the principle of constructive interference in space-time. Section III considers the identities that the solution to the EHJ equation must satisfy. Section IV describes the principle of constructive interference in superspace. In order to be able to use this principle in Sec. V, we introduce Tomonaga's "many-time" parametrization. Section VI presents the semiclassical approximation to quantum geometrodynamics. Section VII gives a method for testing the completeness of the solution to the EHJ equation, while Sec. VIII gives the derivation of the covariant-Hamiltonian equations that describe the history of a three-dimensional spacelike hypersurface. Section IX shows the consistency of these equations with the EHJ equation. Section X points out how one obtains the Einstein vacuum field equations from the covariant-Hamiltonian equations.

II. AN ANALOGY: A FREE PARTICLE IN SPACE-TIME

Consider the HJ equation for a particle in an empty space-time, whose metric is $g^{\mu\nu}$ $(\mu, \nu = 0, 1, \cdots, 3)$,

$$\mathcal{H} \equiv m^2c^2 + g^{\mu\nu}(\partial S/\partial x^\mu)(\partial S/\partial x^\nu) = 0. \tag{4}$$

($\mathcal{H}$ is the relativistically invariant "super-Hamiltonian"; see, for example, Kramers[11] or Landau and Lifschitz.[12])

The solution to this equation, the phase function S, is a function that is defined on all space-time. Different solutions to the HJ equation are characterized by different sets of integration constants. Denote the set of integration constants by α_i, $i=1, 2, 3$; they are usually associated with the energy and the momentum of the particle. In the semiclassical approximation to quantum mechanics one associates with a particular solution a wave function

$$\Psi \sim e^{iS/\hbar}.$$

The regions in space-time that are characterized by $S(x,\alpha)=19.1$, $S(x,\alpha)=19.2$, etc., are the de Broglie wave-front histories[13] of the wave function Ψ.

Now consider a slightly different solution to the HJ equation, say,

$$S(x, \alpha+\delta\alpha).$$

Once again one has wave-front histories such as $S(x, \alpha+\delta\alpha)=19.1$, $S(x, \alpha+\delta\alpha)=19.2$, etc. The wave

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[5] A three-geometry, denoted by ${}^{(3)}\mathcal{G}$, is a class of ${}^{(3)}g_{ij}$'s whose members have the property that one can be transformed into any other by a suitable coordinate transformation.

[6] E. A. Power and J. A. Wheeler, in *Geometrodynamics*, edited by J. A. Wheeler (Academic Press Inc., New York, 1962), p. 221.

[7] A closed three-geometry is assumed here to keep the discussion in closest accord with Einstein's ideas about the structure of space in the large. See A. Einstein, *The Meaning of Relativity* (Princeton University Press, Princeton, New Jersey, 1955), pp. 103, 104. See also J. A. Wheeler, in *Mach's Principle as Boundary Condition for Einstein's Equations in Gravitation and Relativity*, edited by H. Y. Chiu and W. F. Hoffman (W. A. Benjamin, Inc., New York, 1964); D. Brill and J. Cohen, Phys. Rev. **143**, 1011 (1966).

[8] These functions are known as the "lapse" and "shift" functions. See second reference in Ref. 7. See also J. A. Wheeler, in *Geometrodynamics and the Issue of the Final State in Relativity Groups and Topology*, edited by C. De Witt and B. De Witt (Gordon and Breach Science Publishers, Inc., New York, 1964).

[9] S. Tomonaga, Progr. Theoret. Phys. (Kyoto) **1**, 27 (1946); J. Schwinger, Phys. Rev. **74**, 1449 (1948). The first paper is also reprinted in *Quantum Electrodynamics*, edited by J. Schwinger (Dover Publications, Inc., New York, 1958). See also S. Tomonaga, Phys. Today **19**, No. 9, 25 (1966).

[10] This state of affairs is in marked contrast to a way of arriving at the Einstein field equations that merely depends upon the coordinate covariance of these equations. In this case one's starting hypothesis is that the initial value equations of general relativity, $G_{\mu 0}=0$, hold on every slice through space-time. By considering the normals n^ν to any of these slices, one obtains $G_{\mu\nu}n^\nu=0$. Arbitrary slices are under consideration. Consequently, the normals n^ν are arbitrary, and hence the Einstein equations $G_{\mu\nu}=0$ follow. Observe that here no new principles have been invoked: The demonstration only involves the equations; nothing has been said about the solution to these equations.

[11] H. A. Kramers, *Quantum Mechanics* (Interscience Publishers, Inc., New York, 1957), pp. 44, 84.

[12] L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields* (Addison-Wesley Publishing Co., Inc., Reading, Mass., 1962), p. 30.

[13] Synge calls them "de Broglie three-waves" or the "history of a de Broglie two-wave." See J. L. Synge, *Geometrical Mechanics and de Broglie Waves* (Cambridge University Press, New York, 1954).

fronts $S=19.1$ characterized by slightly different integration constants interfere constructively at a single point in space-time determined by

$$\begin{aligned}19.1&=S(x;\alpha_1,\alpha_2,\alpha_3)=S(x;\alpha_1+\delta\alpha_1,\alpha_2,\alpha_3)\\&=S(x;\alpha_1,\alpha_2+\delta\alpha_2,\alpha_3)=S(x;\alpha_1,\alpha_2,\alpha_3+\delta\alpha_3)\end{aligned}\quad(5)$$

(four equations for the four x^μ). The particle is understood to be located[6] at this point, according to the principle of constructive interference. Similarly, wave fronts $S=19.2$, etc., interfere constructively at another point in space-time. The four coordinates of that point are found by solving the four equations

$$\begin{aligned}19.2&=S(x+\Delta x;\alpha_1,\alpha_2,\alpha_3)=S(x+\Delta x;\alpha_1+\delta\alpha_1,\alpha_2,\alpha_3)\\&=S(x+\Delta x;\alpha_1,\alpha_2+\delta\alpha_2,\alpha_3)\\&=S(x+\Delta x;\alpha_1,\alpha_2,\alpha_3+\delta\alpha_3).\end{aligned}\quad(6)$$

The set of interference points characterized by $S=19.1$, $S=19.2$, etc., describe the path of a particle through space-time.

Is it possible to determine the instantaneous direction of the particle path in space-time? Yes. Subtract Eq. (5) from Eq. (6) and obtain

$$\begin{aligned}p_\mu(\alpha_1,\alpha_2\alpha_3)\Delta x^\mu&=p_\mu(\alpha_1+\delta\alpha_1,\alpha_2,\alpha_3)\Delta x^\mu\\&=p_\mu(\alpha_1,\alpha_2+\delta\alpha_2,\alpha_3)\Delta x^\mu=p_\mu(\alpha_1,\alpha_2,\alpha_3+\delta\alpha_3)\Delta x^\mu.\end{aligned}\quad(7)$$

Here the momentum is defined as the gradient of S:

$$p_\mu\equiv\partial S/\partial x^\mu.$$

Instead of specifying the direction of the particle world line by the separation Δx^μ between the two locations of the wave packet at two nearby instants, it is more appropriate to introduce a parameter continuous along the world line and specify the tangent:

$$\Delta x^\mu\rightarrow dx^\mu/ds.\quad(8)$$

Different choices of parametrization will give different magnitudes for the four-vector dx^μ/ds, but always the same direction (at a *given* point in space-time). Subtracting $p_\mu(\alpha_1,\alpha_2,\alpha_3)\Delta x^\mu$ from each term in Eq. (7) and going to the limit yields

$$\delta p_\mu\frac{dx^\mu}{ds}=0.\quad(9)$$

Here the variation δp_μ in p_μ is due to arbitrary infinitesimal variations in the integration constants α_1, α_2, and α_3.

The solution $S(x;\alpha_1,\alpha_2,\alpha_3)$ that we are considering is by assumption a *complete*[14] one (it has the maximum number of integration constants). Consequently, the variations δp_μ (defined over all space-time) due to the variations $\delta\alpha_i$, $i=1, 2, 3$, are all those that satisfy

$$\begin{aligned}0=\mathcal{H}(p_\mu(\alpha+\delta\alpha))&=\mathcal{H}(p_\mu+\delta p_\mu)\\&=\mathcal{H}(p_\mu)+(\partial\mathcal{H}/\partial p_\mu)\delta p_\mu,\end{aligned}\quad(10)$$

[14] R. Courant and D. Hilbert, *Methods of Mathematical Physics II* (Interscience Publishers, Inc., New York, 1962), Chap. 1, para. 4.

so that one can consider $\mathcal{H}(x^\nu,p_\mu)$ as a function of the independent variables $\{x^\nu,p_\mu\}$ and characterize the δp_μ which enter Eq. (9) by the condition

$$(\partial\mathcal{H}/\partial p_\mu)\delta p_\mu=0.\quad(11)$$

Therefore, Eq. (9) can be restated as follows: A necessary condition for dx^μ/ds to lie on a particle world line is that there exists a momentum p_μ such that the quantity

$$p_\mu\frac{dx^\mu}{ds}\quad(12)$$

is an extremum with respect to infinitesimal variations in p_μ. Here the typical variation is

$$p_\mu\rightarrow p_\mu'=p_\mu+\delta p_\mu.$$

It is understood, but it does not have to be said, that the change $\delta p_\mu(x^\sigma)$ is brought about by a change in the integration constants α_i; neither does one have to know the precise functional dependence of the p_μ's upon the α_i's. In other words, the variation from the original p_μ to the new momentum p_μ' can be treated as a quantity in its own right. The only restriction on this variation is that it satisfy Eq. (10), i.e., the p_μ and p_μ' must satisfy the HJ equation (4).

The HJ equation (4), together with the restatement of the condition for constructive interference as the extremum condition on expression (12), contains all the classical equations of motion for the particle in a nutshell.

The extremum principle is easily put into operation with the method of Lagrange multipliers:

$$\delta p_\mu\left(\frac{dx^\mu}{ds}-N\frac{\partial\mathcal{H}}{\partial p_\mu}\right)=0.\quad(13)$$

Arbitrary variations δp_μ fall into two classes: those that do and those that do not satisfy

$$\delta p_\mu\frac{\partial\mathcal{H}}{\partial p_\mu}=0.$$

With a suitable choice for N, the coefficients of both classes of variations in Eq. (13) must vanish. Thus we have the first half of Hamilton's equations ("velocity equations") of motion

$$\frac{dx^\mu}{ds}=N\frac{\partial\mathcal{H}}{\partial p_\mu}$$

expressed in covariant form. To obtain the other half, use the fact that the super-Hamiltonian $\mathcal{H}$ of Eq. (4) is everywhere zero, so that its derivative is also everywhere zero; thus

$$0=\frac{\partial\mathcal{H}}{\partial x^\nu}+\frac{\partial\mathcal{H}}{\partial p_\mu}\frac{\partial p_\mu}{\partial x^\nu}=\frac{\partial\mathcal{H}}{\partial x^\nu}+\frac{\partial\mathcal{H}}{\partial p_\mu}\frac{\partial p_\nu}{\partial x^\mu}.$$

Now find the rate of change of the momentum; thus

$$\frac{dp_\nu}{ds}=\frac{\partial p_\nu}{\partial x^\mu}\frac{dx^\mu}{ds}$$

or, with the help of the velocity equation,

$$\frac{dp_\nu}{ds}=N\frac{\partial p_\nu}{\partial x^\mu}\frac{\partial \mathcal{H}}{\partial p_\mu}=-N\frac{\partial \mathcal{H}}{\partial x^\nu},$$

thus completing the derivation of the Hamilton equations of motion from the HJ equation.

To go from the Hamiltonian form to the Lagrangian form of the equations of motion, solve the velocity equation for p_ν and substitute into the last equation, obtaining

$$0=\frac{d^2x^\mu}{ds^2}+\tfrac{1}{2}g^{\mu\nu}\left(\frac{\partial g_{\nu\alpha}}{\partial x^\beta}+\frac{\partial g_{\nu\beta}}{\partial x^\alpha}-\frac{\partial g_{\alpha\beta}}{\partial x^\nu}\right)\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds}+\frac{N_{,\alpha}}{N}\frac{dx^\alpha}{ds}\frac{dx^\mu}{ds}$$

$$=\frac{d^2x^\mu}{ds^2}+\Gamma_{\alpha\beta}{}^{\mu}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds}+\frac{1}{N}\frac{dN}{ds}\frac{dx^\mu}{ds},$$

the equation for the world line of the particle. It is possible to make the last term vanish by an appropriate choice of the parameters. This derivation shows that the classical equations of motion may be obtained from the semiclassical approximation of quantum mechanics. Observe that the super-Hamiltonian $\mathcal{H}$ for the particle is "constrained" to be zero and that, according to the equations of motion, it stays zero all along the world line of the particle.

III. A RESTRICTION ON THE SOLUTION TO THE EHJ EQUATION

Returning now to the EHJ equation, note that the dynamical phase S, ostensibly a functional of the six metric components,

$$S=S[g_{ij}],$$

we require to be a functional of the three-geometry alone, regardless of all transformations of coordinates (and indeed, in principle, regardless of whether we do or do not choose to use any coordinates at all in describing this three-geometry); thus

$$S=S[{}^{(3)}\mathcal{G}].$$

In consequence it follows that S, expressed as a functional of the g_{ij}'s, must everywhere satisfy the three identities[15]

$$[\delta S/\delta g_{ij}(x)]_{|j}=0, \tag{14}$$

where the vertical bar indicates the three-dimensional covariant derivative.

Proof. Given the ${}^{(3)}\mathcal{G}$ expressed in terms of coordinates x^i by the metric coefficients g_{ij}, go to infinitesimally different coordinates

$$x'^i=x^i+\epsilon\xi^i(x)$$

and have the same three-geometry expressed in terms of the metric coefficients

$$g_{ij}'(x)=g_{ij}(x)+\delta g_{ij}(x),$$

with

$$\delta g_{ij}=-\epsilon(\xi_{i|j}+\xi_{j|i}).$$

The ostensible change in S brought about by this change in the metric coefficients is

$$\delta S=\int\frac{\delta S}{\delta g_{ij}(x)}\delta g_{ij}(x)d^3x$$

$$=-2\epsilon\int\frac{\delta S}{\delta g_{ij}(x)}\xi_{i|j}d^3x$$

$$=2\epsilon\int\frac{\delta S}{\delta g_{ij}(x)}{}_{|j}\xi_i d^3x.$$

However, the ${}^{(3)}\mathcal{G}$ itself has not changed at all. Hence δS must vanish, and vanish, moreover, for arbitrary $\xi_i(x)$, in consequence of which Eq. (14) follows at once.

IV. PRINCIPLE OF CONSTRUCTIVE INTERFERENCE. A

Just as the phase function for a single particle is defined on space-time, so the phase functional of the EHJ equation is defined on superspace. Superspace[16] is the set of equivalence classes of all spacelike $g_{ij}(x)$'s that can be transformed into each other by means of spatial coordinate transformations.

Consider a solution to the EHJ equation satisfying Eq. (14):

$$S[{}^{(3)}\mathcal{G};\alpha(u),\beta(u)]. \tag{15}$$

Here α and β are integration constants identified by the parameters $u=(u_1,u_2,u_3)$. In the linearized theory of gravitation, α and β may be identified with the initial two polarization amplitudes, (u_1,u_2,u_3) being either the spatial coordinates or the wave-vector coordinates. Note, however, that the existence of the constants of integration depends in no way upon the existence of the linear approximation to the exact theory. Although the constants $\alpha(u)$ and $\beta(u)$ have been explicitly indicated in the solution S, no attempt is made in this paper to actually prove the existence of these constants. For the

[15] The essential ideas of this fact were already known to P. Higgs, Phys. Rev. Letters **1**, 373 (1958); **3**, 66 (1959). See also E. Schrödinger, *Space-Time Structure* (Cambridge University Press, New York, 1950), Chap. XI.

[16] Observe that although superspace is infinite-dimensional, it is a space of countable dimension. The reason is that to specify a *continuous* function $g_{ij}(x)$, one merely has to specify it on the points of the spatial hypersurface that have rational coordinates. See L. Streit, in *Proceedings of the Fourth Internationale Universitätswochen für Kernphysik 1965 der Karl Franzens-Universität Graz: Quantum Electrodynamics*, Acta Physica Austriaca Suppl. II, p. 3. Superspace and its relevance to quantum geometrodynamics are discussed in the first reference of Ref. 1.

purpose of formulating the principle of constructive interference, we *assume* their existence. Observe, on the other hand, that the deduction of the ten Einstein field equations (Secs. VIII and X) from this principle is independent of the knowledge of these constants. The isograms (subspaces that are characterized by the same value) of the phase functional are the histories of the de Broglie wave fronts.

Now consider the phase functionals obtained by changing slightly each of the $2\times\infty^3$ constants of motion:

$$\alpha(u)\rightarrow\alpha(u)+\delta\alpha(u),$$
$$\beta(u)\rightarrow\beta(u)+\delta\beta(u).$$

The de Broglie wave-front histories (isograms with a fixed phase value, say, $S=19.1$) of the phase functionals interfere (intersect) to form a packet of three geometries such as ${}^{(3)}\mathcal{G}_A$, ${}^{(3)}\mathcal{G}_B$, ${}^{(3)}\mathcal{G}_C$, etc., in Fig. 1. One ${}^{(3)}\mathcal{G}$—say, ${}^{(3)}\mathcal{G}_A$—can be described in a given coordinate system x^ν by means of a particular set of metric coefficients g_{ij}. However, under a coordinate transformation these metric coefficients are transformed to other metric coefficients. The totality of all g_{ij}'s obtainable under the action of the entire group of three-coordinate transformations corresponds to the ${}^{(3)}\mathcal{G}_A$ and to the one point in superspace in question.

Similarly, from the de Broglie wave-front histories, one constructs a stack of wave packets with $S=19.2$, $19.3,\cdots$, etc. The set of wave packets thus constructed lie on a subset set of points in superspace, the history of a three-geometry. See Fig. 1.

This shows how the principle of constructive interference together with the EHJ equation yields a classical history through superspace.

In order to cast the principle into suitable mathematical language, consider a wave packet; such a wave packet is a superposition of Ψ functions. The phases of the several individual Ψ waves are given slightly varied HJ functionals $S[{}^{(3)}\mathcal{G};\alpha,\beta]$. A three-geometry ${}^{(3)}\mathcal{G}$ lies in this packet if this ${}^{(3)}\mathcal{G}$ satisfies the conditions ($2\times\infty^3$ of them[17])

$$\begin{aligned}19.1&=S[{}^{(3)}\mathcal{G};\alpha(u)+\delta\alpha(u),\beta(u)+\delta\beta(u)]\\&=S[{}^{(3)}\mathcal{G};\alpha(u);\beta(u)]\end{aligned}\tag{16a}$$

for all small variations $\delta\alpha(u)$ and $\delta\beta(u)$. In this event we may say that ${}^{(3)}\mathcal{G}$ is a "yes" point in superspace, in the sense that this ${}^{(3)}\mathcal{G}$ "occurs in the classical history of space" (Fig. 1). Consider a "nearby" allowed ${}^{(3)}\mathcal{G}$—that is to say, a nearby "yes" point in superspace—which we may denote symbolically by ${}^{(3)}\mathcal{G}+d^{(3)}\mathcal{G}$. It satisfies the equation of constructive interference,[17]

$$\begin{aligned}19.2&=S[{}^{(3)}\mathcal{G}+d^{(3)}\mathcal{G};\alpha(u)+\delta\alpha(u),\beta(u)+\delta\beta(u)]\\&=S[{}^{(3)}\mathcal{G}+d^{(3)}\mathcal{G};\alpha(u),\beta(u)],\end{aligned}\tag{16b}$$

[17] One should note here that there is nothing special about the value $S=19.1$ and $S=19.2$. A different choice of a solution S (different in that S is a different functional of the constants) would have resulted in different phase values for the wave packet.

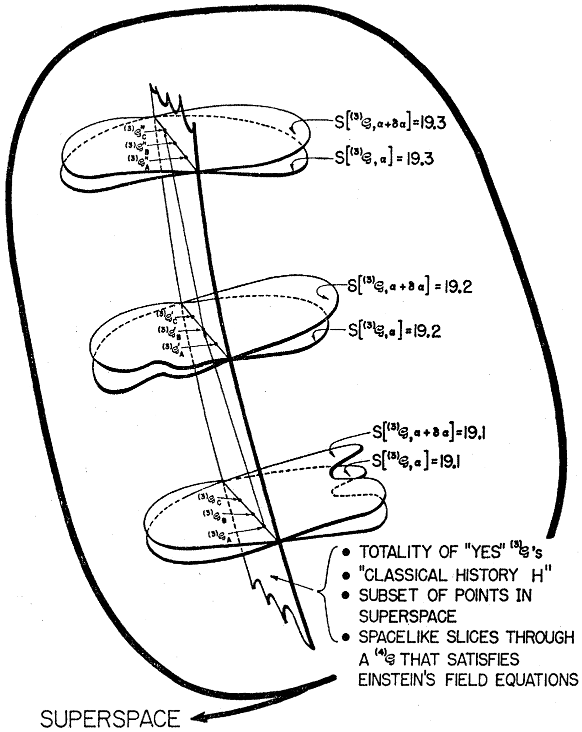

FIG. 1. de Broglie waves in superspace interfering constructively to produce a four-geometry. Each history of a de Broglie wave front is a subspace in superspace on which the phase functional S is constant. Now focus attention on all those de Broglie wave-front histories (in single-particle mechanics, Synge calls them de Broglie three-waves, see Ref. 12) that are characterized by the phase functionals that have $S=19.1$. These phase functionals differ from each other by virtue of the fact that their constants of motion ($2\times\infty^3$ of them) are different for different phase functionals. In the above figure all these constants have been condensed into the letter α. The phase functionals that differ slightly from each other in α but have the same value $S=19.1$ intersect each other in a set of "yes" three-geometries (three typical ones are labelled by ${}^{(3)}\mathcal{G}_A$, ${}^{(3)}\mathcal{G}_B$, ${}^{(3)}\mathcal{G}_C$). Observe that these "yes" points in superspace are characterized by $2\times\infty^3$ equations such as Eq. (16a); however, a point in superspace is characterized by $3\times\infty^3$ quantities. It is therefore quite obvious that there is more than one "yes" three-geometry characterized by $S=19.1$. Similarly for $S=19.2$, 19.3, etc., one has subsets of points which are "yes" three-geometries. As indicated in the figure, the totality of subsets form a four-geometry that satisfies Einstein's field equations. Each point on this totality of points is a spacelike slice through this ${}^{(4)}\mathcal{G}$. As a solution to the Einstein field equations, this ${}^{(4)}\mathcal{G}$ is usually represented as a particular sequence of spacelike slices. In this figure such a representation is a one-parameter curve through a particular sequence of "yes" points, such as ${}^{(3)}\mathcal{G}_A\rightarrow{}^{(3)}\mathcal{G}_{B}'\rightarrow{}^{(3)}\mathcal{G}_{C}''$.

again for all choices of the arbitrary small variations $\delta\alpha(u)$ and $\delta\beta(u)$ in the integration constants. The difference between the three-geometries associated with two wave packets is denoted symbolically by

$$d^{(3)}\mathcal{G}.$$

The difference between Eqs. (16a) and (16b) yields

$$\frac{\delta S}{\delta^{(3)}\mathcal{G}}(\alpha+\delta\alpha,\beta+\delta\beta)d^{(3)}\mathcal{G}=\frac{\delta S}{\delta^{(3)}\mathcal{G}}(\alpha,\beta)d^{(3)}\mathcal{G}.\tag{17}$$

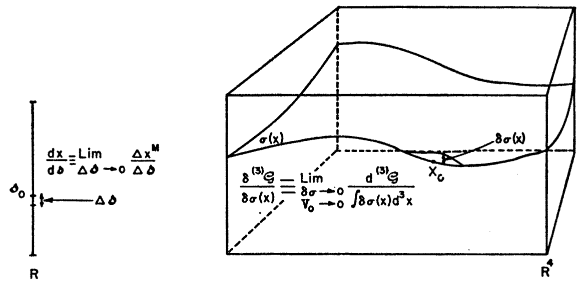

FIG. 2. Tomonaga's infinite-dimensional parametrization applied to a three-geometry. In particle dynamics, as shown on the left, a single parameter is associated with the particle. A mapping from the real line R into space-time is the curve along which the partlcle travels. In particular, given two close-by points in space-time separated by an infinitesimal vector Δx^μ, one can introduce a curve parameter by associating with this vector an infinitesimal interval Δs. The tangent vector obtained by going to the limit $\lim \Delta x^\mu/\Delta s$ as $\Delta s \to 0$ defines the curve in an infinitesimal neighborhood. This tangent vector is merely a linear mapping from the tangent space of R at x_0 into the tangent space of space-time where the particle is located. For a clear and succint discussion of this idea, see for example, J. Milnor, *Morse Theory* (Princeton University Press, Princeton, N. J., 1963), p. 45.

Tomonaga's parametrization as shown on the right is a generalization of the above procedure. A classical history of a spatial hypersurface has to be parametrized by $R\times R\times R\times R \equiv R^4$. A mapping from R^4 into superspace is the history along which the hypersurface travels. A particular hypersurface is labelled by a particular three-dimensional section through R^4 (in order not to strain one's mind only a two-dimensional section through R^3 has actually been drawn). Such a particular three-dimensional section is represented as a function of three variables, $\sigma(x_1,x_2,x_3)$. Is it just as straightforward to define a tangent vector for geometrodynamics as for particle dynamics? Not quite. Consider two close-by geometries that are separated by ${}^{(3)}\mathcal{G}' - {}^{(3)}\mathcal{G} = d^{(3)}\mathcal{G}$. With $d^{(3)}\mathcal{G}$ one associates a variation $\delta\sigma(x_1,x_2,x_3)$ which is zero everywhere except for a small region V_0 around $\{x_{01},x_{02},x_{03}\}$ on the surface $\sigma(x)$. Then the tangent is defined as the limit of $d^{(3)}\mathcal{G}/\int\delta\sigma(x)d^3x$ as shown in the figure. This limit is a distribution. Furthermore, observe the difference in the nature of the tangent vector in particle dynamics and in the geometrodynamics. In the former only the choice of the parameter difference Δs was arbitrary. The effect of that arbitrariness resulted in an arbitrariness in the length of the tangent vector, but not its direction. In geometrodynamics, on the other hand, the choice of $\delta\sigma(x_1,x_2,x_3)$ at each point on the hypersurface is arbitrary. The result of this fact is that $\int(\delta^{(3)}\mathcal{G}/\delta\sigma(x))\delta\sigma(x)d^3x = d^{(3)}\mathcal{G}$ is arbitrary to the extent that even though $d^{(3)}\mathcal{G}$ lies on a "classical history H" (see Fig. 1), $d^{(3)}\mathcal{G}$ may be any one of the vectors $d^{(3)}\mathcal{G} = {}^{(3)}\mathcal{G}_i' - {}^{(3)}\mathcal{G}_j$, i, $j = A$, B, C, etc.; in other words, the various $d^{(3)}\mathcal{G}$ resulting from various choices of $\delta\sigma(x)$ are *not* linearly related in general. Observe that the "number" of linearly independent $d^{(3)}\mathcal{G}$'s is ∞^3.

These infinitesimal "vectors" $d^{(3)}\mathcal{G}$ can be described more precisely as follows: Consider the set of functions $\{\sigma(x)\}$ from closed three-dimensional manifolds into the reals. Now consider a function that takes the space $\{\sigma(x)\}$ into superspace. Let $\mathcal{G}[\sigma]$ be the image of a particular $\sigma(x)$. To describe the infinitesimal "vectors" $d^{(3)}\mathcal{G}$ focus attention on (1) the tangent space to $\{\sigma(x)\}$ at $\sigma(x)$, (2) the tangent space to superspace at $\mathcal{G}[\sigma]$, and (3) the mapping, $\delta^{(3)}\mathcal{G}/\delta\sigma(x)$, which maps linearly an element $\delta\sigma(x)$ in the first tangent space into an element $d^{(3)}\mathcal{G}$ in the second tangent space. The appropriate equation for this mapping is Eq. (18a).

Before continuing with the principle of constructive interference, we must take time to describe the parametrization of ${}^{(3)}\mathcal{G}$.

V. TOMONAGA'S PARAMETRIZATION

In one-particle dynamics, where Eq. (5) was the analog of Eq. (16a), a definite choice for the value of S resulted in a definite point x^m along the dynamical history H of the particle. No corresponding result applies here. For a given value of the HJ phase function, $S = 19.1$, there is not a single "yes" ${}^{(3)}\mathcal{G}$, but a whole set of "yes" ${}^{(3)}\mathcal{G}$'s, as indicated symbolically by such "points of superspace" as A, B, C, etc., in Fig. 1. Consequently, no single quantity S is adequate any longer to parametrize the allowed ${}^{(3)}\mathcal{G}$'s in the dynamical arena (superspace). Instead, it turns out that we must use an infinite-dimensional parametrization of a kind first made familiar to physics by Tomonaga.[9] See Fig. 2.

How can one see qualitatively that the "yes" ${}^{(3)}\mathcal{G}$'s need Tomonaga's σ parametrization to distinguish one from another? Compare the count of unknowns in a three-geometry with the count of equations in (16a). Denote the number of points on a spatial hypersurface by ∞^3. The function $g_{ij}(x_1x_2x_3)$ is determined by specifying its value for each $(x_1x_2x_3)$. Consequently, $6\times\infty^3$ numbers[18] must be specified to single out a particular function $g_{ij}(x)$. The number of possible functions is therefore $\infty^{6\times\infty^3}$. Similarly, the number of coordinate functions that transforms one $g_{ij}(x)$ into an equivalent one is $\infty^{3\times\infty^3}$. Consequently, the number of nonequivalent metrics[18] is $\infty^{6\times\infty^3}/\infty^{3\times\infty^3} = \infty^{3\times\infty^3}$. In other words, it takes $3\times\infty^3$ numbers to specify a three-geometry.

Furthermore, observe that the phase functional S is defined on the space of all g_{ij}'s. The dimensionality of this space is $6\times\infty^3$. The number of equations for S is $(3+1)\times\infty^3$. One can see that there are $(6-4)\times\infty^3 = 2\times\infty^3$ constants of motion.[18] For a particular S value there are just as many equations [Eq. (16a)] that determine the set of "yes" points.

Using the fact that it takes $3\times\infty^3$ numbers to specify a three-geometry, one concludes that the number of "yes" points for a particular S value is $(3-2)\times\infty^3 = 1\times\infty^3$.

For two values, say, $S = 19.1$ and 19.2 in Fig. 1, one has two sets of "yes" points in superspace: $\{{}^{(3)}\mathcal{G}_A, {}^{(3)}\mathcal{G}_B, {}^{(3)}\mathcal{G}_C, \text{etc.}\}$ and $\{{}^{(3)}\mathcal{G}_A', {}^{(3)}\mathcal{G}_B', {}^{(3)}\mathcal{G}_C', \text{etc.}\}$. Consequently, the number of vectors connecting an unprimed ${}^{(3)}\mathcal{G}$ with any one of the primed ${}^{(3)}\mathcal{G}$'s is ∞^3. This is precisely the number of parameters $\sigma(x)$ (Tomonaga's parametrization) necessary to label a classical history.

Before leaving the subject of parametrization, we can appropriately remind ourselves again that the item of physical concern in one-particle dynamics is never a parametr*ization*-dependent quantity such as dp^μ/ds or dx^μ/ds by itself, but always such a parametr*ization*-independent (parameter-dependent!) quantity as $(dp^\mu/ds)ds$ or $(dx^\mu/ds)ds$.

[18] The counting processes here are by no means mathematically rigorous. In particular: (1) Finite multiples and powers of the same cardinal number still have the same cardinality. (2) The division of cardinal numbers is an undefined process. See G. Birkhoff and S. MacLane, *A Survey of Modern Algebra* (The MacMillan Company, New York, 1961), p. 366. However, no better-defined counting procedure seems to exist, and this procedure allows one to communicate what one is talking about. Consequently, we shall use it. See also Ref. 16.

Similarly, in geometrodynamics we are not concerned with the parametrization-dependent quantity $\delta^{(3)}\mathcal{G}/\delta\sigma$ (or with $\delta g_{ij}/\delta\sigma$), but with such a parametrization-independent quantity as

$$\delta^{(3)}\mathcal{G}=\int \frac{\delta^{(3)}\mathcal{G}}{\delta\sigma}\delta\sigma d^3x \tag{18a}$$

or, equivalently, with any particular member of the equivalence class $\delta^{(3)}\mathcal{G}$,

$$\delta g_{ij}=\int \frac{\delta g_{ij}}{\delta\sigma}\delta\sigma d^3x. \tag{18b}$$

Note that the definition of the infinitesimal test function $\delta\sigma$ is such that $\delta\sigma d^3x$ is the volume generated by the displacement $\delta\sigma$ imparted to the spatial hypersurface element d^3x. See Fig. 2. Thus the appropriate focus of attention is δg_{ij} itself, rather than $\delta g_{ij}/\delta\sigma$. This circumstance means that one can forego getting into all the details of this or that conceivable scheme of parametrization.

VI. PRINCIPLE OF CONSTRUCTIVE INTERFERENCE. B

Instead of specifying two close-by geometries by

$$^{(3)}\mathcal{G} \quad \text{and} \quad ^{(3)}\mathcal{G}'={}^{(3)}\mathcal{G}+d^{(3)}\mathcal{G},$$

introduce Tomonaga's parametrization and thus consider $\delta^{(3)}\mathcal{G}/\delta\sigma(x)$ instead of $d^{(3)}\mathcal{G}$. Then subtracting the right side of Eq. (17) from both sides of Eq. (17) yields to first order a necessary condition for the "vector"

$$\delta^{(3)}\mathcal{G}/\delta\sigma(x)$$

to be tangent to a history through $^{(3)}\mathcal{G}$:

$$\delta\left(\frac{\delta S}{\delta^{(3)}\mathcal{G}}\right)\frac{\delta^{(3)}\mathcal{G}}{\delta\sigma}=0. \tag{19}$$

Here

$$\delta\left(\frac{\delta S}{\delta^{(3)}\mathcal{G}}\right)$$

denotes the change in $\delta S/\delta^{(3)}\mathcal{G}$ due to an arbitrary infinitesimal variation in

$$\{\alpha(u),\beta(u)\}.$$

The EHJ equation together with the principle of constructive interference as exhibited in Eq. (19),

$$^{(3)}R-\left(\frac{\delta S}{\delta^{(3)}\mathcal{G}}\right)\left(\frac{\delta S}{\delta^{(3)}\mathcal{G}}\right)^*=0, \tag{20}$$

$$\delta\left(\frac{\delta S}{\delta^{(3)}\mathcal{G}}\right)\frac{\delta^{(3)}\mathcal{G}}{\delta\sigma}=0, \tag{19}$$

contains all of general relativity in a nutshell. The starred vector $(\delta S/\delta^{(3)}\mathcal{G})^*$ is the dual (with respect to the De Witt metric[19]) counterpart of the vector

$$\delta S/\delta^{(3)}\mathcal{G}$$

defined on superspace, the space of three-geometries.

An easy way of obtaining the ten Einstein field equations is to use the language of classical tensor calculus. In that case the tangent vector in Eq. (19) becomes

$$\frac{\delta^{(3)}\mathcal{G}}{\delta\sigma(x')}\rightarrow\frac{\delta g_{ij}(x;\sigma(x'))}{\delta\sigma(x')}. \tag{21}$$

The derivative of S with respect to $^{(3)}\mathcal{G}$ becomes

$$\frac{\delta S}{\delta^{(3)}\mathcal{G}}\rightarrow\frac{\delta S}{\delta g_{ij}(x)}\equiv\pi^{ij}(x),$$

where $\pi^{ij}(x)$ must satisfy

$$\pi^{ij}{}_{|j}=0 \tag{22}$$

in order for S to depend only on $^{(3)}\mathcal{G}$ [Eq. (2)].

With this notation, Eqs. (19) and (20) become

$$\int \delta\pi^{ij}(x)\frac{\delta g_{ij}(x)}{\delta\sigma(x')}d^3x=0, \tag{23}$$

$$^{(3)}R+g^{-1}(\tfrac{1}{2}g_{ij}g_{kl}-g_{ik}g_{jl})\pi^{ij}\pi^{kl}=0. \tag{24}$$

In order to make the principle of constructive interference as stated in Eq. (23) more amenable to mathematical treatment, let us restate it as follows: In order that a change δg_{ij}, or, equivalently, that $\delta g_{ij}/\delta\sigma$, be a vector tangent to a history, it is necessary that there exist a $\pi^{ij}(x)$ with the property that

$$\int \pi^{ij}(x)\frac{\delta g_{ij}(x)}{\delta\sigma(x')}d^3x=\text{extremum} \tag{25}$$

if one changes the integration constants $\alpha(u)$ and $\beta(u)$ slightly.

Since there are six functions $\pi^{ij}(x)$ and only two independent functions $\alpha(u)$ and $\beta(u)$, the $\pi^{ij}(x)$ cannot be varied arbitrarily. This nonarbitrariness is also reflected in the fact that $\pi^{ij}(x)$ must satisfy Eqs. (23) and (24). Although these equations put restrictions on the allowable momentum densities $\pi^{ij}(x)$ at each point in the space of $g_{ij}(x)$'s, there is still a copious amount of freedom for having this, that, or the other $\pi^{ij}(x)$. This freedom in the choice of the momentum density can be associated with the freedom that one has in adjusting the integration constants that result from solving the EHJ equation.

VII. COMPLETENESS OF THE SOLUTION OF THE EHJ EQUATION

In order to clarify the connection between these two freedoms, it is appropriate to ask and answer the following question: Given a solution to Eqs. (1) and (14)

[19] The geometry and topology implied by this metric [i.e., $^{(3)}g^{-1/2}(\frac{1}{2}g_{ij}g_{kl}-g_{ik}g_{jl})$] have been analyzed by B. De Witt, Phys. Rev. **160**, 1113 (1967).

$$S[g_{ij};\alpha(u),\beta(u)], \tag{26}$$

the possibility may arise that this solution is a pathological one, since, although it ostensibly depends upon two free functions α and β ("constants of integration"), it might turn out that for some (u_{01},u_{02},u_{03}) this given solution in reality is independent of $\alpha(u)$ for $u=(u_{01},u_{02},u_{03})$: What assurance does one have that the solution is a *complete* one in the sense that the solution S is a functional of the maximum number of independent constants possible?

To answer that question, denote for the time being the momentum density obtained from Eq. (26) by

$$\bar{\pi}^{ij}(x)=\frac{\delta S}{\delta g_{ij}(x)}[\alpha(u),\beta(u)]. \tag{27}$$

Then consider the linearly independent variations in $\bar{\pi}^{ij}$ due to infinitesimal variations in each constant $\alpha(u)$ and $\beta(u)$:

$$\delta\bar{\pi}^{ij}(x)=\left(\int\frac{\delta^2 S}{\delta g_{ij}(x)\delta\alpha(u)}\delta\alpha(u)+\frac{\delta^2 S}{\delta g_{ij}(x)\delta\beta(u)}\delta\beta(u)\right)du. \tag{28}$$

It is clear that both $\bar{\pi}^{ij}$ and $\bar{\pi}^{ij}+\delta\bar{\pi}^{ij}$ satisfy Eqs. (22) and (24). The problem that we are faced with now is this: Do we get a *maximum* number of independent solutions

$$\bar{\pi}^{ij}+\delta\bar{\pi}^{ij}$$

by slightly varying the integration constants $\{\alpha(u),\beta(u)\}$? In other words, how do we test the solutions

$$\bar{\pi}^{ij}+\delta\bar{\pi}^{ij}$$

to see that we have a *complete* set?

First, we must consider the variation equations associated with Eqs. (22) and (24),

$$\delta\pi^{ij}{}_{|j}=0, \tag{29}$$

$$K_{ij}\delta\pi^{ij}=0. \tag{30}$$

Here K_{ij} is the "extrinsic curvature"[20] of a three-dimensional manifold imbedded in a four-dimensional one:

$$K_{ij}=({}^{(3)}g)^{-1/2}(\tfrac{1}{2}g_{ij}\pi_k{}^k-\pi_{ij}).$$

Since we are examining the completeness of the solution in Eq. (27), we set

$$\pi^{ij}(x)=\bar{\pi}^{ij}(x)$$

and examine all the solutions that differ infinitesimally from $\bar{\pi}^{ij}(x)$. This means that we must examine the solutions to the Eqs. (29) and (30), which are linear in the unknown $\delta\pi^{ij}$. The result of this examination (see the Appendix) will be that Eq. (29) has a solution that is parametrized by three functions, and that, as a consequence of Eq. (30), $\delta\pi^{ij}$ depends upon only two arbitrary functions $\delta a(x)$ and $\delta b(x)$ defined on the three-dimensional submanifold:

$$\delta\pi^{ij}(x)=\delta\pi^{ij}[\delta a(x),\delta b(x)]. \tag{31}$$

Consequently, the question of the completeness of the solution S, Eq. (26), reduces merely to the existence of a one-to-one correspondence between all possible $\delta\pi^{ij}$, Eq. (31), and the $\delta\bar{\pi}^{ij}$ in Eq. (28).

In other words, in order that the solution S be a *complete* solution, it is necessary and sufficient that there exist a nonsingular linear transformation

$$G_{k'l'}{}^{ij}(x,x')$$

that maps the set of linearly independent variations, Eq. (31), onto the set $\delta\bar{\pi}^{ij}(x)$:

$$\delta\bar{\pi}^{ij}(x)=\int G_{k'l'}{}^{ij}(x,x')\delta\pi^{k'l'}(x')d^3x'.$$

In general, $G_{kl}{}^{ij}(x,x')$ is not the identity transformation, because $\delta\bar{\pi}^{ij}$ may be a variation in a "collective mode" [in this case, (u_1,u_2,u_3) in Eq. (15) is a wave vector], whereas $\delta\pi^{ij}$ is a spatially localized function.

If S is not a *complete* solution, the $G_{k'l'}{}^{ij}(x,x')$ is singular. This means that some $\delta\pi^{k'l'}(x')\neq 0$ is mapped into zero. The existence of a nonsingular transformation assures us that the number of constants $\{\alpha(u),\beta(u)\}$, Eq. (27), is equal the number of degrees of freedom in $\{\delta a(x),\delta b(x)\}$, Eq. (31), which are responsible for all possible variations in the π^{ij} that satisfy Eqs. (22) and (24).

Consequently, if one has found a complete solution of the EHJ equation, then the variations of the momentum density [in Eq. (19)] are equivalent to variations in π^{ij} that satisfy Eqs. (22) and (24).

VIII. DERIVATION OF THE DYNAMICAL EQUATIONS

Let us return our attention to proving the proposition relating the EHJ equation to the ten Einstein field equations. As a first step, it is necessary to obtain a relationship between $\delta g_{ij}/\delta\sigma$ and π^{ij}. According to our previous discussion, we replace Eq. (25) with the help of Eq. (18b) by

$$\int\pi^{ij}\delta g_{ij}d^3x. \tag{32}$$

This expression must be an extremum with respect to variations in $\pi^{ij}(x)$ subject to the restrictions

$$R_0\equiv(g)^{1/2}[{}^{(3)}R+g^{-1}(\tfrac{1}{2}g_{ij}g_{kl}-g_{ik}g_{jl})\pi^{ij}\pi^{kl}]=0, \tag{33}$$

$$\pi^{ij}{}_{|j}=0, \tag{22}$$

where

$$\pi^{ij}(x)=\delta S/\delta g_{ij}(x). \tag{34}$$

[20] See Ref. 8. K_{ij} is also called the "second fundamental form" [R. Arnowitt, S. Deser, and C. W. Misner, Phys. Rev. **116**, 1322 (1959)]. See also L. Eisenhart, *Riemannian Geometry* (Princeton University Press, Princeton, New Jersey, 1964), p. 343.

The restrictions on the phase functional S can be taken into account in the extremum principle. Multiply them by yet-to-be-determined functions $\delta M(x)$ and $2\delta M_i(x)$, respectively, and add the product to the integrand in Eq. (32):

$$\int (\pi^{ij}\delta g_{ij}+\delta M R_0+2\delta M_i \pi^{ij}{}_{|j})d^3x.$$

Now consider the changes in this integral due to arbitrary variations in π^{ij}:

$$\int\left[\delta g_{ij}\delta\pi^{ij}+\delta M\left(\frac{\partial R_0}{\partial \pi^{ij}}\right)\delta\pi^{ij}-2\delta M_{i|j}\delta\pi^{ij}\right]d^3x + \int_\partial \delta M_i \delta\pi^{ij} dS_j. \quad (35)$$

Here we performed an integration by parts. The surface term vanishes because of the boundary conditions. Before proceeding further, it may be appropriate to remind ourselves again that the change in g_{ij}, δg_{ij} has nothing to do with the variations in π^{ij}, $\delta\pi^{ij}$. As a matter of fact, while we are considerlng arbitrary $\delta\pi^{ij}$, the change δg_{ij} stays fixed. The arbitrary variations in π^{ij} fall into two classes: (a) those that do satisfy the variation equations of Eqs. (24) and (22) and (b) those that do not satisfy these equations.

The principle of constructive interference requires that the variations of the integral, Eq. (35), vanish for class-(a) variations. Consequently, the coefficients of these variations must vanish. Now adjust the functions $\delta M(x)$ and $\delta M_i(x)$ so that the coefficients of the class-(b) variations also vanish. Evidently the result of these considerations is

$$\delta g_{ij}=-2\delta M(g)^{-1/2}(\tfrac{1}{2}g_{ij}\pi_k{}^k-\pi_{ij})+2\delta M_{(i|j)}. \quad (36)$$

Here $\delta M_{(i|j)}=\frac{1}{2}(\delta M_{i|j}+\delta M_{j|i})$, and g_{ij} lowers the indices of the tensor density π^{kl}, so that $\pi=\pi_i{}^i$. Equation (36) relates the change in g_{ij} between two close-by three-geometries to the "momentum" π^{ij}. The second term on the right-hand side is the change in g_{ij} due to a mere coordinate transformation, while the first term is the change due to an actual motion in superspace. The infinitesimal quantities δM and $\delta M_{i|j}$ are the "proper" parameter differences that separate two close-by three-geometries. (In particle dynamics the quantity corresponding to δM and $\delta M_{i|j}$ is Nds.) Observe that Eq. (36) holds on some spatial hypersurface, and that this equation is covariant with respect to coordinate transformations in space-time.

In order to cast Eq. (36) into a more familiar form observe that

$$\delta M=\int \frac{\delta M}{\delta\sigma(x')}\delta\sigma(x')d^3x',$$

$$\delta M_i=\int \frac{\delta M_i}{\delta\sigma(x')}\delta\sigma(x')d^3x'.$$

Consequently,

$$\frac{\delta g_{ij}(x,\sigma)}{\delta\sigma(x')}=-2\frac{\delta M(x,\sigma)}{\delta\sigma(x')}(g)^{-1/2}(\tfrac{1}{2}g_{ij}\pi_k{}^k-\pi_{ij})+2\frac{\delta M_{(i|j)}}{\delta\sigma(x')} \quad (37a)$$

or

$$\frac{\delta M(x,\sigma)}{\delta\sigma(x')}(\tfrac{1}{2}g_{ij}\pi_k{}^k-\pi_{ij})=\frac{1}{2}\left(\frac{\delta M_{i|j}}{\delta\sigma(x')}+\frac{\delta M_{j|i}}{\delta\sigma(x')}-\frac{\delta g_{ij}}{\delta\sigma(x')}\right)(g)^{1/2}. \quad (37b)$$

These equations are still manifestly covariant. By introducing

$$H_0(x')=-\int \frac{\delta M(x)}{\delta\sigma(x')}R_0(x)d^3x, \quad (38a)$$

$$H_1(x')=-2\int \frac{\delta M_i(x)}{\delta\sigma(x')}\pi^{ij}(x)_{|j}d^3x, \quad (38b)$$

we can rewrite Eq. (37a) as

$$\frac{\delta g_{ij}(x,\sigma)}{\delta\sigma(x')}=\frac{\delta(H_0(x')+H_1(x'))}{\delta\pi^{ij}(x)}. \quad (39)$$

Observe that $g_{ij}(x,\sigma)$ is a *functional* of $\sigma(x')$.

Now introduce a particular parameter for the hypersurface, say,

$$\sigma(x')=t.$$

In that event we have

$$\int \frac{\delta g_{ij}(x,\sigma)}{\delta\sigma(x')}d^3x'=\frac{\partial g_{ij}(x,t)}{\partial t}. \quad (40)$$

This equation relates (a) the change in g_{ij} at point x due to pushing "forward in time" the whole hypersurface $\sigma(x)=t$ to (b) the change in g_{ij} at point x^i due to pushing "forward in time" each small hypersurface element around each point x'^i.

Upon integrating Eqs. (37a) and (37b) with respect to x'^i, we have

$$\frac{\partial g_{ij}}{\partial t}=-2N(g)^{-1/2}(\tfrac{1}{2}g_{ij}\pi_k{}^k-\pi_{ij})+2N_{i|j} \quad (41a)$$

or

$$\tfrac{1}{2}g_{ij}\pi_k{}^k-\pi_{ij}=(g)^{1/2}(N_{i|j}+N_{j|i}-\partial g_{ij}/\partial t)/2N, \quad (41b)$$

where we set

$$N_{i|j}\equiv\int \frac{\delta M_{i|j}}{\delta\sigma(x')}d^3x',$$

$$N=\int \frac{\delta M}{\delta\sigma(x')}d^3x'.$$

Two conclusions can be drawn from Eq. (36), and hence from Eqs. (41):

(a) The term $N_{i|j}$ transforms like a three-tensor, i.e., N_i is a covariant three-vector.

(b) The factor N transforms like a three-scalar.

In addition, the two equations serve two purposes:

(a) Equation (36) reveals how a "tangent vector" $\delta^{(3)}\mathcal{G}/\delta\sigma(x')$ must be related to

$$\pi^{ij}=\delta S/\delta g_{ij}$$

if this vector is tangent to a history.

(b) Equation (41) serves as a definition of the extrinsic curvature[20] if one sets

$$(g)^{-1/2}(\tfrac{1}{2}g_{ij}\pi_k{}^k-\pi_{ij})=K_{ij}=\text{extrinsic curvature},$$

provided that one identifies the hypersurface parameter t with the fourth coordinate and the functions $N(x)$ and $N_i(x)$ with the "lapse" and "shift" functions[8] in the metric, Eq. (3).

Having determined how g_{ij} varies along a classical history (half the dynamical equations), we proceed now to do the same thing for π^{ij} (the other half of the dynamical equations). The change in π^{ij} as one goes from one three-geometry to another is

$$\begin{aligned}\frac{\delta\pi^{kl}(x)}{\delta\sigma}&=\int\frac{\delta\pi^{kl}(x)}{\delta g_{ij}(x')}\frac{\delta g_{ij}(x')}{\delta\sigma}d^3x'\\&=\int\frac{\delta\pi^{ij}(x')}{\delta g_{kl}(x)}\frac{\delta g_{ij}(x')}{\delta\sigma}d^3x'.\end{aligned}\tag{42}$$

The EHJ equation (1) holds for all $^{(3)}\mathcal{G}$'s. Consequently, the functional derivatives of Eqs. (1) and (14) with respect to $g_{ij}(x)$ must vanish at all functions g_{ij}:

$$0=\int\frac{\delta H_0}{\delta\pi^{ij}(x')}\frac{\delta\pi^{ij}(x')}{\delta g_{kl}(x)}d^3x'+\frac{\delta H_0}{\delta g_{kl}(x)},\tag{43a}$$

$$0=\int\frac{\delta H_1}{\delta\pi^{ij}(x')}\frac{\delta\pi^{ij}(x')}{\delta g_{kl}(x)}d^3x'+\frac{\delta H_1}{\delta g_{kl}(x)},\tag{43b}$$

respectively. To evaluate the expression on the right-hand side of Eq. (42), substitute Eq. (39) for $\delta g_{ij}/\delta\sigma$:

$$\frac{\delta\pi^{kl}(x)}{\delta\sigma}=\int\frac{\delta\pi^{ij}(x')}{\delta g_{kl}(x)}\left(\frac{\delta H_0}{\delta\pi^{ij}(x')}+\frac{\delta H_1}{\delta\pi^{ij}(x')}\right)d^3x'.$$

However, according to Eqs. (42), the right-hand side of this equation reduces to (the second half of the dynamical equations)

$$\frac{\delta\pi^{kl}(x)}{\delta\sigma}=-\frac{\delta(H_0+H_1)}{\delta g_{kl}(x)}.\tag{44}$$

Consequently, the change in π^{kl} for a given test function $\delta\sigma(x')$ is

$$\delta\pi^{kl}(x)=-\int\frac{\delta[H_0(x')+H_1(x')]}{\delta g_{kl}(x)}\delta\sigma(x')d^3x'.\tag{45}$$

Observe that, like Eqs. (36) and (37), the "momentum" equations (44) and (45) are also manifestly covariant.

IX. CONSISTENCY REQUIREMENTS

Starting with (a) the EHJ equation (1), (b) the requirement that its solutions be only functions of the three-geometry [Eq. (2)], and (c) the condition of constructive interference [Eq. (19)], we have obtained three constraint conditions [Eq. (14)] and two sets of equations [Eqs. (39) and (44)]. It now only remains to be shown that all of these equations, including the EHJ equation, are equivalent to the ten Einstein field equations. However, before pursuing this goal it is appropriate to answer a possible objection that may be raised about the equations for $\delta\pi^{ij}$. Will the "momentum"

$$\pi^{ij}+\delta\pi^{ij}$$

as computed according to Eq. (45) still satisfy Eqs. (1) and (14)? In other words, is it true that

$$\delta H_0(x)/\delta\sigma=0\tag{46}$$

and

$$\delta H_1(x)/\delta\sigma=0?\tag{47}$$

The affirmative answer to these questions follows from a direct computation. For example, using Eqs. (39) and (44), one has

$$\begin{aligned}\frac{\delta}{\delta\sigma}&=\int\left(\frac{\delta g_{ij}(x)}{\delta\sigma}\frac{\delta}{\delta g_{ij}(x)}+\frac{\delta\pi^{ij}(x)}{\delta\sigma}\frac{\delta}{\delta\pi^{ij}(x)}\right)d^3x\\&=\int\left(\frac{\delta(H_0+H_1)}{\delta\pi^{ij}(x)}\frac{\delta}{\delta g_{ij}(x)}-\frac{\delta(H_0+H_1)}{\delta g_{ij}(x)}\frac{\delta}{\delta\pi^{ij}(x)}\right)d^3x.\end{aligned}\tag{48}$$

Now consider the scalar functionals

$$X=\int\pi^{ij}{}_{|j}\xi_i d^3x',$$

$$Y_0=\int H_0(x')\delta\sigma(x')d^3x',$$

$$Y_1=\int H_1(x')\delta\sigma(x')d^3x',$$

where $\xi_i(x')$ and $\delta\sigma$ are a vector and a scalar field on the three-dimensional hypersurface, respectively. The functional derivatives of the first expression are

$$\begin{aligned}\delta X/\delta g_{ij}(x)&=\pi^{ki}\xi^j{}_{,k}+\pi^{jk}\xi^i{}_{,k}-\pi^{ij}{}_{,k}\xi^k\\&\quad-\pi^{ij}\xi^k{}_{,k}\equiv-\mathcal{L}_\xi\pi^{ij}(x),\end{aligned}$$

$$\delta X/\delta\pi^{ij}(x)=-2\xi_{(i|j)}\equiv\mathcal{L}_\xi g_{ij}(x).$$

The operator $\mathcal{L}_\xi$ may be identified with the Lie derivative associated with the infinitesimal transformation

$$x^i \to x^i + \epsilon \xi^i(x). \tag{49}$$

Consequently, upon substituting X into Eq. (48),[21] one has

$$\delta X = \int \frac{\delta X}{\delta\sigma(x)}\delta\sigma(x)d^3x = -\int\left(\frac{\delta(Y_0+Y_1)}{\delta\pi^{ij}(x)}\mathcal{L}_\xi \pi^{ij}(x) + \frac{\delta(Y_0+Y_1)}{\delta g_{ij}(x)}\mathcal{L}_\xi g_{ij}(x)\right)d^3x. \tag{50}$$

The right-hand side of this equation,

$$-\mathcal{L}_\xi(Y_0+Y_1) + \int [(-\delta M R_0 - 2\delta M_{,i}\pi^{ij}{}_{|j})\xi^k]_{,k}d^3x,$$

is the rate of change (Lie derivative[22]) in Y_0+Y_1 due to the transformation (49). Since Y_0 and Y_1 are independent of the coordinates on the three-dimensional hypersurface, the right-hand side of Eq. (50) vanishes (Noether's theorem[23]), i.e.,

$$\mathcal{L}_\xi(Y_0+Y_1)=0.$$

Since both $\xi^i(x)$ and $\delta\sigma(x)$ are arbitrary, we have, finally,

$$\delta\pi^{ij}(x)_{|j}/\delta\sigma(x')=0.$$

To show that Eq. (46) holds, substitute $R_0(x')$, Eq. (33), into Eq. (48) and calculate the Poisson bracket

$$\begin{aligned}\delta R_0(x')/\delta\sigma &= [H_0+H_1, R_0(x')]\\ &= [H_0,R_0(x')]+[H_1,R_0(x')].\end{aligned} \tag{51}$$

By considering the scalar functional

$$Z=\int R_0(x')\varphi(x')d^3x',$$

where $\varphi(x')$ is a scalar function, and by using the same arguments as above, one obtains

$$[Y_1,Z] = -[Z,Y_1] = -\mathcal{L}_{\delta M_i}(Z) = 0.$$

The second equals sign follows from Eq. (38b) and the definition of δM_i. Since $\varphi(x')$ and $\delta\sigma(x)$ are arbitrary, we have

$$[H_1(x),R_0(x')]=0.$$

The other Poisson bracket is obtained by calculating first

$$[Y_0,R_0(x')] = -2\delta M_{,i}(x')\pi^{ij}{}_{|j}(x') - \delta M \pi^{ij}{}_{|i|j}(x') - \int [\delta M(x),R_0(x')]R_0(x)d^3x.$$

This result can be obtained by explicit functional differentiation. It is useful in this process to use

$$\delta(g^{1/2}R) = g^{1/2}g^{jm}g^{ki}(\delta g_{ij|m} - \delta g_{jm|i})_{|k} + (R_{ij} - \tfrac{1}{2}g_{ij}R)g^{1/2}\delta g^{ij}$$

and to observe that terms in Y_0 or $R_0(x')$ that have undifferentiated π^{ij}'s and g_{ij}'s commute with each other. Since the test function $\delta\sigma(x)$ in Y_0 is arbitrary, we have

$$[H_0(x),R_0(x')]=0.$$

Consequently, the right-hand side of Eq. (51) vanishes.

The lesson to be learned from the above exercise in functional differentiation is this: The EHJ equation and the coordinate invariance of its solution hold if the g_{ij}'s and π^{ij}'s (along a classical history in superspace) are calculated from Eqs. (36) and (45), respectively. One must be sure, however, that the EHJ equation and the coordinate invariance are satisfied at the initial point in superspace.

X. TEN VACUUM FIELD EQUATIONS

Returning now to the derivation of the ten Einstein field equations, we observe that the equations at our disposal are

$${}^{(3)}R + g^{-1}(\tfrac{1}{2}g_{ij}g_{kl} - g_{ik}g_{jl})\pi^{ij}\pi^{kl} = 0, \tag{24}$$

$$\pi^{ij}{}_{|j}=0, \tag{22}$$

$$\frac{\delta g_{ij}(x)}{\delta\sigma(x')} = \frac{\delta[H_0(x')+H_1(x')]}{\delta\pi^{ij}(x)}, \tag{39}$$

$$\frac{\delta\pi^{ij}(x)}{\delta\sigma(x')} = -\frac{\delta[H_0(x')+H_1(x')]}{\delta g_{ij}(x)}. \tag{44}$$

Observe, however, that, as shown in Sec. IX, the first two equations are "essentially" contained in the last two equations already. "Essentially" here means that, when using Eqs. (39) and (44), we must make sure that Eqs. (24) and (22) are satisfied at the initial point in superspace.

Equations (37) and (44) are covariant and hold on every three-dimensional slice through space-time. They are the covariant-Hamiltonian equations of the 3+1 formulation of general relativity.[24] It is interesting to note that these equations have the structure of a many-particle generalization of the single-particle problem treated in Sec. II. That the above four equations imply the ten Einstein field equations can be seen best by observing that these equations can be derived from a

[21] As pointed out by P. W. Higgs [Phys. Rev. Letters 1, 373 (1958)] X is the generator of spatial coordinate transformations.

[22] A. Trautman, *Brandeis Summer Institute, Lectures on General Relativity, 1964* (Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1965), Vol. I, Chap. 7.6.

[23] A. Trautman, in *Gravitation: An Introduction to Current Research*, edited by L. Witten (John Wiley & Sons, Inc., New York, 1962), Chap. 5; see also Ref. 22.

[24] R. Arnowitt, S. Deser, and C. W. Misner, in *Gravitation: An Introduction to Current Research*, edited by L. Witten (John Wiley & Sons, Inc., New York, 1962), Chap. 7.

variational principle whose Lagrangian density is

$$\mathcal{L}=\int\Big[\Big[\frac{\delta g_{ij}}{\delta\sigma(x')}\pi^{ij}+\frac{\delta M}{\delta\sigma(x')}$$

$$\times[g^{1/2}\,{}^{(3)}R+g^{-1/2}(\tfrac{1}{2}\pi_i{}^i\pi_j{}^j-\pi_{ij}\pi^{ij})]+2\frac{\delta M_i}{\delta\sigma(x')}\pi^{ij}{}_{|j}$$

$$-2\Big(\pi^{ij}\frac{\delta M_j}{\delta\sigma(x')}-\tfrac{1}{2}\pi_k{}^k\frac{\delta M^i}{\delta\sigma(x')}+g^{1/2}\frac{\delta M^{,i}}{\delta\sigma(x')}\Big)_{,i}\Big]d^3x'$$

$$=(\partial g_{ij}/\partial t)\pi^{ij}+N[g^{1/2}\,{}^{(3)}R+g^{-1/2}(\tfrac{1}{2}\pi_i{}^i\pi_j{}^j-\pi_{ij}\pi^{ij})]$$

$$-2N_i\pi^{ij}{}_{|j}-2(\pi^{ij}N_j-\tfrac{1}{2}\pi_k{}^kN^i+g^{1/2}N^{,i})_{,i}.$$

This Lagrangian has been written down for the 3+1 formulation of general relativity.[24] It is equal to

$$\mathcal{L}=(-{}^{(4)}g)^{1/2}\,{}^{(4)}R.$$

The necessary identifications with the four-geometry are [see Eq. (3)]

$$g_{ij}={}^{(4)}g_{ij},\quad N=(-{}^{(4)}g^{00})^{1/2},\quad N_i={}^{(4)}g_{0i},$$

$$\pi^{ij}=g({}^{(4)}\Gamma_{mn}{}^0-g_{mn}\,{}^{(4)}\Gamma_{kl}{}^0g^{kl})\,{}^{(4)}g^{im}\,{}^{(4)}g^{jn},$$

$$(Ng)^{1/2}=(-{}^{(4)}g)^{1/2}.$$

Denote the Einstein field equations by

$$G_{\mu\nu}=0,\quad \mu,\nu=0,1,2,3.$$

Then Eqs. (24) and (22) are

$$G^0{}_\nu=0$$

and Eq. (44) is a linear combination of these equations together with the remaining six Einstein field equations,[24] with Eq. (39) serving as the definition of $\pi^{ij}(x)$. QED.

XI. CONCLUSION

By considering waves

$$\Psi\sim e^{iS/\hbar}$$

in superspace, we derived the ten Einstein field equations. The wave function Ψ is the semiclassical approximation to quantum mechanics. Its phase functional S is the solution to the EHJ equation

$$^{(3)}R-\Big(\frac{\delta S}{\delta^{(3)}\mathcal{G}}\Big)\Big(\frac{\delta S}{\delta^{(3)}\mathcal{G}}\Big)^*=0.$$

The history of a three-dimensional hypersurface is the set of three-geometries which is the locus of points in superspace where various waves Ψ interfere constructively; the different waves Ψ are associated with different solutions of the EHJ equation.

The set of equations directly obtained from (a) the EHJ equation and (b) the principle of constructive interference are the covariant-Hamiltonian equations of motion with Tomonaga's many-time taking the place of a timelike parameter. The form of these Hamiltonian equations is independent of one's choice of the three-dimensional spatial hypersurface. Consequently, these equations are more general than the usual equations that arise in the 3+1 formulation of general relativity.[24] The usual equations can be obtained from the covariant-Hamiltonian equations by integrating the former on some special three-dimensional hypersurface as was done in Eq. (40).

As a by-product of the derivation of the ten Einstein field equations, from the EHJ equation we gave a prescription that one can use to test the completeness of the solution to the EHJ equation.

XII. DISCUSSION

Although the principle of constructive interference and the EHJ equation could be brought into operation only by resorting to the language of classical tensor calculus, it would be highly desirable to eliminate "*les dêbauches d'indices*"[25] and give an "intrinsic" proof in which it would not be necessary to refer to the coordinate components of a tensor field at all.[26] In such a proof an immediate payoff would be the fact that it would no longer be necessary to refer to the coordinate invariance condition, Eq. (22), explicitly; one could focus one's attention solely on the fundamental concept in geometrodynamics, the three-geometry.

Quantum geometrodynamics in its classical approximation, exhibited by the wave functional

$$\Psi\sim e^{iS/\hbar}$$

and Eqs. (19) and (20), contains all of classical geometrodynamics, the ten Einstein field equations. Using the semiclassical approximation in particle quantum mechanics, one can describe such quantum phenomena as zero-point fluctuation, elementary excitations in solids, and an electron not spiraling into a hydrogen atom, to name a few. Can one describe quantum geometrodynamical phenomena[1] such as fluctuation in the geometry, elementary particles ("geometrodynamical excitations"), or a star "not collapsing"?

To obtain an exact theory of quantum geometrodynamics it is necessary to formulate—and solve—the appropriate "Einstein-Schrödinger" equation. We have written the EHJ equation

$$^{(3)}R-g^{-1}(g_{ik}g_{jl}-\tfrac{1}{2}g_{ij}g_{kl})\frac{\delta S}{\delta g_{ij}}\frac{\delta S}{\delta g_{kl}}=0$$

in the symbolic form

$$^{(3)}R-\Big(\frac{\delta S}{\delta\mathcal{G}}\Big)\Big(\frac{\delta S}{\delta\mathcal{G}}\Big)^*=0. \qquad (20)$$

[25] E. Cartan, *Leçons sur la géométrie des espaces de Riemann* (Gauthier Villars, Paris, 1963), preface.

[26] For an "intrinsic" formulation of infinite-dimensional Hamiltonian systems, see J. E. Marsden, Arch. Ratl. Mech. Anal. **28**, 362 (1968).

In the HJ equation in the form (1), one can use what coordinates one pleases and still get, as we have seen, the same physical results. The form (20) dramatizes this coordinate independence by eliminating all reference to coordinates [as one today uses the notation $\nabla S\cdot\nabla S$ in place of the old-fashioned $(\partial S/\partial x)^2+(\partial S/\partial y)^2+(\partial S/\partial z)^2$].

In the same spirit, it is tempting to symbolize the desired—and today not yet definitely formulated—Einstein-Schrödinger equation by the formula

$$^{(3)}R\Psi+\nabla^2\Psi(\delta\mathcal{G})^2=0. \qquad (52)$$

However, objections have been raised[27] to taking (1) as the prototype for a suitable Hamiltonian operator because the "Hamiltonian" is constrained to be zero ("frozen formalism"). The objection traces back in the last analysis, it would appear, to the well-justified conclusion that zero Hamiltonian implies zero dynamics.

However, according to the considerations of Secs. II, VI, and VIII we have to do here, not with a Hamiltonian, but a super-Hamiltonian.[28] Moreover, wehappily have enough experience by now with the formalism of the super-Hamiltonian (Sec. II and Refs. 10, 11, and 12) to recognize that in particle physics it *does* give dynamics—and all the dynamics there is. Similarly we find that all of classical geometrodynamics falls out of the super-Hamiltonian of general relativity, or the HJ Eq. (1). Therefore, it is hard to escape the conclusion that all of quantum geometrodynamics must come out of an "Einstein-Schrödinger" or "Einstein-Klein-Gordon" equation of the form (52). If one can introduce spinors, one may go even one step further and convert the Einstein-Klein-Gordon equation into an Einstein-Dirac equation.

APPENDIX

To complete the prescription for testing the completeness of the functional S in Sec. VII, we must examine the existence and uniqueness of the solution $\delta\pi^{ij}$ to the two equations

$$\delta\pi^{ij}{}_{|j}=0, \qquad (29)$$

$$K_{ij}\delta\pi^{ij}=0, \qquad (30)$$

and demonstrate the existence of the two arbitrary functions $\delta a(x)$ and $\delta b(x)$ that label the solution to Eqs. (29) and (30).

Equation (29) is a differential equation, whose solutions Deser[29] has found by a covariant decomposition of a symmetric tensor field. One considers an arbitrary symmetric tensor δT^{ij} which one decomposes uniquely into

$$\delta T^{ij}=\delta\pi^{tij}+\delta w^{(i|j)},$$

where δw^i is a vector field that satisfies

$$\delta T^{ij}{}_{|j}=\delta w^{(i|j)}{}_{|j}. \qquad (53)$$

This equation is an elliptic system that has a unique solution. Consequently,

$$\delta\pi^{tij}=\delta T^{ij}-\delta w^{(i|j)} \qquad (54)$$

satisfies Eq. (29).

Observe that the solutions to Eq. (29) are labelled by δT^{ij}, which has six degrees of freedom at each point of the spacelike manifold. Because of Eq. (53), $\delta w^{(i|j)}$ has only three degrees of freedom. Therefore, $\delta\pi^{tij}$, Eq. (54), has only $6-3=3$ degrees of freedom. From the set of solutions $\{\delta\pi^{tij}\}$ to Eq. (29) it is necessary to select the subset $\{\delta\pi^{ij}\}$, each member of which satisfies also Eq. (30). It follows that the solutions $\delta\pi^{ij}$ to Eqs. (29) and (30) are labelled by only two parameters at each point of the spacelike manifold on which Eqs. (29) and (30) are defined. These two parameters we call $\delta a(x)$ and $\delta b(x)$.

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[27] A. Komar, Phys. Rev. **153**, 1385 (1966); P. G. Bergmann and A. Komar, *Status Report on the Quantization of the Gravitational Field in Recent Developments in General Relativity* (Pergamon Press, Inc., New York, 1962).

[28] An analogous situation exists for a free particle in space-time; its "Hamiltonian" also vanishes, $\mathcal{H}\equiv m^2c^2+g^{\mu\nu}(\partial S/\partial x^\mu)(\partial S/\partial x^\nu)=0$. However, the appropriate wave equation is the Klein-Gordon equation, or rather the Dirac equation if one introduces spinors.

[29] S. Deser, Ann. Inst. Henri Poincaré A**VII**, 149 (1967).