

MATH 5101:  
**Linear Mathematics in Finite  
Dimensions**

Lectures:  
Summaries, Notes, and Text Books

U. H. Gerlach

Autumn 2015

# Contents

<b>1</b>	<b>VECTOR SPACES</b>	<b>6</b>
1.1	Lecture 1 (Wednesday)	6
1.1.1	Three archetypical equations of linear algebra	6
1.1.2	Vector as an aggregate of entities	6
1.1.3	Vector space: Definition and Examples	6
1.1.4	Subspace of a vector space	6
1.2	Lecture 2 (Friday)	6
1.2.1	The Subspace Theorem	6
1.2.2	Spanning set	6
1.2.3	Linear independence ( <i>material for next lecture</i> )	6
1.2.4	Basis; coordinates ( <i>material for next lecture</i> )	6
1.2.5	Basis-induced isomorphism ( <i>material for next lecture</i> )	6
1.3	Lecture 3 (Monday)	6
1.3.1	Spanning set example	6
1.3.2	Linear independence	6
1.3.3	Basis; coordinates	6
1.3.4	Basis-induced isomorphism	6
1.4	Lecture 4 (Wednesday)	7
1.4.1	Coordinate representative of a vector relative to a given basis (Existence and uniqueness)	7
1.4.2	Coordinates	7
1.5	Lecture 5 (Friday)	7
1.5.1	Basis-induced correspondence between $V$ and $R^p$ as structure preserving	7
1.5.2	Preservation of linear independence and dependence of a set of sets of vectors	7
1.5.3	Isomorphism and its basis independence	7
1.6	Lecture 6 (Wednesday)	7
1.6.1	Structure preservation	7
1.6.2	Dimension of $V$ : its coordinate invariance	7
1.6.3	Mutual implication of linear independence and spanning property (in an appropriate context)	7
1.7	Lecture 7 (Friday)	7
1.7.1	Linear functions on a vector space $V$	7

1.7.2	The dual vector space $V^*$ . . . . .	7
1.7.3	Dirac's bracket notation . . . . .	7
1.7.4	Basis (and their coordinate lines) vs. dual basis (and their coordinate surfaces); vectors vs. covectors . . . . .	7
1.8	Lecture 8 (Monday) . . . . .	7
1.8.1	The Duality Principle . . . . .	7
1.8.2	Example for $R^n$ : $\{\text{Column vectors}\}=V$ ; $\{\text{Row vectors}\}=V^*$ . . . . .	7
1.8.3	Addition of vectors and covectors . . . . .	7
1.8.4	APPENDIX for Lecture 8: Four Important Examples of Linear Functions: . . . . .	8
1.9	Lecture 9 (Wednesday) . . . . .	8
1.9.1	Bilinear functional . . . . .	8
1.9.2	Metric as an inner product . . . . .	8
1.9.3	Metric as a <i>natural</i> isomorphism between $V$ and $V^*$ . . . . .	8
1.10	Lecture 10 (Friday) . . . . .	8
1.10.1	Reciprocal vector basis for $V$ . . . . .	8
1.10.2	Its relation to the dual basis for $V^*$ . . . . .	8
1.10.3	Geometrical relation between vectors and their linear functionals in $V^*$ . . . . .	8
<b>2</b>	<b>LINEAR TRANSFORMATIONS</b> . . . . .	<b>9</b>
2.1	Lecture 11 (Monday) . . . . .	9
2.1.1	Transformations: who needs them? . . . . .	9
2.1.2	Linear transformations: their geometric and algebraic definition . . . . .	9
2.1.3	Onto and one-to-one transformations . . . . .	9
2.1.4	Null Space and Range Space: their importance . . . . .	9
2.1.5	Example . . . . .	9
2.2	Lecture 12 (Wednesday) . . . . .	9
2.2.1	T-induced basis properties . . . . .	9
2.2.2	Dimensional consistency . . . . .	9
2.2.3	Direct sum of two vector spaces . . . . .	9
2.3	Lecture 13 (Friday) . . . . .	9
2.3.1	Operations with linear transformations . . . . .	9
2.4	Lecture 14 (Monday) . . . . .	10
2.4.1	Two mathematical sins . . . . .	10
2.4.2	Invertibility . . . . .	10
2.4.3	Appendix: Two good auxiliary texts from Math 2568 and Math 568 . . . . .	10
2.5	Lecture 15 (Wednesday) . . . . .	10
2.5.1	Vector spaces:retail vs. wholesale . . . . .	10
2.5.2	Isomorphism: What is it . . . . .	10
2.5.3	Examples; MATLAB's <i>kron</i> function . . . . .	10
2.5.4	Isomorphism via equal dimensions . . . . .	10
2.5.5	Coordinate representation of a linear transformation . . . . .	10
2.6	Lecture 16 (Friday) . . . . .	10

2.6.1	<i>Basis</i> of a vector space <i>as the mooring</i> that ties abstract vector spaces and their linear transformations to the computationally real world . . . . .	10
2.6.2	<i>The Representation Theorem</i> induced from an illustrative example . . . . .	10
2.6.3	Computations via the Representation Theorem . . . . .	10
2.7	Lecture 17 (Monday) . . . . .	10
2.7.1	Coordinate representation for the Composite of Transformations . . . . .	10
2.7.2	For Next Lecture: The Inverse of a Transformation . . . . .	10
2.8	Lecture 18 (Wednesday) . . . . .	11
2.8.1	Basis Representation of an Invertible Transformation . . . . .	11
2.8.2	Example . . . . .	11
2.8.3	Epistemological Role of the Representation Concept and the Representation Theorem . . . . .	11
2.9	Lecture 19 (Friday) . . . . .	11
2.9.1	Transition Matrix as a change of coordinates . . . . .	11
2.10	Lecture 20 (Monday) . . . . .	11
2.10.1	Linear Transformation: Its coordinate representation . . . . .	11
2.11	Lecture 21 (Wednesday) . . . . .	11
2.11.1	The Levi-Ci vita Epsilon . . . . .	11
2.11.2	Algebraic Index Calculus applied to orthogonal transitions . . . . .	11
2.11.3	APPENDIX (for Lecture 21) . . . . .	11
2.12	Lecture 22 (Friday) . . . . .	12
2.12.1	Invariant Subspace . . . . .	12
2.12.2	Direct Sum: a reminder . . . . .	12
2.12.3	Coordinate Representation Relative to Invariant Subspace . . . . .	12
2.13	Lecture 23 (Monday) . . . . .	12
2.13.1	Existence of convectors on non-trivial subspace invariant under a linear transformation . . . . .	12
2.13.2	Commuting pairs of transformations: . . . . .	12
2.14	Lecture 24 (Wednesday) . . . . .	12
2.14.1	Invariance via Commutativity . . . . .	12
2.14.2	Eigenvalues of commuting transformations . . . . .	12
2.14.3	The “All or Nothing” Theorem . . . . .	12
<b>3</b>	<b>INNER PRODUCT SPACES</b>	<b>13</b>
3.1	Lecture 25 (Friday) . . . . .	13
3.1.1	Inner Product Spaces . . . . .	13
3.2	Lecture 26 (Monday) . . . . .	13
3.2.1	Summary of Inner product vs. Metric vs. Duality . . . . .	13
3.2.2	Gram-Schmidt Normalization . . . . .	13
3.2.3	Relation between Oblique and Orthogonal Basis; the QR decomposition . . . . .	13
3.3	Lecture 27 (Wednesday) . . . . .	13
3.3.1	Orthogonal (Orthonormal) Matrices . . . . .	13

3.3.2	Right vs. Left Inverses . . . . .	13
3.3.3	Application to Orthogonal Matrices . . . . .	13
3.3.4	Rigid Transformation (or next lecture) . . . . .	13
3.4	Lecture 28 (Friday) . . . . .	14
3.4.1	Rigid Transformation . . . . .	14
3.4.2	<i>Active</i> vs. <i>Passive</i> characterization . . . . .	14
3.4.3	Orthogonal Projections . . . . .	14
3.4.4	Least squares Problem . . . . .	14
3.4.5	Least Squares Approximation . . . . .	14
3.5	Lecture 29 (Monday) . . . . .	14
3.5.1	The Least Squares Approximation: Its cognitive value in data mining . . . . .	14
3.5.2	Examples of Least Squares Problems . . . . .	14
3.5.3	The Fundamental Theorem: Orthogonal Projection = Optimal Projection . . . . .	14
3.6	Lecture 30 (Wednesday) . . . . .	14
3.6.1	Least Squares Solution . . . . .	14
3.7	Lecture 31 (Friday) . . . . .	14
3.7.1	The four Fundamental Subspaces of Matrix . . . . .	14
3.7.2	Row rank = Column Rank . . . . .	14
3.7.3	Pairwise Orthogonality . . . . .	14
3.7.4	Direct Sum Decomposition of Domain and Target Space . . . . .	14
3.7.5	Orthogonal Complementarity . . . . .	14
<b>4</b>	<b>EIGENVALUES AND EIGENVECTORS IN REAL VECTOR SPACES</b>	<b>15</b>
4.1	Lecture 32 (Monday) . . . . .	15
4.1.1	Time Invariant (=“Autonomous”) Linear systems . . . . .	15
4.2	Lecture 33 (Wednesday) . . . . .	15
4.2.1	The Diagonal Form of a Matrix . . . . .	15
4.2.2	Hamilton’s Theorem . . . . .	15
4.2.3	The Internal Input-Output Structure of a Matrix . . . . .	15
4.2.4	Diagonalizable Matrices . . . . .	15
4.2.5	APPENDIX for Lecture 33: . . . . .	15
4.3	Lecture 34 (Friday) . . . . .	16
4.3.1	Decoupling a Diagonalizable System . . . . .	16
4.3.2	Linear System with a Defective Generator . . . . .	16
4.3.3	APPENDIX for Lecture 34: Time invariant linear system $\frac{d\vec{u}}{dt} = A\vec{u}$ with defective generator $A$ . . . . .	16
<b>5</b>	<b>EIGENVALUES AND EIGENVECTORS IN COMPLEX VECTOR SPACES</b>	<b>17</b>
5.1	Lecture 35 (Monday) . . . . .	17
5.1.1	Hermetian Adjoint . . . . .	17
5.1.2	Hermetian Operators . . . . .	17

5.1.3	Four of their Properties . . . . .	17
5.2	Lecture 36 (Wednesday) . . . . .	17
5.2.1	Skew Hermetian Matrices and their Properties . . . . .	17
5.2.2	Unitary Matrices and their Properties . . . . .	18
5.2.3	Unitary Dynamical Systems . . . . .	18
5.2.4	Similarity Transformations (revisited) . . . . .	18
5.2.5	APPENDIX for Lecture 36 . . . . .	18
5.2.6	Proof of $e^a e^b = e^{a+b}$ whenever . . . . .	18
5.3	Lecture 37 (Friday) . . . . .	18
5.3.1	Triangular Form via Unitary Matrix . . . . .	18
5.4	Lecture 38 (Monday) . . . . .	18
5.4.1	Normal Matrices . . . . .	18
5.4.2	Normality $\Leftrightarrow$ Diagonalizability via Unitary Similarity Transformation . . . . .	18
5.5	Lecture 39 (Wednesday) . . . . .	18
5.5.1	The Universe, Mathematics, and Quadratic Forms . . . . .	18
5.5.2	Positive Definite Matrices . . . . .	18
5.5.3	The Three Faces of Positive Definiteness . . . . .	18
5.5.4	Geometry of Quadratic Forms . . . . .	19
5.5.5	APPENDIX A for Lecture 39: System of Four Coupled Masses . . . . .	19
5.5.6	APPENDIX B for Lecture 39: System of Four Coupled Pendulums . . . . .	19
5.6	Lecture 40 (Friday) . . . . .	19
5.6.1	Extremum Principle for Normal Modes . . . . .	19
5.6.2	Geometrization via Concentric Ellipsoids . . . . .	19
5.6.3	Geometrization via Concentric Hyperbolas . . . . .	19
5.6.4	The Extremum Principle Geometrized . . . . .	19
5.6.5	Simultaneous Diagonalization of Two quadratic Forms (for next lecture) . . . . .	19
5.7	Lecture 41 (Monday) . . . . .	19
5.7.1	Simultaneous Diagonalization of Two Quadratic Forms . . . . .	19
5.7.2	APPENDIX to Lecture 41: Periodic Resonant Systems . . . . .	20
5.7.3	TEXTBOOKS REFERENCES FOR MATH 5101 . . . . .	20

# Chapter 1

## VECTOR SPACES

### 1.1 Lecture 1 (Wednesday)

1.1.1 Three archetypical equations of linear algebra

1.1.2 Vector as an aggregate of entities

1.1.3 Vector space: Definition and Examples

1.1.4 Subspace of a vector space

### 1.2 Lecture 2 (Friday)

1.2.1 The Subspace Theorem

1.2.2 Spanning set

1.2.3 Linear independence (*material for next lecture*)

1.2.4 Basis; coordinates (*material for next lecture*)

1.2.5 Basis-induced isomorphism (*material for next lecture*)

### 1.3 Lecture 3 (Monday)

1.3.1 Spanning set example

1.3.2 Linear independence

1.3.3 Basis; coordinates

1.3.4 Basis-induced isomorphism

## 1.4 Lecture 4 (Wednesday)

1.4.1 Coordinate representative of a vector relative to a given basis (Existence and uniqueness)

1.4.2 Coordinates

## 1.5 Lecture 5 (Friday)

1.5.1 Basis-induced correspondence between  $V$  and  $R^p$  as structure preserving

1.5.2 Preservation of linear independence and dependence of a set of sets of vectors

1.5.3 Isomorphism and its basis independence

## 1.6 Lecture 6 (Wednesday)

1.6.1 Structure preservation

1.6.2 Dimension of  $V$ : its coordinate invariance

1.6.3 Mutual implication of linear independence and spanning property (in an appropriate context)

## 1.7 Lecture 7 (Friday)

1.7.1 Linear functions on a vector space  $V$

1.7.2 The dual vector space  $V^*$

1.7.3 Dirac's bracket notation

1.7.4 Basis (and their coordinate lines) vs. dual basis (and their coordinate surfaces); vectors vs. covectors

## 1.8 Lecture 8 (Monday)

1.8.1 The Duality Principle

1.8.2 Example for  $R^n$ : {Column vectors} $=V$ ; {Row vectors} $=V^*$

1.8.3 Addition of vectors and covectors



#### 1.8.4 APPENDIX for Lecture 8: Four Important Examples of Linear Functions:

- (i) Dollar Values of Fruit Inventories
- (ii) Interpolation of sampled data: Lagrangian interpolation via quadratic polynomials
- (iii) Interpolation of sampled data via Roof functions
- (iv) Interpolation of sampled data via Band-limited basis functions

### 1.9 Lecture 9 (Wednesday)

#### 1.9.1 Bilinear functional

#### 1.9.2 Metric as an inner product

#### 1.9.3 Metric as a *natural* isomorphism between $V$ and $V^*$

### 1.10 Lecture 10 (Friday)

#### 1.10.1 Reciprocal vector basis for $V$

#### 1.10.2 Its relation to the dual basis for $V^*$

#### 1.10.3 Geometrical relation between vectors and their linear functionals in $V^*$

## Chapter 2

# LINEAR TRANSFORMATIONS

### 2.1 Lecture 11 (Monday)

2.1.1 Transformations: who needs them?

2.1.2 Linear transformations: their geometric and algebraic definition

2.1.3 Onto and one-to-one transformations

2.1.4 Null Space and Range Space: their importance

2.1.5 Example

### 2.2 Lecture 12 (Wednesday)

2.2.1 T-induced basis properties

2.2.2 Dimensional consistency

2.2.3 Direct sum of two vector spaces

### 2.3 Lecture 13 (Friday)

2.3.1 Operations with linear transformations

Sum of two transformations

Composition of linear transformations

Inverse of an invertible transformation

## 2.4 Lecture 14 (Monday)

### 2.4.1 Two mathematical sins

### 2.4.2 Invertibility

is unique

holds also for non-linear xformations

### 2.4.3 Appendix: Two good auxiliary texts from Math 2568 and Math 568

used in Math 2568: Johnson, Riess, and Arnold: Ch. 5, 1 (Matrix theory), 2 (Vectors in  $R^2$  and  $R^3$ ), 3 (Lin. Alg. in  $R^n$ )

used in Math 568: David Poole: Ch. 6 (Vector Spaces)

## 2.5 Lecture 15 (Wednesday)

### 2.5.1 Vector spaces: retail vs. wholesale

### 2.5.2 Isomorphism: What is it

### 2.5.3 Examples; MATLAB's *kron* function

### 2.5.4 Isomorphism via equal dimensions

### 2.5.5 Coordinate representation of a linear transformation

## 2.6 Lecture 16 (Friday)

2.6.1 *Basis* of a vector space *as the mooring* that ties abstract vector spaces and their linear transformations to the computationally real world

2.6.2 *The Representation Theorem* induced from an illustrative example

2.6.3 Computations via the Representation Theorem

## 2.7 Lecture 17 (Monday)

2.7.1 Coordinate representation for the Composite of Transformations

2.7.2 For Next Lecture: The Inverse of a Transformation

## 2.8 Lecture 18 (Wednesday)

2.8.1 Basis Representation of an Invertible Transformation

2.8.2 Example

2.8.3 Epistemological Role of the Representation Concept and the Representation Theorem

## 2.9 Lecture 19 (Friday)

2.9.1 Transition Matrix as a change of coordinates

Exemplified

Defined

Discussed

Its effect on a Linear Transformation

## 2.10 Lecture 20 (Monday)

2.10.1 Linear Transformation: Its coordinate representation

motivated

illustrated

constructed

applied to orthogonal coordinate changes

## 2.11 Lecture 21 (Wednesday)

2.11.1 The Levi-Civita Epsilon

2.11.2 Algebraic Index Calculus applied to orthogonal transitions

2.11.3 APPENDIX (for Lecture 21)

Determinants

Calculation of the Inverse of a Matrix

Orthogonal Coordinate Representative

## **2.12 Lecture 22 (Friday)**

**2.12.1 Invariant Subspace**

**2.12.2 Direct Sum: a reminder**

**2.12.3 Coordinate Representation Relative to Invariant Subspace**

## **2.13 Lecture 23 (Monday)**

**2.13.1 Existence of convectors on non-trivial subspace invariant under a linear transformation**

**2.13.2 Commuting pairs of transformations:**

Their mutual invariant subspace (Consigned to Lecture 24)

Their common convectors (Consigned to Lecture 24)

## **2.14 Lecture 24 (Wednesday)**

**2.14.1 Invariance via Commutativity**

**2.14.2 Eigenvalues of commuting transformations**

**2.14.3 The “All or Nothing” Theorem**

## Chapter 3

# INNER PRODUCT SPACES

### 3.1 Lecture 25 (Friday)

#### 3.1.1 Inner Product Spaces

Motivation

Definition

Examples

Orthogonality and Normalization

### 3.2 Lecture 26 (Monday)

#### 3.2.1 Summary of Inner product vs. Metric vs. Duality

#### 3.2.2 Gram-Schmidt Normalization

#### 3.2.3 Relation between Oblique and Orthogonal Basis; the QR decomposition

### 3.3 Lecture 27 (Wednesday)

#### 3.3.1 Orthogonal (Orthonormal) Matrices

#### 3.3.2 Right vs. Left Inverses

#### 3.3.3 Application to Orthogonal Matrices

#### 3.3.4 Rigid Transformation (or next lecture)

### 3.4 Lecture 28 (Friday)

#### 3.4.1 Rigid Transformation

#### 3.4.2 *Active* vs. *Passive* characterization

#### 3.4.3 Orthogonal Projections

#### 3.4.4 Least squares Problem

#### 3.4.5 Least Squares Approximation

### 3.5 Lecture 29 (Monday)

#### 3.5.1 The Least Squares Approximation: Its cognitive value in data mining

#### 3.5.2 Examples of Least Squares Problems

Algebraic solutions

Geometric solutions

#### 3.5.3 The Fundamental Theorem: Orthogonal Projection = Optimal Projection

### 3.6 Lecture 30 (Wednesday)

#### 3.6.1 Least Squares Solution

Two Solution Methods

The Normal Equations

The subspace Matrix

Solution via Subspace Matrix

### 3.7 Lecture 31 (Friday)

#### 3.7.1 The four Fundamental Subspaces of Matrix

#### 3.7.2 Row rank = Column Rank

#### 3.7.3 Pairwise Orthogonality

#### 3.7.4 Direct Sum Decomposition of Domain and Target Space

#### 3.7.5 Orthogonal Complementarity

## Chapter 4

# EIGENVALUES AND EIGENVECTORS IN REAL VECTOR SPACES

### 4.1 Lecture 32 (Monday)

#### 4.1.1 Time Invariant (=“Autonomous”) Linear systems

Solution via Taylor Series

Solution via Eigenvalues and Eigenvectors:

- (i) Algebraic Analysis
- (ii) Comments
- (iii) Geometrical Analysis
- (iv) Recommendation (Use the “quiver” command in MATLAB)

Diagonal Form of a Matrix (next lecture)

### 4.2 Lecture 33 (Wednesday)

#### 4.2.1 The Diagonal Form of a Matrix

#### 4.2.2 Hamilton’s Theorem

#### 4.2.3 The Internal Input-Output Structure of a Matrix

#### 4.2.4 Diagonalizable Matrices

#### 4.2.5 APPENDIX for Lecture 33:

Linear independence of the set of eigenvectors of a matrix with distinct eigenvalues



### 4.3 Lecture 34 (Friday)

#### 4.3.1 Decoupling a Diagonalizable System

#### 4.3.2 Linear System with a Defective Generator

#### 4.3.3 APPENDIX for Lecture 34: Time invariant linear system $\frac{d\vec{u}}{dt} = A\vec{u}$ with defective generator $A$

## Chapter 5

# EIGENVALUES AND EIGENVECTORS IN COMPLEX VECTOR SPACES

### 5.1 Lecture 35 (Monday)

#### 5.1.1 Hermetian Adjoint

Basis Independent Definition

Matrix Definition

Matrix Element Definition

#### 5.1.2 Hermetian Operators

#### 5.1.3 Four of their Properties

### 5.2 Lecture 36 (Wednesday)

#### 5.2.1 Skew Hermetian Matrices and their Properties

Their eigenvalues

Their eigenvectors

## 5.2.2 Unitary Matrices and their Properties

Their eigenvalues

Their eigenvectors

## 5.2.3 Unitary Dynamical Systems

## 5.2.4 Similarity Transformations (revisited)

## 5.2.5 APPENDIX for Lecture 36

## 5.2.6 Proof of $e^a e^b = e^{a+b}$ whenever

$e^a$  and  $e^b$  and

$ab = ba$

## 5.3 Lecture 37 (Friday)

### 5.3.1 Triangular Form via Unitary Matrix

Partially Decoupled Time Invariant System

General Time Invariant Linear System

The Triangularization Theorem

The Triangularization Theorem: Applications

The Triangularization Theorem: Its Proof

## 5.4 Lecture 38 (Monday)

### 5.4.1 Normal Matrices

### 5.4.2 Normality $\Leftrightarrow$ Diagonalizability via Unitary Similarity Transformation

## 5.5 Lecture 39 (Wednesday)

### 5.5.1 The Universe, Mathematics, and Quadratic Forms

### 5.5.2 Positive Definite Matrices

### 5.5.3 The Three Faces of Positive Definiteness

#### 5.5.4 Geometry of Quadratic Forms

Vibrating System: Energy Conservation

Vibrating System: Normal Modes and Their Energy (for next lecture)

Extremum Principle for Normal Modes (for next lecture)

Geometrization via Concentric Ellipsoids (for next lecture)

Geometrization via Concentric Hyperbolas (for next lecture)

The Extremum Principle Geometrized (for next lecture)

5.5.5 APPENDIX A for Lecture 39: System of Four Coupled Masses

5.5.6 APPENDIX B for Lecture 39: System of Four Coupled Pendulums

#### 5.6 Lecture 40 (Friday)

5.6.1 Extremum Principle for Normal Modes

5.6.2 Geometrization via Concentric Ellipsoids

5.6.3 Geometrization via Concentric Hyperbolas

5.6.4 The Extremum Principle Geometrized

5.6.5 Simultaneous Diagonalization of Two quadratic Forms (for next lecture)

#### 5.7 Lecture 41 (Monday)

5.7.1 Simultaneous Diagonalization of Two Quadratic Forms

General Coupled System

Normal Modes

Eigenvalue Problem: (i) Eigenvalue equation

(ii) Eigenvectors

(iii) B-Orthonormality

Solution to the Eigenvalue Problem: Its Matrix Formulation

Geometrization via Concentric Ellipses

## 5.7.2 APPENDIX to Lecture 41: Periodic Resonant Systems

Stage I: Set up the Equations

Stage II: Solving the Equations Using the Cyclic symmetry of the resonant System:

- (i) The Cyclic Permutation Matrix
- (ii) The normal Modes: Time Dependence and Amplitude Profile
- (iii) Toroidal Geometry

## 5.7.3 TEXTBOOKS REFERENCES FOR MATH 5101

1. L.W. Johnson, R.D. Riess, & Arnold “Introduction to Linear Algebra” (primarily Chapter 5 in the 5th Edition, or equivalently, Chapter 4 in the 4th, 3rd, and 2nd Edition)
  - (a) *Nota bene:* The concepts and ideas developed in this chapter are time- less. Consequently, it matters little whether one has the 2nd, the 3rd, the 4th or 5th edition.
  - (b) The very attractive feature of this text is that it develops the fundamentals and the geometry of Linear Algebra Theory twice: first relative to  $R^n$  (Chap. 2 in the 2nd, 3rd, and 4th Editions, and Chap. 3 in the 5th Edition) and then again in the abstract coordinate independent framework (Chap. 4 in the 2nd, 3rd, and 4th Editions, and Chap. 5 in the 5th Edition). This is done section by section. Thus by referring to the  $R^n$  sections whenever necessary, one learns not only the subject matter, but also the the hierarchical structure inherent in the acquisition of (mathematical) knowledge.
  - (c) However, the 1st edition is pretty worthless by comparison.
2. David Poole, “Linear Algebra: A Modern Introduction” (Second Edition). Here the abstract and  $R^n$  formulation of linear algebra are developed very thoroughly and nicely in Chap. 6. The difference between JR&A and Poole is that JR&A can serve as the text for the students while JR&A together with Poole provides the wider context for the instructor.
3. I have not found a single text which integrates into a coherent whole the geometry of a vector space in its relation to its dual space, the duality of their respective bases, and the role of the inner product and the reciprocal basis in all this. Part of the to-be-integrated components can be found in the following texts:
  - (a) C.W. Misner, K.S. Thorne, and J.A.Wheeler, “Gravitation”. Part of Chap. 2 develops the geometrical meanings of the vector space dual to a given vector space  $R^n$
  - (b) Paul R. Halmos, “FINITE-DIMENSIONAL VECTOR SPACES” (Second Edition). Sections 13, 14, 15 develop clearly the algebraic aspects

of dual spaces, the Dirac bracket notation, dual bases, but not the novel features that come with the introduction of an inner product.

- (c) W. Pauli, "THEORY OF RELATIVITY". Section 10 introduces the concept of a *reciprocal basis* in the context of what used to be called the "contravariant" and "covariant" components of a vector, but what nowadays are simply the coordinates of a vector and its dual image as induced a given inner product.
  - (d) S.H. Friedberg, A.J. Insel, and L.E. Spence, "Linear Algebra" (4th Edition). Section 2.6 gives a telegraphic treatment of the role dual spaces in the context of a linear transformation between vector spaces.
4. G. Strang, "Linear Algebra and Its Applications", Third Edition. In this edition Chap. 5 and 6 highlight the key properties and applications of square matrices. Elsewhere one finds the characterization of linear transformation in terms of their four fundamental subspaces. Furthermore, in one of the Appendices one finds a clear development of the singular value decomposition, which in a later edition has been moved into one of the main body of the book.
  5. D. Larson and B. Edwards: "Elementary Linear Algebra", 3rd Edition. Selected sections on complex inner product spaces (Chapter 8).