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From discrete- to continuous-time ergodic theorems

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In memory of Dan Rudolph

Abstract. We introduce methods that allow us to derive continuous-time versions of various discrete-time ergodic theorems. We then illustrate these methods by giving simple proofs and refinements of some known results as well as establishing new results of interest.

1. *Introduction*

The goal of this paper is to introduce methods that allow one to obtain continuous-time versions of various discrete-time ergodic results. While the classical von Neumann's and Birkhoff's ergodic theorems were dealing with continuous families of invertible measure-preserving transformations, it was very soon observed that ergodic theorems for $\mathbb{Z}$-actions hold true as well, are somewhat easier to handle, and, moreover, can be used as an auxiliary tool for the derivation of the corresponding continuous-time results. (See, for example, the formulation of the so-called Birkhoff's fundamental lemma in [**BiKo**]. See also [**Ko**] and [**H**, §8].) Moreover, since not every measure-preserving $\mathbb{Z}$-action imbeds in a continuous measure-preserving $\mathbb{R}$-flow, and since there are various important classes of non-invertible measure-preserving transformations, it became, over the years, more fashionable to study ergodic theorems for measure-preserving $\mathbb{Z}$- and $\mathbb{N}$-actions. Numerous multiple recurrence and convergence results obtained in the framework of the ergodic Ramsey theory also focused (mainly due to combinatorial and number theoretical applications) on $\mathbb{Z}$-actions and, more generally, actions of various discrete semigroups.

There are, however, questions in modern ergodic theory pertaining to measure-preserving $\mathbb{R}$-actions that naturally present themselves and are connected with interesting applications but do not seem to easily follow from the corresponding results for $\mathbb{Z}$-actions. To better explain our point, let us consider some examples. We start with the $\mathbb{R}$-version of von Neumann's ergodic theorem [**vN**]: if T^t, $t \in \mathbb{R}$, is an ergodic one-parameter group of measure-preserving transformations of a probability measure space (X, μ), then, for any $f \in L^2(X)$, $\lim_{b-a\longrightarrow\infty}(1/(b-a))\int_a^b T^t f\,dt = \int_X f\,d\mu$† in $L^2(X)$. An easy trick shows that this result immediately follows from the corresponding theorem for $\mathbb{Z}$-actions, which says that if T is an invertible ergodic measure-preserving transformation of a probability measure space (X, μ), then, for any $f \in L^2(X)$,

$$\lim_{N-M\longrightarrow\infty} \frac{1}{N-M} \sum_{n=M+1}^{N} T^n f = \int_X f\,d\mu \quad \text{in } L^2(X).$$

Indeed, all one has to do is to apply the $\mathbb{Z}$-version of von Neumann's theorem to the function $\tilde{f} = \int_0^1 T^t f\,dt$ and the transformation T^1, utilizing the fact that, for any $a, b \in \mathbb{R}$,

$$\int_a^b T^t f\,dt = \sum_{n=[a]}^{[b]-1} T^n \tilde{f} - \int_{[a]}^{a} T^t f\,dt + \int_{[b]}^{b} T^t f\,dt.$$

(The $\mathbb{R}$-version of Birkhoff's pointwise ergodic theorem can be derived from its $\mathbb{Z}$-version in a similar way.) This argument is no longer applicable to 'multiple ergodic averages'

$$\frac{1}{b-a}\int_a^b T^{\alpha_1 t} f_1 \cdot \cdots \cdot T^{\alpha_r t} f_r\,dt, \tag{1}$$

where $r \geq 2$, $\alpha_i \in \mathbb{R}$, and $f_i \in L^\infty(X)$; however, it can be modified so that one is still able to show that the averages (1) converge in L^2-norm as $b - a \longrightarrow \infty$ as long as it is known that for arbitrarily small $u > 0$ the averages $(1/(N-M))\sum_{n=M+1}^{N} T^{\alpha_1 un} f_1 \cdot \cdots \cdot T^{\alpha_r un} f_r$ converge as $N - M \longrightarrow \infty$. (See, for example, [**Au1**].) Indeed, given $\varepsilon > 0$, find $\delta > 0$ such that $\|T^{\alpha_i t} f - f\| < \varepsilon$ for all $t \in (0, \delta)$, where $\|\cdot\| = \|\cdot\|_{L^2(X)}$; then, assuming without loss of generality that $\sup|f_i| \leq 1$, $i = 1, \ldots, r$, we have, for any $t \in \mathbb{R}$, $\|T^{\alpha_i t} f_i - T^{\alpha_i \delta[t/\delta]} f_i\| < \varepsilon$, $i = 1, \ldots, r$, and so

$$\left\| \prod_{i=1}^{r} T^{\alpha_i t} f_i - \prod_{i=1}^{r} T^{\alpha_i \delta[t/\delta]} f_i \right\| < r\varepsilon.$$

Hence,

$$\begin{aligned}
&\limsup_{b-a\longrightarrow\infty} \left\| \frac{1}{b-a}\int_a^b \prod_{i=1}^{r} T^{\alpha_i t} f_i\,dt - \frac{1}{[b/\delta]-[a/\delta]} \sum_{n=[a/\delta]}^{[b/\delta]-1} \prod_{i=1}^{r} T^{\alpha_i \delta n} f_i \right\| \\
&\quad \leq \limsup_{b-a\longrightarrow\infty} \left(\frac{1}{b-a} \sum_{n=[a/\delta]}^{[b/\delta]-1} \int_{n\delta}^{(n+1)\delta} \left\| \prod_{i=1}^{r} T^{\alpha_i t} f_i - \prod_{i=1}^{r} T^{\alpha_i \delta n} f_i \right\| dt \right. \\
&\qquad \left. + \frac{1}{b-a}\int_{[a/\delta]\delta}^{a} \left\| \prod_{i=1}^{r} T^{\alpha_i t} f_i \right\| dt + \frac{1}{b-a}\int_{[b/\delta]\delta}^{b} \left\| \prod_{i=1}^{r} T^{\alpha_i t} f_i \right\| dt \right) \leq r\varepsilon.
\end{aligned}$$

Since

$$\lim_{N-M\longrightarrow\infty} \frac{1}{N-M} \sum_{n=M+1}^{N} \prod_{i=1}^{r} T^{\alpha_i \delta n} f_i$$

† Here and below, $T^t f(\omega) = f(T^t \omega)$, $t \in \mathbb{R}$, $\omega \in X$, and the integral $\int_a^b T^t f\,dt$ is understood in the sense of Bochner.

exists in $L^2(X)$, and since ε is arbitrary, we get that

$$\lim_{b-a\longrightarrow\infty}\frac{1}{b-a}\int_a^b\prod_{i=1}^r T^{\alpha_i t}f_i\,dt$$

also exists.

However, even this argument stops working if we consider, say, the 'polynomial averages' $(1/(b-a))\int_a^b T^{p(t)}f\,dt$, where p is a polynomial or, more generally, the 'polynomial multiple averages'

$$\frac{1}{b-a}\int_a^b T^{p_1(t)}f_1\cdot\dots\cdot T^{p_r(t)}f_r\,dt, \tag{2}$$

where p_i are polynomials, since in this case the function $\varphi(t)=T^{p(t)}f$ from $\mathbb{R}$ to $L^2(X)$ is no longer uniformly continuous. The convergence of the corresponding discrete-time averages

$$\frac{1}{N-M}\sum_{n=M+1}^N T^{p_1(n)}f_1\cdot\dots\cdot T^{p_r(n)}f_r \tag{3}$$

is known (see [**HoK2, L3**]) but, to establish the convergence in $L^2(X)$ of the averages (2), one either has to go through all the main stages of the proof of the convergence of averages (3) and verify the validity of the corresponding $\mathbb{R}$-statements (see, for example, [**P**], where the existence of the non-uniform limits $(1/b)\int_0^b T^{p_1(t)}f_1\cdot\dots\cdot T^{p_r(t)}f_r\,dt$ is established), or may try to find some alternative general method connecting the convergence of discrete- and continuous-time averages. (Yet another approach to proving convergence of multiple polynomial averages, utilized in [**Au2**], is based on a 'change of variables' trick and usage of equivalent methods of summation; this method allows one to treat expressions like

$$\frac{1}{b}\int_0^b\prod_{j=1}^k T_j^{p_{j,1}(t)}f_1\cdot\dots\cdot\prod_{j=1}^k T_j^{p_{j,r}(t)}f_r\,dt,$$

where T_j are commuting measure-preserving transformations. However, this method gives no information about what the limits of such averages are and, also, it is not clear whether it can be extended to obtain convergence of uniform averages (2).)

As another example where a passage from a discrete to a continuous setup is desirable but not a priori obvious, let us mention the problem of the study of the distribution of values of generalized polynomials. A generalized polynomial is a function that is obtained from conventional polynomials of one or several variables by applying the operations of taking the integer part, addition, and multiplication; for example, if $p_i(x)$ are conventional polynomials, then

$$u(x)=[[p_1(x)]p_2(x)+p_3(x)]p_4(x)+[p_5(x)[p_6(x)]]^2p_7(x)$$

is a generalized polynomial. It was shown in [**BL**] that the values of any bounded vector-valued generalized polynomial of integer argument are well distributed on a piecewise polynomial surface, with respect to a natural measure on this surface. The proof was based on the theorem on well distribution of polynomial orbits on nilmanifolds; since such a theorem for continuous polynomial flows on nilmanifolds was not known at the time of writing [**BL**], we could not prove that bounded generalized polynomials of continuous argument are well distributed on piecewise polynomial surfaces. (See [**BL**, Theorem B_c]. As a matter of fact, the problem of extending the results from [**BL**] to the case of continuous parameters served as an impetus for the present paper.)

In this paper we introduce two simple but quite general methods that allow one to deduce continuous-time ergodic theorems from their discrete-time counterparts. To deliver the zest of these methods, we will formulate now two easy to state theorems. Let $F(t)$ be a bounded measurable function from $[0, \infty)$ to a Banach space. (In our applications, F will usually be 'an ergodic expression' that depends on a continuous parameter t and takes values in a functional space, say $F(t) = T_1^t f_1 \cdot \cdots \cdot T_k^t f_k \in L^1(X)$, $t \in [0, \infty)$, where T_i are one-parameter groups of measure-preserving transformations of a measure space X and $f_i \in L^\infty(X)$.)

PROPOSITION 1.1. (Additive method) *If the limit* $\lim_{N\longrightarrow\infty}(1/N)\sum_{n=0}^{N-1} F(t+n) = A_t$ *exists for almost every* $t \in [0, 1)$, *then the limit* $\lim_{b\longrightarrow\infty}(1/b)\int_0^b F(t)\,dt$ *also exists and equals* $\int_0^1 A_t\,dt$.

PROPOSITION 1.2. (Multiplicative method) *If the limit* $\lim_{N\longrightarrow\infty}(1/N)\sum_{n=0}^{N-1} F(nt) = L_t$ *exists for almost every* $t \in (0, 1)$, *then the limit* $L = \lim_{b\longrightarrow\infty}(1/b)\int_0^b F(t)\,dt$ *also exists and, moreover,* $L_t = L$ *for almost every* $t \in (0, 1]$.

Each of these 'methods' has its pros and cons. The 'additive' method is very easy to substantiate. However, it has the disadvantage that, being non-homogeneous, it 'desynchronizes' the expression $F(t)$, which may be an obstacle for certain applications. Consider, for example, the expression $F(t) = T_1^t f \cdot \cdots \cdot T_k^t f$ appearing in the formulation of the $\mathbb{R}$-version of the ergodic Szemerédi theorem (see §8.4 below). In this case, $F(t+n) = T_1^n f_1 \cdot \cdots \cdot T_k^n f_k$, where $f_i = T_i^t f$, $i = 1, \ldots, k$, are, generally speaking, distinct functions, which complicates application of the 'discrete' ergodic Szemerédi theorem. The 'multiplicative method' is quite a bit harder to establish, but it preserves the 'structure' of $F(t)$: for $F(t) = T_1^t f \cdot \cdots \cdot T_k^t f$, we now have $F(nt) = (T_1^t)^n f \cdot \cdots \cdot (T_k^t)^n f$. An additional advantage of the multiplicative method is that it guarantees the equality of almost all 'discrete' limits L_t, and therefore gives more information about the 'continuous' limit L. (See Theorem 8.11 below.)

When it comes to convergence on average, there are many types of it (uniform, strong Cesàro, etc) which naturally appear in various situations in classical analysis, number theory, and ergodic theory, and for each of them one can provide a statement that connects discrete and continuous averages. We therefore present several similar results; their proofs are based on similar ideas, but utilizing these ideas in diverse situations we obtain a variety of useful theorems. Here is the descriptive list of various kinds of averaging schemes we will be dealing with. (In what follows, V stands for an abstract Banach space.)

- *One-parameter standard Cesàro limits.* The Cesàro limit of a sequence (v_n) in V is $\lim_{N\longrightarrow\infty}(1/N)\sum_{n=1}^{N} v_n$ and, for a measurable function $f : [0, \infty) \longrightarrow V$, it is $\lim_{b\longrightarrow\infty}(1/b)\int_0^b f(x)\,dx$.

- *One-parameter uniform Cesàro limits.* The uniform Cesàro limit for a sequence (v_n) in V is $\lim_{N-M\longrightarrow\infty}(1/(N-M))\sum_{n=M+1}^{N} v_n$ and, for a measurable function $f : [0, \infty) \longrightarrow V$, it is $\lim_{b-a\longrightarrow\infty}(1/(b-a))\int_a^b f(x)\,dx$.

(The 'one-parameter averaging schemes' above are, of course, a special case of the corresponding 'multiparameter schemes' below, but we start with the one-parameter case to make our proofs more transparent.)

An $\mathbb{N}^d$-sequence (v_n) in V is a mapping $\mathbb{N}^d \longrightarrow V$, $n \mapsto v_n$. For a parallelepiped $P = \prod_{i=1}^d [a_i, b_i] \subset \mathbb{R}^d$, we define $l(P) = \min_{1\le i\le d}(b_i - a_i)$ and $w(P) = \prod_{i=1}^d (b_i - a_i)$.

• *Multiparameter standard Cesàro limits.* The Cesàro limit of an $\mathbb{N}^d$-sequence (v_n) in V is $\lim_{l(P)\longrightarrow\infty}(1/w(P))\sum_{n\in\mathbb{N}^d\cap P} v_n$ and, for a measurable function $f : [0, \infty)^d \longrightarrow V$, it is $\lim_{l(P)\longrightarrow\infty}(1/w(P))\int_P f(x)\,dx$, where, in both cases, P runs over the set of parallelepipeds of the form $\prod_{i=1}^d [0, b_i]$ in $[0, \infty)^d$.

• *Multiparameter uniform Cesàro limits.* The uniform Cesàro limit of an $\mathbb{N}^d$-sequence (v_n) in V is $\lim_{l(P)\longrightarrow\infty}(1/w(P))\sum_{n\in\mathbb{Z}^d\cap P} v_n$ and, for a measurable function $f : [0, \infty)^d \longrightarrow V$, it is $\lim_{l(P)\longrightarrow\infty}(1/w(P))\int_P f(x)\,dx$, where, in both cases, P runs over the set of parallelepipeds of the form $\prod_{i=1}^d [a_i, b_i]$ in $[0, \infty)^d$.

• *Two-sided* (or, rather, *all-sided*) *standard and uniform Cesàro limits.* Instead of $\mathbb{N}^d$-sequences, functions on $[0, \infty)^d$, and parallelepipeds in $[0, \infty)^d$, we deal with $\mathbb{Z}^d$-sequences, functions on $\mathbb{R}^d$, and parallelepipeds in $\mathbb{R}^d$.

• *Limits of averages along general Følner sequences.* Instead of averaging over parallelepipeds of the form $P = \prod_{i=1}^d [a_i, b_i]$, we consider averages over elements of a general Følner sequence $(\Phi_N)_{N=1}^\infty$ in $\mathbb{R}^d$, $\lim_{N\longrightarrow\infty}(1/w(\Phi_N))\int_{\Phi_N} f(x)\,dx$, where w stands for the Lebesgue measure on $\mathbb{R}^d$.

• *Liminf and limsup versions for standard and uniform averages.* When the limits above do not (or are not known to) exist, but (v_n) is a real-valued sequence and f is a real-valued function, we consider the corresponding liminfs and limsups.

• *Lim-limsup versions.* If the limits $\lim_{N\longrightarrow\infty}(1/N)\sum_{n=0}^{N-1} F(nt)$ do not, or are not known to, exist, it may still be possible that for some $L \in V$,

$$\lim_{t\longrightarrow 0^+} \limsup_{N\longrightarrow\infty} \left\| \frac{1}{N} \sum_{n=0}^{N-1} F(nt) - L \right\| = 0;$$

it turns out that this suffices for the multiplicative method to work.

After proving several versions of Propositions 1.1 and 1.2, corresponding to different averaging schemes, we will apply them to re-prove some known and establish some new ergodic-theoretical results; here is a list of the applications that we obtain in §8.

∘ In §8.1, we show that characteristic factors for averages of the form

$$\frac{1}{w(\Phi_N)} \int_{\Phi_N} T^{p_1(t)} f_1 \cdot \dots \cdot T^{p_r(t)} f_r \, dt, \tag{4}$$

where $T^t, t \in \mathbb{R}$, is a continuous one-parameter group of measure-preserving transformations of a probability measure space X, p_i are polynomials $\mathbb{R}^d \longrightarrow \mathbb{R}$, $f_i \in L^\infty(X)$, and (Φ_N) is a Følner sequence in $\mathbb{R}^d$, are Host–Kra–Ziegler factors of X. (A non-uniform version of this result was obtained in [**P**].)

∘ In §8.2, we prove that, for any $d \in \mathbb{N}$, a d-parameter polynomial flow on a nilmanifold X is well distributed on a subnilmanifold of X. (This result is new and refines the fact that any such flow is uniformly distributed in a subnilmanifold of X.)

∘ In §8.3, we prove the convergence of averages (4). (This result is new, strengthening the results obtained in [**P**] and the one-parameter case of the results obtained in [**Au2**].) We also prove that the averages $(1/(b-a))\int_a^b T_1^t f_1 \cdot \cdots \cdot T_r^t \, dt$ converge, where T_i are pairwise-commuting measure-preserving transformations. (This strengthens the linear case of the results obtained in [**Au2**].)

∘ In §8.4, we obtain a continuous-time version of the polynomial ergodic Szemerédi theorem.

∘ In §8.5, we prove that the values of bounded vector-valued generalized polynomials are well distributed on a piecewise polynomial surface. This establishes the continuous version of the well-distribution result from [**BL**] that we discussed above.

∘ In §8.6, we derive, from the corresponding discrete-time results in [**BK, F**], convergence of multiple averages (2) with p_i being functions of polynomial growth.

∘ Finally, in §8.7, we apply our methods to obtain continuous-time theorems dealing with almost-everywhere convergence of certain ergodic averages.

2. *A Fatou lemma and a dominated convergence theorem*

Throughout §§2–7, V stands for a separable Banach space. We will repeatedly use the following Fatou-like lemma and its corollary.

LEMMA 2.1. *Let (X, μ) be a finite measure space and let (f_n) be a sequence of uniformly bounded measurable functions from X to V. Then*

$$\limsup_{n\longrightarrow\infty}\left\|\int_X f_n \, d\mu\right\| \leq \int_X \limsup_{n\longrightarrow\infty} \|f_n\| \, d\mu.$$

Proof. Let $M > 0$ be such that $\|f_n(x)\| \leq M$ for all $x \in X$ and $n \in \mathbb{N}$, and let $s(x) = \limsup_{n\longrightarrow\infty} \|f_n(x)\|$, $x \in X$. Fix $\varepsilon > 0$. For each $x \in X$, let $n(x) \in \mathbb{N}$ be such that $\|f_n(x)\| < s(x) + \varepsilon$ for all $n \geq n(x)$. For each $n \in \mathbb{N}$, let $A_n = \{x \in X : n(x) \leq n\}$. Then $A_1 \subseteq A_2 \subseteq \cdots$ and $\bigcup_{n=1}^{\infty} A_n = X$, so $\lim_{n\longrightarrow\infty} \mu(X\backslash A_n) = 0$. Let N be such that $\mu(X\backslash A_N) < \varepsilon$. Then, for any $n \geq N$,

$$\begin{aligned}
\left\|\int_X f_n \, d\mu\right\| \leq \int_X \|f_n\| \, d\mu &= \int_{A_N} \|f_n\| \, d\mu + \int_{X\backslash A_N} \|f_n\| \, d\mu \\
&\leq \int_{A_N} (s+\varepsilon) \, d\mu + M\varepsilon \\
&\leq \int_X s \, d\mu + \varepsilon(\mu(X) + M).
\end{aligned}$$

Since this is true for any positive ε, $\| \int_X f_n \, d\mu\| \leq \int_X s \, d\mu$. □

As a corollary, we get the following lemma.

LEMMA 2.2. *Let (X, μ) be a finite measure space. If a sequence (f_n) of uniformly bounded measurable functions from X to V converges to a function $f : X \longrightarrow V$ almost everywhere on X, then $\int_X f_n \, d\mu \longrightarrow \int_X f \, d\mu$.*

Remark. Of course, a more general dominated convergence theorem, where the $\| f_n \|$ are not assumed to be bounded but only dominated by an integrable function, like in the case of real-valued functions, also holds, but we will only need its special case given by Lemma 2.2.

3. *Additive method*

When a and b are positive real numbers, we define

$$\sum_{n>a}^{b} v_n = \begin{cases} \displaystyle\sum_{n \in (a,b] \cap \mathbb{N}} v_n & \text{if } a < b, \\ 0 & \text{if } a \geq b. \end{cases}$$

3.1. *Standard Cesàro limits.*

THEOREM 3.1. *Let $f : [0, \infty) \longrightarrow V$ be a bounded measurable function such that the limit*

$$A_t = \lim_{b \longrightarrow \infty} \frac{1}{b} \sum_{n>0}^{b} f(t+n)$$

exists for almost every $t \in [0, 1]$. Then $\lim_{b \longrightarrow \infty} (1/b) \int_0^b f(x) \, dx$ also exists and is equal to $\int_0^1 A_t \, dt$.

Proof. We may assume that the parameter b is an integer. For any $b \in \mathbb{N}$, we have

$$\frac{1}{b} \int_0^b f(x) \, dx = \frac{1}{b} \sum_{n=0}^{b-1} \int_0^1 f(t+n) \, dt = \int_0^1 \frac{1}{b} \sum_{n=0}^{b-1} f(t+n) \, dt.$$

Since, for almost every $t \in [0, 1]$, $(1/b) \sum_{n=0}^{b-1} f(t+n) \, dt \longrightarrow A_t$ as $b \longrightarrow \infty$, by Lemma 2.2,

$$\lim_{b \longrightarrow \infty} \frac{1}{b} \int_0^b f(x) \, dx = \int_0^1 A_t \, dt. \qquad \square$$

Remark. Of course, in the formulation of Theorem 3.1, the interval $[0, 1]$ and the expression $f(t+n)$ can be replaced by the interval $[0, \delta]$ and the expression $f(t+n\delta)$ for any positive δ.

3.2. *Uniform Cesàro limits.*

THEOREM 3.2. *Let $f : [0, \infty) \longrightarrow V$ be a bounded measurable function such that the limit*

$$A_t = \lim_{b-a \longrightarrow \infty} \frac{1}{b-a} \sum_{n>a}^{b} f(t+n)$$

exists for almost every $t \in [0, 1]$. *Then* $\lim_{b-a\longrightarrow\infty}(1/(b-a))\int_a^b f(x)\,dx$ *also exists and is equal to* $\int_0^1 A_t\,dt$.

Proof. We may assume that the parameters a, b are integers. For any sequences (a_k), (b_k) of non-negative integers with $b_k - a_k \longrightarrow +\infty$, we have

$$\frac{1}{b_k - a_k}\int_{a_k}^{b_k} f(x)\,dx = \frac{1}{b_k - a_k}\sum_{n=a_k}^{b_k-1}\int_0^1 f(t+n)\,dt = \int_0^1 \frac{1}{b_k - a_k}\sum_{n=a_k}^{b_k-1} f(t+n)\,dt.$$

Since, for almost every $t \in [0, 1]$,

$$\frac{1}{b_k - a_k}\sum_{n=a_k}^{b_k-1} f(t+n)\,dt \longrightarrow A_t \quad \text{as } k \longrightarrow \infty,$$

by Lemma 2.2,

$$\lim_{k\longrightarrow\infty}\frac{1}{b_k - a_k}\int_{a_k}^{b_k} f(x)\,dx = \int_0^1 A_t\,dt. \qquad \square$$

3.3. *Multiparameter standard Cesàro limits.* Let $d \in \mathbb{N}$. We will call a mapping $\mathbb{N}^d \longrightarrow V$, $n \mapsto v_n$ *an* $\mathbb{N}^d$*-sequence in* V. We write $\mathbb{R}_+$ for $[0, \infty)$. We will now introduce notation that will allow us to formulate and prove the d-parameter versions of the above theorems in complete analogy with the case $d = 1$.

For $a, b \in \mathbb{R}_+^d$, $a = (a_1, \dots, a_d)$, $b = (b_1, \dots, b_d)$, we write $a \le b$ if $a_i \le b_i$ for all $i = 1, \dots, d$ and $a < b$ if $a_i < b_i$ for all i. Under $\min(a, b)$ and $\max(a, b)$, we will understand $(\min(a_1, b_1), \dots, \min(a_d, b_d))$ and $(\max(a_1, b_1), \dots, \max(a_d, b_d))$, respectively. For $a = (a_1, \dots, a_d) \in \mathbb{R}_+^d$ and $b = (b_1, \dots, b_d) \in \mathbb{R}_+^d$, we define $ab = (a_1b_1, \dots, a_db_d)$ and, if $b > 0$, $a/b = (a_1/b_1, \dots, a_d/b_d)$ and $b^\alpha = (b_1^\alpha, \dots, b_d^\alpha)$, $\alpha \in \mathbb{R}$.

For $a = (a_1, \dots, a_d) \in \mathbb{R}_+^d$, we define $w(a) = a_1 \cdots a_d$ and $l(a) = \min\{a_1, \dots, a_d\}$. Note that if $a, b \in \mathbb{R}_+^d$ and $0 < a \le b$, then $w(a)/w(b) \le l(a)/l(b)$.

For $a, b \in \mathbb{R}_+^d$, $a \le b$, we define *intervals* $[a, b] = \{x \in \mathbb{R}_+^d : a \le x \le b\}$ and $(a, b] = \{x \in \mathbb{R}_+^d : a < x \le b\}$.

For $a, b \in \mathbb{R}_+^d$, under $\sum_{n=a}^b v_n$, we will understand $\sum_{n\in\mathbb{N}^d\cap[a,b]} v_n$ if $a \le b$ and 0 otherwise, under $\sum_{n>a}^b v_n$ we will understand $\sum_{n\in\mathbb{N}^d\cap(a,b]} v_n$ if $a \le b$ and 0 otherwise, and under $\int_a^b v(x)\,dx$ we will understand $\int_{[a,b]} v(x)\,dx$.

Finally, for $c \in \mathbb{R}_+$, by $\bar{c}$, we will denote $(c, \dots, c) \in \mathbb{R}_+^d$.

THEOREM 3.3. *Let* $f : \mathbb{R}_+^d \longrightarrow V$ *be a bounded measurable function such that the limit*

$$A_t = \lim_{l(b)\longrightarrow\infty}\frac{1}{w(b)}\sum_{n>0}^{b} f(t+n)$$

exists for almost every $t \in [0, 1]^d$. *Then* $\lim_{l(b)\longrightarrow\infty}(1/w(b))\int_0^b f(x)\,dx$ *also exists and is equal to* $\int_{[0,1]^d} A_t\,dt$.

Proof. We may assume that $b \in \mathbb{N}^d$. Let (b_k) be a sequence in $\mathbb{N}^d$ with $l(b_k) \longrightarrow \infty$ as $k \longrightarrow \infty$. For any $k \in \mathbb{N}$, we have

$$\frac{1}{w(b_k)} \int_0^{b_k} f(x)\,dx = \frac{1}{w(b_k)} \sum_{n=0}^{b_k-\bar{1}} \int_{[0,1]^d} f(t+n)\,dt = \int_{[0,1]^d} \frac{1}{w(b_k)} \sum_{n=0}^{b_k-\bar{1}} f(t+n)\,dt.$$

Since, for almost every $t \in [0, 1]^d$,

$$\frac{1}{w(b_k)} \sum_{n=0}^{b_k-\bar{1}} f(t+n)\,dt \longrightarrow A_t \quad \text{as } k \longrightarrow \infty,$$

by Lemma 2.2,

$$\lim_{k\longrightarrow\infty} \frac{1}{w(b_k)} \int_0^{b_k} f(x)\,dx = \int_{[0,1]^d} A_t\,dt. \qquad \square$$

3.4. *Multiparameter uniform Cesàro limits.*

THEOREM 3.4. *Let $f : \mathbb{R}_+^d \longrightarrow V$ be a bounded measurable function such that the limit*

$$A_t = \lim_{l(b-a)\longrightarrow\infty} \frac{1}{w(b-a)} \sum_{n>a}^{b} f(t+n)$$

exists for almost every $t \in [0, 1]^d$. Then

$$\lim_{l(b-a)\longrightarrow\infty} \frac{1}{w(b-a)} \int_a^b f(x)\,dx$$

also exists and is equal to $\int_{[0,1]^d} A_t\,dt$.

Proof. We may assume that $a, b \in \mathbb{N}^d$. Let (a_k), (b_k) be sequences in $\mathbb{N}^d$ with $a_k < b_k$ and $l(b_k - a_k) \longrightarrow \infty$ as $k \longrightarrow \infty$. For any $k \in \mathbb{N}$, we have

$$\begin{aligned}\frac{1}{w(b_k-a_k)} \int_{a_k}^{b_k} f(x)\,dx &= \frac{1}{w(b_k-a_k)} \sum_{n=a_k}^{b_k-\bar{1}} \int_{[0,1]^d} f(t+n)\,dt \\ &= \int_{[0,1]^d} \frac{1}{w(b_k-a_k)} \sum_{n=a_k}^{b_k-\bar{1}} f(t+n)\,dt.\end{aligned}$$

Since, for almost every $t \in [0, 1]^d$,

$$\frac{1}{w(b_k-a_k)} \sum_{n=0}^{b_k-\bar{1}} f(t+n)\,dt \longrightarrow A_t \quad \text{as } k \longrightarrow \infty,$$

by Lemma 2.2,

$$\lim_{k\longrightarrow\infty} \frac{1}{w(b_k-a_k)} \int_{a_k}^{b_k} f(x)\,dx = \int_{[0,1]^d} A_t\,dt. \qquad \square$$

3.5. *Liminf and limsup versions.* In the case where f is a real-valued function, we may obtain similar results involving liminfs of limsups, even if the limits A_t do not exist.

THEOREM 3.5. *If $f : \mathbb{R}_+^d \longrightarrow \mathbb{R}$ is a bounded measurable function, then*

$$\liminf_{l(b)\longrightarrow\infty} \frac{1}{w(b)} \int_0^b f(x)\,dx \geq \int_{[0,1]^d} \liminf_{l(b)\longrightarrow\infty} \frac{1}{w(b)} \sum_{n>0}^{b} f(t+n)\,dt$$

and

$$\limsup_{l(b)\longrightarrow\infty} \frac{1}{w(b)} \int_0^b f(x)\,dx \leq \int_{[0,1]^d} \limsup_{l(b)\longrightarrow\infty} \frac{1}{w(b)} \sum_{n>0}^{b} f(t+n)\,dt.$$

Proof. After adding a constant to f, we may assume that $f \geq 0$. We may also assume that $b \in \mathbb{N}^d$. Let (b_k) be a sequence in $\mathbb{N}^d$ with $l(b_k) \longrightarrow \infty$ as $k \longrightarrow \infty$. For any $k \in \mathbb{N}$, we have

$$\frac{1}{w(b_k)} \int_0^{b_k} f(x)\,dx = \frac{1}{w(b_k)} \sum_{n=0}^{b_k-\bar{1}} \int_{[0,1]^d} f(t+n)\,dt = \int_{[0,1]^d} \frac{1}{w(b_k)} \sum_{n=0}^{b_k-\bar{1}} f(t+n)\,dt.$$

By (the classical, real-valued) Fatou's theorem,

$$\begin{aligned}\liminf_{k\longrightarrow\infty} \frac{1}{w(b_k)} \int_0^{b_k} f(x)\,dx &\geq \int_{[0,1]^d} \liminf_{k\longrightarrow\infty} \frac{1}{w(b_k)} \sum_{n=0}^{b_k-\bar{1}} f(t+n)\,dt \\ &= \int_{[0,1]^d} \liminf_{k\longrightarrow\infty} \frac{1}{w(b_k)} \sum_{n>0}^{b_k} f(t+n)\,dt. \qquad \square\end{aligned}$$

And, similarly, we have the following theorem.

THEOREM 3.6. *If $f : \mathbb{R}_+^d \longrightarrow \mathbb{R}$ is a bounded measurable function, then*

$$\liminf_{l(b-a)\longrightarrow\infty} \frac{1}{w(b-a)} \int_a^b f(x)\,dx \geq \int_{[0,1]^d} \liminf_{l(b-a)\longrightarrow\infty} \frac{1}{w(b-a)} \sum_{n>a}^{b} f(t+n)\,dt$$

and

$$\limsup_{l(b-a)\longrightarrow\infty} \frac{1}{w(b-a)} \int_a^b f(x)\,dx \leq \int_{[0,1]^d} \limsup_{l(b-a)\longrightarrow\infty} \frac{1}{w(b-a)} \sum_{n>a}^{b} f(t+n)\,dt.$$

4. *Multiplicative method—the one-parameter case*

4.1. *Standard Cesàro limits.*

THEOREM 4.1. *Let a bounded measurable function $f : [0,\infty) \longrightarrow V$ be such that, for some $c > 0$, the limit $L_t = \lim_{b\longrightarrow\infty}(1/b)\sum_{n>0}^{b} f(nt)$ exists for almost every $t \in [0, c]$. Then L_t is almost everywhere constant, $L_t = L \in V$ almost everywhere on $[0, c]$, and $\lim_{b\longrightarrow\infty}(1/b)\int_0^b f(x)\,dx$ exists and equals L.*

Following the referee's suggestion, we will derive Theorem 4.1 from the following classical fact. The proof of this result which we provide for the reader's convenience has an advantage of being easily extendible to the multiparameter case (Lemma 5.2 below).

LEMMA 4.2. *Let (v_n) be a sequence in V such that $\|v_{n+1} - v_n\| = O(1/n)$ and the Cesáro limit $L = \lim_{N\longrightarrow\infty}(1/N)\sum_{n=1}^{N} v_n$ exists. Then $\lim_{n\longrightarrow\infty} v_n = L$.*

Proof. We may assume that $L = 0$, that is, $\lim_{N\longrightarrow\infty}(1/N)\sum_{n=1}^{N} v_n = 0$. Assume that $v_n \not\longrightarrow 0$; let $\varepsilon > 0$ be such that, for any $N \in \mathbb{N}$, there exists $n > N$ such that $\|v_n\| > \varepsilon$. Let $\alpha > 0$ be such that $\|v_{n+1} - v_n\| < \alpha/n$ for all n; put $\delta = \varepsilon^2/16(1 + \varepsilon/2\alpha)\alpha$. Find $N \in \mathbb{N}$ such that $\|(1/M)\sum_{n=1}^{M} v_n\| < \delta$ for all $M > N$. Find $M > N$ such that $\|v_M\| > \varepsilon$ and $1/M < \varepsilon/4\alpha$. Let, by the Hahn–Banach theorem, $\varphi \in V^*$ be such that $|\varphi(v)| \le \|v\|$ for all $v \in V$ and $\varphi(v_M) = \|v_M\|$. Then, for any $n > M$,

$$\varphi(v_n) = \varphi(v_M) + \sum_{m=M}^{n-1} \varphi(v_{m+1} - v_m) \ge \|v_M\| - \sum_{m=M}^{n-1} \|v_{m+1} - v_m\| > \varepsilon - (n - M)\frac{\alpha}{M},$$

which is $\ge \varepsilon/2$ when $(n - M)\alpha/M \le \varepsilon/2$, that is, when $M < n \le M + \varepsilon M/2\alpha$. Put $K = M + \lfloor \varepsilon M/2\alpha \rfloor$; then $\varphi(v_n) > \varepsilon/2$ for $n = M + 1, \ldots, K$. Thus,

$$\begin{aligned}
\left\| \frac{1}{K}\sum_{n=1}^{K} v_n \right\| &\ge \frac{1}{K}\left(\left\| \sum_{n=M+1}^{K} v_n \right\| - \left\| \sum_{n=1}^{M} v_n \right\| \right) \ge \frac{1}{K}\left(\varphi\left(\sum_{n=M+1}^{K} v_n \right) - \delta M \right) \\
&= \frac{1}{K}\left(\sum_{n=M+1}^{K} \varphi(v_n) - \delta M \right) > \frac{1}{K}\left((K - M)\frac{\varepsilon}{2} - \delta M \right) \\
&\ge \frac{1}{M(1 + \varepsilon/2\alpha)}\left(\left(\frac{\varepsilon M}{2\alpha} - 1 \right)\frac{\varepsilon}{2} - \delta M \right) \ge \frac{1}{1 + \varepsilon/2\alpha}\left(\left(\frac{\varepsilon}{2\alpha} - \frac{1}{M} \right)\frac{\varepsilon}{2} - \delta \right) \\
&\ge \frac{\varepsilon^2}{(1 + \varepsilon/2\alpha)8\alpha} - \delta = \delta,
\end{aligned}$$

which contradicts the choice of N. □

Proof of Theorem 4.1. Let $v_n = (1/nc)\int_0^{nc} f(x)\,dx$, $n \in \mathbb{N}$. Then, for any n,

$$\begin{aligned}
\|v_{n+1} - v_n\| &= \frac{1}{c}\left\| \left(\frac{1}{n+1} - \frac{1}{n} \right) \int_0^{nc} f(x)\,dx + \frac{1}{n+1}\int_{nc}^{(n+1)c} f(x)\,dx \right\| \\
&\le \frac{cn \sup\|f\|}{cn(n+1)} + \frac{c \sup\|f\|}{c(n+1)} = O(1/n).
\end{aligned}$$

Also, we have, for any $N \in \mathbb{N}$,

$$\begin{aligned}
\frac{1}{N}\sum_{n=0}^{N-1} v_n &= \frac{1}{N}\sum_{n=0}^{N-1} \frac{1}{nc}\int_0^{nc} f(x)\,dx \\
&= \frac{1}{N}\sum_{n=0}^{N-1} \frac{1}{c}\int_0^{c} f(nt)\,dt = \frac{1}{c}\int_0^{c} \frac{1}{N}\sum_{n=0}^{N-1} f(nt)\,dt.
\end{aligned}$$

Since $(1/N)\sum_{n=0}^{N-1} f(nt) \longrightarrow L_t$ as $N \longrightarrow \infty$ for almost every $t \in [0, c]$, by Lemma 2.2,

$$\frac{1}{N}\sum_{n=0}^{N-1} v_n \longrightarrow \frac{1}{c}\int_0^{c} L_t\,dt.$$

By Lemma 4.2, $v_n \longrightarrow (1/c)\int_0^c L_t\,dt$. On the other hand,

$$\lim_{n\longrightarrow\infty} v_n = \lim_{b\longrightarrow\infty} \frac{1}{b}\int_0^{b} f(x)\,dx.$$

So, $L = \lim_{b\longrightarrow\infty}(1/b)\int_0^b f(x)\,dx$ exists and equals $(1/c)\int_0^c L_t\,dt$.

Next, for any $z \in (0, c)$, we also have $L = (1/z)\int_0^z L_t\, dt$. So, for any $z \in [0, c]$, $\int_0^z L_t\, dt = zL$, which implies that $L_t = L$ almost everywhere on $[0, c]$. □

4.2. *Uniform Cesàro limits.*

THEOREM 4.3. *Let a bounded measurable function $f : [0, \infty) \longrightarrow V$ be such that for some $c > 0$, for almost every $t \in (0, c]$, the limit $L_t = \lim_{b-a\longrightarrow\infty}(1/(b-a))\sum_{n>a}^{b} f(nt)$ exists. Then L_t is constant almost everywhere on $(0, c]$, $L_t = L \in V$ for almost every $t \in (0, c]$, and $\lim_{b-a\longrightarrow\infty}(1/(b-a))\int_a^b f(x)\, dx$ exists and equals L.*

LEMMA 4.4. *Let (v_n) be a bounded sequence in V such that $\lim_{b-a\longrightarrow\infty}(1/(b-a)) \sum_{n>a}^{b} v_n = 0$, let (b_k) be a sequence of positive real numbers with $b_k \longrightarrow \infty$, and let (α_k), (β_k) be sequences of real numbers such that $0 < \beta_k - \alpha_k \le b_k$ for all k. Then $\lim_{k\longrightarrow\infty}(1/b_k)\sum_{n>\alpha_k}^{\beta_k} v_n = 0$.*

Proof. Assume that $\|v_n\| \le 1$ for all n. Let $\varepsilon > 0$. Let $B > 1$ be such that $\|(1/(b-a)) \sum_{n>a}^{b} v_n\| < \varepsilon$ whenever $b - a > B$. Let K be such that $b_k > (B + 1)/\varepsilon$ for all $k > K$. Then, for any $k > K$, if $\beta_k - \alpha_k > B$, then

$$\left\|\frac{1}{b_k}\sum_{n>\alpha_k}^{\beta_k} v_n\right\| \le \left\|\frac{1}{\beta_k-\alpha_k}\sum_{n>\alpha_k}^{\beta_k} v_n\right\| < \varepsilon$$

and, if $\beta_k - \alpha_k \le B$, then also

$$\left\|\frac{1}{b_k}\sum_{n>\alpha_k}^{\beta_k} v_n\right\| \le \frac{\beta_k-\alpha_k+1}{b_k} < \varepsilon. \qquad \square$$

Proof of Theorem 4.3. Since, in particular, $L_t = \lim_{b\longrightarrow\infty}(1/b)\sum_{n>0}^{b} f(nt)$ for almost every $t \in (0, c]$, we have $L_t = \text{const} = L$ for almost every $t \in (0, c]$ by Theorem 4.1. Replacing f by $f - L$, we may assume that $L = 0$. After replacing $f(x)$ by $f(cx/2)$, we assume that $c = 2$. Let $a \ge 0$, $b \ge a + 1$. For any $n \in \mathbb{N}$, $(1/n)\int_a^b f(x)\, dx = \int_{a/n}^{b/n} f(nt)\, dt$. Adding these equalities for all $n \in (b/2, b]$, and taking into account that $b/n < 2$ for $n > b/2$, we get

$$\lambda\int_a^b f(x)\, dx = \int_0^2 \sum_{n>\alpha(a,b,t)}^{\beta(a,b,t)} f(nt)\, dt,$$

where $\lambda = \sum_{n>b/2}^{b} 1/n \ge 1/2$, and, for every $t \in (0, 2]$, $\alpha(a, b, t) = \max\{b/2, a/t\}$ and $\beta(a, b, t) = \min\{b, b/t\}$. Thus,

$$\left\|\frac{1}{b-a}\int_a^b f(x)\, dx\right\| \le 2\left\|\int_0^2 f_{a,b}(t)\, dt\right\|, \tag{5}$$

where $f_{a,b}(t) = (1/(b-a))\sum_{n>\alpha(a,b,t)}^{\beta(a,b,t)} f(nt)$, $t \in (0, 2]$.

We will now show that the functions $f_{a,b}$, for $a \ge 0$, $b \ge a + 1$, are uniformly bounded. Let us assume that $\sup_{x\in(0,\infty)} \|f(x)\| \le 1$. If $a \le b/2$, then $b - a \ge b/2$ and, since $\beta(a, b, t) - \alpha(a, b, t) \le b/2$, for any $t \in (0, 2]$, we have $\| f_{a,b}(t)\| \le (1/(b/2))(b/2 + 1) \le 3$. If $a > b/2$, then, for any $t \in (0, 1/2]$, we have $\alpha(a, b, t) \ge a/t \ge 2a > b \ge \beta(a, b, t)$, so $f_{a,b}(t) = 0$; and, since, for any $t \in (0, 2]$, $\beta(a, b, t) - \alpha(a, b, t) \le (b - a)/t$, we have $\| f_{a,b}(t)\| \le (1/(b - a))((b - a)/t + 1) \le 3$ for $t \in [1/2, 2]$.

For almost every $t \in (0, 2]$, since $\beta(a, b, t) - \alpha(a, b, t) \le (b - a)/t$ for all $a \ge 0$, $b \ge a + 1$, by Lemma 4.4,

$$\lim_{b-a\longrightarrow\infty} f_{a,b}(t) = \frac{1}{t} \lim_{b-a\longrightarrow\infty} \frac{t}{b-a} \sum_{n>\alpha(a,b,t)}^{\beta(a,b,t)} f(nt) = 0.$$

Hence, by Lemma 2.2, $\int_0^2 f_{a,b}(t)\, dt \longrightarrow 0$ as $b - a \longrightarrow \infty$. So, by (5),

$$\frac{1}{b-a}\int_a^b f(x)\, dx \longrightarrow 0 \quad \text{as } b - a \longrightarrow \infty. \qquad \square$$

4.3. *Liminf and limsup versions for uniform averages.* In the case where f is a real-valued function and the limits L_t do not exist, we have liminf/limsup versions of the above theorems. We start with the uniform case.

THEOREM 4.5. *For any bounded measurable function $f : [0, \infty) \longrightarrow \mathbb{R}$ and any $c > 0$,*

$$\liminf_{b-a\longrightarrow\infty} \frac{1}{b-a}\int_a^b f(x)\, dx \ge \frac{1}{c}\int_0^c \liminf_{b-a\longrightarrow\infty} \frac{1}{b-a}\sum_{n>a}^{b} f(nt)\, dt$$

and

$$\limsup_{b-a\longrightarrow\infty} \frac{1}{b-a}\int_a^b f(x)\, dx \le \frac{1}{c}\int_0^c \limsup_{b-a\longrightarrow\infty} \frac{1}{b-a}\sum_{n>a}^{b} f(nt)\, dt.$$

Proof. We will only prove the first inequality. We may assume that $f \ge 0$. Let $L = \liminf_{b-a\longrightarrow\infty}(1/(b-a))\int_a^b f(x)\, dx$; find a sequence of intervals $[a_k, b_k]$ with $b_k - a_k \longrightarrow \infty$ such that $L = \lim_{k\longrightarrow\infty}(1/(b_k - a_k))\int_{a_k}^{b_k} f(x)\, dx$. We may assume that $a_k \longrightarrow \infty$ (after replacing each a_k by $\max\{a_k, \sqrt{b_k}\}$) and that $b_k/a_k \longrightarrow 1$ (after replacing each interval $[a_k, b_k]$ by a suitable subinterval).

For any $t > 0$, we have

$$\liminf_{b-a\longrightarrow\infty} \frac{1}{b-a}\sum_{n>a}^{b} f(nt) \le \liminf_{k\longrightarrow\infty} \frac{1}{b_k/t - a_k/t}\sum_{n>a_k/t}^{b_k/t} f(nt),$$

so

$$\frac{1}{c}\int_0^c \liminf_{b-a\longrightarrow\infty} \frac{1}{b-a}\sum_{n>a}^{b} f(nt)\, dt \le \frac{1}{c}\int_0^c \liminf_{k\longrightarrow\infty} \frac{1}{b_k/t - a_k/t}\sum_{n>a_k/t}^{b_k/t} f(nt)\, dt.$$

By (the classical) Fatou's lemma, we have

$$\begin{aligned}\frac{1}{c}\int_0^c & \liminf_{k\longrightarrow\infty} \frac{1}{b_k/t - a_k/t}\sum_{n>a_k/t}^{b_k/t} f(nt)\, dt \\ &\le \frac{1}{c}\liminf_{k\longrightarrow\infty}\int_0^c \frac{1}{b_k/t - a_k/t}\sum_{n>a_k/t}^{b_k/t} f(nt)\, dt\end{aligned}$$

and

$$\frac{1}{c}\liminf_{k\longrightarrow\infty}\int_0^c \frac{1}{b_k/t-a_k/t}\sum_{n>a_k/t}^{b_k/t} f(nt)\,dt = \frac{1}{c}\liminf_{k\longrightarrow\infty}\frac{1}{b_k-a_k}\int_0^c \sum_{n>a_k/t}^{b_k/t} tf(nt)\,dt$$
$$\leq \frac{1}{c}\liminf_{k\longrightarrow\infty}\frac{1}{b_k-a_k}\sum_{n>a_k/c}\int_{a_k/n}^{b_k/n} tf(nt)\,dt.$$

For every $k\in\mathbb{N}$, for each n we have

$$I_n=\int_{a_k/n}^{b_k/n} tf(nt)\,dt=\frac{1}{n^2}\int_{a_k}^{b_k} xf(x)\,dx=\frac{1}{n^2}\alpha_k\int_{a_k}^{b_k} f(x)\,dx$$

with $\alpha_k\in[a_k,b_k]$, so

$$\sum_{n>a_k/c} I_n=\alpha_k\int_{a_k}^{b_k} f(x)\,dx\sum_{n>a_k/c}\frac{1}{n^2}=\alpha_k s_k\int_{a_k}^{b_k} f(x)\,dx,$$

where $s_k=\sum_{n>a_k/c}(1/n^2)$ satisfies $s_k a_k/c\longrightarrow 1$ as $k\longrightarrow\infty$. Since, by our assumption, also $\alpha_k/a_k\longrightarrow 1$, we get

$$\frac{1}{c}\lim_{k\longrightarrow\infty}\frac{1}{b_k-a_k}\sum_{n>a_k/c}\int_{a_k/n}^{b_k/n} tf(nt)\,dt=\lim_{k\longrightarrow\infty}\frac{\alpha_k s_k}{c}\cdot\frac{1}{b_k-a_k}\int_{a_k}^{b_k} f(x)\,dx=L.$$

So, $(1/c)\int_0^c \liminf_{b-a\longrightarrow\infty}(1/(b-a))\sum_{n>a}^{b} f(nt)\,dt\leq L$. □

4.4. *Liminf and limsup versions for standard averages.*

THEOREM 4.6. *For any bounded measurable function $f:[0,\infty)\longrightarrow\mathbb{R}$ and any $c>0$,*

$$\liminf_{b\longrightarrow\infty}\frac{1}{b}\int_0^b f(x)\,dx\geq\frac{1}{c}\int_0^c\liminf_{b\longrightarrow\infty}\frac{1}{b}\sum_{n>0}^{b} f(nt)\,dt$$

and

$$\limsup_{b\longrightarrow\infty}\frac{1}{b}\int_0^b f(x)\,dx\leq\frac{1}{c}\int_0^c\limsup_{b\longrightarrow\infty}\frac{1}{b}\sum_{n>0}^{b} f(nt)\,dt.$$

Proof. We may assume that $f\geq 0$. Let $L=\liminf_{b\longrightarrow\infty}(1/b)\int_0^b f(x)\,dx$; choose a sequence (b_k), with $b_k\longrightarrow\infty$ as $k\longrightarrow\infty$, such that $\lim_{k\longrightarrow\infty}(1/b_k)\int_0^{b_k} f(x)\,dx=L$. Then also $\lim_{k\longrightarrow\infty}(1/(b_k-a_k))\int_{a_k}^{b_k} f(x)\,dx=L$, where $a_k=\sqrt{b_k}$, $k\in\mathbb{N}$. For all $t>0$, we have

$$\liminf_{b\longrightarrow\infty}\frac{1}{b}\sum_{n>0}^{b} f(nt)=\liminf_{b\longrightarrow\infty}\frac{1}{b-\sqrt{b}}\sum_{n>\sqrt{b}}^{b} f(nt)\leq\liminf_{k\longrightarrow\infty}\frac{1}{b_k-a_k}\sum_{n>a_k}^{b_k} f(nt)$$

and, as in the proof of Theorem 4.5, by Fatou's lemma,

$$\frac{1}{c}\int_0^c\liminf_{k\longrightarrow\infty}\frac{1}{b_k-a_k}\sum_{n>a_k}^{b_k} f(nt)\,dt\leq\frac{1}{c}\liminf_{k\longrightarrow\infty}\frac{1}{b_k-a_k}\int_0^c\sum_{n>a_k/t}^{b_k/t} tf(nt)\,dt,$$

so it suffices to show that this last expression is $\leq L$.

For every $k \in \mathbb{N}$, put $M_k = \lfloor b_k^{2/3} \rfloor$ and subdivide the interval $[a_k, b_k]$ into M_k equal parts: put $b_{k,j} = a_k + j(b_k - a_k)/M_k$, $j = 0, \dots, M_k$. As in the proof of Theorem 4.5, for any k and j we have

$$\frac{1}{b_{k,j} - b_{k,j-1}} \int_0^c \sum_{n > b_{k,j-1}/t}^{b_{k,j}/t} t f(nt)\, dt \le \frac{\alpha_{k,j} s_{k,j}}{b_{k,j} - b_{k,j-1}} \int_{b_{k,j-1}}^{b_{k,j}} f(x)\, dx, \tag{6}$$

where $\alpha_{k,j} \in [b_{k,j-1}, b_{k,j}]$ and $s_{k,j} = \sum_{n > b_{k,j-1}/c} (1/n^2)$. Since the function $\varphi(t) = (t+\delta)/(t-1)$ with $\delta > 0$ is decreasing, for any k and any j we have

$$\alpha_{k,j} s_{k,j} < \frac{b_{k,j}}{(b_{k,j-1} - 1)/c} \le c \frac{a_k + b_k/M_k}{a_k - 1} = c \frac{b_k^{1/2} + b_k/\lfloor b_k^{2/3} \rfloor}{b_k^{1/2} - 1} =: r_k,$$

which tends to c as $k \longrightarrow \infty$. Replacing $\alpha_{k,j} s_{k,j}$ by r_k and taking the average of both sides of the inequality (6) for a fixed k and $j = 1, \dots, M_k$, we get

$$\frac{1}{b_k - a_k} \int_0^c \sum_{n > a_k/t}^{b_k/t} t f(nt)\, dt < \frac{r_k}{b_k - a_k} \int_{a_k}^{b_k} f(x)\, dx,$$

so

$$\frac{1}{c} \liminf_{k \longrightarrow \infty} \frac{1}{b_k - a_k} \int_0^c \sum_{n > a_k/t}^{b_k/t} t f(nt)\, dt \le \lim_{k \longrightarrow \infty} \frac{r_k}{c(b_k - a_k)} \int_{a_k}^{b_k} f(x)\, dx = L. \qquad \square$$

5. *Multiplicative method—the multiparameter case*

5.1. *Standard Cesàro limits.* We will use the notation introduced in §3.3.

THEOREM 5.1. *Let a bounded measurable function $f : \mathbb{R}_+^d \longrightarrow V$ be such that for some $c \in \mathbb{R}_+^d$, $c > 0$, the limit $L_t = \lim_{l(b) \longrightarrow \infty} (1/w(b)) \sum_{n>0}^{b} f(nt)$ exists for almost every $t \in [0, c]$. Then L_t is almost everywhere constant, $L_t = L \in V$ almost everywhere on $[0, c]$, and $\lim_{l(b) \longrightarrow \infty} (1/w(b)) \int_0^b f(x)\, dx$ exists and equals L.*

Let $e_1 = (1, 0, \dots, 0, 0), \dots, e_d = (0, 0, \dots, 0, 1)$.

LEMMA 5.2. *Let (v_n) be an $\mathbb{N}^d$-sequence in V such that the limit*

$$v = \lim_{l(N) \longrightarrow \infty} \frac{1}{w(N)} \sum_{n \le N} v_n$$

exists and, for some $\alpha > 0$, for any $n = (n_1, \dots, n_d) \in \mathbb{N}^d$, and for any i, one has $\|v_{n+e_i} - v_n\| < \alpha n_i$. Then $\lim_{l(n) \longrightarrow \infty} v_n = v$.

Proof. We may assume that $v = 0$, that is, $\lim_{l(N) \longrightarrow \infty} (1/w(N)) \sum_{n \le N} v_n = 0$. Assume that $v_n \not\longrightarrow 0$ as $l(n) \longrightarrow \infty$; let $\varepsilon > 0$ be such that for any $N \in \mathbb{N}^d$, there exists $n > N$ such that $\|v_n\| > \varepsilon$. Put

$$\delta = \frac{\varepsilon(\varepsilon/4\alpha d)^d}{2(2^d + 1)(1 + \varepsilon/2\alpha d)^d}.$$

Find $N \in \mathbb{N}^d$ such that $\|(1/w(M))\sum_{n\le M} v_n\| < \delta$ for all $M > N$. Find $M = (M_1, \ldots, M_d) > N$, such that $\|v_M\| > \varepsilon$ and $1/M_i < \varepsilon/4\alpha d$ for all i. Let, by the Hahn–Banach theorem, $\varphi \in V^*$ be such that $|\varphi(v)| \le \|v\|$ for all $v \in V$ and $\varphi(v_M) = \|v_M\|$. Then, for any $n = (n_1, \ldots, n_d) > M$,

$$\begin{aligned}\varphi(v_n) &= \varphi(v_M) + \sum_{i=1}^{d}\sum_{m=M_i}^{n_i-1} \varphi(v_{(n_1,\ldots,n_{i-1},m+1,M_{i+1},\ldots,M_d)} - v_{(n_1,\ldots,n_{i-1},m,M_{i+1},\ldots,M_d)})\\ &\ge \|v_M\| - \sum_{m=M}^{n-1} \|v_{(n_1,\ldots,n_{i-1},m+1,M_{i+1},\ldots,M_d)} - v_{(n_1,\ldots,n_{i-1},m,M_{i+1},\ldots,M_d)}\|\\ &> \varepsilon - \sum_{i=1}^{d}(n_i - M_i)\frac{\alpha}{M_i},\end{aligned}$$

which is $\ge \varepsilon/2$ when $((n_i - M_i)\alpha)/M_i \le \varepsilon/2d$ for all i, that is, when $M_i < n_i \le M_i + \varepsilon M_i/2\alpha d$ for all i. Put $K_i = M_i + \lfloor \varepsilon M_i/2\alpha d \rfloor$ and $K = (K_1, \ldots, K_d)$; then $\varphi(v_n) > \varepsilon/2$ for $M \le n \le K$ and $w(K) \le w(M)(1 + \varepsilon/2\alpha d)^d$, $w(K - M) \ge w(M)\prod_{i=1}^{d}(\varepsilon/2\alpha d - 1/M_i) \ge w(M)(\varepsilon/4\alpha d)^d$. Now, we can represent $\sum_{n\le K} v_n$ as an alternating sum

$$\sum_{n\le K} v_n = \sum_{j=1}^{2^d-1}\Big(\pm \sum_{n\le R_j} v_n\Big) + \sum_{n>M}^{K} v_n,$$

where, for each j, for every i, the ith entry of R_j is equal to either M_i or to K_i. (For $d = 2$, for instance, the formula is $\sum_{n\le K} v_n = \sum_{n\le(M_1,K_2)} v_n + \sum_{n\le(K_1,M_2)} v_n - \sum_{n\le(M_1,M_2)} v_n + \sum_{n>M}^{K} v_n$.) For each j, $\|\sum_{n\le R_j} v_n\| < w(R_j)\delta \le w(K)\delta$; thus,

$$\begin{aligned}\Big\|\frac{1}{w(K)}\sum_{n\le K} v_n\Big\| &\ge \frac{1}{w(K)}\Big(\Big\|\sum_{n>M}^{K} v_n\Big\| - 2^d w(K)\delta\Big) \ge \frac{1}{w(K)}\,\varphi\Big(\sum_{n=M+1}^{K} v_n\Big) - 2^d\delta\\ &= \frac{1}{w(K)}\sum_{n=M+1}^{K}\varphi(v_n) - 2^d\delta > \frac{w(K-M)}{w(K)}\cdot\frac{\varepsilon}{2} - 2^d\delta\\ &\ge \frac{w(M)(\varepsilon/4\alpha d)^d}{w(M)(1+\varepsilon/2\alpha d)^d}\cdot\frac{\varepsilon}{2} - 2^d\delta = \frac{\varepsilon(\varepsilon/4\alpha d)^d}{2(1+\varepsilon/2\alpha d)^d} - 2^d\delta = \delta,\end{aligned}$$

which contradicts the choice of N. □

Proof of Theorem 5.1. Let

$$v_n = \frac{1}{w(nc)}\int_0^{nc} f(x)\,dx, \quad n \in \mathbb{N}^d.$$

Then, for any $n = (n_1, \ldots, n_d) \in \mathbb{N}^d$ and any $i \in \{1, \ldots, d\}$,

$$\begin{aligned}\|v_{n+e_i} - v_n\| &= \frac{1}{w(c)}\Big\|\Big(\frac{1}{w(n+e_i)} - \frac{1}{w(n)}\Big)\int_0^{nc} f(x)\,dx\\ &\qquad + \frac{1}{w(n+e_i)}\int_{nc}^{(n+e_i)c} f(x)\,dx\Big\|\\ &\le \frac{w(cn)\sup\|f\|}{w(c)w(n)w(n+e_i)} + \frac{w(c)\sup\|f\|}{w(c)w(n+e_i))} = 2\sup\|f\|/(n_i+1).\end{aligned}$$

Also, we have, for any $N \in \mathbb{N}^d$,

$$\frac{1}{w(N)} \sum_{n \le N} v_n = \frac{1}{w(N)} \sum_{n \le N} \frac{1}{w(nc)} \int_0^{nc} f(x)\,dx = \frac{1}{w(N)} \sum_{n \le N} \frac{1}{w(c)} \int_0^{c} f(nt)\,dt$$
$$= \frac{1}{w(c)} \int_0^{c} \frac{1}{w(N)} \sum_{n \le N} f(nt)\,dt.$$

Since $(1/w(N))\sum_{n \le N} f(nt) \longrightarrow L_t$ as $l(N) \longrightarrow \infty$ for almost every $t \in [0, c]$, by Lemma 2.2, $(1/w(N))\sum_{n \le N} v_n \longrightarrow (1/w(c))\int_0^c L_t\,dt$. By Lemma 5.2, $v_n \longrightarrow (1/w(c))\int_0^c L_t\,dt$. On the other hand,

$$\lim_{l(n) \longrightarrow \infty} v_n = \lim_{l(b) \longrightarrow \infty} \frac{1}{w(b)} \int_0^b f(x)\,dx.$$

So, $L = \lim_{l(b) \longrightarrow \infty}(1/w(b))\int_0^b f(x)\,dx$ exists and equals $(1/w(c))\int_0^c L_t\,dt$.

Next, for any $z \in (0, c)$, we also have $L = (1/w(z))\int_0^z L_t\,dt$. So, for any $z \in (0, c]$, $\int_0^z L_t\,dt = w(z)L$, which implies that $L_t = L$ almost everywhere on $[0, c]$. □

5.2. *Uniform Cesàro limits.*

THEOREM 5.3. *Let a bounded measurable function $f : \mathbb{R}_+^d \longrightarrow V$ be such that for some $c \in \mathbb{R}_+^d$, $c > 0$, for almost every $t \in (0, c]$, the limit $L_t = \lim_{b-a \longrightarrow \infty}(1/(b-a)) \sum_{n>a}^{b} f(nt)$ exists. Then L_t is constant almost everywhere on $(0, c]$, $L_t = L \in V$ for almost every $t \in (0, c]$, and $\lim_{b-a \longrightarrow \infty}(1/(b-a))\int_a^b f(x)\,dx$ exists and equals L.*

LEMMA 5.4. *Let (v_n) be a bounded $\mathbb{N}^d$-sequence in V such that*

$$\lim_{l(b-a) \longrightarrow \infty} \frac{1}{w(b-a)} \sum_{n>a}^{b} v_n = 0,$$

and let (b_k), (α_k), and (β_k) be sequences in $\mathbb{R}_+^d$ such that $0 < \beta_k - \alpha_k \le b_k$ for all k and $l(b_k) \longrightarrow \infty$. Then $\lim_{k \longrightarrow \infty}(1/w(b_k))\sum_{n>\alpha_k}^{\beta_k} v_n = 0$.

Proof. Assume that $\|v_n\| \le 1$ for all n. Let $\varepsilon > 0$. Let $B > 1$ be such that $\|(1/w(b-a)) \sum_{n>a}^{b} v_n\| < \varepsilon$ whenever $l(b-a) > B$. Let K be such that $l(b_k) > (B+1)/\varepsilon$ for all $k > K$. Then, for any $k > K$, if $l(\beta_k - \alpha_k) > B$, then

$$\left\| \frac{1}{w(b_k)} \sum_{n>\alpha_k}^{\beta_k} v_n \right\| \le \left\| \frac{1}{w(\beta_k - \alpha_k)} \sum_{n>\alpha_k}^{\beta_k} v_n \right\| < \varepsilon$$

and, if $l(\beta_k - \alpha_k) \le B$, then also

$$\left\| \frac{1}{w(b_k)} \sum_{n>\alpha_k}^{\beta_k} v_n \right\| \le \frac{l(\beta_k - \alpha_k) + 1}{l(b_k)} < \varepsilon. \qquad \square$$

Proof of Theorem 5.3. Since, in particular, $L_t = \lim_{l(b) \longrightarrow \infty}(1/w(b))\sum_{n>0}^{b} f(nt)$ for almost every $t \in (0, c]$, we have $L_t = \text{const} = L$ for almost every $t \in (0, c]$ by Theorem 5.1. Replacing f by $f - L$, we may assume that $L = 0$. After replacing $f(x)$ by $f(cx/2)$, we assume that $c = \bar{2}$. Let $a \geq 0, b \geq a + 1$. For any $n \in \mathbb{N}$, $(1/w(n))\int_a^b f(x)\,dx = \int_{a/n}^{b/n} f(nt)\,dt$. Adding these equalities for all $n \in (b/2, b]$, and taking into account that $b/n < \bar{2}$ for $n > b/2$, we get $\lambda \int_a^b f(x)\,dx = \int_0^{\bar{2}} \sum_{n>\alpha(a,b,t)}^{\beta(a,b,t)} f(nt)\,dt$, where $\lambda = \sum_{n>b/2}^{b}(1/w(n)) \geq 1/2^d$ and, for every $t \in (0, 2]^d$, $\alpha(a, b, t) = \max\{b/2, a/t\}$ and $\beta(a, b, t) = \min\{b, b/t\}$. Thus,

$$\left\| \frac{1}{w(b-a)} \int_a^b f(x)\,dx \right\| \leq 2^d \left\| \int_0^{\bar{2}} f_{a,b}(t)\,dt \right\|, \tag{7}$$

where $f_{a,b}(t) = (1/w(b-a))\sum_{n>\alpha(a,b,t)}^{\beta(a,b,t)} f(nt)$, $t \in (0, 2]^d$.

We will now show that the functions $f_{a,b}$, for $a \geq 0$, $b \geq a + \bar{1}$, are uniformly bounded. Let us assume that $\sup_{x\in(0,\infty)} \|f(x)\| \leq 1$. If $a \leq b/2$, then $b - a \geq b/2$ and, since $\beta(a, b, t) - \alpha(a, b, t) \leq b/2$, for any $t \in (0, 2]^d$, we have

$$\|f_{a,b}(t)\| \leq \frac{1}{w(b/2)} w(b/2 + \bar{1}) \leq 3^d.$$

If $a > b/2$, then, for any $t \in (0, 2]^d$ with $t_i < 1/2$ for some i, we have $\alpha(a, b, t)_i \geq a_i/t_i \geq 2a_i > b_i \geq \beta(a, b, t)_i$, so $f_{a,b}(t) = 0$; and, since, for any $t \in (0, 2]^d$, $\beta(a, b, t) - \alpha(a, b, t) \leq (b - a)/t$, we have

$$\|f_{a,b}(t)\| \leq \frac{1}{w(b-a)} w((b-a)/t + \bar{1}) \leq 3^d \quad \text{for all } t \in [1/2, 2]^d.$$

For almost every $t \in (0, 2]^d$, since $\beta(a, b, t) - \alpha(a, b, t) \leq (b - a)/t$ for all $a, b \in \mathbb{R}^d_+$, $b \geq a + \bar{1}$, by Lemma 5.4,

$$\lim_{l(b-a)\longrightarrow\infty} f_{a,b}(t) = \frac{1}{w(t)} \lim_{l(b-a)\longrightarrow\infty} \frac{1}{w((b-a)/t)} \sum_{n>\alpha(a,b,t)}^{\beta(a,b,t)} f(nt) = 0.$$

Hence, by Lemma 2.2, $\int_0^{\bar{2}} f_{a,b}(t)\,dt \longrightarrow 0$ as $l(b - a) \longrightarrow \infty$. So, by (7),

$$\frac{1}{w(b-a)} \int_a^b f(x)\,dx \longrightarrow 0 \quad \text{as } l(b-a) \longrightarrow \infty. \qquad \square$$

5.3. *Liminf and limsup versions for uniform limits.*

THEOREM 5.5. *For any bounded measurable function $f : \mathbb{R}^d_+ \longrightarrow \mathbb{R}$ and any $c \in \mathbb{R}^d_+$, $c > 0$, one has*

$$\liminf_{l(b-a)\longrightarrow\infty} \frac{1}{w(b-a)} \int_a^b f(x)\,dx \geq \frac{1}{w(c)} \int_0^c \liminf_{l(b-a)\longrightarrow\infty} \frac{1}{w(b-a)} \sum_{n>a}^{b} f(nt)\,dt$$

and

$$\limsup_{l(b-a)\longrightarrow\infty} \frac{1}{w(b-a)} \int_a^b f(x)\,dx \leq \frac{1}{w(c)} \int_0^c \limsup_{l(b-a)\longrightarrow\infty} \frac{1}{w(b-a)} \sum_{n>a}^{b} f(nt)\,dt.$$

Proof. We will only prove the first inequality. We may assume that $f \geq 0$. Let

$$L = \liminf_{l(b-a)\longrightarrow\infty} \frac{1}{w(b-a)} \int_a^b f(x)\,dx;$$

find a sequence of intervals $[a_k, b_k] \subset \mathbb{R}_+^d$ with $l(b_k - a_k) \longrightarrow \infty$ such that

$$L = \lim_{k\longrightarrow\infty} \frac{1}{w(b_k - a_k)} \int_{a_k}^{b_k} f(x)\,dx.$$

We may assume that $l(a_k) \longrightarrow \infty$ (after replacing each a_k by $\max\{a_k, \sqrt{b_k}\}$) and that $w(b_k)/w(a_k) \longrightarrow 1$ (after replacing each interval $[a_k, b_k]$ by a suitable subinterval).

For any $t > 0$, we have

$$\liminf_{l(b-a)\longrightarrow\infty} \frac{1}{w(b-a)} \sum_{n>a}^{b} f(nt) \leq \liminf_{k\longrightarrow\infty} \frac{1}{w(b_k/t - a_k/t)} \sum_{n>a_k/t}^{b_k/t} f(nt),$$

so

$$\begin{aligned}
&\frac{1}{w(c)} \int_0^c \liminf_{l(b-a)\longrightarrow\infty} \frac{1}{w(b-a)} \sum_{n>a}^{b} f(nt)\,dt \\
&\leq \frac{1}{w(c)} \int_0^c \liminf_{k\longrightarrow\infty} \frac{1}{w(b_k/t - a_k/t)} \sum_{n>a_k/t}^{b_k/t} f(nt)\,dt.
\end{aligned}$$

By (the classical) Fatou's lemma, we have

$$\begin{aligned}
&\frac{1}{w(c)} \int_0^c \liminf_{k\longrightarrow\infty} \frac{1}{w(b_k/t - a_k/t)} \sum_{n>a_k/t}^{b_k/t} f(nt)\,dt \\
&\leq \frac{1}{w(c)} \liminf_{k\longrightarrow\infty} \int_0^c \frac{1}{w(b_k/t - a_k/t)} \sum_{n>a_k/t}^{b_k/t} f(nt)\,dt
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{w(c)} \liminf_{k\longrightarrow\infty} \int_0^c \frac{1}{w(b_k/t - a_k/t)} \sum_{n>a_k/t}^{b_k/t} f(nt)\,dt \\
&= \frac{1}{w(c)} \liminf_{k\longrightarrow\infty} \frac{1}{w(b_k - a_k)} \int_0^c \sum_{n>a_k/t}^{b_k/t} t f(nt)\,dt \\
&\leq \frac{1}{w(c)} \liminf_{k\longrightarrow\infty} \frac{1}{w(b_k - a_k)} \sum_{n>a_k/c} \int_{a_k/n}^{b_k/n} t f(nt)\,dt.
\end{aligned}$$

For every $k \in \mathbb{N}$, for each n we have

$$I_n = \int_{a_k/n}^{b_k/n} t f(nt)\,dt = \frac{1}{w(n^2)} \int_{a_k}^{b_k} x f(x)\,dx = \frac{1}{w(n^2)} w(\alpha_k) \int_{a_k}^{b_k} f(x)\,dx$$

with $\alpha_k \in [a_k, b_k]$, so

$$\sum_{n>a_k/c} I_n = w(\alpha_k) \int_{a_k}^{b_k} f(x)\,dx \sum_{n>a_k/c} \frac{1}{w(n^2)} = w(\alpha_k) s_k \int_{a_k}^{b_k} f(x)\,dx,$$

where $s_k = \sum_{n>a_k/c}(1/w(n^2))$ satisfies $s_k w(a_k/c) \longrightarrow 1$ as $k \longrightarrow \infty$. Since, by our assumption, also $w(\alpha_k)/w(a_k) \longrightarrow 1$, we get

$$\begin{aligned}
&\frac{1}{w(c)} \lim_{k\longrightarrow\infty} \frac{1}{w(b_k - a_k)} \sum_{n>a_k/c} \int_{a_k/n}^{b_k/n} tf(nt)\,dt \\
&= \lim_{k\longrightarrow\infty} \frac{w(\alpha_k)s_k}{w(c)} \cdot \frac{1}{w(b_k - a_k)} \int_{a_k}^{b_k} f(x)\,dx = L.
\end{aligned}$$

So,

$$\frac{1}{w(c)} \int_0^c \liminf_{l(b-a)\longrightarrow\infty} \frac{1}{w(b-a)} \sum_{n>a}^{b} f(nt)\,dt \le L. \qquad \square$$

5.4. *Liminf and limsup versions for standard averages.*

THEOREM 5.6. *For any bounded measurable function* $f : \mathbb{R}_+^d \longrightarrow \mathbb{R}$ *and any* $c \in \mathbb{R}_+^d$, $c > 0$, *one has*

$$\liminf_{l(b)\longrightarrow\infty} \frac{1}{w(b)} \int_0^b f(x)\,dx \ge \frac{1}{w(c)} \int_0^c \liminf_{l(b)\longrightarrow\infty} \frac{1}{w(b)} \sum_{n>0}^{b} f(nt)\,dt$$

and

$$\limsup_{l(b)\longrightarrow\infty} \frac{1}{w(b)} \int_0^b f(x)\,dx \le \frac{1}{w(c)} \int_0^c \limsup_{l(b)\longrightarrow\infty} \frac{1}{w(b)} \sum_{n>0}^{b} f(nt)\,dt.$$

Proof. We may assume that $f \ge 0$. Let $L = \liminf_{b\longrightarrow\infty}(1/b)\int_0^b f(x)\,dx$; choose a sequence (b_k) in $\mathbb{R}_+^d$, with $l(b_k) \longrightarrow \infty$ as $k \longrightarrow \infty$, such that

$$\lim_{k\longrightarrow\infty} \frac{1}{w(b_k)} \int_0^{b_k} f(x)\,dx = L.$$

Then also $\lim_{k\longrightarrow\infty}(1/w(b_k - a_k))\int_{a_k}^{b_k} f(x)\,dx = L$, where $a_k = \sqrt{b_k}$, $k \in \mathbb{N}$. For all $t > 0$, we have

$$\begin{aligned}
\liminf_{l(b)\longrightarrow\infty} \frac{1}{w(b)} \sum_{n>0}^{b} f(nt) &= \liminf_{l(b)\longrightarrow\infty} \frac{1}{w(b-\sqrt{b})} \sum_{n>\sqrt{b}}^{b} f(nt) \\
&\le \liminf_{k\longrightarrow\infty} \frac{1}{w(b_k - a_k)} \sum_{n>a_k}^{b_k} f(nt)
\end{aligned}$$

and, as in the proof of Theorem 5.5, by Fatou's lemma,

$$\begin{aligned}
&\frac{1}{w(c)} \int_0^c \liminf_{k\longrightarrow\infty} \frac{1}{w(b_k - a_k)} \sum_{n>a_k}^{b_k} f(nt)\,dt \\
&\le \frac{1}{w(c)} \liminf_{k\longrightarrow\infty} \frac{1}{w(b_k - a_k)} \int_0^c \sum_{n>a_k/t}^{b_k/t} tf(nt)\,dt,
\end{aligned}$$

so it suffices to show that this last expression is $\le L$.

For every $k \in \mathbb{N}$, put $M_k = \lfloor b_k^{2/3} \rfloor$ and subdivide the interval $[a_k, b_k]$ into $w(M_k)$ equal parts: put $b_{k,j} = a_k + j(b_k - a_k)/M_k$, $j \in (\{0\} \cup \mathbb{N})^d \cap [0, M_k]$. As in the proof of Theorem 5.5, for any k and j we have

$$\frac{1}{w(b_{k,j} - b_{k,j-1})} \int_0^c \sum_{n > b_{k,j-1}/t}^{b_{k,j}/t} t f(nt)\, dt \le \frac{w(\alpha_{k,j}) s_{k,j}}{w(b_{k,j} - b_{k,j-1})} \int_{b_{k,j-1}}^{b_{k,j}} f(x)\, dx, \tag{8}$$

where $\alpha_{k,j} \in [b_{k,j-1}, b_{k,j}]$ and $s_{k,j} = \sum_{n > b_{k,j-1}/c} (1/w(n^2))$. For any k and any j,

$$\begin{aligned} w(\alpha_{k,j}) s_{k,j} < \frac{w(b_{k,j})}{w(b_{k,j-1} - \bar{1})/w(c)} &\le w(c) \frac{w(a_k + b_k/M_k)}{w(a_k - \bar{1})} \\ &= w(c) \frac{w(b_k^{1/2} + b_k/\lfloor b_k^{2/3} \rfloor)}{w(b_k^{1/2} - \bar{1})} =: r_k, \end{aligned}$$

which tends to $w(c)$ as $k \longrightarrow \infty$. Replacing $w(\alpha_{k,j}) s_{k,j}$ by r_k and taking the average of both sides of the inequality (8) for a fixed k and $j = 1, \dots, M_k$, we get

$$\frac{1}{w(b_k - a_k)} \int_0^c \sum_{n > a_k/t}^{b_k/t} t f(nt)\, dt < \frac{r_k}{w(b_k - a_k)} \int_{a_k}^{b_k} f(x)\, dx,$$

so

$$\begin{aligned} &\frac{1}{w(c)} \liminf_{k \longrightarrow \infty} \frac{1}{w(b_k - a_k)} \int_0^c \sum_{n > a_k/t}^{b_k/t} t f(nt)\, dt \\ &\quad \le \lim_{k \longrightarrow \infty} \frac{r_k}{w(c) w(b_k - a_k)} \int_{a_k}^{b_k} f(x)\, dx = L. \end{aligned}$$

□

5.5. *A lim-limsup version for standard averages.* It turns out that if the limits $\lim_{l(b) \longrightarrow \infty} (1/w(b)) \sum_{n>0}^{b} f(nt)$, $t > 0$, do not exist, but, for some $L \in V$, one has $\lim_{t \longrightarrow 0^+} \limsup_{l(b) \longrightarrow \infty} \|(1/w(b)) \sum_{n>0}^{b} f(nt) - L\| = 0$, we still have the result. For a function $h : (0, r)^d \longrightarrow V$, $r > 0$, we write ess-$\lim_{t \longrightarrow 0^+} h(t) = h_0$ if, for any $\varepsilon > 0$, there exists $\delta \in \mathbb{R}_+^d$ such that $\|h(t) - h_0\| < \varepsilon$ for almost every $t \in (0, \delta]$.

THEOREM 5.7. *Let $f : \mathbb{R}_+^d \longrightarrow V$ be a bounded measurable function satisfying, for some $L \in V$,*

$$\operatorname*{ess\text{-}lim}_{t \longrightarrow 0^+} \limsup_{l(b) \longrightarrow \infty} \left\| \frac{1}{w(b)} \sum_{n>0}^{b} f(nt) - L \right\| = 0.$$

Then $\lim_{l(b) \longrightarrow \infty} (1/w(b)) \int_0^b f(x)\, dx = L$.

LEMMA 5.8. *Let (v_n) be a bounded $\mathbb{N}^d$-sequence in V and let (b_k), (β_k) be sequences in $\mathbb{R}_+^d$ with $0 < \beta_k \le b_k$ for all k and $l(b_k) \longrightarrow \infty$. Then*

$$\limsup_{k \longrightarrow \infty} \left\| \frac{1}{w(b_k)} \sum_{n>0}^{\beta_k} v_n \right\| \le \limsup_{l(b) \longrightarrow \infty} \left\| \frac{1}{w(b)} \sum_{n>0}^{b} v_n \right\|$$

and

$$\limsup_{k\longrightarrow\infty}\left\|\frac{1}{w(b_k)}\sum_{n>b_k/2}^{\beta_k} v_n\right\| \le 2^d \limsup_{l(b)\longrightarrow\infty}\left\|\frac{1}{w(b)}\sum_{n>0}^{b} v_n\right\|.$$

Proof. Assume that $\|v_n\|\le 1$ for all n. Let

$$s=\limsup_{l(b)\longrightarrow\infty}\left\|\frac{1}{w(b)}\sum_{n>0}^{b} v_n\right\|.$$

Let $\varepsilon>0$ and let $B>1$ be such that $\|(1/w(b))\sum_{n>0}^{b} v_n\|<s+\varepsilon$ whenever $l(b)>B$. Let K be such that $l(b_k)>B/\varepsilon$ for all $k>K$. Then, for any $k>K$, if $l(\beta_k)>B$, then

$$\left\|\frac{1}{w(b_k)}\sum_{n>0}^{\beta_k} v_n\right\| \le \left\|\frac{1}{w(\beta_k)}\sum_{n>0}^{\beta_k} v_n\right\| < s+\varepsilon$$

and, if $l(\beta_k)\le B$, then also

$$\left\|\frac{1}{w(b_k)}\sum_{n>0}^{\beta_k} v_n\right\| \le \frac{w(\beta_k)}{w(b_k)} \le \frac{l(\beta_k)}{l(b_k)} < \varepsilon \le s+\varepsilon.$$

So, $\limsup_{k\longrightarrow\infty}\|(1/w(b_k))\sum_{n>0}^{\beta_k} v_n\|\le s$.

For any $S\subseteq\{1,\ldots,d\}$ and $a=(a_1,\ldots,a_d)$, $b=(b_1,\ldots,b_d)\in\mathbb{R}^d$, let $\sigma_S(a,b)=(c_1,\ldots,c_d)$, where, for each i, $c_i=a_i$ if $i\in S$ and $c_i=b_i$ if $i\notin S$. Then, for any $a,b\in\mathbb{R}_+^d$ with $a\le b$, we have

$$\sum_{n>a}^{b} v_n=\sum_{S\subseteq\{1,\ldots,d\}}(-1)^{|S|}\sum_{n>0}^{\sigma_S(a,b)} v_n.$$

Since, for any $S\subseteq\{1,\ldots,d\}$,

$$\limsup_{k\longrightarrow\infty}\left\|\frac{1}{w(b_k)}\sum_{n>0}^{\sigma_S(b_k/2,\beta_k)} v_n\right\|\le s,$$

we also get that

$$\limsup_{k\longrightarrow\infty}\left\|\frac{1}{w(b_k)}\sum_{n>b_k/2}^{\beta_k} v_n\right\|\le 2^d s. \qquad\square$$

Proof of Theorem 5.7. We may assume that $L=0$. Fix $\varepsilon>0$. Find $\delta\in\mathbb{R}_+^d$, $\delta>0$, such that $\limsup_{b\longrightarrow\infty}\|(1/b)\sum_{n>0}^{b} f(nt)\|<\varepsilon$ for almost every $t\in[0,\delta]$, and define $g(x)=f((\delta/2)x)$, $x\in\mathbb{R}_+^d$. Then $\limsup_{b\longrightarrow\infty}\|(1/b)\sum_{n>0}^{b} g(nt)\|<\varepsilon$ for almost every $t\in[0,2]^d$.

Let $b\in\mathbb{R}_+^d$, $b\ge\bar{1}$. For any $n\in\mathbb{N}^d$, $(1/w(n))\int_0^b g(x)\,dx=\int_0^{b/n} g(nt)\,dt$. Adding these equalities for all $n\in(b/2,b]$, and taking into account that $b/n<\bar{2}$ for $n>b/2$, we get $\lambda\int_0^b g(x)\,dx=\int_0^{\bar{2}}\sum_{n>b/2}^{\beta(b,t)} g(nt)\,dt$, where $\lambda=\sum_{n>b/2}^{b}(1/w(n))\ge 1/2^d$, and, for every $t\in(0,2]^d$, $\beta(b,t)=\min\{b,b/t\}$. Thus,

$$\left\|\frac{1}{w(b)}\int_0^b g(x)\,dx\right\|\le 2^d\left\|\int_0^{\bar{2}} g_b(t)\,dt\right\|, \tag{9}$$

where $g_b(t)=(1/w(b))\sum_{n>b/2}^{\beta(b,t)} g(nt)$, $t\in(0,2]^d$.

Let us assume that $\sup_{x\in\mathbb{R}_+^d}\|g(x)\| = \sup_{x\in\mathbb{R}_+^d}\|f(x)\| \le 1$. Then, for any $b \ge \bar{1}$, for every $t \in (0,2]^d$, we have $\|g_b(t)\| \le (1/w(b))w(\beta(b,t)) \le 1$, so the functions g_b are uniformly bounded. For almost every $t \in (0,2]^d$, since $\beta(b,t) \le b$, by Lemma 5.8, we have $\limsup_{l(b)\longrightarrow\infty}\|g_b(t)\| \le 2^d\varepsilon$. Hence, by Lemma 2.1, $\limsup_{b\longrightarrow\infty}\|\int_0^{\bar{2}} g_b(t)\,dt\| \le 2^{2d}\varepsilon$. So, by (9), $\limsup_{b\longrightarrow\infty}\|(1/b)\int_0^b g(x)\,dx\| \le 2^{3d}\varepsilon$. Since, for any $b \in \mathbb{R}_+$, $b > 0$, we have

$$\frac{1}{w(b)}\int_0^b g(x)\,dx = \frac{1}{w(b\delta/2)}\int_0^{b\delta/2} f(x)\,dx,$$

we get $\limsup_{l(b)\longrightarrow\infty}\|(1/w(b))\int_0^b f(x)\,dx\| \le 2^{3d}\varepsilon$. Since this is true for any positive ε, $\lim_{l(b)\longrightarrow\infty}(1/w(b))\int_0^b f(x)\,dx = 0$. □

5.6. *A lim-limsup version for uniform averages.*

THEOREM 5.9. *Let $f : \mathbb{R}_+^d \longrightarrow V$ be a bounded measurable function satisfying, for some $L \in V$,*

$$\operatorname*{ess\text{-}lim}_{t\longrightarrow 0^+}\ \limsup_{l(b-a)\longrightarrow\infty}\left\|\frac{1}{w(b-a)}\sum_{n>a}^{b} f(nt) - L\right\| = 0.$$

Then $\lim_{l(b-a)\longrightarrow\infty}(1/w(b-a))\int_a^b f(x)\,dx = L$.

LEMMA 5.10. *Let (v_n) be a bounded $\mathbb{N}^d$-sequence in V and let (b_k), (α_k), and (β_k) be sequences in $\mathbb{R}_+^d$ such that $0 < \beta_k - \alpha_k \le b_k$ for all k and $l(b_k) \longrightarrow \infty$. Then*

$$\limsup_{k\longrightarrow\infty}\left\|\frac{1}{w(b_k)}\sum_{n>\alpha_k}^{\beta_k} v_n\right\| \le \limsup_{l(b-a)\longrightarrow\infty}\left\|\frac{1}{w(b-a)}\sum_{n>a}^{b} v_n\right\|.$$

Proof. Assume that $\|v_n\| \le 1$ for all n. Let

$$s = \limsup_{l(b-a)\longrightarrow\infty}\left\|\frac{1}{w(b-a)}\sum_{n>a}^{b} v_n\right\|.$$

Let $\varepsilon > 0$. Let $B > 1$ be such that

$$\left\|\frac{1}{w(b-a)}\sum_{n>a}^{b} v_n\right\| < s + \varepsilon$$

whenever $l(b-a) > B$. Let K be such that $l(b_k) > (B+1)/\varepsilon$ for all $k > K$. Then, for any $k > K$, if $l(\beta_k - \alpha_k) > B$, then

$$\left\|\frac{1}{w(b_k)}\sum_{n>\alpha_k}^{\beta_k} v_n\right\| \le \left\|\frac{1}{w(\beta_k-\alpha_k)}\sum_{n>\alpha_k}^{\beta_k} v_n\right\| < s + \varepsilon,$$

and, if $l(\beta_k - \alpha_k) \le B$, then also

$$\left\|\frac{1}{w(b_k)}\sum_{n>\alpha_k}^{\beta_k} v_n\right\| \le \frac{l(\beta_k-\alpha_k)+1}{l(b_k)} < \varepsilon \le s + \varepsilon.$$

So, $\limsup_{k\longrightarrow\infty}\|(1/w(b_k))\sum_{n>\alpha_k}^{\beta_k} v_n\| \le s$. □

Proof of Theorem 5.9. We may assume that $L = 0$. Fix $\varepsilon > 0$. Find $\delta \in \mathbb{R}_+^d$, $\delta > 0$, such that

$$\limsup_{l(b-a)\longrightarrow\infty} \left\| \frac{1}{w(b-a)} \sum_{n>a}^{b} f(nt) \right\| < \varepsilon \quad \text{for a.e. } t \in [0, \delta]$$

and define $g(x) = f((\delta/2)x)$, $x \in \mathbb{R}_+^d$. Then

$$\limsup_{l(b-a)\longrightarrow\infty} \left\| \frac{1}{w(b-a)} \sum_{n>a}^{b} g(nt) \right\| < \varepsilon \quad \text{for a.e. } t \in [0, 2]^d.$$

Let $a, b \in \mathbb{R}_+^d$, $a \geq 0$, $b \geq a + \bar{1}$. For any $n \in \mathbb{N}^d$,

$$\frac{1}{w(n)} \int_a^b g(x)\, dx = \int_{a/n}^{b/n} g(nt)\, dt.$$

Adding these equalities for all $n \in (b/2, b]$, and taking into account that $b/n < \bar{2}$ for $n > b/2$, we get

$$\lambda \int_a^b g(x)\, dx = \int_0^{\bar{2}} \sum_{n>\alpha(a,b,t)}^{\beta(a,b,t)} g(nt)\, dt,$$

where $\lambda = \sum_{n>b/2}^{b} (1/w(n)) \geq 1/2^d$, and for, every $t \in (0, 2]^d$, $\alpha(a, b, t) = \max\{b/2, a/t\}$ and $\beta(a, b, t) = \min\{b, b/t\}$. Thus,

$$\left\| \frac{1}{w(b-a)} \int_a^b g(x)\, dx \right\| \leq 2^d \left\| \int_0^{\bar{2}} g_{a,b}(t)\, dt \right\|, \tag{10}$$

where $g_{a,b}(t) = (1/w(b-a)) \sum_{n>\alpha(a,b,t)}^{\beta(a,b,t)} g(nt)$, $t \in (0, 2]^d$.

We will now show that the functions $g_{a,b}$, for $a, b \in \mathbb{R}_+^d$, $b \geq a + 1$, are uniformly bounded. Let us assume that $\sup_{x\in\mathbb{R}_+^d} \|g(x)\| = \sup_{x\in\mathbb{R}_+^d} \|f(x)\| \leq 1$. If $a \leq b/2$, then $b - a \geq b/2$ and, since $\beta(a, b, t) - \alpha(a, b, t) \leq b/2$, for any $t \in (0, 2]^d$ we have

$$\|g_{a,b}(t)\| \leq \frac{1}{w(b/2)} w(b/2 + \bar{1}) \leq 3^d.$$

If $a > b/2$, then, for any $t = (t_1, \ldots, t_d) \in (0, 2]^d$ with $t_i < 1/2$ for some i, we have $\alpha(a, b, t)_i \geq a_i/t_i \geq 2a_i > b_i \geq \beta(a, b, t)_i$, so $f_{a,b}(t) = 0$; and, since, for any $t \in (0, 2]^d$, $\beta(a, b, t) - \alpha(a, b, t) \leq (b - a)/t$, we have

$$\|f_{a,b}(t)\| \leq \frac{1}{w(b-a)} w((b-a)/t + \bar{1}) \leq 3^d \quad \text{for all } t \in [1/2, 2]^d.$$

For almost every $t \in (0, 2]^d$, since

$$\beta(a, b, t) - \alpha(a, b, t) \leq (b - a)/t \quad \text{for all } a, b \in \mathbb{R}_+^d,\ b \geq a + \bar{1},$$

by Lemma 5.10,

$$\limsup_{l(b-a)\longrightarrow\infty} \|g_{a,b}(t)\| \leq \frac{1}{w(t)} \limsup_{l(b-a)\longrightarrow\infty} \left\| \frac{1}{w((b-a)/t)} \sum_{n>\alpha(a,b,t)}^{\beta(a,b,t)} g(nt) \right\| < \frac{\varepsilon}{w(t)}.$$

Since also $\limsup_{l(b-a)\longrightarrow\infty} \|g_{a,b}(t)\| \le 3^d$, we obtain that

$$\int_0^{\bar{2}} \limsup_{l(b-a)\longrightarrow\infty} \|g_{a,b}(t)\|\, dt \le \int_{[0,2]^d\setminus[\varepsilon,2]^d} 3^d\, dt + \int_{[\varepsilon,2]^d} \frac{\varepsilon}{w(t)}\, dt = c_\varepsilon,$$

where $c_\varepsilon \le d\varepsilon 2^{d-1}3^d + \varepsilon(\log(2/\varepsilon))^d$. Hence, by Lemma 2.1,

$$\limsup_{l(b-a)\longrightarrow\infty} \left\| \int_0^{\bar{2}} g_{a,b}(t)\, dt \right\| \le c_\varepsilon.$$

So, by (10),

$$\limsup_{l(b-a)\longrightarrow\infty} \left\| \frac{1}{w(b-a)} \int_a^b g(x)\, dx \right\| \le 2^d c_\varepsilon.$$

Since, for any $0 < a < b$,

$$\frac{1}{w(b-a)} \int_a^b g(x)\, dx = \frac{1}{w(b\delta/2 - a\delta/2)} \int_{a\delta/2}^{b\delta/2} f(x)\, dx,$$

we get

$$\limsup_{l(b-a)\longrightarrow\infty} \left\| \frac{1}{w(b-a)} \int_a^b f(x)\, dx \right\| \le 2^d c_\varepsilon.$$

Since this is true for any positive ε and $c_\varepsilon \longrightarrow 0$ as $\varepsilon \longrightarrow 0^+$, we obtain that $\lim_{l(b-a)\longrightarrow\infty}(1/w(b-a))\int_a^b f(x)\, dx = 0$. □

6. *Two-sided limits and limits with respect to Følner sequences*

6.1. *Two-sided multiparameter limits.* We will now pass from the $(\mathbb{N}^d, \mathbb{R}^d_+)$ setup to the $(\mathbb{Z}^d, \mathbb{R}^d)$ setup. We adapt the notation introduced above to this new situation: for $a, b \in \mathbb{R}^d$, $a = (a_1, \ldots, a_d)$, $b = (b_1, \ldots, b_d)$, we write $a \le b$ if $a_i \le b_i$ for all $i = 1, \ldots, d$, and $a < b$ if $a_i < b_i$ for all i. When writing $l(b)$ or $w(b)$, we will always assume that $b > 0$. As before, under $\sum_{n>a}^b v_n$ we understand $\sum_{\substack{n\in\mathbb{Z}^d\\ a<n\le b}} v_n$, and under $\int_a^b v(x)\, dx$ we understand $\int_{a\le x\le b} v(x)\, dx$.

Theorem 3.3 clearly implies the following.

THEOREM 6.1. *Let $f : \mathbb{R}^d \longrightarrow V$ be a bounded measurable function such that the limit*

$$A_t = \lim_{l(b)\longrightarrow\infty} \frac{1}{w(2b)} \sum_{n>-b}^{b} f(t+n)$$

exists for almost every $t \in [0, 1]^d$. Then $\lim_{l(b)\longrightarrow\infty}(1/w(2b))\int_{-b}^b f(x)\, dx$ also exists and is equal to $\int_{[0,1]^d} A_t\, dt$.

Theorem 3.4 can also be easily adapted to the $\mathbb{R}^d$ case.

THEOREM 6.2. *Let $f : \mathbb{R}^d \longrightarrow V$ be a bounded measurable function such that the limit*

$$A_t = \lim_{l(b-a)\longrightarrow\infty} \frac{1}{w(b-a)} \sum_{n>a}^{b} f(t+n)$$

exists for almost every $t \in [0, 1]^d$. Then $\lim_{l(b-a)\longrightarrow\infty}(1/w(b-a))\int_a^b f(x)\, dx$ also exists and is equal to $\int_{[0,1]^d} A_t\, dt$.

The derivation of Theorem 6.2 from Theorem 3.4 is based on the following fact.

LEMMA 6.3. *For any* $s = (s_1, \dots, s_d) \in \{+, -\}^d = S$*, let* $\mathbb{R}^d_s = \mathbb{R}_{s_1} \times \cdots \times \mathbb{R}_{s_d}$*. Let* $f : \mathbb{R}^d \longrightarrow V$ *be a bounded function and let* L *be an element of* V *such that, for any* $s \in S$*,*

$$\lim_{\substack{a,b \in \mathbb{R}_s \\ l(b-a) \longrightarrow \infty}} \frac{1}{w(b-a)} \int_a^b f(x)\, dx = L.$$

Then

$$\lim_{l(b-a) \longrightarrow \infty} \frac{1}{w(b-a)} \int_a^b f(x)\, dx = L.$$

Proof. We will assume that $\sup |f| \le 1$ and that $L = 0$. Given $\varepsilon > 0$, find $l \in \mathbb{R}$ such that $\|(1/w(b-a)) \int_a^b f(x)\, dx\| < \varepsilon$ whenever $a, b \in \mathbb{R}^d_s$ for some $s \in S$ and $l(b-a) \ge l$.

Now let $a, b \in \mathbb{R}^d$, $b > a$, and $l(b-a) > l/\varepsilon$. Let $a = (a_1, \dots, a_d)$ and $b = (b_1, \dots, b_d)$. For each i such that $a_i < 0 < b_i$, partition the interval $[a_i, b_i]$ into subintervals $[a_i, 0]$ and $[0, b_i]$, and thus partition the d-dimensional interval $[a, b] = \{x : a \le x \le b\}$ into $\le 2^d$ d-dimensional subintervals $[p_j, q_j]$ such that, for each j, $[p_j, q_j] \subseteq \mathbb{R}^d_s$ for some $s \in S$. Then, for each j, if $l(q_j - p_j) \ge l$, then

$$\left\| \frac{1}{w(b-a)} \int_{p_j}^{q_j} f(x)\, dx \right\| \le \left\| \frac{1}{w(q_j - p_j)} \int_{p_j}^{q_j} f(x)\, dx \right\| < \varepsilon$$

and, if $l(q_j - p_j) < l$, then

$$\left\| \frac{1}{w(b-a)} \int_{p_j}^{q_j} f(x)\, dx \right\| \le \frac{w(q_j - p_j)}{w(b-a)} \le \frac{l(q_j - p_j)}{l(b-a)} < \frac{l}{l/\varepsilon} = \varepsilon;$$

so,

$$\left\| \frac{1}{w(b-a)} \int_a^b f(x)\, dx \right\| = \sum_j \left\| \frac{1}{w(b-a)} \int_{p_j}^{q_j} f(x)\, dx \right\| < 2^d \varepsilon. \qquad \square$$

In the case where f is a real-valued function, the same proof gives a stronger result.

LEMMA 6.4. *For any bounded measurable function* $f : \mathbb{R}^d \longrightarrow \mathbb{R}$*,*

$$\liminf_{l(b-a) \longrightarrow \infty} \frac{1}{w(b-a)} \int_a^b f(x)\, dx = \min_{s \in S} \left\{ \liminf_{\substack{a,b \in \mathbb{R}_s \\ l(b-a) \longrightarrow \infty}} \frac{1}{w(b-a)} \int_a^b f(x)\, dx \right\}$$

and

$$\limsup_{l(b-a) \longrightarrow \infty} \frac{1}{w(b-a)} \int_a^b f(x)\, dx = \max_{s \in S} \left\{ \limsup_{\substack{a,b \in \mathbb{R}_s \\ l(b-a) \longrightarrow \infty}} \frac{1}{w(b-a)} \int_a^b f(x)\, dx \right\}.$$

Lemma 6.4 allows us to derive the 'two-sided' version of Theorems 3.5 and 3.6.

THEOREM 6.5. *If* $f : \mathbb{R}^d \longrightarrow \mathbb{R}$ *is a bounded measurable function, then*

$$\liminf_{l(b) \longrightarrow \infty} \frac{1}{w(2b)} \int_{-b}^b f(x)\, dx \ge \int_{[0,1]^d} \liminf_{l(b) \longrightarrow \infty} \frac{1}{w(2b)} \sum_{n > -b}^{b} f(t+n)\, dt$$

and

$$\limsup_{l(b-a)\longrightarrow\infty} \frac{1}{w(2b)} \int_{-b}^{b} f(x)\,dx \le \int_{[0,1]^d} \limsup_{l(b)\longrightarrow\infty} \frac{1}{w(2b)} \sum_{n>-b}^{b} f(t+n)\,dt.$$

THEOREM 6.6. *If $f:\mathbb{R}^d \longrightarrow \mathbb{R}$ is a bounded measurable function, then*

$$\liminf_{l(b-a)\longrightarrow\infty} \frac{1}{w(b-a)} \int_{a}^{b} f(x)\,dx \ge \int_{[0,1]^d} \liminf_{l(b-a)\longrightarrow\infty} \frac{1}{w(b-a)} \sum_{n>a}^{b} f(t+n)\,dt$$

and

$$\limsup_{l(b-a)\longrightarrow\infty} \frac{1}{w(b-a)} \int_{a}^{b} f(x)\,dx \le \int_{[0,1]^d} \limsup_{l(b-a)\longrightarrow\infty} \frac{1}{w(b-a)} \sum_{n>a}^{b} f(t+n)\,dt.$$

For $a, b \in \mathbb{R}^d$ with $a < 0 < b$, let us call 'the interval' $(a, b) = \{t \in \mathbb{R}^d : a < t < b\}$ *a P-neighborhood of* 0 *in* $\mathbb{R}^d$ and, for $b \in \mathbb{R}^d_+$ with $b > 0$, let us call 'the interval' $[0, b) = \{t \in \mathbb{R}^d_+ : t < b\}$ *a P-neighborhood of* 0 *in* $\mathbb{R}^d_+$.

The 'multiplicative' theorems for $\mathbb{Z}^d$ and $\mathbb{R}^d$-actions take the following form.

THEOREM 6.7. *Let a bounded measurable function $f:\mathbb{R}^d \longrightarrow V$ be such that for some P-neighborhood of* 0 *in $\mathbb{R}^d$, for almost every $t \in P$, the limit*

$$L_t = \lim_{l(b)\longrightarrow\infty} \frac{1}{w(2b)} \sum_{n>-b}^{b} f(nt)$$

exists. Then $L_t = \mathrm{const} = L$ almost everywhere on P and $\lim_{l(b)\longrightarrow\infty}(1/w(2b)) \int_{-b}^{b} f(x)\,dx = L$.

THEOREM 6.8. *Let a bounded measurable function $f:\mathbb{R}^d \longrightarrow V$ be such that for some P-neighborhood of* 0 *in $\mathbb{R}^d$, for almost every $t \in P$, the limit*

$$L_t = \lim_{l(b-a)\longrightarrow\infty} \frac{1}{w(b-a)} \sum_{n>a}^{b} f(nt)$$

exists. Then $L_t = \mathrm{const} = L$ almost everywhere on P and

$$\lim_{l(b-a)\longrightarrow\infty} \frac{1}{w(b-a)} \int_{a}^{b} f(x)\,dx = L.$$

THEOREM 6.9. *For any bounded measurable function $f:\mathbb{R}^d \longrightarrow \mathbb{R}$ and any $c \in \mathbb{R}^d_+$, $c > 0$,*

$$\liminf_{l(b)\longrightarrow\infty} \frac{1}{w(2b)} \int_{-b}^{b} f(x)\,dx \ge \frac{1}{w(c)} \int_{0}^{c} \liminf_{l(b)\longrightarrow\infty} \frac{1}{w(2b)} \sum_{n>-b}^{b} f(nt)\,dt$$

and

$$\limsup_{l(b)\longrightarrow\infty} \frac{1}{w(2b)} \int_{-b}^{b} f(x)\,dx \le \frac{1}{w(c)} \int_{0}^{c} \limsup_{l(b)\longrightarrow\infty} \frac{1}{w(2b)} \sum_{n>-b}^{b} f(nt)\,dt.$$

THEOREM 6.10. *For any bounded measurable function* $f : \mathbb{R}^d \longrightarrow \mathbb{R}$ *and any* $c \in \mathbb{R}^d_+$, $c > 0$,

$$\liminf_{l(b-a)\longrightarrow\infty} \frac{1}{w(b-a)} \int_a^b f(x)\,dx \geq \frac{1}{w(c)} \int_0^c \liminf_{l(b-a)\longrightarrow\infty} \frac{1}{w(b-a)} \sum_{n>a}^{b} f(nt)\,dt$$

and

$$\limsup_{l(b-a)\longrightarrow\infty} \frac{1}{w(b-a)} \int_a^b f(x)\,dx \leq \frac{1}{w(c)} \int_0^c \limsup_{l(b-a)\longrightarrow\infty} \frac{1}{w(b-a)} \sum_{n>a}^{b} f(nt)\,dt.$$

THEOREM 6.11. *Let* $f : \mathbb{R}^d \longrightarrow V$ *be a bounded measurable function satisfying*

$$\operatorname*{ess\text{-}lim}_{t\longrightarrow 0^+} \limsup_{l(b)\longrightarrow\infty} \left\| \frac{1}{w(2b)} \sum_{n>-b}^{b} f(nt) \right\| = 0.$$

Then $\lim_{l(b)\longrightarrow\infty}(1/w(2b))\int_{-b}^{b} f(x)\,dx = 0$.

THEOREM 6.12. *Let* $f : \mathbb{R}^d \longrightarrow V$ *be a bounded measurable function satisfying, for some* $L \in V$,

$$\operatorname*{ess\text{-}lim}_{t\longrightarrow 0^+} \limsup_{l(b-a)\longrightarrow\infty} \left\| \frac{1}{w(b-a)} \sum_{n>a}^{b} f(nt) - L \right\| = 0.$$

Then $\lim_{l(b-a)\longrightarrow\infty}(1/w(b-a))\int_a^b f(x)\,dx = L$.

6.2. *Limits with respect to an arbitrary Følner sequence.* Let us denote by w the standard Lebesgue measure on $\mathbb{R}^d$ (this agrees with the notation used in the previous sections). *A Følner sequence in* $\mathbb{R}^d$ is a sequence $(\Phi_N)_{N=1}^{\infty}$ of subsets of finite measure such that, for any $y \in \mathbb{R}^d$, $w(\Phi_N \triangle (\Phi_N + y))/w(\Phi_N) \longrightarrow 0$ as $N \longrightarrow \infty$.

LEMMA 6.13. *Let* $f : \mathbb{R}^d \longrightarrow V$ *be a bounded measurable function with the property that*

$$\lim_{l(b-a)\longrightarrow\infty} \frac{1}{w(b-a)} \int_a^b f(x)\,dx = L \in V.$$

Then, for any Følner sequence (Φ_N) *in* $\mathbb{R}^d$,

$$\lim_{N\longrightarrow\infty} \frac{1}{w(\Phi_N)} \int_{\Phi_N} f(x)\,dx = L.$$

Proof. We will assume that $L = 0$ and that $\sup|f| \leq 1$. Let $\varepsilon > 0$, and let Q be a d-dimensional interval $\{x \in \mathbb{R}^d : 0 \leq x \leq c\}$ with $l(c)$ large enough so that $\|(1/w(Q))\int_{Q+y} f(x)\,dx\| < \varepsilon$ for any $y \in \mathbb{R}^d$. Let (Φ_N) be a Følner sequence in $\mathbb{R}^d$. For any $y \in Q$, we have

$$\begin{aligned} 2 &\geq \frac{1}{w(\Phi_N)} \left\| \int_{\Phi_N} f(x+y)\,dx - \int_{\Phi_N} f(x)\,dx \right\| \\ &\leq \frac{1}{w(\Phi_N)} \left(\left\| \int_{(\Phi_N+y)\setminus\Phi_N} f(x)\,dx \right\| + \left\| \int_{\Phi_N\setminus(\Phi_N+y)} f(x)\,dx \right\| \right) \\ &\leq \frac{w(\Phi_N \triangle (\Phi_N + y))}{w(\Phi_N)} \longrightarrow 0 \end{aligned}$$

as $N \longrightarrow \infty$. So, by Lemma 2.2,

$$\frac{1}{w(Q)w(\Phi_N)} \int_Q \left(\int_{\Phi_N} f(x+y)\,dx - \int_{\Phi_N} f(x)\,dx \right) dy \longrightarrow 0 \quad \text{as } N \longrightarrow \infty.$$

But

$$\frac{1}{w(Q)w(\Phi_N)} \int_Q \int_{\Phi_N} f(x)\,dx\,dy = \frac{1}{w(\Phi_N)} \int_{\Phi_N} f(x)\,dx \quad \text{for all } N,$$

whereas

$$\begin{aligned}\left\| \frac{1}{w(Q)w(\Phi_N)} \int_Q \int_{\Phi_N} f(x+y)\,dx\,dy \right\| &\le \frac{1}{w(\Phi_N)} \int_{\Phi_N} \left\| \frac{1}{w(Q)} \int_Q f(x+y)\,dy \right\| dx \\ &= \frac{1}{w(\Phi_N)} \int_{\Phi_N} \left\| \frac{1}{w(Q)} \int_{Q-x} f(y)\,dy \right\| dx \\ &\le \frac{1}{w(\Phi_N)} \int_{\Phi_N} \varepsilon\,dx = \varepsilon\end{aligned}$$

for all N. Hence,

$$\limsup_{N \longrightarrow \infty} \left\| \frac{1}{w(\Phi_N)} \int_{\Phi_N} f(x)\,dx \right\| < \varepsilon. \qquad \square$$

Lemma 6.13 allows us to strengthen Theorems 6.2, 6.8, and 6.12.

THEOREM 6.14. *Let $f : \mathbb{R}^d \longrightarrow V$ be a bounded measurable function such that the limit*

$$A_t = \lim_{l(b-a) \longrightarrow \infty} \frac{1}{w(b-a)} \sum_{n>a}^{b} f(t+n)$$

exists for almost every $t \in [0, 1]^d$. Then, for any Følner sequence (Φ_N) in $\mathbb{R}^d$,

$$\lim_{N \longrightarrow \infty} \frac{1}{w(\Phi_N)} \int_{\Phi_N} f(x)\,dx = \int_{[0,1]^d} A_t\,dt.$$

THEOREM 6.15. *Let $f : \mathbb{R}^d \longrightarrow V$ be a bounded measurable function such that for some P-neighborhood P of 0 in $\mathbb{R}^d$, for almost every $t \in P$, the limit*

$$L_t = \lim_{l(b-a) \longrightarrow \infty} \frac{1}{w(b-a)} \sum_{n>a}^{b} f(nt)$$

exists. Then $L_t = \mathrm{const} = L$ almost everywhere on P and, for any Følner sequence (Φ_N) in $\mathbb{R}^d$,

$$\lim_{N \longrightarrow \infty} \frac{1}{w(\Phi_N)} \int_{\Phi_N} f(x)\,dx = L.$$

THEOREM 6.16. *Let $f : \mathbb{R}^d \longrightarrow V$ be a bounded measurable function satisfying, for some $L \in V$,*

$$\operatorname*{ess\text{-}lim}_{t \longrightarrow 0^+} \limsup_{l(b-a) \longrightarrow \infty} \left\| \frac{1}{w(b-a)} \sum_{n>a}^{b} f(nt) - L \right\| = 0.$$

Then, for any Følner sequence (Φ_N) in $\mathbb{R}^d$,

$$\lim_{N \longrightarrow \infty} \frac{1}{w(\Phi_N)} \int_{\Phi_N} f(x)\,dx = L.$$

In the case where f is a real-valued function, we can get the following version of Lemma 6.13.

LEMMA 6.17. *For any bounded measurable function $f : \mathbb{R}^d \longrightarrow \mathbb{R}$, for any Følner sequence (Φ_N) in $\mathbb{R}^d$,*

$$\liminf_{N\longrightarrow\infty} \frac{1}{w(\Phi_N)} \int_{\Phi_N} f(x)\,dx \geq \liminf_{l(b-a)\longrightarrow\infty} \frac{1}{w(b-a)} \int_a^b f(x)\,dx$$

and

$$\limsup_{N\longrightarrow\infty} \frac{1}{w(\Phi_N)} \int_{\Phi_N} f(x)\,dx \leq \limsup_{l(b-a)\longrightarrow\infty} \frac{1}{w(b-a)} \int_a^b f(x)\,dx.$$

Using Lemma 6.17, we may also strengthen Theorems 6.6 and 6.10.

THEOREM 6.18. *For any bounded measurable function $f : \mathbb{R}^d \longrightarrow \mathbb{R}$ and any Følner sequence (Φ_N) in $\mathbb{R}^d$,*

$$\liminf_{N\longrightarrow\infty} \frac{1}{w(\Phi_N)} \int_{\Phi_N} f(x)\,dx \geq \int_{[0,1]^d} \liminf_{l(b-a)\longrightarrow\infty} \frac{1}{w(b-a)} \sum_{n>a}^{b} f(t+n)\,dt$$

and

$$\limsup_{N\longrightarrow\infty} \frac{1}{w(\Phi_N)} \int_{\Phi_N} f(x)\,dx \leq \int_{[0,1]^d} \limsup_{l(b-a)\longrightarrow\infty} \frac{1}{w(b-a)} \sum_{n>a}^{b} f(t+n)\,dt.$$

THEOREM 6.19. *For any bounded measurable function $f : \mathbb{R}^d \longrightarrow \mathbb{R}$, any $c \in \mathbb{R}^d_+$, $c > 0$, and any Følner sequence (Φ_N) in $\mathbb{R}^d$,*

$$\liminf_{N\longrightarrow\infty} \frac{1}{w(\Phi_N)} \int_{\Phi_N} f(x)\,dx \geq \frac{1}{w(c)} \int_0^c \liminf_{l(b-a)\longrightarrow\infty} \frac{1}{w(b-a)} \sum_{n>a}^{b} f(nt)\,dt$$

and

$$\limsup_{N\longrightarrow\infty} \frac{1}{w(\Phi_N)} \int_{\Phi_N} f(x)\,dx \leq \frac{1}{w(c)} \int_0^c \limsup_{l(b-a)\longrightarrow\infty} \frac{1}{w(b-a)} \sum_{n>a}^{b} f(nt)\,dt.$$

7. *Density of sets and convergence in density*

We will now formulate some special cases of the theorems above. For a set $S \subseteq \mathbb{N}^d$, *the density* of S is

$$\mathrm{D}(S) = \lim_{l(b)\longrightarrow\infty} \frac{1}{w(b)} |S \cap (0, b]|,$$

if it exists; for a measurable set $S \subseteq \mathbb{R}^d_+$, *the density* of S is

$$\mathrm{D}(S) = \lim_{l(b)\longrightarrow\infty} \frac{1}{w(b)} w(S \cap [0, b]),$$

if it exists. (As before, w stands for the standard Lebesgue measure on $\mathbb{R}^d$.) *The lower density* $\underline{\mathrm{D}}(S)$ and *the upper density* $\overline{\mathrm{D}}(S)$ of a set $S \subseteq \mathbb{N}^d$ or $S \subseteq \mathbb{R}^d_+$ are defined as the lim inf and, respectively, the lim sup of the above expressions.

Taking $f = 1_S$ in Theorems 3.5 and 3.3 and in Theorems 5.6 and 5.1, we get, respectively, the following theorems.

THEOREM 7.1. *Let S be a measurable subset of $\mathbb{R}_+^d$ and, for each $t \in [0, 1]^d$, let $S_t = \{n \in \mathbb{N}^d : t + n \in S\}$. Then $\underline{\mathrm{D}}(S) \geq \int_{[0,1]^d} \underline{\mathrm{D}}(S_t)\, dt$ and $\overline{\mathrm{D}}(S) \leq \int_{[0,1]^d} \overline{\mathrm{D}}(S_t)\, dt$. If $\mathrm{D}(S_t)$ exists for almost every $t \in [0, 1]^d$, then $\mathrm{D}(S)$ also exists and equals $\int_{[0,1]^d} \mathrm{D}(S_t)\, dt$.*

THEOREM 7.2. *Let S be a measurable subset of $\mathbb{R}_+^d$ and, for each $t \in \mathbb{R}_+^d$, let $S_t = \{n \in \mathbb{N}^d : nt \in S\}$. Then, for any $c \in \mathbb{R}_+^d$, $c > 0$, one has $\underline{\mathrm{D}}(S) \geq \int_{[0,c]} \underline{\mathrm{D}}(S_t)\, dt$ and $\overline{\mathrm{D}}(S) \leq \int_{[0,c]} \overline{\mathrm{D}}(S_t)\, dt$. If $\mathrm{D}(S_t)$ exists for almost every t in a P-neighborhood P of 0 in $\mathbb{R}_+^d$, then $\mathrm{D}(S(t)) = \mathrm{const} = D$ for almost every $t \in P$ and $\mathrm{D}(S) = D$.*

The uniform (or *Banach*) *density* of a set $S \subseteq \mathbb{N}^d$ is

$$\mathrm{UD}(S) = \lim_{l(b-a)\longrightarrow\infty} \frac{1}{w(b-a)} |S \cap (a, b]|,$$

if it exists; for a measurable set $S \subseteq \mathbb{R}_+^d$, *the uniform density* of S is

$$\mathrm{UD}(S) = \lim_{l(b-a)\longrightarrow\infty} \frac{1}{w(b-a)} w(S \cap [a, b]),$$

if it exists. (And, it follows from (an $\mathbb{R}_+^d$-version of) Lemma 6.13 that for $S \subseteq \mathbb{R}_+^d$, if $\mathrm{UD}(S)$ exists, then, for any Følner sequence (Φ_N) in $\mathbb{R}_+^d$, $\lim_{N\longrightarrow\infty}(1/|\Phi_N|)w(S \cap \Phi_N) = \mathrm{UD}(S)$.) *The lower uniform density* $\underline{\mathrm{UD}}(S)$ and *the upper uniform density* $\overline{\mathrm{UD}}(S)$ of a set $S \subseteq \mathbb{N}^d$ or $S \subseteq \mathbb{R}_+^d$ are the lim inf and, respectively, the lim sup of the above expressions.

From Theorems 3.6, 3.4, 5.5, and 5.3 we get, respectively, the following theorems.

THEOREM 7.3. *Let S be a measurable subset of $\mathbb{R}_+^d$ and, for each $t \in [0, 1]^d$, let $S_t = \{n \in \mathbb{N}^d : n + t \in S\}$. Then*

$$\underline{\mathrm{UD}}(S) \geq \int_{[0,1]^d} \underline{\mathrm{UD}}(S_t)\, dt \quad \textit{and} \quad \overline{\mathrm{UD}}(S) \leq \int_{[0,1]^d} \overline{\mathrm{UD}}(S_t)\, dt.$$

If $\mathrm{UD}(S_t)$ exists for almost every $t \in [0, 1]^d$, then $\mathrm{UD}(S)$ also exists and equals $\int_{[0,1]^d} \mathrm{UD}(S_t)\, dt$.

THEOREM 7.4. *Let S be a measurable subset of $\mathbb{R}_+^d$ and, for each $t \in \mathbb{R}_+^d$, let $S_t = \{n \in \mathbb{N}^d : nt \in S\}$. Then, for any $c \in \mathbb{R}_+^d$, $c > 0$, one has*

$$\underline{\mathrm{UD}}(S) \geq \frac{1}{w(c)} \int_{[0,c]} \underline{\mathrm{UD}}(S_t)\, dt \quad \textit{and} \quad \overline{\mathrm{UD}}(S) \leq \frac{1}{w(c)} \int_{[0,c]} \overline{\mathrm{UD}}(S_t)\, dt.$$

If $\mathrm{UD}(S_t)$ exists for almost every t in a P-neighborhood P of 0 in $\mathbb{R}_+^d$, then $\mathrm{UD}(S_t) = \mathrm{const} = D$ in P and $\mathrm{UD}(S) = D$.

Of course, the 'two-sided' versions of Theorems 7.1–7.4, where one deals with $\mathbb{Z}^d$-sequences and functions on $\mathbb{R}^d$ instead of $\mathbb{N}^d$-sequences and functions on $\mathbb{R}_+^d$, are also true.

We will now bring in two theorems that deal with limits in density instead of Cesàro limits. We say that an $\mathbb{N}^d$-sequence (v_n) in V *converges in density to* $L \in V$ if, for any $\varepsilon > 0$, the set $S_\varepsilon = \{n \in \mathbb{N}^d : \|v_n - L\| > \varepsilon\}$ has zero density, $\mathrm{D}(S_\varepsilon) = 0$, and *converges to* L *in uniform density* if, for any $\varepsilon > 0$, $\mathrm{UD}(S_\varepsilon) = 0$. We say that a (measurable) function

$f : \mathbb{R}_+^d \longrightarrow V$ converges to $L \in V$ in density if, for any $\varepsilon > 0$, the set $S_\varepsilon = \{x \in \mathbb{R}_+^d : \|f(x) - L\| > \varepsilon\}$ has zero density, $\mathrm{D}(S_\varepsilon) = 0$, and converges to L in uniform density if, for any $\varepsilon > 0$, $\mathrm{UD}(S_\varepsilon) = 0$. Applying Theorems 7.1–7.4 to the real-valued function $\|f(x) - L\|$, we obtain the following theorem.

THEOREM 7.5. *Let $f : \mathbb{R}_+^d \longrightarrow V$ be a bounded measurable function such that for some $L \in V$, for almost every $t \in [0, 1]^d$, the $\mathbb{N}^d$-sequence $f(n + t)$, $n \in \mathbb{N}^d$, converges to L in density (respectively, in uniform density). Then f converges to L in density (respectively, in uniform density).*

THEOREM 7.6. *Let $f : \mathbb{R}_+^d \longrightarrow V$ be a bounded measurable function such that for some $L \in V$, for almost every t in a P-neighborhood of 0 in $\mathbb{R}_+^d$, the $\mathbb{N}^d$-sequence $f(nt)$, $n \in \mathbb{N}^d$, converges to L in density (respectively, in uniform density). Then f converges to L in density (respectively, in uniform density).*

Of course, the two-sided versions of Theorems 7.5 and 7.6 also hold.

8. *Applications*

8.1. *Characteristic factors for multiple averages along polynomials.* Let X be a probability measure space; we will always assume that X is sufficiently regular so that $L^1(X)$ is separable.

Let G be a group of measure-preserving transformations of X and let $g_1(n), \dots, g_r(n)$, $n \in \mathbb{Z}^d$, be (d-parameter) sequences of elements of G. A factor Z of the system (X, G) is said to be *characteristic* for the averages $(1/|\Psi_N|)\sum_{n\in\Psi_N} g_1(n)f_1 \cdot \dots \cdot g_r(n)f_r$, where (Ψ_N) is a Følner sequence in $\mathbb{Z}^d$, if, for any $f_1, \dots, f_r \in L^\infty(X)$,

$$\lim_{N\longrightarrow\infty} \frac{1}{|\Psi_N|} \sum_{n\in\Psi_N} (g_1(n)f_1 \cdot \dots \cdot g_r(n)f_r - g_1(n)E(f_1|Z) \cdot \dots \cdot g_r(n)E(f_r|Z)) = 0$$

in $L^1(X)$ (where $E(f|Z)$ stands for the conditional expectation of f with respect to Z). An analogous notion can be introduced for averages $(1/w(\Phi_N))\int_{\Phi_N} g_1(x)f_1 \cdot \dots \cdot g_r(x)f_r\,dx$, where $g_1, \dots, g_r$ are functions $\mathbb{R}^d \longrightarrow G$ and (Φ_N) is a Følner sequence in $\mathbb{R}^d$.

Let T be an ergodic invertible measure-preserving transformation of X. *The kth Host–Kra–Ziegler factor $Z_k(T)$* of (X, T) is the minimal characteristic factor for the averages $(1/|\Psi_N|)\sum_{n\in\Psi_N} \prod_{\emptyset\neq\sigma\subseteq\{0,\dots,k\}} T^{n_\sigma} f_\sigma$, where $n_\sigma = \sum_{i\in\sigma} n_i$, and (Ψ_N) are Følner sequences in $\mathbb{Z}^{k+1}$. $Z_k(T)$ is the maximal factor of (X, T) isomorphic to a k-step pro-nilmanifold (an inverse limit of compact k-step nilmanifolds) on which T acts as a translation. (See [**HoK1, Z**].) The factors $Z_k(T)$ turn out to be characteristic for any system of polynomial powers of T:

THEOREM 8.1. [**L3**] *For any system of polynomials $p_1, \dots, p_r : \mathbb{Z}^d \longrightarrow \mathbb{Z}$, there exists $k \in \mathbb{N}$ such that for any measure-preserving transformation of a probability measure space X, $Z_k(T)$ is a characteristic factor for the averages $(1/|\Phi_N|)\sum_{n\in\Phi_N} T^{p_1(n)} f_1 \cdot \dots \cdot T^{p_r(n)} f_r$.*

It is easy to see (see, for example, [**FK**]) that if S is another ergodic transformation of X commuting with T, then, for all k, $Z_k(S) = Z_k(T)$. Thus, if T is a family of pairwise-commuting ergodic transformations of X, we may denote by $Z_k(T)$ the kth Host–Kra–Ziegler factor of any (and so, of every) element of T. This allows one to generalize Theorem 8.1 in the following way.

THEOREM 8.2. [**J**] *For any finite system of polynomials $p_i : \mathbb{Z}^d \longrightarrow \mathbb{Z}^c$, $i = 1, \dots, r$, there exists $k \in \mathbb{N}$ such that, given any totally ergodic† discrete c-parameter commutative group T^m, $m \in \mathbb{Z}^c$, of measure-preserving transformations T of a probability measure space X, the factor $Z_k(T)$ is characteristic for the averages $(1/|\Psi_N|)\sum_{n\in\Psi_N} T^{p_1(n)} f_1 \cdot \dots \cdot T^{p_r(n)} f_r$, where (Ψ_N) are Følner sequences in $\mathbb{Z}^d$.*

Now let $T^t, t \in \mathbb{R}$, be a continuous one-parameter group of measure-preserving transformations of X and assume that it is ergodic on X. Then, for almost all (actually, for all but countably many) $t \in \mathbb{R}$, the transformation T^t is ergodic, so, for any k, $Z_k(T^t)$ coincide for almost every t; we will denote this factor by $Z_k(T)$. We can now prove the following fact (obtained in [**P**] for non-uniform averages).

THEOREM 8.3. *For any system of polynomials $p_1, \dots, p_r : \mathbb{R}^d \longrightarrow \mathbb{R}$, there exists $k \in \mathbb{N}$ such that for any continuous one-parameter group $T^t, t \in \mathbb{R}$, of measure-preserving transformations of a probability measure space X, $Z_k(T)$ is a characteristic factor for the averages $(1/w(\Phi_N))\int_{\Phi_N} T^{p_1(x)} f_1 \cdot \dots \cdot T^{p_r(x)} f_r \, dx$.*

Proof. Given polynomials $p_1, \dots, p_r$ on $\mathbb{R}^d$, find monomials $q_\lambda(x) = c_\lambda x^{\alpha_\lambda}$, $\lambda = 1, \dots, \Lambda$, where $c_\lambda \in \mathbb{R}$ and α_λ are multi-indices, that are $\mathbb{Q}$-linearly independent and such that each of the polynomials p_i is a sum of the monomials q_λ with integer coefficients, $p_i = \sum_{\lambda=1}^{\Lambda} b_{i,\lambda} q_\lambda$, $b_{i,\lambda} \in \mathbb{Z}$. Then, for any $x \in \mathbb{R}^d$, any $n \in \mathbb{Z}^d$, and any i, $T^{p_i(nx)} = \prod_{\lambda=1}^{\Lambda} T_{x,\lambda}^{b_{i,\lambda} n^{\alpha_\lambda}}$, where $T_{x,\lambda} = T^{c_\lambda x^{\alpha_\lambda}}$, and, since T^t is ergodic for almost every $t \in \mathbb{R}$, the Λ-parameter group generated by the transformations $T_{x,\lambda}$, $\lambda = 1, \dots, \Lambda$, satisfies the assumptions of Theorem 8.2 for almost every $x \in \mathbb{R}^d$. Find k which, by Theorem 8.2, corresponds to the polynomials $b_{i,\lambda} n^{\alpha_\lambda}$, $i = 1, \dots, r$, $\lambda = 1, \dots, \Lambda$, so that for almost every $x \in \mathbb{R}^d$,

$$\lim_{N\longrightarrow\infty} \frac{1}{|\Psi_N|} \sum_{n\in\Psi_N} (T^{p_1(nx)} f_1 \cdot \dots \cdot T^{p_r(nx)} f_r \\ - T^{p_1(nx)} E(f_1|Z_k(T)) \cdot \dots \cdot T^{p_r(nx)} E(f_r|Z_k(T))) = 0$$

for any Følner sequence (Ψ_N) in $\mathbb{Z}^d$. Then, by Theorem 6.15,

$$\lim_{N\longrightarrow\infty} \frac{1}{w(\Phi_N)} \int_{\Phi_N} (T^{p_1(x)} f_1 \cdot \dots \cdot T^{p_r(x)} f_r \\ - T^{p_1(x)} E(f_1|Z_k(T)) \cdot \dots \cdot T^{p_r(x)} E(f_r|Z_k(T))) \, dx = 0$$

for any Følner sequence (Φ_N) in $\mathbb{R}^d$, which proves Theorem 8.3. □

† A group G of measure-preserving transformations of a measure space is *totally ergodic* if every non-identical element of G is totally ergodic.

8.2. *Polynomial orbits on nilmanifolds.* Let X be a topological space with a probability Borel measure μ. We say that a d-parameter sequence $g(n)$, $n \in \mathbb{Z}^d$, *is well distributed with respect to* μ if, for any $h \in C(X)$ and any Følner sequence (Ψ_N) in $\mathbb{Z}^d$, one has

$$\lim_{N \longrightarrow \infty} \frac{1}{|\Psi_N|} \sum_{n \in \Psi_N} h(g(n)) = \int_X h \, d\mu.$$

We also say that a measurable function $g(t)$, $t \in \mathbb{R}^d$, in X is well distributed with respect to μ if, for any $h \in C(X)$ and any Følner sequence (Φ_N) in $\mathbb{R}^d$,

$$\lim_{N \longrightarrow \infty} \frac{1}{w(\Phi_N)} \int_{\Phi_N} h(g(t)) \, dt = \int_X h \, d\mu.$$

The following proposition is an immediate corollary of Theorem 6.14, applied to the functions $h \circ g$, $h \in C(X)$.

PROPOSITION 8.4. *Let X be a topological space and let $g : \mathbb{R}^d \longrightarrow X$ be a function such that for almost every $t \in [0, 1]^d$ the sequence $g(n + t)$, $n \in \mathbb{Z}^d$, is well distributed in X with respect to a probability Borel measure μ_t. Then g is well distributed with respect to the measure $\mu = \int_{[0,1]^d} \mu_t \, dt$.*

From Theorem 6.15, we get the following proposition.

PROPOSITION 8.5. *Let X be a compact Hausdorff space for which $C(X)$ is separable and let $g : \mathbb{R}^d \longrightarrow X$ be a function such that for almost every t in a P-neighborhood P of* 0 *in $\mathbb{R}^d_+$ the sequence $g(nt)$, $n \in \mathbb{Z}^d$, is well distributed in X with respect to a probability Radon measure μ_t. Then $\mu_t = \mathrm{const} = \mu$ for almost every $t \in P$ and g is well distributed with respect to the measure μ.*

Proof. By Theorem 6.15, applied to the function $h \circ g$, for any $h \in C(X)$ we have $\mu_t(h) = \mathrm{const} = \mu(h)$ for almost every $t \in P$ and $\lim_{N \longrightarrow \infty}(1/w(\Phi_N))\int_{\Phi_N} h(g(t)) \, dt = \mu(h)$ for any Følner sequence (Φ_N) in $\mathbb{R}^d$. Excluding those t for which $\mu_t(h) \neq \mu(h)$ for all functions h from a fixed countable subset of $C(X)$, we obtain that $\mu_t = \mathrm{const} = \mu$ for almost every $t \in P$ and g is well distributed with respect to μ. (The assumption that the μ_t are Radon measures allows us to identify them with continuous linear functionals on $C(X)$.) □

We will apply these propositions in the following situation. Let X be a compact *nilmanifold*, that is, a homogeneous space of a nilpotent Lie group G, and let $g : \mathbb{R}^d \longrightarrow X$ be a *polynomial mapping*, that is, $g(t) = a_1^{p_1(t)} \ldots a_k^{p_k(t)} \omega$, $t \in \mathbb{R}^d$, where $a_1, \ldots, a_k \in G$, $p_1, \ldots, p_k$ are polynomials $\mathbb{R}^d \longrightarrow \mathbb{R}$, and $\omega \in X$. Let $Y = \overline{\{g(t), t \in \mathbb{R}^d\}}$. It follows from a general result obtained in [**Sh**] that Y is a connected sub-nilmanifold of X (that is, a closed subset of X of the form $H\omega$, where H is a connected closed subgroup of G and $\omega \in X$), and g is uniformly distributed in Y in the following sense: for any $h \in C(Y)$, $\lim_{R \longrightarrow \infty}(1/w(B_R))\int_{B_R} h(g(t)) \, dw(t) = \int_Y h \, d\mu$, where w is the Lebesgue measure on $\mathbb{R}^d$, B_R, $R > 0$, is the ball $\{t \in \mathbb{R}^d : |t| \leq R\}$, and μ is the Haar measure on Y. We would like to have a stronger result which states that g is not only uniformly distributed, but is well distributed in Y. A discrete analogue of this fact, which we will presently formulate, was obtained in [**L2, L4**], but before formulating it we need to introduce some terminology. We call a finite disjoint union of connected subnilmanifolds of X *a FU subnilmanifold*. We say that an element ω' of X is *rational* with respect to an element $\omega \in X$ if $\omega' = a\omega$ for some $a \in G$ such that $a^m\omega = \omega$ for some $m \in \mathbb{N}$. We say that a subnilmanifold Y of X is *rational with respect to* ω if Y contains an element ω' rational with respect to ω. (Then such elements ω' are dense in Y.) Finally, we say that a FU subnilmanifold of X is rational with respect to ω if all connected components of Y are subnilmanifolds rational with respect to ω.

PROPOSITION 8.6. (See [**L2, L4**].) *Let g be a d-parameter polynomial sequence in X, that is, $g(n) = a_1^{p_1(n)} \dots a_k^{p_k(n)}\omega$, where $a_1, \dots, a_k \in G$, $p_1, \dots, p_k$ are polynomials $\mathbb{Z}^d \longrightarrow \mathbb{R}$, and $\omega \in X$. Then the closure $Y = \overline{\{g(n), n \in \mathbb{Z}^d\}}$ of g is a FU subnilmanifold of X rational with respect to the point $g(0)$. If Y is connected, then the sequence $g(n)$, $n \in \mathbb{Z}^d$, is well distributed in Y (with respect to the Haar measure on Y).*

We may now use Theorem 6.15 or Theorem 6.14 to deduce from Proposition 8.6 its continuous analogue. We will also need the following fact.

PROPOSITION 8.7. [**L5**, Theorem 2.1] *Let M be a set and let $\varphi : \mathbb{R}^d \times M \longrightarrow X$ be a mapping such that for every $m \in M$, $\varphi(\cdot, m)$ is a polynomial mapping $\mathbb{R}^d \longrightarrow X$, and there exists $\omega \in X$ such that for each $t \in \mathbb{R}^d$, the set $Y_t = \overline{\varphi(t, M)}$ is a FU subnilmanifold of X rational with respect to ω. Then there exists a FU subnilmanifold Y of X such that $Y_t \subseteq Y$ for all $t \in \mathbb{R}^d$ and $Y_t = Y$ for almost every $t \in \mathbb{R}^d$.*

Now let $g : \mathbb{R}^d \longrightarrow X$ be a polynomial mapping. By Proposition 8.6, the mapping $\varphi : \mathbb{R}^d \times \mathbb{Z}^d \longrightarrow X$ defined by $\varphi(t, n) = g(nt)$ satisfies the assumptions of Proposition 8.7 (with $\omega = g(0)$); thus, there exists a FU subnilmanifold Y such that $\overline{\{g(nt), n \in \mathbb{Z}^d\}} \subseteq Y$, for all t and $= Y$ for almost every $t \in \mathbb{R}^d$. But then $Y = \overline{\{g(t), t \in \mathbb{R}^d\}}$, and so Y is a connected subnilmanifold; by the second part of Proposition 8.6, the sequence $g(nt)$, $n \in \mathbb{Z}^d$, is well distributed in Y for almost every $t \in \mathbb{R}^d$. Applying Proposition 8.5, we get the following theorem.

THEOREM 8.8. *Let X be a compact nilmanifold and $g : \mathbb{R}^d \longrightarrow X$ be a polynomial mapping. Then $Y = \overline{\{g(t), t \in \mathbb{R}^d\}}$ is a connected subnilmanifold of X and $g(t)$ is well distributed in Y (with respect to the Haar measure in Y).*

Remark. If we were only interested in proving the well distribution of g in a subnilmanifold Y, we could avoid the usage of Proposition 8.7; we need it to show that $g(t) \in Y$ for all t.

8.3. *Convergence of multiple averages.* Combining Theorems 8.3 and 8.8, we can now get the following theorem.

THEOREM 8.9. *Let T^t, $t \in \mathbb{R}$, be a continuous one-parameter group of measure-preserving transformations of a probability measure space X and let $p_1, \dots, p_r$ be polynomials $\mathbb{R}^d \longrightarrow \mathbb{R}$. Then, for any $f_1, \dots, f_r \in L^\infty(X)$ and any Følner sequence*

(Φ_N) in $\mathbb{R}^d$, the limit

$$\lim_{N\longrightarrow\infty}\frac{1}{w(\Phi_N)}\int_{\Phi_N}T^{p_1(x)}f_1\cdot\dots\cdot T^{p_r(x)}f_r\,dx$$

exists in L^1-norm.

(In [**P**], a version of Theorem 8.9 was obtained for 'standard' Cesàro averages (that is, for the case $\Phi_N=\prod_{i=1}^d[0,b_{i,N}]$, $N\in\mathbb{N}$, with $b_{i,N}\longrightarrow\infty$ as $N\longrightarrow\infty$ for all $i=1,\dots,d$). In [**Au2**], a multidimensional (that is, for $T:\mathbb{R}^d\longrightarrow\mathbb{R}^c$ with $c\geq 1$) version of this result was obtained, again, for the standard Cesàro averages.)

Proof. We may assume that T is ergodic. Applying Theorem 8.3, we can then replace (X,T) by a pro-nilmanifold $Z_k(T)$. Now, given the functions $f_1,\dots,f_r\in L^\infty(X)$, we can approximate them in L^1-norm by functions that come from a factor Y of $Z_k(T)$ which is a nilmanifold, and replace $Z_k(T)$ by Y and T by a nilrotation a on it. Next, we note that it is enough to assume that $f_1,\dots,f_r$ are continuous functions on Y. Then an application of Theorem 8.8 to the polynomial flow $(a^{p_1(x)}y,\dots,a_r^{p_r(x)}y)$, $x\in\mathbb{R}^d$, on the nilmanifold Y^r and the function $f_1(y_1)\cdot\dots\cdot f_r(y_r)\in C(Y^r)$ proves that the limit

$$\lim_{N\longrightarrow\infty}\frac{1}{w(\Phi_N)}\int_{\Phi_N}f_1(a^{p_1(t)}y)\cdot\dots\cdot f_r(a^{p_r(t)}y)\,dt$$

exists for all $y\in Y$, and so, in $L^1(Y)$. □

Another way to prove Theorem 8.9 is to deduce it, with the help of either Theorem 6.14 or Theorem 6.15, from the following discrete-time theorem.

THEOREM 8.10. [**J**] *For any totally ergodic discrete c-parameter commutative group T^m, $m\in\mathbb{Z}^c$, of measure-preserving transformations of a probability measure space X, any finite system of polynomials $p_i:\mathbb{Z}^d\longrightarrow\mathbb{Z}^c$, $i=1,\dots,r$, any $f_1,\dots,f_r\in L^\infty(X)$, and any Følner sequence (Ψ_N) in $\mathbb{Z}^d$, the limit*

$$\lim_{N\longrightarrow\infty}\frac{1}{|\Psi_N|}\sum_{n\in\Psi_N}T^{p_1(n)}f_1\cdot\dots\cdot T^{p_r(n)}f_r$$

exists in L^1-norm.

Applying Theorem 6.15, we obtain from Theorem 8.10 the following refinement of Theorem 8.9.

THEOREM 8.11. *Under the assumptions of Theorem 8.9,*

$$\lim_{N\longrightarrow\infty}\frac{1}{w(\Phi_N)}\int_{\Phi_N}T^{p_1(x)}f_1\cdot\dots\cdot T^{p_r(x)}f_r\,dx$$
$$=\lim_{N\longrightarrow\infty}\frac{1}{|\Psi_N|}\sum_{n\in\Psi_N}T^{p_1(nt)}f_1\cdot\dots\cdot T^{p_r(nt)}f_r$$

for almost every $t\in\mathbb{R}^d$ and any Følner sequences (Φ_N) in $\mathbb{R}^d$ and (Ψ_N) in $\mathbb{Z}^d$.

As for the actions of several commuting operators, the following 'linear' result has been recently obtained.

THEOREM 8.12. ([**Au1**]; see also [**Ho1**]) *Let $T_1, \dots, T_r$ be pairwise-commuting measure-preserving transformations of a probability measure space X. Then, for any $f_1, \dots, f_r \in L^\infty(X)$, the limit*

$$\lim_{b-a\longrightarrow\infty} \frac{1}{b-a} \sum_{n=a+1}^{b} T_1^n f_1 \cdot \dots \cdot T_r^n f_r$$

exists in L^1-norm.

Applying Theorem 6.15, we obtain the following theorem.

THEOREM 8.13. *Let $T_1^t, \dots, T_r^t$, $t \in \mathbb{R}$, be pairwise-commuting continuous one-parameter groups of measure-preserving transformations of a probability measure space X. Then, for any $f_1, \dots, f_r \in L^\infty(X)$, the limit*

$$\lim_{b-a\longrightarrow\infty} \frac{1}{b-a} \int_a^b T_1^x f_1 \cdot \dots \cdot T_r^x f_r \, dx$$

exists in L^1-norm and equals

$$\lim_{b-a\longrightarrow\infty} \frac{1}{b-a} \sum_{n=a+1}^{b} T_1^{nt} f_1 \cdot \dots \cdot T_r^{nt} f_r \quad \text{for a.e. } t \in \mathbb{R}.$$

8.4. *The polynomial Szemerédi theorem.* The 'multiparameter multidimensional polynomial ergodic Szemerédi theorem' says the following.

THEOREM 8.14. (See [**BM**] or [**BLM**].) *Let T^m, $m \in \mathbb{Z}^c$, be a discrete c-parameter commutative group of measure-preserving transformations of a probability measure space (X, μ), let $p_i : \mathbb{Z}^d \longrightarrow \mathbb{Z}^c$, $i = 1, \dots, r$, be a system of polynomials with $p_i(0) = 0$ for all i, and let $A \subseteq X$, $\mu(A) > 0$. Then, for any Følner sequence (Ψ_N) in $\mathbb{Z}^d$,*

$$\liminf_{N\longrightarrow\infty} \frac{1}{|\Psi_N|} \sum_{n\in\Psi_N} \mu(T^{p_1(n)}(A) \cap \dots \cap T^{p_r(n)}(A)) > 0.$$

Since the convergence of the averages

$$\lim_{N\longrightarrow\infty} \frac{1}{|\Psi_N|} \sum_{n\in\Psi_N} \mu(T^{p_1(n)}(A) \cap \dots \cap T^{p_r(n)}(A))$$

is unknown, we cannot apply Theorem 6.14 or Theorem 6.15 to get a continuous-time version of Theorem 8.14; however, it can be obtained with the help of either Theorem 6.18 or Theorem 6.19.

THEOREM 8.15. *Let T^t, $t \in \mathbb{R}^c$, be a c-parameter commutative group of measure-preserving transformations of a probability measure space (X, μ), let $p_i : \mathbb{R}^d \longrightarrow \mathbb{R}^c$, $i = 1, \dots, r$, be a system of polynomials with $p_i(0) = 0$ for all i, and let $A \subseteq X$, $\mu(A) > 0$. Then, for any Følner sequence (Φ_N) in $\mathbb{R}^d$,*

$$\liminf_{N\longrightarrow\infty} \frac{1}{w(\Phi_N)} \int_{\Phi_N} \mu(T^{p_1(x)}(A) \cap \dots \cap T^{p_r(x)}(A)) \, dx > 0.$$

A (d-parameter) *polynomial sequence* in a group G is a sequence of the form $g(n) = \prod_{j=1}^{k} v_j^{p_j(n)}$, where v_j are elements of G and p_j are integer-valued polynomials on $\mathbb{Z}^d$. Theorem 8.14 was extended in [**L1**] to the nilpotent setup as follows.

THEOREM 8.16. *Let G be a nilpotent group of measure-preserving transformations of a probability measure space (X, μ), let $g_i : \mathbb{Z}^d \longrightarrow G$, $i = 1, \ldots, r$, be a system of d-parameter polynomial sequences in G with $g_i(0) = 1_G$ for all i, and let $A \subseteq X$, $\mu(A) > 0$. Then*

$$\liminf_{l(b-a)\longrightarrow\infty} \frac{1}{w(b-a)} \sum_{n>a}^{b} \mu((g_1(n))(A) \cap \ldots \cap (g_r(n))(A)) > 0.$$

If G is a connected nilpotent Lie group, then, for any $v \in G$, there exists a one-parameter subgroup v^t, $t \in \mathbb{R}$, of G such that $v^1 = v$; this allows one to define v^t for all $t \in \mathbb{R}$. Let us call *a polynomial mapping* $g : \mathbb{R}^d \longrightarrow G$ a mapping of the form $g(x) = \prod_{j=1}^{k} v_j^{p_j(x)}$, where v_j are elements of G and p_j are polynomials on $\mathbb{R}^d$. Applying one of Theorems 6.18 or 6.19, we get the following ‘continuous-time nilpotent polynomial Szemerédi theorem’.

THEOREM 8.17. *Let G be a nilpotent Lie group of measure-preserving transformations of a probability measure space (X, μ), let $g_i : \mathbb{R}^d \longrightarrow G$, $i = 1, \ldots, r$, be a system of polynomial mappings with $g_i(0) = 1_G$ for all i, and let $A \subseteq X$, $\mu(A) > 0$. Then, for any Følner sequence (Φ_N) in $\mathbb{R}^d$,*

$$\liminf_{N\longrightarrow\infty} \frac{1}{w(\Phi_N)} \int_{\Phi_N} \mu((g_1(x))(A) \cap \cdots \cap (g_r(x))(A))\, dx > 0.$$

8.5. *Distribution of values of generalized polynomials.* Another application of Theorem 8.8 is a sharpening of the results from [**BL**] about the distribution of values of bounded generalized polynomials. Recall that a generalized polynomial is a function from $\mathbb{R}^d$ or from $\mathbb{Z}^d$ to $\mathbb{R}$ that is constructed from conventional polynomials by applying the operations of addition, multiplication, and taking the integer part. We call a function $u : \mathbb{R}^d \longrightarrow \mathbb{R}^c$ a *generalized polynomial mapping* if all components of u are generalized polynomials. Under a *piecewise polynomial surface* $\mathcal{S} \subseteq \mathbb{R}^c$, we understand the image $\mathcal{S} = S(Q)$ of the cube $Q = [0, 1]^s$, where S is a *piecewise polynomial mapping*, which means that Q can partitioned into a finite union $Q = \bigcup_{i=1}^{l} Q_i$ of subsets so that for each i, Q_i is defined by a system of polynomial inequalities and $S_{|Q_i}$ is a polynomial mapping. We endow $\mathcal{S}$ with the measure $\mu_{\mathcal{S}} = S_*(w)$, the push-forward of the standard Lebesgue measure w on Q. In [**BL**], the following theorem was proved.

THEOREM 8.18. [**BL**] *Let $u : \mathbb{Z}^d \longrightarrow \mathbb{R}^c$ be a bounded generalized polynomial mapping. Then the sequence $u(n)$, $n \in \mathbb{Z}^d$ is well distributed with respect to $\mu_{\mathcal{S}}$ on a piecewise polynomial surface $\mathcal{S} \subset \mathbb{R}^c$.*

(Note that it is not claimed in this theorem that $u(n) \in \mathcal{S}$ for all n; it follows however that the set $\{n : u(n) \notin \mathcal{S}\}$ has zero uniform density in $\mathbb{Z}^d$.)

Applying Proposition 8.5, we may now obtain the $\mathbb{R}$-version of Theorem 8.18.

THEOREM 8.19. *Any bounded generalized polynomial mapping $u : \mathbb{R}^d \longrightarrow \mathbb{R}^c$ is well distributed on a piecewise polynomial surface $\mathcal{S} \subset \mathbb{R}^c$.*

An application of the spectral theorem gives, as a corollary, the following proposition.

PROPOSITION 8.20. *Let U^t, $t \in \mathbb{R}^c$, be a continuous c-parameter group of unitary operators on a Hilbert space $\mathcal{H}$ and let $u : \mathbb{R}^d \longrightarrow \mathbb{R}^c$ be a generalized polynomial mapping. Then, for any $v \in \mathcal{H}$ and any Følner sequence (Φ_N) in $\mathbb{R}^d$, the limit $\lim_{N\longrightarrow\infty}(1/w(\Phi_N))\int_{\Phi_N} U^{g(x)}v\,dx$ exists.*

8.6. *Ergodic theorems along functions from Hardy fields.* We will now deal with a situation where our 'uniform Cesàro theorems' are not applicable, but the 'standard Cesàro' Theorem 4.1 is; namely, we will deal with multiple ergodic averages along (not necessarily) polynomial functions of polynomial growth. Such averages for functions of integer argument were considered in [**BK, F**].

To state the results obtained in [**BK**], we first need to introduce some notation:

$\mathcal{T}$ is the set of real-valued C^∞ functions g defined on intervals $[a, \infty)$, $a \in \mathbb{R}$, such that a finite $\lim_{x\longrightarrow+\infty} xg^{(j+1)}(x)/g^{(j)}(x)$ exists for all $j \in \mathbb{N}$ and there exist an integer $i \geq 0$ and $\alpha \in (i, i+1]$ such that $\lim_{x\longrightarrow+\infty} xg'(x)/g(x) = \alpha$ and $\lim_{x\longrightarrow+\infty} g^{(i+1)}(x) = 0$;

$\mathcal{P}$ is the set of real-valued C^∞ functions g defined on intervals $[a, \infty)$, $a \in \mathbb{R}$ such that for some integer $i \geq 0$ a finite non-zero $\lim_{x\longrightarrow+\infty} g^{(i+1)}(x)$ exists and $\lim_{x\longrightarrow+\infty} x^j g^{(i+j+1)}(x) = 0$ for all $j \in \mathbb{N}$;

$\mathcal{G} = \mathcal{T} \cup \mathcal{P}$;

$\mathcal{L}$ is the Hardy field of logarithmico-exponential functions, that is, the minimal field of real-valued functions defined on intervals $[a, \infty)$, $a \in \mathbb{R}$, that contains polynomials and is closed under the operations of taking exponent and logarithm-of-modulus;

for $\alpha > 0$, $\mathcal{G}(\alpha)$ is the set of functions $g \in \mathcal{G}$ with $\lim_{x\longrightarrow+\infty} xg'(x)/g(x) = \alpha$, $\mathcal{T}(\alpha)$ is the set of functions $g \in \mathcal{T}$ with $\lim_{x\longrightarrow+\infty} xg'(x)/g(x) = \alpha$, and, for any $G \subseteq \mathcal{G}$, $G(\alpha) = G \cap \mathcal{G}(\alpha)$;

a finite family $G \subset \mathcal{G}$ with $g_1 - g_2 \in \mathcal{G}$ for all $g_1, g_2 \in \mathcal{G}$ is said to have *R-property* if, for any $\alpha > 0$, any $g_1, g_2 \in (G(\alpha) \cup (G(\alpha) - G(\alpha)))\backslash\{0\}$, any integer $l \geq 0$, and $\beta \in (0, \alpha)$ such that $g_1^{(l)}, g_2 \in \mathcal{T}(\beta)$, a finite non-zero $\lim_{x\longrightarrow+\infty} g_1^{([\beta]+l+1)}(x)/g_2^{([\beta]+1)}(x)$ exists.

The following theorem was proved in [**BK**].

THEOREM 8.21. [**BK**] *Let $g_1, \dots, g_r \in \mathcal{G}$ be such that $g_i - g_j \in \mathcal{G}$ for all $i \neq j$, and also either $g_1, \dots, g_r \in \mathcal{L}$ or the family $\{g_1, \dots, g_r\}$ has the R-property. Then, for any invertible weakly mixing transformation T of a probability measure space (X, μ) and any $f_1, \dots, f_r \in L^\infty(X)$, the sequence $F_n = T^{[g_1(n)]}f_1 \cdot \dots \cdot T^{[g_r(n)]}f_r$, $n \in \mathbb{N}$, tends in density in L^1-norm to $\prod_{i=1}^r \int f_i\,d\mu$.*

The statement 'F_n tends in density in L^1-norm' means that

$$\lim_{N\longrightarrow\infty} \frac{1}{N}\sum_{n=1}^{N}\left\| T^{[g_1(n)]}f_1 \cdot \dots \cdot T^{[g_r(n)]}f_r - \prod_{i=1}^{r}\int f_i\,d\mu \right\|_{L^1(X)} = 0.$$

From this and Theorem 4.1 we get that, under the assumptions of Theorem 8.21,

$$\lim_{b\longrightarrow\infty} \frac{1}{b}\int_0^b \left\| T^{[g_1(x)]}f_1 \cdot \dots \cdot T^{[g_r(x)]}f_r\,dx - \prod_{i=1}^{r}\int f_i\,d\mu \right\|_{L^1(X)} dx = 0,$$

that is, the function $F_x = T^{[g_1(x)]} f_1 \cdot \dots \cdot T^{[g_r(x)]} f_r$, $x \in [0, \infty)$ (whose range is in $L^1(X)$) tends in density in L^1-norm to $\prod_{i=1}^r \int f_i \, d\mu$. Hence, we obtain the following theorem.

THEOREM 8.22. *Let $g_1, \dots, g_r \in \mathcal{G}$ be such that $g_i - g_j \in \mathcal{G}$ for all $i \neq j$, and also either $g_1, \dots, g_r \in \mathcal{L}$ or the family $\{g_1, \dots, g_r\}$ has the R-property. Then, for any invertible weakly mixing transformation T of a probability measure space (X, μ) and any $f_1, \dots, f_r \in L^\infty(X)$, the function $F_x = T^{[g_1(x)]} f_1 \cdot \dots \cdot T^{[g_r(x)]} f_r$, $x \in [0, \infty)$, tends in density in L^1-norm to $\prod_{i=1}^r \int f_i \, d\mu$.*

Actually, one can eliminate the brackets appearing in the exponents in the expression for F_x. Indeed, put $G_x = T^{g_1(x)} f_1 \cdot \dots \cdot T^{g_r(x)} f_r$, $x \in [0, \infty)$, and let $L = \prod_{i=1}^r \int f_i \, d\mu$. Assume that $\|f_i\| \le 1$ for all i. Fix any $\varepsilon > 0$ and, for each $i = 1, \dots, r$, choose functions $g_{i,j} \in L^\infty(X)$, $j = 1, \dots, k$, that form an ε-net in the (compact) set $\{T^t f_i, t \in [0, 1]\} \subset L^1(X)$. For any $J = (j_1, \dots, j_r) \in \{1, \dots, k\}^r$, the function $(F_J)_x = T^{[g_1(x)]} f_{1,j_1} \dots T^{[g_r(x)]} f_{r,j_r}$ tends in density to L and, for any $x \in [0, \infty)$, there exists $J = (j_1, \dots, j_r) \in \{1, \dots, k\}^r$ such that

$$\|T^{g_i(x)} f_i - T^{[g_i(x)]} f_{i,j_i}\| = \|T^{\{g_i(x)\}} f_i - f_{i,j_i}\| < \varepsilon$$

for all i and, so, $\|G_x - (F_J)_x\| < 2^{2r}\varepsilon$. This implies that

$$\limsup_{N \longrightarrow \infty} \frac{1}{N} \sum_1^N \|G_x - L\| < 2^{2r}\varepsilon.$$

Since this holds for any positive ε, we see that G_x also tends in density to L. So, we have the following result.

THEOREM 8.23. *Let $g_1, \dots, g_r \in \mathcal{G}$ be such that $g_i - g_j \in \mathcal{G}$ for all $i \neq j$, and also either $g_1, \dots, g_r \in \mathcal{L}$ or the family $\{g_1, \dots, g_r\}$ has the R-property. Then, for any weakly mixing continuous one-parameter group T^t, $t \in \mathbb{R}$, of measure-preserving transformations of a probability measure space (X, μ) and any $f_1, \dots, f_r \in L^\infty(X)$, the function $G_x = T^{g_1(x)} f_1 \cdot \dots \cdot T^{g_r(x)} f_r$, $x \in [0, \infty)$, tends in density in L^1-norm to $\prod_{i=1}^r \int f_i \, d\mu$.*

Another paper dealing with multiple-ergodic averages along non-polynomial functions of polynomial growth is [**F**]. Let $\mathcal{H}$ denote the union of all Hardy fields of real-valued functions.

THEOREM 8.24. [**F**] *Let $g \in \mathcal{H}$ satisfy $\lim_{x \longrightarrow +\infty} g(x)/x^j = 0$ for some $j \in \mathbb{N}$, and assume that one of the following is true: $\lim_{x \longrightarrow +\infty} (g(x) - cp(x))/\log x = \infty$ for all $c \in \mathbb{R}$ and $p \in \mathbb{Z}[x]$; or $\lim_{x \longrightarrow +\infty} (g(x) - cp(x)) = d$ for some $c, d \in \mathbb{R}$ and $p \in \mathbb{Z}[x]$; or $(g(x) - x/m)/\log x$ is bounded on $[2, \infty)$ for some $m \in \mathbb{Z}$. Then, for any invertible measure-preserving transformation of a probability measure space X,*

$$\lim_{N \longrightarrow \infty} \frac{1}{N} \sum_{n=1}^N T^{[g(n)]} f_1 \cdot T^{2[g(n)]} f_2 \cdot \dots \cdot T^{r[g(n)]} f_r$$

exists in $L^1(X)$ for any $r \in \mathbb{N}$ and any $f_1, \dots, f_r \in L^\infty(X)$.

THEOREM 8.25. [**F**] *Let $g_1, \dots, g_r \in \mathcal{L}$ be logarithmico-exponential functions satisfying*

$$\lim_{x \longrightarrow +\infty} g_i(x)/x^{k_i+1} = \lim_{x \longrightarrow +\infty} x^{k_i+\varepsilon_i}/g_i(x) = 0$$

for some integer $k_i \geq 0$ and $\varepsilon_i > 0$, $i = 1, \dots, r$, and $\lim_{x \longrightarrow +\infty} g_i(x)/g_j(x) = 0$ or ∞ for any $i \neq j$. Then, for any invertible ergodic measure-preserving transformation of a probability measure space X,

$$\lim_{N \longrightarrow \infty} \frac{1}{N} \sum_{n=1}^{N} T^{[g_1(n)]} f_1 \cdot \dots \cdot T^{[g_r(n)]} f_r = \prod_{i=1}^{r} \int_X f_i \, d\mu$$

in $L^1(X)$ for any $f_1, \dots, f_r \in L^\infty(X)$.

From this and Theorem 4.1, we get the following theorem.

THEOREM 8.26. *Let $g \in \mathcal{H}$ satisfy $\lim_{x \longrightarrow +\infty} g(x)/x^j = 0$ for some $j \in \mathbb{N}$, and assume that one of the following is true: $\lim_{x \longrightarrow +\infty} (g(x) - cp(x))/\log x = \infty$ for all $c \in \mathbb{R}$ and $p \in \mathbb{Z}[x]$; or $\lim_{x \longrightarrow +\infty} (g(x) - cp(x)) = d$ for some $c, d \in \mathbb{R}$ and $p \in \mathbb{Z}[x]$; or $(g(x) - x/m)/\log x$ is bounded on $[2, \infty)$ for some $m \in \mathbb{Z}$. Then, for any invertible measure-preserving transformation of a probability measure space X,*

$$\lim_{b \longrightarrow \infty} \frac{1}{b} \int_0^b T^{[g(x)]} f_1 \cdot T^{2[g(x)]} f_2 \cdot \dots \cdot T^{r[g(x)]} f_r \, dx$$

exists in $L^1(X)$ for any $r \in \mathbb{N}$ and any $f_1, \dots, f_r \in L^\infty(X)$.

THEOREM 8.27. *Let $g_1, \dots, g_r \in \mathcal{L}$ be logarithmico-exponential functions satisfying*

$$\lim_{x \longrightarrow +\infty} g_i(x)/x^{k_i+1} = \lim_{x \longrightarrow +\infty} x^{k_i+\varepsilon_i}/g_i(x) = 0$$

for some integer $k_i \geq 0$ and $\varepsilon_i > 0$, $i = 1, \dots, r$, and $\lim_{x \longrightarrow +\infty} g_i(x)/g_j(x) = 0$ or ∞ for any $i \neq j$. Then, for any invertible ergodic measure-preserving transformation of a probability measure space X,

$$\lim_{b \longrightarrow \infty} \frac{1}{b} \int_0^b T^{[g_1(x)]} f_1 \cdot \dots \cdot T^{[g_r(x)]} f_r \, dx = \prod_{i=1}^{r} \int_X f_i \, d\mu$$

in $L^1(X)$ for any $f_1, \dots, f_r \in L^\infty(X)$.

8.7. *Pointwise ergodic theorems.* Here are two theorems of Bourgain dealing with pointwise convergence.

THEOREM 8.28. [**Bo1**] *Let T be a measure-preserving transformation of a probability measure space X. Then, for any $f_1, f_2 \in L^2(X)$, the sequence $(1/N)\sum_{n=1}^{N} T^n f_1 \cdot T^{2n} f_2$, $N \in \mathbb{N}$, converges almost everywhere.*

THEOREM 8.29. [**Bo2**] *Let $T_1, \dots, T_r$ be commuting invertible measure-preserving transformations of a probability measure space X. Then, for any $f \in L^2(X)$ and any polynomials $p_1, \dots, p_r : \mathbb{Z} \longrightarrow \mathbb{Z}$, the sequence $(1/N)\sum_{n=1}^{N} (\prod_{i=1}^{r} T_i^{p_i(n)}) f$, $N \in \mathbb{N}$, converges almost everywhere.*

We now have the following theorem.

THEOREM 8.30. *Let T^t, $t \in \mathbb{R}$, be a continuous action of the semigroup $[0, \infty)$ by measure-preserving transformations on a probability measure space X. Then, for any $f_1, f_2 \in L^2(X)$, $\lim_{b \longrightarrow \infty}(1/b)\int_0^b T^t f_1 \cdot T^{2t} f_2\, dt$ exists almost everywhere.*

Proof. By Theorem 8.28, for every $t \in \mathbb{R}$, the sequence $(1/N)\sum_{n=1}^{N} T^{nt} f_1(\omega) \cdot T^{2nt} f_2(\omega)$, $N \in \mathbb{N}$, converges for almost every $\omega \in X$; let $S_t \subset X$ be the set of points ω for which this is not so. Then $\{(t, \omega) : \omega \in S_t\}$ is a null-subset of $\mathbb{R} \times X$; thus, for almost every $\omega \in X$, the limit $\lim_{N \longrightarrow \infty}(1/N)\sum_{n=1}^{N} T^{nt} f_1(\omega) \cdot T^{2nt} f_2(\omega)$ exists for almost every $t \in \mathbb{R}$. By (the scalar version of) Theorem 4.1, the limit $\lim_{b \longrightarrow \infty}(1/b)\int_0^b T^t f_1(\omega) \cdot T^{2t} f_2(\omega)\, dt$ exists for almost every $\omega \in X$. □

In the same way, from Theorem 8.29 we get the following theorem.

THEOREM 8.31. *Let T^t, $t \in \mathbb{R}^c$, be a continuous c-parameter group of measure-preserving transformations of a probability measure space X. Then, for any $f \in L^2(X)$ and any polynomial $p : \mathbb{R} \longrightarrow \mathbb{R}^c$, $\lim_{b \longrightarrow \infty}(1/b)\int_0^b T^{p(t)} f\, dt$ exists almost everywhere.*

Here are two more pointwise theorems, established by Assani.

THEOREM 8.32. [**A**] *Let T be a weakly mixing measure-preserving transformation of a probability measure space X, let (P, S) be the Pinsker factor of (X, T), and assume that the spectrum of S is singular. Then, for any $f_1, \ldots, f_r \in L^\infty(X)$, the sequence $(1/N)\sum_{n=1}^{N} T^n f_1 \cdot \cdots \cdot T^{rn} f_r$, $N \in \mathbb{N}$, converges to $\prod_{i=1}^{r} \int f_i\, d\mu$ almost everywhere on X.*

THEOREM 8.33. [**A**] *Let T be a weakly mixing measure-preserving transformation of a probability measure space X, let (P, S) be the Pinsker factor of (X, T), and let $L \subseteq L^2(P)$ be the space of functions on P whose spectral measure under the action of S is absolutely continuous with respect to the Lebesgue measure. For any function $f \in L^2(X)$, let $\hat{f}$ denote the projection of $E(f|P)$ to L. Then, for any $f_1, f_2, f_3 \in L^\infty(X)$,*

$$\lim_{N \longrightarrow \infty}\left(\frac{1}{N}\sum_{n=1}^{N} T^n f_1 \cdot T^{2n} f_2 \cdot T^{3n} f_3 - \frac{1}{N}\sum_{n=1}^{N} T^n \hat{f}_1 \cdot T^{2n} \hat{f}_2 \cdot T^{3n} \hat{f}_3\right)$$
$$= 0 \textit{ almost everywhere.}$$

Let T^t, $t \in \mathbb{R}$, be a continuous action of $\mathbb{R}$ by measure-preserving transformations on a measure space X. Then, with the help of either Theorem 3.1 or Theorem 4.1, repeating (the first two phrases from) the proof of Theorem 8.30, and taking into account that (i) if T is weakly mixing, then T^t is weakly mixing for all $t \neq 0$; (ii) the Pinsker algebra of T is the Pinsker algebra of T^t for all $t \neq 0$; and (iii) if the spectrum of T is singular (respectively, absolutely continuous), then the spectrum of T^t is singular (respectively, absolutely continuous) for all $t \neq 0$, we obtain the following theorem.

THEOREM 8.34. *Let T be a continuous action of $\mathbb{R}$ on a probability measure space X by weakly mixing measure-preserving transformations, let (P, S) be the Pinsker factor of*

(X, T), and assume that the spectrum of S is singular. Then, for any $f_1, \ldots, f_r \in L^\infty(X)$, one has

$$\lim_{b\longrightarrow\infty} \frac{1}{b}\int_0^b T^t f_1 \cdot \cdots \cdot T^{rt} f_r \, dt = \prod_{i=1}^r \int f_i \, d\mu \text{ almost everywhere.}$$

THEOREM 8.35. *Let T be a continuous action of $\mathbb{R}$ on a probability measure space X by weakly mixing measure-preserving transformations, let (P, S) be the Pinsker factor of (X, T), and let $L \subseteq L^2(P)$ be the space of functions on P whose spectral measure under the action of S is absolutely continuous with respect to the Lebesgue measure. For any function $f \in L^2(X)$, let $\hat{f}$ denote the projection of $E(f|P)$ to L. Then, for any $f_1, f_2, f_3 \in L^\infty(X)$,*

$$\lim_{b\longrightarrow\infty}\left(\frac{1}{b}\int_0^b T^t f_1 \cdot T^{2t} f_2 \cdot T^{2t} f_3 \, dt - \frac{1}{b}\int_0^b T^t \hat{f}_1 \cdot T^{2t} \hat{f}_2 \cdot T^{3t} \hat{f}_3 \, dt\right)$$
$$= 0 \text{ almost everywhere.}$$

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