Taken from "Methods of Theoretical Physics, Vol I" by P. Morse and H. Feshback

The derivatives of y turn out to be just as required by the Cauchy-Riemann conditions $\partial x/\partial u = \partial y/\partial v$, etc. It is also possible now to evaluate

$$\frac{dz}{df} = \frac{\partial x}{\partial u} + i\frac{\partial y}{\partial u} = \frac{\partial x}{\partial u} - i\frac{\partial x}{\partial v} = \frac{1}{(\partial u/\partial x) - i(\partial u/\partial y)} = \frac{1}{df/dz}$$

proving the final statement of the theorem. We shall devote a considerable portion of the chapter later to develop the subject of conformal representation further, for it is very useful in application.

Integration in the Complex Plane. The theory of integration in the complex plane is just the theory of the line integral. If C is a possible contour (to be discussed below), then from the analysis of page 354 [see material above Eq. (4.1.12)] it follows that

$$\int_C \bar{E}\,dz = \int_C E_t\,ds + i\int_C E_n\,ds; \quad ds = |dz|$$

where E_t is the component of the vector **E** along the path of integration while E_n is the normal component. Integrals of this kind appear frequently in physics. For example, if **E** is any force field, then the integral $\int_C E_t\,ds$ is just the work done against the force field in moving along the contour C. The second integral measures the total flux passing through the contour. If **E** were the velocity vector in hydrodynamics, then the second integral would be just the total fluid current through the contour.

In order for both of these integrals to make physical (and also mathematical) sense, it is necessary for the contour to be sufficiently smooth. Such a *smooth curve* is composed of arcs which join on continuously, each arc having a continuous tangent. This last requirement eliminates some pathological possibilities, such as a contour sufficiently irregular that is of infinite length. For purposes of convenience, we shall also insist that each arc have no multiple points, thus eliminating loops. However, loops may be easily included in the theory, for any contour containing a loop may be decomposed into a closed contour (the loop) plus a smooth curve, and the theorem to be derived can be applied to each. A *closed contour* is a closed smooth curve. A closed contour is described in a *positive* direction with respect to the domain enclosed by the contour if with respect to some point inside the domain the contour is traversed in a *counterclockwise* direction. The negative direction is then just the clockwise one. Integration along a closed contour will be symbolized by $\oint$.

One fairly obvious result we shall use often in our discussions: If $f(z)$ is an analytic function within and on the contour, and if df/dz is single-valued in the same region,

$$\oint (df/dz)\,dz = 0$$

This result is not necessarily true if df/dz is *not* single-valued.

Contours involving smooth curves may be combined to form new contours. Some examples are shown in Figure 4.4. Some of the contours so formed may no longer be smooth. For example, the boundary b' is not bounded by a smooth curve (for the inner circle and outer circle are not joined) so that this contour is not composed of arcs which join on continuously. Regions of this type are called *multiply connected*, whereas the remaining examples in the figure are *simply connected*. To test for connectivity of a region note that any closed contour drawn within a simply connected region can be shrunk to a point by continuous deformation without crossing the boundary of the region. In b' a

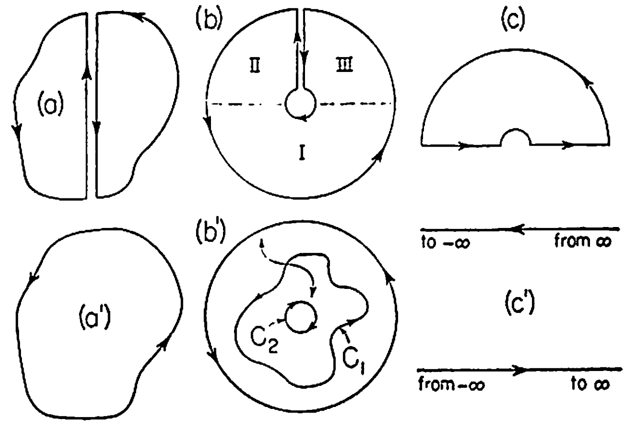

Fig. 4.4 Possible alterations in contours in the complex plane.

curve C_1 intermediate to the two boundary circles cannot be so deformed. The curve b illustrates the fact that any multiply connected surface may be made singly connected if the boundary is extended by means of crosscuts so that it is impossible to draw an irreducible contour. For example, the intermediate contour C_1 drawn in b' would not, if drawn in b, be entirely within the region as defined by the boundary lines. The necessity for the discussion of connectivity and its physical interpretation will become clear shortly.

Having disposed of these geometric matters, we are now able to state the central theorem of the theory of functions of a complex variable.

Cauchy's Theorem. *If a function $f(z)$ is an analytic function, continuous within and on a smooth closed contour C, then*

$$\oint f(z)\,dz = 0 \qquad (4.2.3)$$

For a proof of Cauchy's theorem as stated above, the reader may be referred to several texts in which the Goursat proof is given. The simple proof given earlier assumes that $f'(z)$ not only exists at every point within C but is also continuous therein. It is useful to establish the theorem within a minimum number of assumptions about $f(z)$, for this extends the ease of its applicability. In this section we shall content ourselves with assuming that *C bounds a star-shaped region and that* $f'(z)$ *is bounded everywhere within and on C.*

The geometric concept of "star-shaped" requires some elucidation. A star-shaped region exists if a point O can be found such that every ray from O intersects the bounding curve in precisely one point. A simple example of such a region is the region bounded by a circle. A region which is not star-shaped is illustrated by any annular region. Restricting our proof to a star-shaped region is not a limitation on the theorem, for any simply connected region may be broken up into a number of star-shaped regions and the Cauchy theorem applied to each. This process is illustrated in Fig. 4.4c for the case of a semiannular region. Here the semiannular region is broken up into parts like II and III, each of which is star-shaped. The Cauchy theorem may then be applied to each along the indicated contours so that

$$\oint_{\text{II}} f\,dz + \oint_{\text{III}} f\,dz = 0$$

However, in the sum of these integrals, the integrals over the parts of the contour common to III and II cancel out completely so that the sum of the integrals over I, II, and III just becomes the integral along the solid lines, the boundary of the semiannular contour.

The proof of the Cauchy theorem may now be given. Take the point O of the star-shaped region to be the origin. Define $F(\lambda)$ by

$$F(\lambda) = \lambda \oint f(\lambda z)\,dz; \quad 0 \leq \lambda \leq 1 \tag{4.2.4}$$

The Cauchy theorem is that $F(1) = 0$. To prove it, we differentiate $F(\lambda)$:

$$F'(\lambda) = \oint f(\lambda z)\,dz + \lambda \oint z f'(\lambda z)\,dz$$

Integrate the second of these integrals by parts [which is possible only if $f'(z)$ is bounded]:

$$F'(\lambda) = \oint f(\lambda z)\,dz + \lambda \left\{ \left[\frac{z f(\lambda z)}{\lambda} \right] - \frac{1}{\lambda} \oint f(\lambda z)\,dz \right\}$$

where the square bracket indicates that we take the difference of values at beginning and end of the contour of the quantity within the bracket. Since $z f(\lambda z)$ is a single-valued function, $[z f(\lambda z)/\lambda]$ vanishes for a closed contour so that

$$F'(\lambda) = 0 \quad \text{or} \quad F(\lambda) = \text{constant}$$

To evaluate the constant, let $\lambda = 0$ in Eq. (4.2.4), yielding $F(0) = 0 = F = F(\lambda)$. Therefore $F(1) = 0$, which proves the theorem. This proof, which appears so simple, in reality just transfers the onus to the question as to when an integral can be integrated by parts. The requirements, of course, involve just the ones of differentiability, continuity, and finiteness which characterize analytic functions.

Cauchy's theorem does *not apply to multiply connected regions,* for such regions are not bounded by a smooth contour. The physical reason for this restriction is easy to find. Recall from the discussion of page 354 that the Cauchy theorem, when applied to the electrostatic field, is equivalent to the statement that no charge is included within the region bounded by the contour C. Using Fig. 4.4b′ as an example of a multiply connected region, we see that contours entirely within the region in question exist (for example, contour C_1 in Fig. 4.4b′) to which Cauchy's

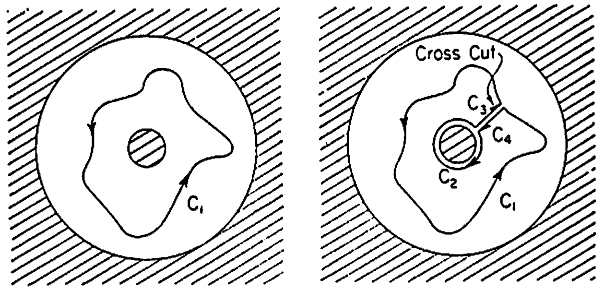

Fig. 4.5 Contours in multiply connected regions.

theorem obviously cannot apply because of the possible presence of charge outside the region in question, *e.g.,* charge within the smaller of the two boundary circles. The way to apply Cauchy's theorem with certainty would be to subtract the contour integral around the smaller circle; *i.e.,*

$$\oint_{C_1} f\,dz - \oint_{C_2} f\,dz = 0 \tag{4.2.5}$$

This may be also shown directly by using crosscuts to reduce the multiply connected domain to a single-connected one. From Fig. 4.5 we see that a contour in such a simply connected domain consists of the old contours C_1 and C_2 (C_1 described in a positive direction, C_2 in a negative direction) plus two additional sections C_3 and C_4. Cauchy's theorem may be applied to such a contour. The sections along C_3 and C_4 will cancel, yielding Eq. (4.2.5).

Some Useful Corollaries of Cauchy's Theorem. From Cauchy's theorem it follows that, *if* $f(z)$ *is an analytic function within a region bounded by closed contour C, then* $\int_{z_1}^{z_2} f(z)\,dz$, along any *contour within C depends only on* z_1 and z_2. That is, $f(z)$ has not only a unique derivative but also a unique integral. The uniqueness requirement is often used as motivation for a discussion of the Cauchy theorem. To prove this, we compare the two integrals $\int_{C_1}$ and $\int_{C_2'}$ in Fig. 4.6, where C_1 and C_2 are two different contours starting at z_1 and going to z_2. According to Cauchy's theorem $\int_{C_1} f(z)\,dz - \int_{C_2} f(z)\,dz = \oint f(z)\,dz$, is zero, proving the corollary.

It is a very important practical consequence of this corollary that *one may deform a contour without changing the value of the integral, provided that the contour crosses no singularity of the integrand during the deformation.* We shall have many occasions to use this theorem in the

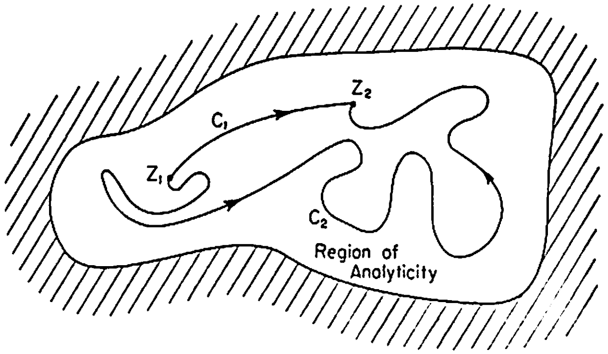

Fig. 4.6 Independence of integral value on choice of path within region of analyticity.

evaluation of contour integrals, for it thus becomes possible to choose a most convenient contour.

Because of the uniqueness of the integral $\int_{z_1}^{z_2} f\,dz$ it is possible to define an *indefinite integral of* $f(z)$ *by*

$$F(z) = \int_{z_1}^{z} f(z)\,dz$$

where the contour is, of course, within the region of analyticity of $f(z)$. It is an interesting theorem that, if $f(z)$ is analytic in a given region, then $F(z)$ is also analytic in the same region. Or, conversely, if $f(z)$ is singular at z_0, so is $F(z_0)$. To prove this result, we need but demonstrate the uniqueness of the derivative of $F(z)$, which can be shown by considering the identity

$$\frac{F(z) - F(\zeta)}{z - \zeta} - f(\zeta) = \frac{\int_{\zeta}^{z} [f(z) - f(\zeta)]\,dz}{z - \zeta}$$

Because of the continuity and single-valuedness of $f(z)$ the right-hand side of the above equation may be made as small as desired as z is made to approach ζ. Therefore in the limit

$$\lim_{z\to\zeta}\left[\frac{F(z) - F(\zeta)}{z - \zeta}\right] = f(\zeta)$$

Since the limit on the left is just the derivative $F'(\zeta)$, the theorem is proved.

We recall from Eqs. (4.1.19) *et seq.* that, if $f(z)$ is the conjugate of the electrostatic field, then the real part of $F(z)$ is the electrostatic potential while the imaginary part is constant along the electric lines of force and is therefore the stream function (see page 355). Therefore the two-dimensional electrostatic potential and the stream function form the real and imaginary parts of an analytic function of a complex variable.

Looking back through the proof of Cauchy's theorem, we see that we used only the requirements that $f(z)$ be continuous, one-valued, and that the integral be unique, with the result that we proved that $F(z)$ was *analytic.* We shall later show that, if a function is analytic in a region, so is its derivative [see Eq. (4.3.1)]. Drawing upon this information in advance of its proof, we see that, once we have found that $F(z)$ is analytic, we also know that $f(z)$ is analytic. This leads to the converse of Cauchy's theorem, known as *Morera's theorem:*

If $f(z)$ is continuous and single-valued within a closed contour C, and if $\oint f(z)\,dz = 0$ for any closed contour within C, then $f(z)$ is analytic within C.

This converse serves as a means for the identification of an analytic function and is thus the integral analogue of the differential requirement given by the Cauchy-Riemann conditions. Since the latter requires continuity in the derivative of f, the integral condition may sometimes be easier to apply.

The physical interpretation of Morera's theorem as given by the electrostatic analogue will strike the physicist as being rather obvious. It states that, if $f(z)$ is an electrostatic field and the net charge within any closed contour [evaluated with the aid of $f(z)$] within C is zero, then the charge density within that region is everywhere zero.

Cauchy's Integral Formula. This formula, a direct deduction from the Cauchy theorem, is the chief tool in the application of the theory of analytic functions to other branches of mathematics and also to physics. Its electrostatic analogue is known as Gauss' theorem, which states that the integral of the normal component of the electric field about a closed contour C equals the net charge within the contour. In electrostatics the proof essentially consists of separating off the field due to sources outside the contour from the field due to sources inside. The first, when integrated, must yield zero, while the second may be found by adding up the contribution due to each source. Cauchy's integral formula applies to the situation in which there is but one source inside C.

Consider the integral

$$J(a) = \oint [f(z)/(z - a)]\,dz \tag{4.2.6}$$

around a closed contour C within and on which $f(z)$ is analytic. The contour for this integral may be deformed into a small circle of radius ρ about the point a according to the corollary of Cauchy's theorem on the deformation of contours. Thus letting $z - a = \rho e^{i\varphi}$,

$$J = i \int_0^{2\pi} f(a + \rho e^{i\varphi})\, d\varphi = if(a) \int_0^{2\pi} d\varphi + i \int_0^{2\pi} [f(a + \rho e^{i\varphi}) - f(a)]\, d\varphi \tag{4.2.7}$$

Taking the limit as $\rho \to 0$, the second integral vanishes because of the continuity of $f(z)$. The Cauchy integral formula states, therefore, that, *if $f(z)$ is analytic inside and on a closed contour C, and if a is a point within C, then*

$$\oint [f(z)/(z - a)]\, dz = 2\pi i f(a) \tag{4.2.8}$$

If a is a point outside of C, then $\oint [f(z)/(z - a)]\, dz = 0$. If a is a point on C, the integral will have a Cauchy principal value[1] equal to $\pi i f(a)$ (just halfway between). The Cauchy principal value corresponds to putting half of the point source inside C and half outside. To summarize:

$$\oint \frac{f(z)}{z - a}\, dz = 2\pi i f(a) \begin{cases} 1; & \text{if } a \text{ within } C \\ \frac{1}{2}; & \text{if } a \text{ on } C \text{ (principal value)} \\ 0; & \text{if } a \text{ outside } C \end{cases} \tag{4.2.9}$$

Cauchy's formula is an *integral representation* of $f(z)$ which permits us to compute $f(z)$ anywhere in the interior of C, knowing only the value of $f(z)$ on C. Representations of this kind occur frequently in physics (particularly in the application of Green's or source functions) with the same sort of discontinuity as is expressed by Eq. (4.2.9). Thus if f is an electrostatic field, Eq. (4.2.8) tells us that the field within C may be computed in terms of the field along C. Similar theorems occur in the theory of wave propagation, where they are known collectively as Huygens' principle.

Cauchy's formula provides us with a very powerful tool for the investigation of the properties of analytic functions. It points up the strong correlation which exists between the values of an analytic function all over the complex plane. For example, using Eq. (4.2.7) we see that $f(a)$ is the *arithmetic average* of the values of f on any circle centered at a. *Therefore $|f(a)| \le M$ where M is the maximum value of $|f|$ on the circle. Equality can occur only if f is constant on the contour,* in which case *f is constant within the contour.* This theorem may be easily extended to a region bounded by any contour C.

[1] The Cauchy principal value is defined as follows: Let $q(x) \to \infty$ as $x \to a$, then the principal value of

$\int_b^c q(x)\, dx$, written $\mathcal{P} \int_b^c q(x)\, dx$, $(b < a < c)$ is $\lim_{\delta \to 0} \left\{ \int_b^{a-\delta} q(x)\, dx + \int_{a+\delta}^c q(x)\, dx \right\}$

In terms of the electrostatic analogue, the largest values of an electrostatic field within a closed contour occur at the boundary. If $f(z)$ has no zeros within C, then $[1/f(z)]$ will be an analytic function inside C and therefore $|1/f(z)|$ will have no maximum within C, taking its maximum value on C. *Therefore $|f(z)|$ will not have a minimum within C but will have its minimum value on the contour C.* The proof and theorem do not hold if $f(z)$ has zeros within C. The absolute value of an analytic function can have neither a maximum nor a minimum within the region of analyticity. If the function assumes either the maximum or minimum value within C, the function is a constant. Points at which $f(z)$ has a zero derivative will therefore be saddle points, rather than true maxima or minima.

Applying these results to the electrostatic field, we see that the field will take on both its minimum and maximum values on the boundary curve.

These theorems apply not only to $|f(z)|$ but also to the real and imaginary parts of an analytic function and therefore to the electrostatic potential V. To see this result rewrite Eq. (4.2.7) as

$$2\pi i f(a) = 2\pi i(u + iv) = i \int_0^{2\pi} f(x + iy)\, d\varphi = i \int_0^{2\pi} (u + iv)\, d\varphi$$

Equating real parts of the second and fourth expressions in this sequence of equations one obtains

$$u = \frac{1}{2\pi} \int^{2\pi} u\, d\varphi \tag{4.2.10}$$

so that u at the center of the circle is the arithmetic average of the values of u on the boundary of the circle. We may now use precisely the same reasoning as was employed in the discussion above for $|f(z)|$ and conclude that u will take on its minimum and maximum value on the boundary curve of a region within which f is analytic.

We therefore have the theorem that the electrostatic potential within a source-free region can have neither a maximum nor a minimum within that region. This fact has already been established in Chap. 1 (page 7) in the discussion of the Laplace equation which the electrostatic potential satisfies. From the theorem it immediately follows that, if V is constant on a contour enclosing a source-free singly connected region, then V is a constant within that region. This is just the well-known electrostatic result which states that the electrostatic field within a perfect conductor forming a closed surface is zero.

From these examples the general usefulness of the Cauchy integral formula should be clear. In addition we have once more demonstrated the special nature of analytic functions. We shall return to a more thorough discussion of these properties later on in this chapter.