MATH 5101:
Linear Mathematics in Finite Dimensions
Lectures:
Summaries, Notes, and Text Books

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VECTOR SPACES

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1.2.4 Basis; coordinates *(material for next lecture)*

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(i) Dollar Values of Fruit Inventories

(ii) Interpolation of sampled data: Lagrangian interpolation via quadratic polynomials

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Chapter 2

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Composition of linear transformations

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2.4 Lecture 14 (Monday)

2.4.1 Two mathematical sins

2.4.2 Invertibility

is unique

holds also for non-linear xformations

2.4.3 Appendix: Two good auxiliary texts from Math 2568 and Math 568

used in Math 2568: Johnson,Riess, and Arnold: Ch. 5, 1 (Matrix theory), 2 (Vectors in R^2 and R^3), 3 (Lin. Alg. in R^n)

used in Math 568: David Poole: Ch. 6 (Vector Spaces)

2.5 Lecture 15 (Wednesday)

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2.5.4 Isomorphism via equal dimensions

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2.8.2 Example

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2.9.1 Transition Matrix as a change of coordinates

Exemplified

Defined

Discussed

Its effect on a Linear Transformation

2.10 Lecture 20 (Monday)

2.10.1 Linear Transformation: Its coordinate representation

motivated

illustrated

constructed

applied to orthogonal coordinate changes

2.11 Lecture 21 (Wednesday)

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2.11.2 Algebraic Index Calculus applied to orthogonal transitions

2.11.3 APPENDIX (for Lecture 21)

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Calculation of the Inverse of a Matrix

Orthogonal Coordinate Representative

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2.12.2 Direct Sum: a reminder

2.12.3 Coordinate Representation Relative to Invariant Subspace

2.13 Lecture 23 (Monday)

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2.13.2 Commuting pairs of transformations:

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2.14.2 Eigenvalues of commuting transformations

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Chapter 3

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3.1 Lecture 25 (Friday)

3.1.1 Inner Product Spaces

Motivation

Definition

Examples

Orthogonality and Normalization

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3.2.2 Gram-Schmidt Normalization

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3.3.2 Right vs. Left Inverses

3.3.3 Application to Orthogonal Matrices

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3.4 Lecture 28 (Friday)

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3.4.2 *Active* vs. *Passive* characterization

3.4.3 Orthogonal Projections

3.4.4 Least squares Problem

3.4.5 Least Squares Approximation

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3.5.2 Examples of Least Squares Problems

Algebraic solutions

Geometric solutions

3.5.3 The Fundamental Theorem: Orthogonal Projection = Optimal Projection

3.6 Lecture 30 (Wednesday)

3.6.1 Least Squares Solution

Two Solution Methods

The Normal Equations

The subspace Matrix

Solution via Subspace Matrix

3.7 Lecture 31 (Friday)

3.7.1 The four Fundamental Subspaces of Matrix

3.7.2 Row rank = Column Rank

3.7.3 Pairwise Orthogonality

3.7.4 Direct Sum Decomposition of Domain and Target Space

3.7.5 Orthogonal Complementarity

Chapter 4

EIGENVALUES AND EIGENVECTORS IN REAL VECTOR SPACES

4.1 Lecture 32 (Monday)

4.1.1 Time Invariant (="Autonomous") Linear systems

Solution via Taylor Series

Solution via Eigenvalues and Eigenvectors:
(i) Algebraic Analysis
(ii) Comments
(iii) Geometrical Analysis
(iv) Recommendation (Use the "quiver" command in MATLAB)

Diagonal Form of a Matrix (next lecture)

4.2 Lecture 33 (Wednesday)

4.2.1 The Diagonal Form of a Matrix

4.2.2 Hamilton's Theorem

4.2.3 The Internal Input-Output Structure of a Matrix

4.2.4 Diagonalizable Matrices

4.2.5 APPENDIX for Lecture 33:

Linear independence of the set of eigenvectors of a matrix with distinct eigenvalues

4.3 Lecture 34 (Friday)

4.3.1 Decoupling a Diagonalizable System

4.3.2 Linear System with a Defective Generator

4.3.3 APPENDIX for Lecture 34: Time invariant linear system $\frac{d\vec{u}}{dt} = A\vec{u}$ with defective generator A

Chapter 5

EIGENVALUES AND EIGENVECTORS IN COMPLEX VECTOR SPACES

5.1 Lecture 35 (Monday)

5.1.1 Hermetian Adjoint

Basis Independent Definition

Matrix Definition

Matrix Element Definition

5.1.2 Hermetian Operators

5.1.3 Four of their Properties

5.2 Lecture 36 (Wednesday)

5.2.1 Skew Hermetian Matrices and their Properties

Their eigenvalues

Their eigenvectors

5.5.4 Geometry of Quadratic Forms

Vibrating System: Energy Conservation

Vibrating System: Normal Modes and Their Energy (for next lecture)

Extremum Principle for Normal Modes (for next lecture)

Geometrization via Concentric Ellipsoids (for next lecture)

Geometrization via Concentric Hyperbolas (for next lecture)

The Extremum Principle Geometrized (for next lecture)

5.5.5 APPENDIX A for Lecture 39: System of Four Coupled Masses

5.5.6 APPENDIX B for Lecture 39: System of Four Coupled Pendulums

5.6 Lecture 40 (Friday)

5.6.1 Extremum Principle for Normal Modes

5.6.2 Geometrization via Concentric Ellipsoids

5.6.3 Geometrization via Concentric Hyperbolas

5.6.4 The Extremum Principle Geometrized

5.6.5 Simultaneous Diagonalization of Two quadratic Forms (for next lecture)

5.7 Lecture 41 (Monday)

5.7.1 Simultaneous Diagonalization of Two Quadratic Forms

General Coupled System

Normal Modes

Eigenvalue Problem: (i) Eigenvalue equation
(ii) Eigenvectors
(iii) B-Orthonormality

Solution to the Eigenvalue Problem: Its Matrix Formulation

Geometrization via Concentric Ellipses

5.7.2 APPENDIX to Lecture 41: Periodic Resonant Systems

Stage I: Set up the Equations

Stage II: Solving the Equations Using the Cyclic symmetry of the resonant System:
(i) The Cyclic Permutation Matrix
(ii) The normal Modes: Time Dependence and Amplitude Profile
(iii) Toroidal Geometry

5.7.3 TEXTBOOKS REFERENCES FOR MATH 5101

1. L.W. Johnson, R.D. Riess, & Arnold "Introduction to Linear Algebra" (primarily Chapter 5 in the 5th Edition, or equivalently, Chapter 4 in the 4th, 3rd, and 2nd Edition)

   (a) *Nota bene:* The concepts and ideas developed in this chapter are time- less. Consequently, it matters little whether one has the 2nd, the 3rd, the 4th or 5th edition.

   (b) The very attractive feature of this text is that it develops the fundamentals and the geometry of Linear Algebra Theory twice: first relative to R^n (Chap. 2 in the 2nd, 3rd, and 4th Editions, and Chap. 3 in the 5th Edition) and then again in the abstract coordinate independent framework (Chap. 4 in the 2nd, 3rd, and 4th Editions, and Chap. 5 in the 5th Edition). This is done section by section. Thus by referring to the R^n sections whenever necessary, one learns not only the subject matter, but also the the hierarchical structure inherent in the acquisition of (mathematical) knowledge.

   (c) However, the 1st edition is pretty worthless by comparison.

2. David Poole, "Linear Algebra: A Modern Introduction" (Second Edition). Here the abstract and R^n formulation of linear algebra are developed very thoroughly and nicely in Chap. 6. The difference between JR&A and Poole is that JR&A can serve as the text for the students while JR&A together with Poole provides the wider context for the instructor.

3. I have not found a single text which integrates into a coherent whole the geometry of a vector space in its relation to its dual space, the duality of their respective bases, and the role of the inner product and the reciprocal basis in all this. Part of the to-be-integrated components can be found in the following texts:

   (a) C.W. Misner, K.S. Thorne, and J.A.Wheeler, "Gravitation". Part of Chap. 2 develops the geometrical meanings of the vector space dual to a given vector space R^n

   (b) Paul R. Halmos, "FINITE-DIMENSIONAL VECTOR SPACES" (Second Edition). Sections 13, 14, 15 develop clearly the algebraic aspects

of dual spaces, the Dirac bracket notation, dual bases, but not the novel features that come with the introduction of an inner product.

(c) W. Pauli, "THEORY OF RELATIVITY". Section 10 introduces the concept of a *reciprocal basis* in the context of what used to be called the "contravariant" and "covariant" components of a vector, but what nowadays are simply the coordinates of a vector and its dual image as induced a given inner product.

(d) S.H. Friedberg, A.J. Insel, and L.E. Spence, "Linear Algebra" (4th Edition). Section 2.6 gives a telegraphic treatment of the role dual spaces in the context of a linear transformation between vector spaces.

4. G. Strang, "Linear Algebra and Its Applications", Third Edition. In this edition Chap. 5 and 6 highlight the key properties and applications of square matrices. Elsewhere one finds the characterization of linear transformation in terms of their four fundamental subspaces. Furthermore, in one of the Appendices one finds a clear development of the singular value decomposition, which in a later edition has been moved into one of the main body of the book.

5. D. Larson and B.Edwards: "Elementary Linear Algebra", 3rd Edition. Selected sections on complex inner product spaces (Chapter 8).