Intersective polynomials and the polynomial Szemerédi theorem

V. Bergelson*, A. Leibman*, and E. Lesigne

July 1, 2008

Abstract

Let $P = \{p_1, \dots, p_r\} \subset \mathbb{Q}[n_1, \dots, n_m]$ be a family of polynomials such that $p_i(\mathbb{Z}^m) \subseteq \mathbb{Z}$, $i = 1, \dots, r$. We say that the family P has *the PSZ property* if for any set $E \subseteq \mathbb{Z}$ with $d^*(E) = \limsup_{N-M\to\infty} \frac{|E\cap[M,N-1]|}{N-M} > 0$ there exist infinitely many $n \in \mathbb{Z}^m$ such that E contains a polynomial progression of the form $\{a, a+p_1(n), \dots, a+p_r(n)\}$. We prove that a polynomial family $P = \{p_1, \dots, p_r\}$ has the PSZ property if and only if the polynomials $p_1, \dots, p_r$ are *jointly intersective*, meaning that for any $k \in \mathbb{N}$ there exists $n \in \mathbb{Z}^m$ such that the integers $p_1(n), \dots, p_r(n)$ are all divisible by k. To obtain this result we give a new ergodic proof of the polynomial Szemerédi theorem, based on the fact that the key to the phenomenon of polynomial multiple recurrence lies with the dynamical systems defined by translations on nilmanifolds. We also obtain, as a corollary, the following generalization of the polynomial van der Waerden theorem: If $p_1, \dots, p_r \in \mathbb{Q}[n]$ are jointly intersective integral polynomials, then for any finite partition $\mathbb{Z} = \bigcup_{i=1}^k E_i$ of $\mathbb{Z}$, there exist $i \in \{1, \dots, k\}$ and $a, n \in E_i$ such that $\{a, a+p_1(n), \dots, a+p_r(n)\} \subset E_i$.

1 Introduction

Let us call a polynomial $p \in \mathbb{Q}[n]$ *integral* if it takes on integer values on the integers. The polynomial Szemerédi theorem ([BeL]) states that if a set $E \subseteq \mathbb{Z}$ has positive upper Banach density, $d^*(E) = \limsup_{N-M\to\infty} \frac{|E\cap[M,N-1]|}{N-M} > 0$, then for any finite family of integral polynomials $P = \{p_1, \dots, p_r\}$ with $p_i(0) = 0$, $i = 1, \dots, r$, one can find an

*The first two authors were supported by NSF grant DMS-0600042

arbitrarily large $n \in \mathbb{N}$ such that, for some $a \in E$, $\{a, a + p_1(n), \ldots, a + p_r(n)\} \subset E$. Moreover, the set

$$N_P(E) = \Big\{n \in \mathbb{Z} : \text{for some } a,\ \{a, a + p_1(n), \ldots, a + p_r(n)\} \subset E\Big\}$$

is syndetic, that is, $N_P(E)$ has a nontrivial intersection with any long enough interval in $\mathbb{Z}$ (see [BeM1]). The polynomial Szemerédi theorem is an extension of Szemerédi's theorem on arithmetic progressions, which corresponds to $p_i(n) = in$, $i = 1, \ldots, r$, (see [Sz] and [Fu1]) and of the Sàrközy–Furstenberg theorem, which corresponds to the case $r = 1$ (see [Sa], [Fu1], [Fu2]).

It is not hard to see that the condition $p_i(0) = 0$, $i = 1, \ldots, r$, in the polynomial Szemerédi theorem is not superfluous. (Consider, for example, $r = 1$, $p(n) = 2n + 1$, $E = 2\mathbb{N}$, or $r = 1$, $p(n) = n^2 + 1$, $E = 3\mathbb{N}$.) On the other hand, it is also clear that this condition is not a necessary one. For example, it is easy to see that it can be replaced by the condition $p_i(n_0) = 0$, $i = 1, \ldots, r$, for some $n_0 \in \mathbb{Z}$. Actually, the latter condition still falls short of being necessary. Let us say that a family of integral polynomials $P = \{p_1, \ldots, p_r\}$ has the PSZ property if for every set $E \subseteq \mathbb{Z}$ with $d^*(E) > 0$ the introduced above set $N_P(E)$ is nonempty, and let us say that P has the SPSZ property if for every set $E \subseteq \mathbb{Z}$ with $d^*(E) > 0$ the set $N_P(E)$ is syndetic. Our goal in this paper is to establish necessary and sufficient conditions for a family of integral polynomials to have the PSZ property. When $r = 1$, such a condition was obtained in [KaM]. Namely, it was proved in [KaM] that a family consisting of a single integral polynomial p has the PSZ property if and only if p is *intersective*, meaning that for any $k \in \mathbb{N}$ the intersection $\{p(n),\ n \in \mathbb{Z}\} \cap k\mathbb{Z}$ is nonempty. It is clear that any integral polynomial with zero constant term, and, more generally, any integral polynomial with an integer root, is intersective. There are also examples of intersective polynomials without rational roots; for example, one can show that the polynomial $p(n) = (n^2 - 5)(n^2 - 41)(n^2 - 205)$ is intersective (see Section 6).

As we will see, our condition for a family P to have the PSZ property is a natural generalization of the Kamae and Mendès France condition from [KaM]. We will say that polynomials $p_1, \ldots, p_r$ are *jointly intersective* if for every $k \in \mathbb{N}$ there exists $n \in \mathbb{Z}$ such that $p_i(n)$ is divisible by k for all $i = 1, \ldots, r$. Here is now the formulation of our main result.

Theorem 1.1. *Let $P = \{p_1, \ldots, p_r\}$ be a system of integral polynomials. The following statements are equivalent:*
(i) *P has the PSZ property;*
(ii) *P has the SPSZ property;*
(iii) *the polynomials $p_1, \ldots, p_r$ are jointly intersective.*

A special case of this theorem for 3 polynomials in one variable was proved by Frantzikinakis ([F], Theorem F).

Remark. One can easily show (see Section 6.1 below) that several integral polynomials of one variable are jointly intersective if and only if they are all divisible by a single intersective polynomial, and thus it follows from Theorem 1.1 that a family P of integral polynomials possesses the PSZ property iff it is of the form $P = \{q_1 p, q_2 p, \ldots, q_r p\}$ where $q_1, \ldots, q_r \in \mathbb{Q}[n]$ and p is an intersective polynomial. In particular, for any intersective polynomial p and any $r \in \mathbb{N}$ the family $P = \{p, 2p, \ldots, rp\}$ has the PSZ property; this result was also obtained in [F].

Theorem 1.1 tells us that the only obstacle for a family of integral polynomials to possess the PSZ property is of *arithmetic* nature. The following direct corollary of Theorem 1.1 gives a precise meaning to this observation:

Theorem 1.2. *If $p_1, \ldots, p_r$ are integral polynomials such that any lattice $k\mathbb{Z}$ in $\mathbb{Z}$ contains a configuration of the form $\{a, a+p_1(n), \ldots, a+p_r(n)\}$ with $a, n \in \mathbb{Z}$, then any set of positive upper Banach density in $\mathbb{Z}$ also contains such a configuration.*

As a matter of fact, we will obtain a "multiparameter" version of Theorem 1.1, that is, we will prove this theorem for polynomials of several variables. (Passing from one to many variables does not make the proof longer, but essentially strengthens the theorem.) We say that a polynomial p of $m \geq 1$ variables with rational coefficients is *integral* if $p(\mathbb{Z}^m) \subseteq \mathbb{Z}$. We will interpret any integral polynomial p of m variables as a mapping $\mathbb{Z}^m \longrightarrow \mathbb{Z}$, and say that p is *an integral polynomial on $\mathbb{Z}^m$*. A set S in $\mathbb{Z}^m$ is said to be *syndetic* if $S + K = \mathbb{Z}^m$ for some finite $K \subset \mathbb{Z}^m$; the other definitions do not change, and from now on we will assume that the polynomials $p_1, \ldots, p_r$ in Theorem 1.1 are integral polynomials on $\mathbb{Z}^m$.

Clearly, (ii) in Theorem 1.1 implies (i); it is also clear that (i) implies (iii): if $p_1, \ldots, p_r$ are not jointly intersective and $k \in \mathbb{N}$ is such that for no $n \in \mathbb{Z}^m$ the integers $p_1(n), \ldots, p_r(n)$ are all divisible by k, then the lattice $k\mathbb{Z}$ does not contain configurations of the form $\{a, a+p_1(n), \ldots, a+p_r(n)\}$. So, it is only the implication (iii) $\Longrightarrow$ (ii) which needs to be proven. We will actually get a stronger result:

Theorem 1.3. *Let $p_1, \ldots, p_r$ be jointly intersective integral polynomials on $\mathbb{Z}^m$ and let $E \subseteq \mathbb{Z}$, $d^*(E) > 0$. Then there exists $\varepsilon > 0$ such that the set*

$$\Big\{ n \in \mathbb{Z}^m : d^*\big(E \cap (E - p_1(n)) \cap \ldots \cap (E - p_r(n))\big) > \varepsilon \Big\}$$

is syndetic.

Like the proof of the polynomial Szemerédi theorem in [BeL], our proof of Theorem 1.3 relies on Furstenberg's correspondence principle. This principle, which plays instrumental role in [Fu1], can be found in the following form in [Be]:
For any set $E \subseteq \mathbb{Z}$ with $d^*(E) > 0$ there exists an invertible probability measure preserving system $(X, \mathcal{B}, \mu, T)$ and a set $A \in \mathcal{B}$ with $\mu(A) = d^*(E)$ such that for any $r \in \mathbb{N}$ and $n_1, n_2, \ldots, n_r \in \mathbb{Z}$ one has

$$d^*\big(E \cap (E - n_1) \cap \ldots \cap (E - n_r)\big) \geq \mu\big(A \cap T^{-n_1}A \cap \ldots \cap T^{-n_r}A\big).$$

For a multiparameter sequence $(a_n)_{n \in \mathbb{Z}^m}$ of real numbers we define $\text{UC-lim}_n\, a_n = \lim_{N\to\infty} \frac{1}{|\Phi_N|}\sum_{n\in\Phi_N} a_n$, if this limit exists for every Følner sequence (Φ_N) in $\mathbb{Z}^m$.[1] (Note that if this limit exists for all Følner sequences, then it does not depend on the choice of the sequence.) In view of Furstenberg's correspondence principle, Theorem 1.3 is a corollary of the following ergodic result.

Theorem 1.4. *Let integral polynomials $p_1, \ldots, p_r$ on $\mathbb{Z}^m$ be jointly intersective. Then for any invertible probability measure preserving system $(X, \mathcal{B}, \mu, T)$ and any set $A \in \mathcal{B}$ with $\mu(A) > 0$,*

$$\underset{n}{\text{UC-lim}}\, \mu\big(A \cap T^{-p_1(n)}A \cap \ldots \cap T^{-p_r(n)}A\big) > 0. \tag{1}$$

We remark that the converse of this theorem is also true: if the polynomials $p_1, \ldots, p_r$ are not jointly intersective, one can construct a (finite) measure preserving system and a set A such that the limit in (1) is equal to 0. We also remark that having "lim inf" instead of "lim" in formula (1) would be quite sufficient to prove Theorem 1.3; but, anyway, it is known that the limit $\text{UC-lim}_n\, \mu\big(A \cap T^{-p_1(n)}A \cap \ldots \cap T^{-p_r(n)}A\big)$ exists, – see [L4].

It is worth noticing that while being ergodic in nature, our proof of Theorem 1.4 is quite different from the ergodic proofs of polynomial Szemerédi theorem in [BeL] and [BeM1]. The reason that we had to resort to a completely different approach lies with the fact that the main ingredients of the proofs in [BeL] and [BeM1], namely the PET induction and combinatorial results such as the polynomial van der Waerden theorem (in [BeL]) and the polynomial Hales–Jewett theorem (in [BeM1]), do not work when the polynomials involved may have a non-zero constant term. In particular, it is not clear how to obtain by purely combinatorial means (or with the help of topological dynamics but without using an invariant measure) the following corollary of Theorem 1.3.

[1] *A Følner sequence* in a (discrete) abelian group G is a sequence (Φ_N) of finite subsets of G with the property that for any $g \in G$, $|(\Phi_N + g)\triangle\Phi_N|/|\Phi_N| \longrightarrow 0$ as $N \longrightarrow \infty$.

Theorem 1.5. *For any finite partition $\mathbb{Z} = \bigcup_{i=1}^{k} E_i$, one of the E_i has the property that for any r, m, and any jointly intersective integral polynomials $p_1, \ldots, p_r$ on $\mathbb{Z}^m$ there exists $\varepsilon > 0$ such that the set $\{n \in \mathbb{Z}^m : d^*\big(E_i \cap (E_i - p_1(n)) \cap \ldots \cap (E_i - p_r(n))\big) > \varepsilon\}$ is syndetic.*

Remarks. 1. One can also show that, for any collection $\{p_1, \ldots, p_r\}$ of integral polynomials of one variable and any partition $\mathbb{Z} = \bigcup_{i=1}^{k} E_i$, one of the E_i contains many configurations of the form $\{a, a + p_1(n), \ldots, a + p_r(n)\}$ with $n \in E_i$; see Theorem 5.5 below.
2. Note that the converse of 1.5 is also true: if $p_1, \ldots, p_r$ are not jointly intersective and $k \in \mathbb{N}$ is such that for no $n \in \mathbb{Z}^m$ the integers $p_1(n), \ldots, p_r(n)$ are all divisible by k, then no element of the partition $\mathbb{Z} = \bigcup_{i=0}^{k-1}(k\mathbb{Z} + i)$ of $\mathbb{Z}$ contains configurations of the form $\{a, a + p_1(n), \ldots, a + p_r(n)\}$.

The proof of Theorem 1.4 is divided into several steps. The first one is a reduction to nilsystems via the Host-Kra–Ziegler machinery. The second step is a differential geometry argument (Lemma 2.3) which allows us to reduce the recurrence problem to properties of the closure of an orbit in a nilsystem (Proposition 2.4). The last step is a description of polynomial orbits on tori (Section 3) and on nilmanifolds (Section 4). In Section 5 we finish the proof of Theorem 1.3 and obtain (an enhanced version of) Theorem 1.5. Section 6 is devoted to concluding remarks and conjectures.

2 Polynomial Szemerédi theorem and polynomial orbits in nilmanifolds

To facilitate the proof of Theorem 1.4 we make some relatively routine reductions. First, we assume that the measure space $(X, \mathcal{B}, \mu)$ is Lebesgue. (To justify this assumption, let us observe that given a set $A \in \mathcal{B}$, we can confine ourselves to the separable T-invariant σ-algebra generated by the family $\{T^n A\}_{n\in\mathbb{Z}}$. Invoking the Carathéodory and von Neumann Theorems (see for example [Roy], Ch. 15, Theorems 4 and 20) we may now assume that $(X, \mathcal{B}, \mu)$ is a Lebesgue space.) Next, we will assume that T is ergodic. (Indeed, by using the ergodic decomposition one can easily show that the validity of Theorem 1.4 follows from its validity in the ergodic case.)

Our next step is to reduce the situation to the case where (X, T) is a nilsystem. An s-step *nilsystem* is a measure preserving system defined by a translation on a compact s-step nilmanifold[2] equipped with the normalized Haar measure. *An s-step pro-nilsystem* is

[2] *An s-step nilmanifold* is the quotient space, $X = G/\Gamma$, of an s-step nilpotent Lie group G by a discrete

the inverse limit of a sequence of s-step nilsystems.[3] Let $p_1, \ldots, p_r$ be integral polynomials on $\mathbb{Z}^m$, $m \geq 1$. It was proved in [L4] (see also [HKr2]) that for an ergodic probability measure preserving system $(X, \mathcal{B}, \mu, T)$, where X is a Lebesgue space, a certain pro-nilsystem $(\widetilde{X}, \widetilde{\mathcal{B}}, \widetilde{\mu}, \widetilde{T})$ is *a characteristic factor* of $(X, \mathcal{B}, \mu, T)$ with respect to the system of polynomial actions $\{T^{p_1(n)}, \ldots, T^{p_r(n)}\}$, which means that $(\widetilde{X}, \widetilde{\mathcal{B}}, \widetilde{\mu}, \widetilde{T})$ is a factor[4] of $(X, \mathcal{B}, \mu, T)$ such that for any $f_0, f_1, \ldots, f_r \in L^\infty(X)$ one has

$$\begin{aligned} &\operatorname*{UC-lim}_n \int_X f_0 \cdot f_1 \circ T^{p_1(n)} \cdot \ldots \cdot f_r \circ T^{p_r(n)} \, d\mu \\ &\qquad = \operatorname*{UC-lim}_n \int_X E(f_0|\widetilde{X}) \cdot E(f_1|\widetilde{X}) \circ \widetilde{T}^{p_1(n)} \cdot \ldots \cdot E(f_r|\widetilde{X}) \circ \widetilde{T}^{p_r(n)} \, d\widetilde{\mu} \end{aligned}$$

(where $E(\cdot|\widetilde{X})$ stands for the conditional expectation[5] with respect to $\widetilde{X}$).

The statement

$$\operatorname*{UC-lim}_n \mu\big(A \cap T^{-p_1(n)} A \cap \ldots \cap T^{-p_r(n)} A\big) > 0 \text{ for any measurable } A \subseteq X \text{ with } \mu(A) > 0 \tag{2}$$

is clearly equivalent to the statement

$$\operatorname*{UC-lim}_n \int_X f \cdot f \circ T^{p_1(n)} \cdot \ldots \cdot f \circ T^{p_r(n)} d\mu > 0 \quad \text{for all } f \in L^\infty(X)$$
$$\text{such that } f \geq 0 \text{ and } \textstyle\int_X f \, d\mu > 0.$$

Thus, in order to prove Theorem 1.4, we have to check (2) for pro-nilsystems only. The following lemma, which appears in [FuK], shows that it is enough to check the result in the case where $(X, \mathcal{B}, \mu, T)$ is a nilsystem.

---

cocompact subgroup Γ of G. *A translation* on the nilmanifold $X = G/\Gamma$ is the mapping $g\Gamma \mapsto ag\Gamma$, $g \in G$, defined by an element $a \in G$.

[3] A measure preserving system $(X, \mathcal{B}, \mu, T)$ is *the inverse limit* of (an increasing) sequence of its factors[4] $(X_\alpha, \mathcal{B}_\alpha, \mu_\alpha, T_\alpha)$, $\alpha \in \mathbb{N}$, if $\mathcal{B}_1 \subseteq \mathcal{B}_2 \subseteq \ldots$ in $\mathcal{B}$ and the σ-alegbra $\bigvee_{\alpha \in \mathbb{N}} \mathcal{B}_\alpha$ generated by the union of $\mathcal{B}_\alpha$ equals $\mathcal{B}$.

[4] *A factor* of a measure preserving system $(X, \mathcal{B}, \mu, T)$ is a measure preserving system $(\widetilde{X}, \widetilde{\mathcal{B}}, \widetilde{\mu}, \widetilde{T})$ together with a measure-preserving mapping $\pi \colon X \longrightarrow \widetilde{X}$ satisfying $\widetilde{T} \circ \pi = \pi \circ T$. In this situation, $\pi^{-1}(\widetilde{\mathcal{B}})$ is a T-invariant sub-σ-algebra of $\mathcal{B}$, which is identified with $\widetilde{\mathcal{B}}$. Another, and in some sense equivalent, way to introduce a factor of $(X, \mathcal{B}, \mu, T)$ is to indicate the corresponding T-invariant sub-σ-algebra of $\mathcal{B}$.

[5] *The conditional expectation* $E(f|\widetilde{X})$, or $E(f|\widetilde{\mathcal{B}})$, of a function $f \in L^1(X)$ with respect to a factor $\pi \colon (X, \mathcal{B}, \mu) \longrightarrow (\widetilde{X}, \widetilde{\mathcal{B}}, \widetilde{\mu})$ is the (uniquely defined) function $\tilde{f} \in L^1(\widetilde{X})$ satisfying $\int_B \tilde{f} \, d\widetilde{\mu} = \int_{\pi^{-1}(B)} f \circ \pi \, d\mu$ for every $B \in \widetilde{\mathcal{B}}$.

Lemma 2.1. *Let $r \in \mathbb{N}$. Let $(X, \mathcal{B}, \mu, T)$ be a measure preserving dynamical system and $(\mathcal{B}_\alpha)_{\alpha\in\mathbb{N}}$ be an increasing sequence of T-invariant sub-σ-algebras such that $\bigvee_{\alpha\in\mathbb{N}} \mathcal{B}_\alpha = \mathcal{B}$. Then, for any $B \in \mathcal{B}$, there exist $\alpha \in \mathbb{N}$ and $B' \in \mathcal{B}_\alpha$ such that $\mu(B') \geq \mu(B)/2$ and, for all $n_1, \dots, n_r \in \mathbb{Z}$,*

$$\mu\left(B \cap T^{-n_1}B \cap \ldots \cap T^{-n_r}B\right) \geq \tfrac{1}{2}\mu\left(B' \cap T^{-n_1}B' \cap \ldots \cap T^{-n_r}B'\right).$$

Proof. We assume that $\mu(B) > 0$. The sequence of conditional probabilities (measures) $\left(\mu\left(B \mid \mathcal{B}_\alpha\right)\right)_{\alpha\in\mathbb{N}}$[6] converges in measure to the characteristic function 1_B. Hence there exists α such that the set $B' := \left\{\mu\left(B \mid \mathcal{B}_\alpha\right) \geq 1 - \frac{1}{2(r+1)}\right\}$ has measure $\geq \frac{1}{2}\mu(B)$. For any $n \in \mathbb{Z}$, we have $T^{-n}B' := \left\{\mu\left(T^{-n}B \mid \mathcal{B}_\alpha\right) \geq 1 - \frac{1}{2(r+1)}\right\}$. Using the fact that $\mu\left(B_0 \cap B_1 \cap \ldots \cap B_r \mid \mathcal{B}_\alpha\right) \geq 1 - (r+1)\varepsilon$ if $\mu\left(B_i \mid \mathcal{B}_\alpha\right) \geq 1 - \varepsilon$, $0 \leq i \leq r$, we get

$$\begin{aligned}
\mu\left(B \cap T^{-n_1}B \cap \ldots \cap T^{-n_r}B\right) &= \int_X \mu\left(B \cap T^{-n_1}B \cap \ldots \cap T^{-n_r}B \mid \mathcal{B}_\alpha\right) d\mu \\
&\geq \int_{B' \cap T^{-n_1}B' \cap \ldots \cap T^{-n_r}B'} \mu\left(B \cap T^{-n_1}B \cap \ldots \cap T^{-n_r}B \mid \mathcal{B}_\alpha\right) d\mu \\
&\geq \frac{1}{2}\mu\left(B' \cap T^{-n_1}B' \cap \ldots \cap T^{-n_r}B'\right).
\end{aligned}$$

□

Thus, Theorem 1.4 is reduced to the following proposition:

Proposition 2.2. *Let integral polynomials $p_1, \dots, p_r$ on $\mathbb{Z}^m$ be jointly intersective. Then for any nilsystem $(X, \mathcal{B}, \mu, T)$ and any set $A \in \mathcal{B}$ with $\mu(A) > 0$,*

$$\operatorname*{UC-lim}_n \mu\left(A \cap T^{-p_1(n)}A \cap \ldots \cap T^{-p_r(n)}A\right) > 0.$$

We will now assume that X is a nilmanifold, $X = G/\Gamma$, and $(X, \mathcal{B}, \mu, T)$ is a nilsystem. *A subnilmanifold* of X is a closed subset of X of the form $D = Kx$, where K is a closed subgroup of G and $x \in X$. A subnilmanifold is a nilmanifold itself under the action of the nilpotent Lie group K, and supports a unique probability Haar measure which we will denote by μ_D. It is known (see [L2], or [Sh] for a much more general result) that if H is a subgroup of G and $x \in X$, then $D = \overline{Hx}$ is a subnilmanifold of X.

A (multiparameter) polynomial sequence in G is a mapping $g\colon \mathbb{Z}^m \longrightarrow G$ of the form $g(n) = a_1^{p_1(n)} \dots a_r^{p_r(n)}$, $n \in \mathbb{Z}^m$, where $a_i \in G$ and p_i are integral polynomials on $\mathbb{Z}^m$. It

[6] $\mu(B|\mathcal{B}_\alpha) = E(1_B|\mathcal{B}_\alpha)$.

is proved in [L3] that if g is a polynomial sequence in G and D is a subnilmanifold of X, then the closure $Y=\overline{\mathrm{Orb}}_g(D)$ of the orbit $\mathrm{Orb}_g(D)=\bigcup_{n\in\mathbb{Z}}g(n)D$ of D is either a subnilmanifold or a finite disjoint union of subnilmanifolds of X. Moreover, the sequence $\{g(n)D\}_{n\in\mathbb{Z}}$ has an asymptotic distribution in Y: we have $\text{UC-lim}_n\, g(n)\mu_D=\mu'_Y$, where μ'_Y is a convex combination of the Haar measures on the connected components of Y. In particular, if Y is connected, then Y is a subnilmanifold, and $\mu'_Y=\mu_Y$ is the Haar measure on Y.

Let $p_1,\dots,p_r$ be integral polynomials on $\mathbb{Z}^m$; consider the polynomial sequence $g(n)=\begin{pmatrix}1_G\\ a^{p_1(n)}\\ \vdots\\ a^{p_r(n)}\end{pmatrix}$, $n\in\mathbb{Z}^m$, in the group G^{r+1}. Let $\Delta_{X^{r+1}}$ be *the diagonal*, $\Delta_{X^{r+1}}=\left\{\bar{x}=\begin{pmatrix}x\\ \vdots\\ x\end{pmatrix}:x\in X\right\}$ in the nilmanifold X^{r+1}, and let $Y=\overline{\mathrm{Orb}}_g(\Delta_{X^{r+1}})$. Then for any continuous functions $f_0,f_1,\dots,f_r$ on X,

$$\begin{aligned}
&\operatorname*{UC-lim}_n\int_X f_0\cdot f_1{\circ}T^{p_1(n)}\cdot\ldots\cdot f_r{\circ}T^{p_r(n)}\,d\mu\\
=\;&\operatorname*{UC-lim}_n\int_{\Delta_{X^{r+1}}} f_0\otimes f_1{\circ}T^{p_1(n)}\otimes\ldots\otimes f_r{\circ}T^{p_r(n)}\,d\mu_{\Delta_{X^{r+1}}}\\
=\;&\operatorname*{UC-lim}_n\int_{\Delta_{X^{r+1}}} \big(f_0\otimes f_1\otimes\ldots\otimes f_r\big)(g(n)\overline{x})\,d\mu_{\Delta_{X^{r+1}}}(\overline{x})\\
=\;&\operatorname*{UC-lim}_n\int_{g(n)\Delta_{X^{r+1}}} f_0\otimes f_1\otimes\ldots\otimes f_r\,d\mu_{g(n)\Delta_{X^{r+1}}}.\\
=\;&\int_Y f_0\otimes f_1\otimes\ldots\otimes f_r\,d\mu'_Y.
\end{aligned}$$

Since $C(X)$ is dense in $L^{r+1}(X,\mu)$ and all the marginals of μ'_Y are equal to μ, we obtain from the multilinearity of the above expressions that

$$\operatorname*{UC-lim}_n\int_X f_0\cdot f_1{\circ}T^{p_1(n)}\cdot\ldots\cdot f_r{\circ}T^{p_r(n)}\,d\mu=\int_Y f_0\otimes f_1\otimes\ldots\otimes f_r\,d\mu'_Y$$

for any $f_0,f_1,\dots,f_r\in L^\infty(X)$. In particular, for any measurable set $A\subseteq X$,

$$\operatorname*{UC-lim}_n\mu\big(A\cap T^{-p_1(n)}A\cap\ldots\cap T^{-p_r(n)}A\big)=\mu'_Y(A^{r+1}\cap Y),$$

and in order to prove Proposition 2.2 we only need to show that $\mu'_Y(A^{r+1}\cap Y)>0$ whenever $\mu(A)>0$.

We claim that this is true as long as $Y\supseteq\Delta_{X^{r+1}}$. Indeed, let us assume that this inclusion holds, and let A be a set of positive measure in X. Let $x\in X$ be a Lebesgue point of A, and let $\bar{x} = \begin{pmatrix} x \\ \vdots \\ x \end{pmatrix} \in \Delta_{X^{r+1}}$. Using a system of Malcev coordinates in G (see Section 4), we identify a connected open neighborhood Ω of x with an open subset of $\mathbb{R}^d$, where $d = \dim X$. Then, under this identification, $Y' = Y \cap \Omega^{r+1}$ is a smooth (polynomial) manifold in $\mathbb{R}^{d(r+1)}$, and the restriction on Y' of the measure μ'_Y is equivalent to the Lebesgue measure (that is, the s-volume[7], where $s = \dim Y$) in Y'. Let S be the connected component of Y' that contains $\Delta_{\Omega^{r+1}}$. Our claim now follows from the following lemma.

Lemma 2.3. *Let Ω be an open subset of $\mathbb{R}^d$ and let S be a connected C^1-manifold in Ω^k with $S \supseteq \Delta_{\Omega^k}$. Let σ be the Lebesgue measure on S. Then for any subset A of Ω with positive Lebesgue measure one has $\sigma(A^k \cap S) > 0$.*

Proof. Let x be a density point of A. For $t > 0$ let Q_t be the cube in $\mathbb{R}^d$ of size t centered at x, and let $P_t = Q_t^k$ (which is the cube in $\mathbb{R}^{dk}$ of size t centered at $\bar{x} = \begin{pmatrix} x \\ \vdots \\ x \end{pmatrix}$). Let π_i, $i = 1, \ldots, k$, be the projection from $\mathbb{R}^{dk} = (\mathbb{R}^d)^k$ onto the ith factor. Since S contains Δ_{Ω^k}, for any i, π_i projects S onto Ω and has full rank at all points of S.

Let $s = \dim S$. Let L be the tangent space to S at the point $\bar{x} = \begin{pmatrix} x \\ \vdots \\ x \end{pmatrix}$ and let λ be the Lebesgue measure on L. Let $1 \leq i \leq k$. To simplify notation, assume that $x = 0$, so that L is a vector subspace of $(\mathbb{R}^d)^k$. Since $S \supseteq \Delta_{\Omega^k}$, $L \supseteq \Delta_{(\mathbb{R}^d)^k}$, and so $L + \big((\mathbb{R}^d)^{i-1} \times \{0\} \times (\mathbb{R}^d)^{k-i}\big) = \mathbb{R}^{dk}$. Thus, there exists a vector subspace $V \subseteq (\mathbb{R}^d)^{i-1} \times \{0\} \times (\mathbb{R}^d)^{k-i}$ such that $L \oplus V = \mathbb{R}^{dk}$. Let η be the projection $S \longrightarrow L$ in the direction of V. η is a diffeomorphism in a neighborhood of $\bar{x}$, and in the coordinate system in $\mathbb{R}^{dk}$ in which L is the first s-dimensional coordinate plane, $D\eta^{-1}(\bar{x}) = (I|0)$. Thus, using (L, η^{-1}) to parametrize S in a neighborhood of $\bar{x}$, we see that $\frac{d\eta(\sigma)}{d\lambda}(z) \to 1$ as $z \to \bar{x}$, and so, there exists a neighborhood U of $\bar{x}$ such that

$$\frac{1}{2}\lambda(\eta(E)) \leq \sigma(E) \leq 2\lambda(\eta(E)) \tag{3}$$

for any measurable set $E \subseteq S \cap U$. For the same reason, for t small enough we have $\eta(S \cap P_t) \subseteq L \cap P_{2t}$ and $L \cap P_t \subseteq \eta(S \cap P_{2t})$. By the definition of η, $\pi_1 \circ \eta = \pi_1$; thus for any $B \subseteq \Omega$, $\eta(\pi_1^{-1}(B) \cap S) = \pi_1^{-1}(B) \cap L$. It follows that for t small enough,

$$\eta\big(\pi_1^{-1}(B) \cap S \cap P_t\big) \subseteq \pi_1^{-1}(B) \cap L \cap P_{2t} \tag{4}$$

[7]For a smooth s-dimensional surface S in $\mathbb{R}^m$, defined (locally) by $y = \tau(u)$, $u \in U \subseteq \mathbb{R}^m$, the Lebesgue measure σ on S is given by $d\sigma = \tau\big(\big(\sum_{|I|=s}\big|\frac{\partial y_I}{\partial u}\big|^2\big)^{1/2} d\lambda\big)$, where λ is the Lebesgue measure on U. (Note that σ does not depend on the choice of parametrization of S.)

and

$$\pi_1^{-1}(B) \cap L \cap P_t \subseteq \eta\big(\pi_1^{-1}(B) \cap S \cap P_{2t}\big) \tag{5}$$

for any measurable set $B \subseteq \Omega$. Combining (4) and (3), (5) and (3), we obtain that for t small enough,

$$\sigma\left(\pi_1^{-1}(B) \cap S \cap P_t\right) \leq 2\lambda\left(\pi_1^{-1}(B) \cap L \cap P_{2t}\right) \tag{6}$$

and

$$\lambda\left(\pi_1^{-1}(B) \cap L \cap P_t\right) \leq 2\sigma\left(\pi_1^{-1}(B) \cap S \cap P_{2t}\right) \tag{7}$$

for any measurable set $B \subseteq \Omega$. Let σ_t and λ_t be the normalized Lebesgue measures on $S \cap P_t$ and on $L \cap P_t$ respectively. Then for t small enough we have from (7) that $\sigma\left(S \cap P_t\right) \geq \frac{1}{2}\lambda(L \cap P_{t/2}) = 2^{-2s-1}\lambda(L \cap P_{2t})$, and thus from (6),

$$\sigma_t(\pi_i^{-1}(B) \cap S \cap P_t) \leq 2^{2s+2}\lambda_{2t}(\pi_i^{-1}(B) \cap L \cap P_{2t}) \tag{8}$$

for any measurable $B \subseteq \Omega$.

For $t > 0$, let ν_t be the normalized Lebesgue measure on the cube $Q_t \subset \mathbb{R}^d$. Since L is an affine space passing through the center of Q_t, and since, for each i, L projects by π_i *onto* $\mathbb{R}^d$, we have $\pi_i(\lambda_t) \leq c_i\nu_t$ with a constant c_i independent on t. Let $c = \max\{c_1, \ldots, c_k\}$, then $\lambda_t(\pi_i^{-1}(B) \cap L) \leq c\nu_t(B)$ for any measurable set $B \subseteq \mathbb{R}^d$ and all i.

Now choose t small enough so that (8) holds for all i and that $\nu_{2t}\left(Q_{2t} \setminus A\right) < 1/(2^{2s+2}kc)$ (which is possible since x is a density point of A). Then

$$\begin{aligned}\sigma_t\left(A^k \cap P_t \cap S\right) \geq 1 - \sum_{i=1}^{k}\sigma_t\left(\pi_i^{-1}(Q_t \setminus A) \cap S\right) &\geq 1 - \sum_{i=1}^{k}2^{2s+2}\lambda_{2t}\left(\pi_i^{-1}(Q_{2t} \setminus A) \cap L\right) \\ &\geq 1 - \sum_{i=1}^{k}2^{2s+2}c\nu_{2t}\left(Q_{2t} \setminus A\right) > 0,\end{aligned}$$

and so $\sigma(A^k \cap S) > 0$. $\square$

Hence, we are done if we prove that $\overline{\mathrm{Orb}}(\bar{x}) \ni \bar{x}$ for every $\bar{x} \in \Delta_{X^{r+1}}$. After considering the new nilmanifold X^{r+1} and changing notation, Proposition 2.2 is now reduced to the following proposition.

Proposition 2.4. *Let $X = G/\Gamma$ be a nilmanifold and let $g(n) = a_1^{p_1(n)} \ldots a_r^{p_r(n)}$ be a polynomial sequence in G such that the polynomials $p_1, \ldots, p_r$ are jointly intersective. Then $\overline{\mathrm{Orb}}_g(x) \ni x$ for any $x \in X$.*

We will prove Proposition 2.4 in Section 4 (see Proposition 4.3).

3 Intersective polynomials and polynomial orbits on tori

Given two integers b, k, we will write $b \vdots k$ if k divides b. We will use the term *lattice* for cosets of subgroups of finite index in $\mathbb{Z}^m$. If Λ is a lattice of $\mathbb{Z}^m$, then Λ is an affine image of $\mathbb{Z}^m$, and the notion of an integral polynomial on Λ is well defined. (Clearly, integral polynomials on Λ are restrictions of polynomials on $\mathbb{Z}^m$ taking on integer values on Λ.) We will say that integral polynomials $p_1, \ldots, p_r$ on Λ are jointly intersective (on Λ) if for any $k \in \mathbb{N}$ there exists $n \in \Lambda$ such that $p_1(n), \ldots, p_r(n) \vdots k$.

Lemma 3.1. *If integral polynomials $p_1, \ldots, p_r$ on a lattice Λ are jointly intersective, then for any sublattice Λ' of Λ there exists $l \in \Lambda$ such that the polynomials $p_1, \ldots, p_r$ are jointly intersective on $\Lambda' + l$.*

Proof. Let $L \subset \Lambda$ be a finite set such that $\Lambda' + L = \Lambda$. For any $k \in \mathbb{N}$ there exists $l_k \in L$ such that $p_i(n + l_k) \vdots k$, $i = 1, \ldots, r$, for some $n \in \Lambda'$. Let l be such that $l_{k!} = l$ for infinitely many k. Then for any $k \in \mathbb{N}$ there exists $k_0 > k$ such that $l_{k_0!} = l$, and thus there exists $n \in \Lambda'$ such that $p_i(n + l) \vdots k_0! \vdots k$, $i = 1, \ldots, r$. □

Lemma 3.2. *Let integral polynomials $p_1, \ldots, p_r$ on a lattice Λ be jointly intersective. For any $k \in \mathbb{N}$ there exists a lattice $\Lambda' \subseteq \Lambda$ such that $p_1, \ldots, p_r$ are jointly intersective on Λ' and $p_1(n), \ldots, p_r(n) \vdots k$ for all $n \in \Lambda'$.*

Proof. Let $d \in \mathbb{N}$ be such that $dp_1, \ldots, dp_r$ have integer coefficients. By Lemma 3.1, there exists $l \in \Lambda$ such that $p_1, \ldots, p_r$ are jointly intersective on $\Lambda' = kd\Lambda + l$. There exists $n_0 \in \Lambda$ such that $p_i(kdn_0 + l) \vdots k$, $i = 1, \ldots, r$. For any $n \in \Lambda$ and every i we have $p_i(kdn + l) = p_i(kdn_0 + l) + q_i(kd(n - n_0))$ where q_i is an integral polynomial with coefficients in $\frac{1}{d}\mathbb{Z}$ and zero constant term. Hence, $q_i(kd(n - n_0)) \vdots k$, $i = 1, \ldots, r$, and so $p_i(kdn + l) \vdots k$, $i = 1, \ldots, r$, for all n. □

Let M be an (additive) torus. *A polynomial sequence* in M is a (multiparameter) sequence of the form $t(n) = \sum_{i=1}^r p_i(n)v_i$, $n \in \mathbb{Z}^m$, where p_i are integral polynomials on $\mathbb{Z}^m$ and $v_i \in M$, $i = 1, \ldots, r$. It is well known (see [W]) that if t is a polynomial sequence in M, then the closure $S = \overline{\{t(n)\}}_{n \in \Lambda}$ of t is a connected component, or a union of several connected components, of a coset $u + N$ for some closed subgroup N of M and an element $u \in M$. In particular, if S is connected, it is a subtorus of M. After choosing coordinates in M we identify M with a standard torus $\mathbb{R}^s/\mathbb{Z}^s$, $s \in \mathbb{N}$. Then any polynomial sequence $t(n) = \sum_{i=1}^r p_i(n)v_i$ in M can be written in the form

$$t(n) = \left[\begin{pmatrix} q_{0,1}(n) \\ \vdots \\ q_{0,s}(n) \end{pmatrix} \frac{1}{k} + \sum_{i=1}^{l} \begin{pmatrix} q_{i,1}(n) \\ \vdots \\ q_{i,s}(n) \end{pmatrix} \alpha_i \right] \mod \mathbb{Z}^s, \tag{9}$$

where $1, \alpha_1, \ldots, \alpha_l \in \mathbb{R}$ are rationally independent, $k \in \mathbb{N}$, and the polynomials $q_{i,j}$ are linear combinations, with integer coefficients, of the polynomials $p_1, \ldots, p_r$.

We first take care of the "irrational" part of t. For any polynomial q let $\hat{q}$ denote the polynomial $q - q(0)$.

Lemma 3.3. (i) *Let* $t(n) = \begin{pmatrix} q_1(n) \\ \vdots \\ q_s(n) \end{pmatrix} \alpha \mod \mathbb{Z}^s$ *where* $\alpha \in \mathbb{R}$ *is irrational and* $q_1, \ldots, q_s$ *are integral polynomials on a lattice* Λ. *Then* $\overline{\{t(n)\}}_{n\in\Lambda}$ *is the (connected) subtorus* $\left[\begin{pmatrix} q_1(0) \\ \vdots \\ q_s(0) \end{pmatrix} \alpha + \mathrm{span}_{\mathbb{R}} \left\{ \begin{pmatrix} \hat{q}_1(n) \\ \vdots \\ \hat{q}_s(n) \end{pmatrix},\ n \in \Lambda \right\} \right] \mod \mathbb{Z}^s$ *of* M.
(ii) *Let* $b_i(n) = \begin{pmatrix} q_{i,1}(n) \\ \vdots \\ q_{i,s}(n) \end{pmatrix} \mod \mathbb{Z}^s$ *and* $t_i = b_i \alpha_i$, $i = 1, \ldots, l$, *where* $1, \alpha_1, \ldots, \alpha_l \in \mathbb{R}$ *are rationally independent and* $q_{i,j}$ *are integral polynomials on a lattice* Λ. *Let* $t = \sum_{i=1}^{l} t_i$; *then* $\overline{\{t(n)\}}_{n\in\Lambda} = \sum_{i=1}^{l} \overline{\{t_i(n)\}}_{n\in\Lambda}$. *In particular,* $\overline{\{t(n)\}}_{n\in\Lambda}$ *is a (connected) subtorus of* M.

Proof. (i) We may assume that $q_j(0) = 0$, $j = 1, \ldots, s$. Let $\widetilde{S} = \mathrm{span}_{\mathbb{R}} \left\{ \begin{pmatrix} q_1(n) \\ \vdots \\ q_s(n) \end{pmatrix},\ n \in \Lambda \right\} \subseteq \mathbb{R}^s$ and $S = \widetilde{S} \mod \mathbb{Z}^s$; since the vectors $\begin{pmatrix} q_1(n) \\ \vdots \\ q_s(n) \end{pmatrix}$ are rational, S is closed in M. Hence S is a subtorus and we have $\overline{\{t(n)\}}_{n\in\Lambda} \subseteq S$. On the other hand, consider an additive character χ on M, $\chi\begin{pmatrix} v_1 \\ \vdots \\ v_s \end{pmatrix} = c_1 v_1 + \ldots + c_s v_s \mod 1$ with $c_1, \ldots, c_s \in \mathbb{Z}$; if $\chi(t(n)) = 0$ for all $n \in \Lambda$, then $(c_1 q_1(n) + \ldots + c_s q_s(n))\alpha \in \mathbb{Z}$ for all $n \in \Lambda$, so $c_1 q_1(n) + \ldots + c_s q_s(n) = 0$ for all $n \in \Lambda$, so $\chi|_S = 0$. Hence, the sequence $\{t(n)\}_{n\in\Lambda}$ is not contained in any proper closed subgroup of S, and thus, is dense in S.
(ii) Again, we may assume that $q_{i,j}(0) = 0$ for all i, j. By (i), $\overline{\{t_i(n)\}}_{n\in\Lambda}$, $i = 1, \ldots, l$, are connected subgroups of M, and such is $N = \sum_{i=1}^{l} \overline{\{t_i(n)\}}_{n\in\Lambda}$. Let $S = \sum_{i=1}^{l} \overline{\{t_i(n)\}}_{n\in\Lambda}$; clearly, $S \subseteq N$. We have $S \ni 0_M$, thus S a union of connected components of a closed subgroup of N.

Let χ be a character on M, $\chi\begin{pmatrix} v_1 \\ \vdots \\ v_s \end{pmatrix} = c_1 v_1 + \ldots + c_s v_s \mod 1$ with $c_1, \ldots, c_s \in \mathbb{Z}$, and let ϕ be the corresponding linear function on $\mathbb{R}^s$, $\phi\begin{pmatrix} v_1 \\ \vdots \\ v_s \end{pmatrix} = c_1 v_1 + \ldots + c_s v_s$. Then $\chi(t(n)) = 0$, $n \in \Lambda$, iff $\sum_{i=1}^{l} \phi(b_i(n))\alpha_i = 0 \mod 1$, $n \in \Lambda$, which, because of the independence of $\alpha_1, \ldots, \alpha_l$ and 1, is equivalent to $\phi(b_i(n)) = 0$ and so, $\chi(t_i(n)) = 0$, $n \in \Lambda$, for all $i = 1, \ldots, l$. Hence, any character vanishing on S also vanishes on N, and so, S is not contained in any proper closed subgroup of N. Thus, $S = N$. □

Lemma 3.4. *Let* $t(n)=\begin{pmatrix}q_1(n)\\ \vdots\\ q_s(n)\end{pmatrix}\alpha \mod \mathbb{Z}^s$ *where* $\alpha\in\mathbb{R}$ *is irrational and* $q_1,\ldots,q_s$ *are integral polynomials on a lattice* Λ. *Then* $\overline{\{t(n)\}}_{n\in\Lambda}\ni 0_M$ *iff no linear combination of* $q_1,\ldots,q_s$ *is a nonzero constant.*

Proof. By Lemma 3.3(i), $\overline{\{t(n)\}}_{n\in\Lambda}\ni 0_M$ iff $\begin{pmatrix}q_1(0)\\ \vdots\\ q_s(0)\end{pmatrix}\in \mathrm{span}_{\mathbb{R}}\left\{\begin{pmatrix}\hat{q}_1(n)\\ \vdots\\ \hat{q}_s(n)\end{pmatrix},\ n\in\Lambda\right\}$. This is so iff any linear function on $\mathbb{R}^s$ vanishing on $\mathrm{span}_{\mathbb{R}}\left\{\begin{pmatrix}\hat{q}_1(n)\\ \vdots\\ \hat{q}_s(n)\end{pmatrix},\ n\in\Lambda\right\}$ vanishes at $\begin{pmatrix}q_1(0)\\ \vdots\\ q_s(0)\end{pmatrix}$ as well. This is equivalent to saying that if $\sum_{i=1}^s c_i\hat{q}_i=0$, with $c_1,\ldots,c_s\in\mathbb{R}$, then also $\sum_{i=1}^s c_iq_i=0$. □

Corollary 3.5. *Let* $t(n)=\begin{pmatrix}q_1(n)\\ \vdots\\ q_s(n)\end{pmatrix}\alpha \mod \mathbb{Z}^s$ *where* $\alpha\in\mathbb{R}$ *is irrational and* $q_1,\ldots,q_s$ *are jointly intersective integral polynomials on a lattice* Λ. *Then* $\overline{\{t(n)\}}_{n\in\Lambda}\ni 0_M$.

Proof. If there exist $c_1,\ldots,c_s\in\mathbb{R}$ and a nonzero $c\in\mathbb{R}$ such that $\sum_{i=1}^s c_iq_i=c$, then, since the polynomials q_i have rational coefficients, there exist $c_1,\ldots,c_s\in\mathbb{Z}$ and a nonzero $c\in\mathbb{Z}$ such that $\sum_{i=1}^s c_iq_i=c$. But this is impossible if q_i are jointly intersective. □

Let now t be a polynomial sequence in M, $t(n)=p_1(n)v_1+\ldots+p_r(n)v_r$, $v_i\in M$, where $p_1,\ldots,p_r$ are jointly intersective polynomials on Λ.

Proposition 3.6. *There exists a sublattice* Λ' *of* Λ *such that* $p_1,\ldots,p_r$ *are jointly intersective on* Λ', $S=\overline{\{t(n)\}}_{n\in\Lambda'}$ *is a connected subtorus of* M, *and* $0_M\in S$.

Example. Consider the polynomial sequence $t(n)=\left(n\alpha+\frac{1}{3}n^2,n\alpha\right) \mod \mathbb{Z}^2$, $n\in\mathbb{Z}$, in the torus $M=\mathbb{R}^2/\mathbb{Z}^2$, where α is an irrational number. The closure $\overline{\{t(n)\}}_{n\in\mathbb{Z}}$ of t is the union of two subtori of M, $S_1=\{(x,x),\ x\in\mathbb{R}/\mathbb{Z}\}$ and $S_2=\left\{\left(x+\frac{1}{3},x\right),\ x\in\mathbb{R}/\mathbb{Z}\right\}$. Passing to the sublattice $3\mathbb{Z}$ of $\mathbb{Z}$, we get $\overline{\{t(n)\}}_{n\in3\mathbb{Z}}=S_1$.

Proof. We represent t in the form (9), where all polynomials $q_{i,j}$ are linear combinations of polynomials p_i and so, are jointly intersective. If a nontrivial "rational" term $\begin{pmatrix}q_{0,1}\\ \vdots\\ q_{0,s}\end{pmatrix}\frac{1}{k}$ is present, by Lemma 3.2 there exists a sublattice $\Lambda'\subset\Lambda$ such that the polynomials $q_{0,1},\ldots,q_{0,r}$ are jointly intersective on Λ' and $q_{0,j}(n)\,\vdots\,k$ for all $n\in\Lambda'$ and $j=1,\ldots,s$. Then $\begin{pmatrix}q_{0,1}(n)\\ \vdots\\ q_{0,s}(n)\end{pmatrix}\frac{1}{k}=0 \mod \mathbb{Z}^s$ for all $n\in\Lambda'$, and we may ignore this term. By Corollary 3.5, for each $i=1,\ldots,l$ and $t_i(n)=\begin{pmatrix}q_{i,1}(n)\\ \vdots\\ q_{i,s}(n)\end{pmatrix}\alpha_i \mod \mathbb{Z}^s$, $S_i=\overline{\{t_i(n)\}}_{n\in\Lambda'}$ is a (connected) subtorus of M with $0_M\in S_i$, and by Lemma 3.3(ii), $S=\overline{\{t(n)\}}_{n\in\Lambda'}=\sum_{i=1}^l S_i$. Thus, S is a (connected) subtorus of M with $0_M\in S$. □

4 Intersective polynomials and polynomial orbits on nilmanifolds

Let P be a ring of integral polynomials on a lattice Λ. We will say that a mapping g from Λ to a nilpotent group G is *a P-polynomial sequence* if g has the form $g(n)=a_1^{p_1(n)}\ldots a_r^{p_r(n)}$ with $r\in\mathbb{N}$, $a_i\in G$ and $p_i\in P$, $i=1,\ldots,r$. The following facts are obvious and will be used repeatedly in the sequel.
(i) if g_1, g_2 are P-polynomial sequences in G, then the sequence $g_1(n)g_2(n)$ is P-polynomial;
(ii) if $\eta\colon G\longrightarrow G'$ is a homomorphism to a nilpotent group G' and g is a P-polynomial sequence in G, then $\eta(g)$ is a P-polynomial sequence in G';
(iii) if $\eta\colon G\longrightarrow G'$ is a homomorphism onto a nilpotent group G' and g' is a P-polynomial sequence in G', then there exists a P-polynomial sequence g in G such that $\eta(g)=g'$.

Proposition 4.1. *Let G be a connected nilpotent Lie group and H be a connected closed subgroup of G. If g is a P-polynomial sequence in G such that $g(n)\in H$ for all $n\in\Lambda$, then g is a P-polynomial sequence in H.*

Remark. Actually, the assertion of Proposition 4.1 holds for any (not necessarily topological) nilpotent group and any subgroup thereof (see [L1]).

Proof. Replacing G by its universal cover we may assume that G is simply-connected. We then may choose a Malcev basis in G, that is, elements $e_1,\ldots,e_k\in G$ such that every element of G is uniquely representable in the form $\prod_{j=1}^k e_j^{y_j}$ with $y_1,\ldots,y_k\in\mathbb{R}$. (See [M]. Elements e_i can be chosen to be of the form $e_i=\exp(\epsilon_i)$ where $(\epsilon_1,\ldots,\epsilon_k)$ is a linear base of the Lie algebra of G.) Moreover, by an elementary linear algebra argument, the basis can be chosen to be compatible with H, so that for some $j_1,\ldots,j_l\in\{1,\ldots,k\}$, the elements $e_{j_1},\ldots,e_{j_l}$ form a basis in H, and thus $\prod_{j=1}^k e_j^{y_j}\in H$ iff $y_j=0$ for all $j\notin\{j_1,\ldots,j_l\}$.

It follows from the Campbell-Hausdorff formula[8] that the multiplication in G is polynomial: in the Malcev basis one has $\left(\prod_{j=1}^k e_j^{y_j}\right)\cdot\left(\prod_{j=1}^k e_j^{z_j}\right)=\prod_{j=1}^k e_j^{Q_j(y_1,\ldots,y_k,z_1,\ldots,z_k)}$ and $\left(\prod_{j=1}^k e_j^{y_j}\right)^n=\prod_{j=1}^k e_j^{R_j(y_1,\ldots,y_k,n)}$ where Q_j and R_j are polynomials vanishing at 0. Thus, any polynomial sequence $g(n)=a_1^{p_1(n)}\ldots a_r^{p_r(n)}$ in G can be uniquely written as $g(n)=\prod_{j=1}^k e_j^{F_j(p_1(n),\ldots,p_r(n))}$ where F_j are polynomials vanishing at 0. If g

[8]The Campbell-Hausdorff formula relates multiplication in a connected Lie group G with a certain operation $P\colon\mathcal{G}\times\mathcal{G}\longrightarrow\mathcal{G}$ on the Lie algebra $\mathcal{G}$ of G: $\exp(u)\exp(v)=\exp(P(u,v))$, $u,v\in\mathcal{G}$. When G is nilpotent, P is a polynomial mapping (that is, a finite linear combination of multilinear forms applied to $(w_1,\ldots,w_d)$ for $w_i=u$ or v, and various values of d).

takes values only in H, $F_j(p_1(n),\dots,p_r(n))=0$ for all $j\notin\{j_1,\dots,j_l\}$, and $g(n)=\prod_{j\in\{j_1,\dots,j_l\}}e_j^{F_j(p_1(n),\dots,p_r(n))}$ is a polynomial sequence in H. The last formula can be rewritten as $g(n)=\prod_{j\in\{j_1,\dots,j_l\}}\prod_{i=1}^{k_j}(e_j^{\alpha_{j,i}})^{F_{j,i}(p_1(n),\dots,p_r(n))}$ where $\alpha_{j,i}\in\mathbb{R}$ and $F_{j,i}$ are nonconstant monomials. Now, if all p_i are in P, the polynomials $q_{j,i}(n)=F_{j,i}(p_1(n),\dots,p_r(n))$ are also in P, and so, g is a P-polynomial sequence in H. $\square$

We will also need the following fact:

Proposition 4.2. ([L2]) *Let G be a connected nilpotent Lie group, let $X=G/\Gamma$ be a nilmanifold, let π be the canonical projection $G\longrightarrow X$, let M be the torus ${}_{[G,G]}\backslash X$, and let $\xi\colon X\longrightarrow M$ be the projection. If a polynomial sequence g in G is such that $\xi(\pi(g(n)))$ is dense in M, then $\pi(g(n))$ is dense in X.*

Now let G be a nilpotent group, Γ a closed cocompact subgroup of G, and $X=G/\Gamma$. Let π be the projection $G\longrightarrow X$, and $1_X=\pi(1_G)\in X$. Proposition 2.4 is a consequence of the following proposition, applied to $g(n)=a_1^{p_1(n)}\dots a_r^{p_r(n)}$:

Proposition 4.3. *Let $p_1,\dots,p_r$ be jointly intersective polynomials on a lattice Λ and let P be the ring generated by $p_1,\dots,p_r$. If g is a P-polynomial sequence in G and $x\in X$, then $\overline{\{g(n)x\}}_{n\in\Lambda}\ni x$.*

Proof. We will proceed as follows: if $\{g(n)x\}_{n\in\Lambda}$ is not dense in X, then, applying Propositions 4.2 and 3.6, we will pass to a certain sublattice Λ' of Λ so that $\{g(n)x\}_{n\in\Lambda'}$ will lie in a proper subnilmanifold of X containing x, and then use induction on the dimension of X.

It is enough to prove that, for any P-polynomial sequence g, we have $\overline{\{\pi(g(n))\}}_{n\in\Lambda}\ni 1_X$. Indeed, if $x=g_0\Gamma\in X$ then $g_0^{-1}gg_0$ is a P-polynomial sequence (this follows from either of (i) and (ii) at the beginning of the section) and $\overline{\{g(n)x\}}_{n\in\Lambda}\ni x$ iff $\overline{\{\pi(g_0^{-1}g(n)g_0)\}}_{n\in\Lambda}\ni 1_X$.

If X is not connected, put $\hat{G}=\pi^{-1}(X^o)$, where X^o is the connected component of X that contains 1_X; then $\hat{G}$ is a subgroup of finite index k in G. By Lemma 3.2, there exists a sublattice Λ' of Λ such that the polynomials $p_1,\dots,p_r$ are jointly intersective on Λ' and for any $n\in\Lambda'$, $p_1(n),\dots,p_r(n):k$. The sequence $g|_{\Lambda'}$ takes values in $\hat{G}$, and after replacing Λ by Λ', G by $\hat{G}$, and X by X^o we may assume that X is connected.

Let G^o be the identity component of G and let θ be the canonical homomorphism $G\longrightarrow G/G^o$. Since X is connected, $\theta(\Gamma)=G/G^o$, and thus there exists a P-polynomial sequence δ in Γ such that $\theta(\delta)=\theta(g)$. The sequence $g'(n)=g(n)\delta(n)^{-1}$ takes values in G^o and satisfies $\pi(g')=\pi(g)$, $n\in\Lambda$. By Proposition 4.1, $g'(n)$ is a P-polynomial sequence in G^o. After replacing g by g' and G by G^o we may assume that G is connected.

Let $V = G/[G,G] = [G,G]\backslash G$ with $\eta\colon G \longrightarrow V$ being the canonical projection. V is a connected commutative Lie group. Let M be the torus $V/\eta(\Gamma) = [G,G]\backslash X$ with $\tau\colon V \longrightarrow M$ being the projection; we will use multiplicative notation for V and M. Let $t(n) = g(n)1_M$, $n \in \Lambda$; in other words, $t = \tau(\eta(g))$ is the projection of g on M. If t is dense in M, then by Proposition 4.2, g is dense in X and we are done. Assume that t is not dense in M. We know that t is a P-polynomial sequence in M. By Proposition 3.6, after replacing Λ by a suitable sublattice, the polynomials $p_1, \ldots, p_r$ remain jointly intersective and $S = \overline{\{t(n)\}_{n\in\Lambda}}$ is a connected proper subtorus of M with $1_M \in S$.

Note that $\tau^{-1}(S)$ is a proper subgroup of V. Let $L \subseteq V$ be the identity component of $\tau^{-1}(S)$:

$$\begin{array}{rcccc}
 & G^o & \ni & g(n) & H \\
 & \downarrow\eta & & \downarrow\eta & \downarrow\eta \\
 & V = G^o/[G^o,G^o] & \ni & \eta(g(n)) & L \\
 & \downarrow\tau & & \downarrow\tau & \downarrow\tau \\
M = & G^o/\left([G^o,G^o](\Gamma\cap G^o)\right) & \ni & t(n) = \tau(\eta(g(n))) \ \in & S
\end{array}$$

We have $\tau(L) = S$. Let u be a P-polynomial sequence in L such that $\tau(u) = t$. Then $\tau(\eta(g)) = \tau(u)$, thus $u(n)^{-1}\eta(g(n)) \in \eta(\Gamma)$, $n \in \Lambda$. The sequence $\lambda(n) = u(n)^{-1}\eta(g(n))$, $n \in \Lambda$, is P-polynomial in $\eta(\Gamma)$; let γ be a P-polynomial sequence in Γ such that $\eta(\gamma) = \lambda$. Put $h(n) = g(n)\gamma(n)^{-1}$, $n \in \Lambda$; then $\pi(h) = \pi(g)$ and $\eta(h) = u$.

Let $H = \eta^{-1}(L)$; then H is a proper closed connected subgroup of G, and $Y = \pi(H)$ is a subnilmanifold of X that contains the sequence $\pi(h) = \pi(g)$. The sequence h takes values in H, thus by Proposition 4.1, h is a P-polynomial sequence in H. By induction on the dimension of H, $\overline{\{\pi(h(n))\}_{n\in\Lambda}} \ni 1_Y = 1_X$. □

5 Polynomial Szemerédi and van der Waerden theorems

Proof of Theorem 1.3. By Furstenberg's correspondence principle, there exist a probability measure preserving system $(X,\mathcal{B},\mu,T)$ and a set $A \in \mathcal{B}$ with $\mu(A) = d^*(E)$ such that for any $n_1,\ldots,n_l \in \mathbb{Z}$ one has $d^*\big(E\cap(E-n_1)\cap\ldots\cap(E-n_l)\big) \geq \mu\big(A\cap T^{-n_1}A\cap \ldots\cap T^{-n_l}A\big)$. Let $c_n = \mu\big(A\cap T^{-p_1(n)}A\cap\ldots\cap T^{-p_r(n)}A\big)$, $n \in \mathbb{Z}^m$. By Theorem 1.4, $\lim_{N-M\to\infty}\frac{1}{(N-M)^m}\sum_{n\in[M,N-1]^m} c_n = C > 0$, and thus $d_*\big(\{n\in\mathbb{Z}^m : c_n > C/2\}\big) > 0$, where $d_*(F) = \liminf_{N-M\to\infty}\frac{|F\cap[M,N-1]^m|}{(N-M)^m}$). This means that the set $\{n \in \mathbb{Z}^m : c_n > C/2\}$ is syndetic. □

A subset F of $\mathbb{Z}^m$ is said to be *thick* if F contains arbitrarily large cubes. A standard argument allows one to obtain the following "finitary" version of Theorem 1.3:

Theorem 5.1. *Let $p_1, \dots, p_r$ be jointly intersective polynomials on $\mathbb{Z}^m$. For any $\delta > 0$ and any thick set $F \subseteq \mathbb{Z}^m$ there exists $N_0 \in \mathbb{N}$ such that if $N \geq N_0$ and E is a subset of $\{1, \dots, N\}$ with $|E| > \delta N$, then there exist $a \in E$ and $n \in F$ such that $a + p_1(n), \dots, a + p_r(n) \in E$.*

Indeed, if the assertion of the theorem is wrong, then there exist $\delta > 0$, a thick set $F \subseteq \mathbb{Z}^m$, a sequence of intervals $I_i = \{1, \dots, N_i\}$ with $N_i \longrightarrow \infty$, and a sequence of sets $E_i \subseteq I_i$ with $|E_i| > \delta |N_i|$, $i = 1, 2, \dots$, such that, for each i, one has $E_i \cap (E_i - p_1(n)) \cap \dots \cap (E_i - p_r(n)) = \emptyset$ for all $n \in F$. Let S be the shift of $\mathbb{Z}$, $S(k) = k + 1$. Using a diagonal process, choose a sequence $(i_j)_{j=1}^{\infty}$ such that the limit $a_{k_1,\dots,k_l} = \lim_{j\to\infty} \frac{1}{N_{i_j}} \left| S^{k_1} E_{i_j} \cap S^{k_2} E_{i_j} \cap \dots \cap S^{k_l} E_{i_j} \right|$ exists for any $l \in \mathbb{N}$ and $k_1, \dots, k_l \in \mathbb{Z}$. One can then construct a probability measure preserving system $(X, \mathcal{B}, \mu, T)$ with a marked set $A \in \mathcal{B}$ such that $\mu\big(T^{k_1} A \cap T^{k_2} A \cap \dots \cap T^{k_l} A\big) = a_{k_1,\dots,k_l}$ for any $k_1, \dots, k_l \in \mathbb{Z}$.[9] Then $\mu(A) \geq \delta$, but $\mu\big(A \cap T^{-p_1(n)} A \cap \dots \cap T^{-p_r(n)} A\big) = 0$ for any $n \in F$, in contradiction with Theorem 1.4.

By utilizing a somewhat more sophisticated argument (see [BeHMP] and [FLeWi]), one actually can get the following result:

Theorem 5.2. *Let $p_1, \dots, p_r$ on $\mathbb{Z}^m$ be jointly intersective polynomials. For any $\delta > 0$ and any thick set $F \subseteq \mathbb{Z}^m$ there exist $\gamma > 0$ and $N_0 \in \mathbb{N}$ such that for any $N \geq N_0$ and any subset $E \subseteq [1, N]$ with $|E| > \delta N$ one has $\big| E \cap (E - p_1(n)) \cap \dots \cap (E - p_r(n)) \big| > \gamma$ for some $n \in F$.*

One can also derive from Theorem 1.4 the following ostensibly stronger result:

Theorem 5.3. *Let integral polynomials $p_1, \dots, p_r$ on $\mathbb{Z}^m$ be jointly intersective, let $(X, \mathcal{B}, \mu, T)$ be an invertible probability measure preserving system, let $f \in L^\infty(X)$ be a nonnegative function with $\int f \, d\mu > 0$, and let $A = \{x \in X : f(x) > 0\}$. Let $\tilde{f} = \text{UC-}\lim_n f \circ T^{p_1(n)} \cdot \ldots \cdot f \circ T^{p_r(n)}$ in L^2-norm; then $\tilde{f} > 0$ a.e. on A.*

Proof. Clearly $\tilde{f} \geq 0$. It suffices to show that for any measurable subset B of A with $\mu(B) > 0$ one has $\int_B \tilde{f} \, d\mu > 0$. Find $\delta > 0$ and a measurable set $B' \subseteq B$ with $\mu(B') > 0$

[9] This is a version of Furstenberg's correspondence principle. The system $(X, \mathcal{B}, \mu, T)$ and the set A can be constructed in the following way: take $X = \{0, 1\}^{\mathbb{Z}}$ with T being the coordinate shift, $(Tx)_i = x_{i+1}$, $x = (\dots, x_{-1}, x_0, x_1, \dots) \in X$, and let $A = \{x \in X : x_0 = 0\}$. Define a premeasure ρ on the cylindrical subsets of X by $\rho\big(T^{k_1} A \cap T^{k_2} A \cap \dots \cap T^{k_l} A\big) = a_{k_1,\dots,k_l}$, $l \in \mathbb{N}$, $k_1, \dots, k_l \in \mathbb{Z}$; one can check that ρ is σ-additive and T-invariant. Finally, let μ be the Borel measure on X induced by ρ.

such that $f \geq \delta 1_{B'}$. Then

$$\begin{aligned}
\int_B \tilde{f}\, d\mu \geq \int_{B'} \tilde{f}\, d\mu &= \int_{B'} \underset{n}{\text{UC-lim}}\, f\circ T^{p_1(n)} \cdot \ldots \cdot f\circ T^{p_r(n)} d\mu \\
&= \underset{n}{\text{UC-lim}} \int_{B'} f\circ T^{p_1(n)} \cdot \ldots \cdot f\circ T^{p_r(n)} d\mu \\
&\geq \delta^r\, \underset{n}{\text{UC-lim}} \int_X 1_{B'} \cdot 1_{B'}\circ T^{p_1(n)} \cdot \ldots \cdot 1_{B'}\circ T^{p_r(n)} d\mu \\
&= \delta^r\, \underset{n}{\text{UC-lim}}\, \mu\big(B' \cap T^{-p_1(n)} B' \cap \ldots \cap T^{-p_r(n)} B'\big) > 0.
\end{aligned}$$

□

The polynomial van der Waerden theorem for jointly intersective polynomials, Theorem 1.5, is an immediate corollary of Theorem 1.3. However, using the "uniformity" in Theorem 1.4 (and following an idea which was utilized in [BeM1]), we can get a stronger version of Theorem 1.5. We start with the following strengthening of Theorem 1.3.

Proposition 5.4. *Let $p_1, \ldots, p_r$ be jointly intersective integral polynomials on $\mathbb{Z}^m$ and let sets $E_1, \ldots, E_s \subseteq \mathbb{Z}$ be such that $d^*(E_i) > 0$ for all $i = 1, \ldots, s$. Then there exists $\varepsilon > 0$ such that the set*

$$S = \bigcap_{i=1}^{s} \Big\{ n \in \mathbb{Z}^m : d^*\big(E_i \cap (E_i - p_1(n)) \cap \ldots \cap (E_i - p_r(n))\big) > \varepsilon \Big\} \tag{10}$$

is syndetic.

Proof. (Cf. the proof of Theorem 0.4 in [BeM1].) Using Furstenberg's correspondence principle, for each $i = 1, \ldots, s$ find a probability measure preserving system $(X_i, \mathcal{B}_i, \mu_i, T_i)$ and a set $A_i \in \mathcal{B}_i$ with $\mu(A_i) = d^*(E_i)$ such that for any $n_1, \ldots, n_l \in \mathbb{Z}$ one has $d^*\big(E_i \cap (E_i - n_1) \cap \ldots \cap (E_i - n_l)\big) \geq \mu_i\big(A_i \cap T_i^{-n_1} A_i \cap \ldots \cap T_i^{-n_l} A_i\big)$. Put $X = X_1 \times \ldots \times X_s$, $T = T_1 \times \ldots \times T_s$, and $A = A_1 \times \ldots \times A_s$. By Theorem 1.4, there exists $\varepsilon > 0$ such that the set

$$\begin{aligned}
&\Big\{ n \in \mathbb{Z}^m : \mu\big(A \cap T^{-p_1(n)} A \cap \ldots \cap T^{-p_r(n)} A\big) > \varepsilon \Big\} \\
&\qquad = \Big\{ n \in \mathbb{Z}^m : \prod_{i=1}^{s} \mu_i\big(A_i \cap T^{-p_1(n)} A_i \cap \ldots \cap T^{-p_r(n)} A_i\big) > \varepsilon \Big\}
\end{aligned}$$

is syndetic, and this is a subset of

$$\bigcap_{i=1}^{s} \Big\{ n \in \mathbb{Z}^m : \mu_i\big(A_i \cap T^{-p_1(n)} A_i \cap \ldots \cap T^{-p_r(n)} A_i\big) > \varepsilon \Big\}.$$

□

We now confine ourselves to the one-parameter situation. A subset E of $\mathbb{Z}$ is said to be *piecewise syndetic* if there exists a sequence of intervals $J_1, J_2, \ldots$ with $|J_j| \longrightarrow \infty$ and a syndetic set $E' \subseteq \mathbb{Z}$ such that $E = E' \cap \bigcup_{j=1}^{\infty} J_j$. It is not hard to see that if a syndetic set is partitioned into finitely many subsets, then one of these subsets is piecewise syndetic.

Theorem 5.5. *Let $p_1, \ldots, p_r$ be jointly intersective integral polynomials. For any finite partition $\mathbb{Z} = \bigcup_{i=1}^{k} E_i$, one of the E_i has the property that, for some $\varepsilon > 0$, the set*

$$\Big\{ n \in E_i : d^*\big(E_i \cap (E_i - p_1(n)) \cap \ldots \cap (E_i - p_r(n))\big) > \varepsilon \Big\}$$

is piecewise syndetic.

Remark. As it was already mentioned above, the fact that for some E_i (and indeed for any E_i that has positive upper density) and some $\varepsilon > 0$ the set

$$\Big\{ n \in \mathbb{Z} : d^*\big(E_i \cap (E_i - p_1(n)) \cap \ldots \cap (E_i - p_r(n))\big) > \varepsilon \Big\}$$

is syndetic is a direct corollary of Theorem 1.3. The delicate point in Theorem 5.5 is that the set of n satisfying the assertion of the theorem is a (large) subset of E_i.

Proof. Re-index $E_1, \ldots, E_k$ so that $d^*(E_i) > 0$ for $i = 1, \ldots, s$ and $d^*(E_i) = 0$ for $i = s+1, \ldots, k$. Choose ε as in Proposition 5.4, and let S be the syndetic set defined by (10). Since the set $\mathbb{Z} \setminus \bigcup_{i=1}^{s} E_i$ has zero upper Banach density, the set $S \cap \bigcup_{i=1}^{s} E_i$ is also syndetic, and thus $S \cap E_i$ is piecewise syndetic for some $i \in \{1, \ldots, s\}$. □

6 Concluding remarks

6.1 Intersective and jointly intersective polynomials

Clearly, every integral polynomial with an integer root is intersective. There are also intersective polynomials without rational roots; one can show that if a_1, a_2 are distinct prime integers such that $a_1 \equiv a_2 \equiv 1 \pmod 4$ and a_1 is a square in $\mathbb{Z}/(a_2\mathbb{Z})$, then the polynomial $p(n) = (n^2 - a_1)(n^2 - a_2)(n^2 - a_1 a_2)$ is intersective. (Such is, for example, the polynomial $p(n) = (n^2 - 5)(n^2 - 41)(n^2 - 205)$, mentioned in the Introduction.) There are similar examples of intersective polynomials of degree 5 (for instance, $p(n) = (n^3 - 19)(n^2 + n + 1)$), but there are no intersective polynomials in one variable of degree less than 5 without rational roots (see [BBi]).

An example of an intersective polynomial of several variables with no rational roots is $p(n_1,\ldots,n_4)=n_1^2+\ldots+n_4^2+b$, where b is an arbitrary positive integer; this polynomial has the property that all its shifts $p+c$, $c\in\mathbb{Z}$, are also intersective. (No intersective polynomials in one variable, except the polynomials $\pm n+b$, $b\in\mathbb{Z}$, have this property. Indeed, if an integral polynomial $p(n)$ is not of the form $\pm n+b$, then there exists $n_0\in\mathbb{Z}$ such that $k=|p(n_0+1)-p(n_0)|\neq 1$. Then p is not one-to-one in $\mathbb{Z}/(k\mathbb{Z})$, so is not onto, and thus there exists $d\in\mathbb{Z}$ such that $p(n)-d\neq 0 \mod k$ for any $n\in\mathbb{Z}$.)

Systems of jointly intersective polynomials in one variable can be easily described:

Proposition 6.1. *Integral polynomials $p_1,\ldots,p_r$ of one variable are jointly intersective iff they all are multiples of an intersective polynomial p.*

(We say that a polynomial q is a multiple of a polynomial p if q is divisible by p in the ring $\mathbb{Q}[n]$.)

Proof. Clearly, if $p\in\mathbb{Q}[n]$ is an intersective polynomial and $p_1,\ldots,p_r \vdots p$ then $p_1,\ldots,p_r$ are jointly intersective.

Let $p_1,\ldots,p_r\in\mathbb{Q}[n]$ be jointly intersective. Let $p\in\mathbb{Z}[n]$ be the greatest common divisor of $p_1,\ldots,p_r$ in $\mathbb{Q}[n]$. Then there exist $h_1,\ldots,h_r\in\mathbb{Q}[n]$ such that $\sum_{i=1}^r h_ip_i=p$. Multiplying all h_i by an integer d if necessary, we may assume that $h_1,\ldots,h_r$ have integer coefficients, and that $\sum_{i=1}^r h_ip_i=dp$. It is then clear that if $p_1,\ldots,p_r$ are jointly intersective, then dp is intersective, and thus p is intersective. □

The natural conjecture that integral polynomials are jointly intersective if any linear combination of these polynomials is intersective, fails to be true. For example, one can show that the polynomials $p_1(n)=n(n+1)(2n+1)$ and $p_2(n)=(n^3+n^2+2)(2n+1)$ satisfy the above condition, but are not jointly intersective (see Appendix in [BeLe]).

Proposition 6.1 is no longer true for jointly intersective polynomials of several variables. If polynomials $p_1,\ldots,p_r$ in m variables are jointly intersective, then the whole ideal I in $\mathbb{Q}[n_1,\ldots,n_m]$ generated by these polynomials consists of jointly intersective polynomials. In the case $m=1$, I is principal, which implies Proposition 6.1. If $m\geq 2$, $\mathbb{Q}[n_1,\ldots,n_m]$ is not a principal ideal domain, and Proposition 6.1 fails. (Consider, for example, the pair of jointly intersective polynomials $p_i(n_1,n_2)=n_i$, $i=1,2$.)

6.2 Total ergodicity

If one deals with totally ergodic dynamical systems (this means that T^k is ergodic for any nonzero integer k), it is not hard to verify (see Proposition 6.2 below) that any integral polynomial is "good" for single recurrence. This is no longer true for multiple recurrence, as the simple example following Proposition 6.2 shows.

Proposition 6.2. *Let $(X, \mathcal{B}, \mu, T)$ be a totally ergodic probability measure preserving dynamical system and let p be an integral polynomial on $\mathbb{Z}^m$. Then, for any set $A \in \mathcal{B}$,* $\text{UC-lim}_n\, \mu\big(A \cap T^{-p(n)}A\big) = \mu(A)^2$.

Proof. Total ergodicity of T is equivalent to the lack of discrete rational spectrum for the unitary operator $f \mapsto f \circ T$ on $L^2(X)$. For any $f \in L^2(X)$ and any Følner sequence $(\Phi_N)_{N=1}^{\infty}$ in $\mathbb{Z}^m$, the convergence in L^2 of the sequence $\big(\frac{1}{|\Phi_N|}\sum_{n\in\Phi_N} f\circ T^{p(n)}\big)_{N=1}^{\infty}$ to the limit $\int f\,d\mu$ is then a consequence of basic spectral theory and Weyl's equidistribution theorem. (Cf. [Fu2], p. 70-71.) □

Example. Consider the totally ergodic probability measure preserving system given by rotation of the one dimensional torus by an irrational number α. The simplest example of a non-intersective polynomial is $2n + 1$. If we choose A to be a sufficiently small interval on the torus, then for any n we will have $A \cap T^{-n}A \cap T^{-(2n+1)}A = \emptyset$.

It is natural to ask what is a necessary and sufficient condition for a family $P = \{p_1, \ldots, p_r\}$ of integral polynomials to have "the multiple recurrence property" (namely, that for any $A \subseteq X$ with $\mu(A) > 0$ one has $\mu(A \cap T^{-p_1(n)}A \cap \ldots \cap T^{-p_r(n)}A) > 0$ for a certain n) in the framework of totally ergodic dynamical systems. We conjecture that the condition that the ring generated by $p_1, \ldots, p_r$ does not contain nonzero constants is a sufficient one. However, this condition is far from being necessary. In order to find a necessary and sufficient condition for a family $P = \{p_1, \ldots, p_r\}$ of polynomials to have the multiple recurrence property under the assumption of total ergodicity one has to take into consideration *the complexity* of the family $\{p_1, \ldots, p_r\}$ (see [BeLLe] and [L5]). (For example, if $\{p_1, \ldots, p_r\}$ has complexity 0, that is, if the polynomials $p_i - p_i(0)$, $i = 1, \ldots, r$, are linearly independent, no additional restrictions on p_i is needed (see [FKr]); for a polynomial family of complexity 1, it suffices that $\text{span}_{\mathbb{Z}}\{p_1, \ldots, p_r\}$ does not contain nonzero constants.) It seems however that this necessary and sufficient condition is too cumbersome to be either of practical or aesthetic value.

6.3 Multidimensional conjecture

The *multidimensional* polynomial Szemerédi theorem states that given a set E of positive upper Banach density in $\mathbb{Z}^k$ and vector-valued polynomials $p_1, \ldots, p_r \colon \mathbb{Z}^m \longrightarrow \mathbb{Z}^k$ with zero constant term, the set

$$N_P(E) = \Big\{n \in \mathbb{Z}^m : \text{for some } a \in \mathbb{Z}^k,\ \{a, a + p_1(n), \ldots, a + p_r(n)\} \subset E\Big\}$$

is infinite, and, moreover, syndetic. (See [BeL] and [BeM2].) It is natural to try to generalize Theorem 1.1 to this multidimensional situation. Let us say that a family

$\{p_1,\dots,p_r\}$ of polynomial mappings $\mathbb{Z}^m \longrightarrow \mathbb{Z}^k$ *has the SPSZ property* if for any set E of positive upper Banach density in $\mathbb{Z}^k$ the set $N_P(E)$ is syndetic in $\mathbb{Z}^m$; let us say that $p_1,\dots,p_r$ are *jointly intersective* if for any subgroup Λ of finite index in $\mathbb{Z}^k$ there exists $n \in \mathbb{Z}^m$ such that $p_1(n),\dots,p_r(n) \in \Lambda$.

Conjecture 6.3. *A set $\{p_1,\dots,p_r\}$ of polynomial mappings $\mathbb{Z}^m \longrightarrow \mathbb{Z}^k$ has the SPSZ property iff the mappings $p_1,\dots,p_r$ are jointly intersective.*

At this stage, we are unable to check this conjecture by methods developed above because of lack of theory of characteristic factors for $\mathbb{Z}^k$-actions, similar to that established in [HKr1] and [Z] for $\mathbb{Z}$-actions.

References

[BBi] D. Berend and Y. Bilu, Polynomials with roots modulo every integer, *Proc. Amer. Math. Soc.* **124** (1996), no. 6, 1663–1671.

[Be] V. Bergelson, Ergodic Ramsey theory, *Contemp. Math.* **65** (1985), 63–87.

[BeHMP] V. Bergelson, B. Host, R. McCutcheon, and F. Parreau, Aspects of uniformity in recurrence, *Colloq. Math.*, **84/85** (2000), part 2, 549–576.

[BeL] V. Bergelson and A. Leibman, Polynomial extensions of van der Waerden's and Szemerédi's theorems, *J. of AMS* **9** (1996), 725–753.

[BeLe] V. Bergelson and E. Lesigne, Van der Corput sets in $\mathbb{Z}^d$, *Colloquium Mathematicum* **110** (2008), 1–49.

[BeLLe] V. Bergelson, A. Leibman, and E. Lesigne, Complexities of finite families of polynomials, Weyl systems, and constructions in combinatorial number theory, *J. d'Analyse Math.* **103** (2007), 47–92.

[BeM1] V. Bergelson and R. McCutcheon, Uniformity in polynomial Szemerédi theorem, *Ergodic Theory of $\mathbb{Z}^d$-actions*, London Math. Soc. Lecture Notes Series **228** (1996), 273–296.

[BeM2] V. Bergelson and R. McCutcheon, An ergodic IP polynomial Szemerédi theorem, *Mem. Amer. Math. Soc.* **146** (2000), no. 695, viii+106.

[F] N. Frantzikinakis, Multiple ergodic averages for three polynomials and applications, to appear in *Trans. of AMS.*

[FKr] N. Frantzikinakis and B. Kra, Polynomial averages converge to the product of integrals, *Israel J. of Math.* **148** (2005), 267–276.

[FLeWi] N. Frantzikinakis, E. Lesigne, and M. Wierdl, Powers of sequences and recurrence, preprint.

[Fu1] H. Furstenberg, Ergodic behavior of diagonal measures and a theorem of Szemerédi on arithmetic progressions, *J. d'Analyse Math.* **31** (1977), 204–256.

[Fu2] H. Furstenberg, *Recurrence in Ergodic Theory and Combinatorial Number Theory*, Princeton Univ. Press, 1981.

[FuK] H. Furstenberg and Y. Katznelson, An ergodic Szemerédi theorem for commuting transformations, *J. d'Analyse Math.* **34** (1978), 275–291.

[HKr1] B. Host and B. Kra, Nonconventional ergodic averages and nilmanifolds, *Annals Math.* **161** (2005), 397–488.

[HKr2] B. Host and B. Kra, Convergence of polynomial ergodic averages, *Israel J. Math.* **149** (2005), 1–19.

[KaM] T. Kamae and M. Mendès France, Van der Corput's difference theorem, *Israel J. Math.* **31** (1977), 335–342.

[L1] A. Leibman, Polynomial mappings of groups, *Israel J. Math.* **129** (2002), 29–60.

[L2] A. Leibman, Pointwise convergence of ergodic averages for polynomial sequences of translations on a nilmanifold, *Ergod. Th. and Dynam. Sys.* **25** (2005), 201–213.

[L3] A. Leibman, Pointwise convergence of ergodic averages for polynomial actions of $\mathbb{Z}^d$ by translations on a nilmanifold, *Ergod. Th. and Dynam. Sys.* **25** (2005), 215–225.

[L4] A. Leibman, Convergence of multiple ergodic averages along polynomials of several variables, *Israel J. Math.* **146** (2005), 303–322.

[L5] A. Leibman, Orbit of the diagonal of a power of a nilmanifold, preprint, available at *http://www.math.ohio-state.edu/~leibman/preprints*.

[M] A. Malcev, On a class of homogeneous spaces, *Amer. Math. Soc. Transl.* **9** (1962), 276–307.

[Roy] H.L. Royden, *Real Analysis* (third edition), Prentice Hall, 1988.

[Sa] A. Sàrközy, On difference sets of sequences of integers, I, *Acta Math. Acad. Sci. Hungar.* **31** (1978), 125–149.

[Sh] N. Shah, Invariant measures and orbit closures on homogeneous spaces for actions of subgroups generated by unipotent elements, *Lie groups and ergodic theory* (Mumbai, 1996), 229–271, Tata Institute of Fundamental Research, Bombay, 1998.

[Sz] E. Szemerédi, On sets of integers containing no k elements in arithmetic progression, *Acta Arith.* **27** (1975), 199–245.

[Z] T. Ziegler, Universal characteristic factors and Furstenberg averages, *J. Amer. Math. Soc.* **20** (2007), no. 1, 53–97.

[W] H. Weyl, Über die Gleichverteilung von Zahlen mod Eins, *Math. Ann.* **77** (1916), 313–352.

V. Bergelson
Department of Mathematics
The Ohio State University
Columbus, OH 43210, USA
vitaly@math.ohio-state.edu

A. Leibman
Department of Mathematics
The Ohio State University
Columbus, OH 43210, USA
leibman@math.ohio-state.edu

E. Lesigne
Laboratoire de Mathématiques et Physique Théorique
Université François-Rabelais Tours
Fédération Denis Poisson – CNRS
Parc de Grandmont, 37200 Tours, France
Emmanuel.Lesigne@lmpt.univ-tours.fr