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POLYNOMIAL RECURRENCE WITH LARGE INTERSECTION OVER COUNTABLE FIELDS

BY

VITALY BERGELSON* AND DONALD ROBERTSON

Department of Mathematics, The Ohio State University
231 West 18th Avenue, Columbus, OH 43210-1174, USA
e-mail: vitaly@math.ohio-state.edu, robertson@math.ohio-state.edu

ABSTRACT

We give a short proof of polynomial recurrence with large intersection for additive actions of finite-dimensional vector spaces over countable fields on probability spaces, improving upon the known size and structure of the set of strong recurrence times.

1. Introduction

Let F be a countable field and let $\phi \in F[x]$ have zero constant term. Given a measure preserving action T of the additive group of F on a probability space $(X, \mathscr{B}, \mu)$, a set $B \in \mathscr{B}$ and $\varepsilon > 0$, we will show that, for any $\varepsilon > 0$, the set

$$\{u \in F : \mu(B \cap T^{\phi(u)} B) \geq \mu(B)^2 - \varepsilon\}$$

of strong recurrence times is large, in the sense of being IP^*_r up to a set of zero Banach density. (These notions of size are defined below.) In fact, we prove a more general result regarding strong recurrence for commuting actions of countable fields along polynomial powers. This strengthens and extends recent results from [MW14] regarding actions of fields having finite characteristic. Here are the relevant definitions.

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Definition 1.1: Let G be an abelian group. An **IP set** or **finite sums set** in G is any subset of G containing a set of the form

$$\mathrm{FS}(x_1, x_2, \ldots) := \left\{ \sum_{n\in\alpha} x_n : \varnothing \neq \alpha \subset \mathbb{N}, |\alpha| < \infty \right\}$$

for some sequence $n \mapsto x_n$ in G. Given $r \in \mathbb{N}$, an **IP**$_r$ **set** in G is any subset of G containing a set of the form

$$\mathrm{FS}(x_1, x_2, \ldots, x_r) := \left\{ \sum_{n\in\alpha} x_n : \varnothing \neq \alpha \subset \{1, \ldots, r\} \right\}$$

for some $x_1, \ldots, x_r$ in G. A subset of G is IP^* if its intersection with every IP set in G is non-empty, and IP_r^* if its intersection with every IP_r set is non-empty. The term IP was introduced in [FW78], the initials standing for "idempotence" or "infinite-dimensional parallelepiped" and IP_r^* sets were introduced in [FK85]. The **upper Banach density** of a subset S of G is defined by

$$\mathrm{d}^*(S) = \sup\{\mathrm{d}^*_\Phi(S) : \Phi \text{ a Følner sequence in } G\}$$

where

$$\mathrm{d}^*_\Phi(S) = \limsup_{N\to\infty} \frac{|S \cap \Phi_N|}{|\Phi_N|}$$

and a **Følner sequence** is a sequence $N \mapsto \Phi_N$ of finite, non-empty subsets of G such that

$$\lim_{N\to\infty} \frac{|(g + \Phi_N) \cap \Phi_N|}{|\Phi_N|} = 1$$

for all g in G. Lastly, $S \subset G$ is said to be **almost** IP^* (written AIP^*) if it is of the form $A \setminus B$ where A is IP^* and $\mathrm{d}^*(B) = 0$, and said to be **almost** IP_r^* (written AIP_r^*) if it is of the form $A \setminus B$ where A is IP_r^* and $\mathrm{d}^*(B) = 0$.

Although when $G = \mathbb{Z}$ any IP set with non-zero generators is unbounded, this is not the case in general. For example, if $G = \mathbb{Q}$ then the IP set generated by the sequence $n \mapsto 1/n^2$ remains bounded.

To state our result we recall some definitions from [BLM05]. Fix a countable field F. By a **monomial** we mean a mapping $F^n \to F$ of the form $(x_1, \ldots, x_n) \mapsto a x_1^{d_1} \cdots x_n^{d_n}$ for some $a \in F$ and integers $d_1, \ldots, d_n \geq 0$ not all zero. Let V and W be finite-dimensional vector spaces over F. A mapping $F^n \to W$ is a **polynomial** if it is a linear combination of vectors with monomial coefficients. A mapping $V \to W$ is a **polynomial** if, in terms of a basis of V

over F, it is a polynomial mapping $F^n \to W$. Note that whether a mapping is polynomial or not is independent of the basis chosen. Here is our main result.

THEOREM 1.2: *Let W be a finite-dimensional vector space over a countable field F and let T be an action of the additive group of W on a probability space $(X, \mathscr{B}, \mu)$. For any polynomial $\phi : F^n \to W$, any $B \in \mathscr{B}$ and any $\varepsilon > 0$ the set*

$$\{u \in F^n : \mu(B \cap T^{\phi(u)}B) > \mu(B)^2 - \varepsilon\} \tag{1.3}$$

is AIP_r^* *for some $r \in \mathbb{N}$.*

Our result implies in particular that (1.3) is syndetic. In fact, as we will show in Section 3, we have generalized [MW14, Corollary 5], where, in the finite characteristic case, the set (1.3) is shown to belong to every essential idempotent ultrafilter on F. This latter notion of largeness, introduced in [BD08], lies between syndeticity and AIP_r^*.

We also remark that, by our definition of polynomial above, all polynomials have zero constant term. Accordingly Theorem 1.2 says nothing about polynomials with non-zero constant term. It would be interesting to know whether Theorem 1.2 can be extended to a larger class of polynomials, as we have recently done [BR15, Theorem 1.17] for polynomials over rings of integers of algebraic number fields. A positive answer to this question would constitute a common generalization of Theorem 1.2 and a theorem of Larick [Lar98] (see also [BLM05, Theorem 3.10]).

The conclusion of Theorem 1.2 is of an additive nature: the notion of being AIP_r^* is only related to the additive structure of F^n. It is natural to ask, when $n = 1$, whether (1.3) is also large in terms of the multiplicative structure of F. We address this question in Section 4, proving that in fact (1.3) intersects any multiplicatively central set that has positive upper Banach density. Multiplicatively central sets are defined in Section 4 and upper Banach density is as defined above.

Theorem 1.2 is proved in Section 3. In Section 2 we prove the facts we will need about IP_r^* sets. Finally, in Section 4 we relate the largeness of the set (1.3) to the multiplicative structure of F in the case $n = 1$.

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2. Finite IP sets

Let $\mathscr{F}$ be the collection of all finite, non-empty subsets of $\mathbb{N}$. Write $\alpha < \beta$ for elements of $\mathscr{F}$ if $\max\alpha < \min\beta$. A subset of $\mathscr{F}$ is an FU set if it contains a sequence $\alpha_1 < \alpha_2 < \cdots$ from $\mathscr{F}$ and all finite unions of sets from the sequence. Write $\mathscr{F}_r$ for all finite, non-empty subsets of $\{1,\dots,r\}$. A subset of $\mathscr{F}_r$ (or of $\mathscr{F}$) is an FU_s set if it contains sets $\alpha_1 < \cdots < \alpha_s$ from $\mathscr{F}_r$ (or from $\mathscr{F}$) and all finite unions. For any IP_r set $A \supset \mathrm{FS}(x_1,\dots,x_r)$ in an abelian group G there is a map $\mathscr{F}_r \to G$ given by $\alpha \mapsto \sum\{x_i : i \in \alpha\}$, and for any IP set in G there is a map $\mathscr{F} \to G$ defined similarly.

Furstenberg and Katznelson [FK85] showed that any IP_r^* set A in $\mathbb{Z}$ satisfies

$$\liminf_{N\to\infty} \frac{|A \cap \{1,\dots,N\}|}{N} \geq \frac{1}{2^{r-1}}$$

so for any $r \in \mathbb{N}$ one can construct an IP^* set that is not IP_r^*. The set $k\mathbb{N}$, with k large enough, is one such example. As the following example shows, by removing well-spread IP_r sets from $\mathbb{Z}$, it is possible to construct a set that is IP^* but never IP_r^*.

Example 2.1: Let A_r be the IP_r set with generators $x_1 = \cdots = x_r = 2^{2^r}$ so that $A_r = \{i \cdot 2^{2^r} : 1 \leq i \leq r\}$. Let A be the union of all the A_r. We claim that A cannot contain an IP set, from which it follows that $\mathbb{N}\backslash A$ is IP^*. Since A contains IP_r sets for arbitrarily large r we also have that $\mathbb{N}\backslash A$ is not IP_r^* for any r.

Suppose that x_n is a sequence generating an IP set in A. If one can find $x_i \in A_r$ and $x_j \in A_s$ with $r < s$, then $x_j + x_i$ does not belong to A because the gaps in A_s are larger than the largest element in A_r. On the other hand, if all x_i belong to the same A_r, then some combination of them is not in A because the gap between A_r and A_{r+1} is too large.

A family $\mathscr{S}$ of subsets of G is said to have the **Ramsey property** if $S_1 \cup S_2$ belonging to $\mathscr{S}$ always implies that at least one of S_1 or S_2 contains a member of $\mathscr{S}$. It follows from the reformulation of Hindman's theorem [Hin74], stated below, that the collection of all IP subsets of a group G has the Ramsey property. A **coloring** of a set A is any map $c : A \to \{1,\dots,k\}$ for some $k \in \mathbb{N}$. Given a coloring of A, a subset B is then called **monochromatic** if c is constant on B.

THEOREM 2.2 ([Hin74, Corollary 3.3]): *For any coloring of $\mathscr{F}$ one can find $\alpha_1 < \alpha_2 < \cdots$ in $\mathscr{F}$ such that the collection of all finite unions of the sets α_i is monochromatic.*

Given a family $\mathscr{I}$ of subsets of G, the **dual family** of $\mathscr{S}$ is the collection $\mathscr{S}^*$ of subsets of G that intersect every member of $\mathscr{S}$ non-emptily. Taking $\mathscr{S}$ to consist of all IP sets, one can deduce that the intersection of an IP* set with an IP set contains an IP set and that the intersection of two IP* sets is again IP*. The collection of all IP$_r$ sets does not have the Ramsey property, but there is a suitable replacement that allows one to deduce results about IP$^*_r$ sets similar to the ones for IP* sets mentioned above.

Proposition 2.3: *For any s and k in $\mathbb{N}$ there is an r such that any k-coloring of any* IP$_r$ *set yields a monochromatic* IP$_s$ *set.*

Proof. Suppose to the contrary that one can find s and k in $\mathbb{N}$ such that, for any r, there is a k-coloring of an IP$_r$ set A_r having no monochromatic IP$_s$ subset. This coloring of A_r gives rise to a coloring c_r of $\mathscr{F}_r$ via the canonical map $\mathscr{F}_r \to A_r$. That no A_r contains a monochromatic IP$_s$ set implies that no $\mathscr{F}_r$ contains a monochromatic FU$_s$ set. We now use Hindman's theorem to reach a contradiction.

Let α_i be an enumeration of $\mathscr{F}$. We construct a coloring $c : \mathscr{F} \to \{1, \ldots, k\}$ by induction on i. To begin, note that $\alpha_1 \in \mathscr{F}_r$ whenever $r > \max \alpha_1$ so we can find a strictly increasing sequence $r(1, n)$ in $\mathbb{N}$ such that $c_{r(1,n)}(\alpha_1)$ takes the same value for all n. Put $c(\alpha_1) = c_{r(1,n)}(\alpha_1)$. Now, assuming that we have found a strictly increasing sequence $r(i, n)$ such that, for each $1 \le j \le i$, the color $c_{r(i,n)}(\alpha_j)$ is constant in n and equal to $c(\alpha_j)$, choose a strictly increasing subsequence $r(i + 1, n)$ of $r(i, n)$ such that $c_{r(i+1,n)}(\alpha_{i+1})$ is constant and let this value be $c(\alpha_{i+1})$. The colors of $\alpha_1, \ldots, \alpha_i$ are unchanged and the induction argument is concluded.

By Hindman's theorem we can find $\beta_1 < \cdots < \beta_s$ in $\mathscr{F}$ such that $B = \mathrm{FU}(\beta_1, \ldots, \beta_s)$ is monochromatic, meaning c is constant on B. Choose i such that $B \subset \{\alpha_1, \ldots, \alpha_i\}$ and then choose n so large that $r(i, n) > \max \beta_s$. It follows that $B \subset \mathscr{F}_{r(i,n)}$ is monochromatic because $c_{r(i,n)}(\beta) = c(\beta)$ for all $\beta \in B$. Thus $\mathscr{F}_{r(i,n)}$ contains a monochromatic FU$_s$ set, which is a contradiction. ∎

With this version of partition regularity for IP$_r$ sets we can deduce some facts about IP$^*_r$ sets.

Proposition 2.4: *Given any $s \in \mathbb{N}$ there is some $r \in \mathbb{N}$ such that any* IP$^*_s$ *set intersects any* IP$_r$ *set in an* IP$_s$ *set.*

Proof. Let A be an IP_s^* set and choose by the previous proposition some r such that any two-coloring of an IP_r set yields a monochromatic IP_s set. Let B be an IP_r set. One of $B \cap A$ and $B \backslash A$ contains an IP_s set. It cannot be $B \backslash A$ because A is IP_s^* and disjoint from it. Thus $A \cap B$ contains an IP_s set as desired. ∎

Proposition 2.5: *Given any r, s in $\mathbb{N}$ there is some $\alpha(r,s) \in \mathbb{N}$ such that if A is IP_r^* and B is IP_s^* then $A \cap B$ is $\mathrm{IP}_{\alpha(r,s)}^*$.*

Proof. Let A be IP_r^* and let B be IP_s^* with $r \geq s$. Choose q so large that $A \cap C$ contains an IP_r set whenever C is an IP_q set. This is possible by the previous result. Since $A \cap C$ contains an IP_r set and $r \geq s$ the set $(A \cap C) \cap B$ must be non-empty. Since C was arbitrary $A \cap B$ is an IP_q^* set. Put $\alpha(r,s) = q$. ∎

3. Proof of Theorem 1.2

First we note that we may assume, by restricting our attention to the sub-σ-algebra generated by the orbit of B, that the probability space $(X, \mathscr{B}, \mu)$ is separable.

We begin with a corollary of the Hales–Jewett theorem. For any $n \in \mathbb{N}$ write $[n] = \{1, \ldots, n\}$. Write $\mathcal{P}A$ for the set of all subsets of a set A. Recall that, given $k, m \in \mathbb{N}$, a **combinatorial line** in $[k]^{[m]}$ is specified by a partition $U_0 \cup U_1$ of $\{1, \ldots, m\}$ with $U_1 \neq \varnothing$ and a function $\varphi : U_0 \to [k]$, and consists of all functions $[m] \to [k]$ that extend φ and are constant on U_1. With these definitions we can state the Hales–Jewett theorem.

Theorem 3.1 ([HJ63]): *For every $d, t \in \mathbb{N}$ there is $r = \mathrm{HJ}(d,t) \in \mathbb{N}$ such that for any t-coloring of $[d]^{[r]}$ one can find a monochromatic combinatorial line.*

Corollary 3.2: *For any $d, t \in \mathbb{N}$ there is $r \in \mathbb{N}$ such that any t-coloring*

$$(\mathcal{P}\{1, \ldots, r\})^d \to \{1, \ldots, t\}$$

contains a monochromatic configuration of the form

$$\{(\alpha_1 \cup \eta_1, \ldots, \alpha_d \cup \eta_d) : (\eta_1, \ldots, \eta_d) \in \{\varnothing, \gamma\}^d\} \tag{3.3}$$

for some $\gamma, \alpha_1, \ldots, \alpha_d \subset \{1, \ldots, r\}$ with γ non-empty and $\gamma \cap \alpha_i = \varnothing$ for each $1 \leq i \leq d$.

Proof. Let $r = \mathrm{HJ}(2^d, t)$. Define a map $\psi : [2^d]^{[r]} \to (\mathcal{P}[r])^d$ by declaring $\psi(w) = (\alpha_1, \ldots, \alpha_d)$ where α_i consists of those $j \in [r]$ for which the binary expansion of $w(j) - 1$ has a 1 in the ith position. Combinatorial lines in $[2^d]^{[r]}$ correspond via this map to configurations of the form (3.3) in $(\mathcal{P}[r])^d$. ∎

We use the above version of the Hales–Jewett theorem to derive the following topological recurrence result. Given $n \in \mathbb{N}$ and a ring R, by a **monomial mapping** from R^n to R we mean any map of the form $(x_1, \ldots, x_n) \mapsto a x_1^{d_1} \cdots x_n^{d_n}$ for some $a \in R$ and some $d_1, \ldots, d_n \geq 0$ not all zero.

Proposition 3.4 (cf. [Ber10, Theorem 7.7]): *Let R be a commutative ring and let T be an action of the additive group of R on a compact metric space (X, d) by isometries. For any monomial mapping $\phi : R^n \to R$, any $x \in X$ and any $\varepsilon > 0$, there is $r \in \mathbb{N}$ such that the set*

$$\{u \in R^n : \mathsf{d}(T^{\phi(u)} x, x) < \varepsilon\}$$

is IP_r^*.

Proof. Write $\phi(x_1, \ldots, x_n) = a x_1^{d_1} \cdots x_n^{d_n}$ for some $a \in R$ and some $d_i \geq 0$ not all zero. Let $d = d_1 + \cdots + d_n$. Put $e_0 = 0$ and $e_i = d_1 + \cdots + d_i$ for each $1 \leq i \leq n$. Fix $x \in X$ and $\varepsilon > 0$. Let $V_1, \ldots, V_t$ be a cover of X by balls of radius $\varepsilon/2^d$. Let $r = r(d, t)$ be as in Corollary 3.2. Fix $u_1, \ldots, u_r$ in R^n. Given $\alpha \subset \{1, \ldots, r\}$ write u_α for $\Sigma\{u_i : i \in \alpha\}$ and $u_\alpha(i)$ for the ith coordinate of u_α. By choosing for each $(\alpha_1, \ldots, \alpha_d) \in (\mathcal{P}\{1, \ldots, r\})^d$ the minimal $1 \leq i \leq t$ such that

$$T(a u_{\alpha_1}(1) \cdots u_{\alpha_{e_1}}(1) \cdots u_{\alpha_{e_{n-1}+1}}(n) \cdots u_{\alpha_{e_n}}(n)) x \in V_i,$$

we obtain via Theorem 3.2 sets $\alpha_1, \ldots, \alpha_d, \gamma \subset \{1, \ldots, r\}$ with γ non-empty and disjoint from all α_i which, combined with the expansion

$$a u_\gamma(1)^{d_1} \cdots u_\gamma(n)^{d_n} = a \prod_{k=1}^{n} \prod_{i=e_{k-1}+1}^{e_k} u_\gamma(k) + u_{\alpha_i}(k) - u_{\alpha_i}(k)$$

and the fact that T is an isometry, yields $\mathsf{d}(T^{\phi(u_\gamma)} x, x) < \varepsilon$ as desired. ∎

Let G be an abelian group. Actions T_1 and T_2 of G are said to **commute** if $T_1^g T_2^h = T_2^h T_1^g$ for all $g, h \in G$. As we now show, iterating the previous result yields a version for commuting actions of rings.

Corollary 3.5: *Let R be a commutative ring and let $T_1, \ldots, T_k$ be commuting actions of the additive group of R on a compact metric space (X, d) by isometries. For any monomial mappings $\phi_1, \ldots, \phi_k : R^n \to R$, any $x \in X$ and any $\varepsilon > 0$, there is $r \in \mathbb{N}$ such that*

$$\{u \in R^n : \mathsf{d}(T_1^{\phi_1(u)} \cdots T_k^{\phi_k(u)} x, x) < \varepsilon\} \tag{3.6}$$

is IP_r^*.

Proof. Fix $1 \le i \le k$. By applying Proposition 3.4 to the R action $r \mapsto T_i^r$, we can find $r_i \in \mathbb{N}$ such that

$$Z_i = \{u \in R^n : \mathsf{d}(T_i^{\phi_i(u)}x, x) < \varepsilon/k\}$$

is $\mathrm{IP}^*_{r_i}$. By Proposition 2.5, the intersection $Z_1 \cap \cdots \cap Z_k$ is IP^*_r for some $r \in \mathbb{N}$. Since the T_i are isometries, it follows that (3.6) is IP^*_r as desired. ∎

Combining the preceding lemma with the following facts from [BLM05] will lead to a proof of Theorem 1.2. Let $\phi : V \to W$ be a polynomial and let T be an action of W on a probability space $(X, \mathscr{B}, \mu)$. Assume that ϕV spans W. As in [BLM05], say that f in $\mathrm{L}^2(X, \mathscr{B}, \mu)$ is **weakly mixing** for (T, ϕ) if $\text{UD-lim}_v \langle T^{\phi(v)} f, g\rangle = 0$ for all g in $\mathrm{L}^2(X, \mathscr{B}, \mu)$, where UD-lim denotes convergence with respect to the filter of sets whose complements have zero upper Banach density. This is the same as strong Cesàro convergence along every Følner sequence in V. Call $f \in \mathrm{L}^2(X, \mathscr{B}, \mu)$ **compact** for T if $\{T^v f : v \in V\}$ is pre-compact in the norm topology. Denote by $\mathscr{H}_{\mathrm{wm}}(T, \phi)$ the closed subspace of $\mathrm{L}^2(X, \mathscr{B}, \mu)$ spanned by functions that are weakly mixing for (T, ϕ), and let $\mathscr{H}_{\mathrm{c}}(T)$ be the closed subspace of $\mathrm{L}^2(X, \mathscr{B}, \mu)$ spanned by functions compact for T. We have $\mathrm{L}^2(X, \mathscr{B}, \mu) = \mathscr{H}_{\mathrm{c}}(T) \oplus \mathscr{H}_{\mathrm{wm}}(T, \phi)$ by [BLM05, Theorem 3.17].

Proof of Theorem 1.2. Write $\phi = \phi_1 w_1 + \cdots + \phi_k w_k$ where the ϕ_i are monomials $F^n \to F$ and the w_i belong to V. Fix B in $\mathscr{B}$ and $\varepsilon > 0$. Let $f = P1_B$ be the orthogonal projection of 1_B on $\mathscr{H}_{\mathrm{c}}(T)$. Let Ω be the orbit closure of f in the norm topology under T. Since f is compact, Ω is a compact metric space. Applying Lemma 3.5 to the F actions $x \mapsto T^{xw_i}$ and monomials ϕ_i for $1 \le i \le k$, we see that

$$\{u \in F^n : \|f - T^{\phi(u)} f\| < \varepsilon/2\}$$

is IP^*_r. We have

$$\langle T^{\phi(u)} 1_B, 1_B\rangle = \langle T^{\phi(u)} f, 1_B\rangle + \langle T^{\phi(u)}(1_B - f), 1_B\rangle$$

so the set

$$\{u \in F^n : \langle T^{\phi(u)} 1_B, 1_B\rangle \ge \langle f, 1_B\rangle - \varepsilon/2 + \langle T^{\phi(u)}(1_B - f), 1_B\rangle$$

is IP^*_r. Since $1_B - f$ is weakly mixing for (T, ϕ) the set

$$\{u \in F^n : \langle T^{\phi(u)} 1_B, 1_B\rangle \ge \langle f, 1_B\rangle - \varepsilon\}$$

is AIP_r^*. Thus (1.3) is AIP_r^* by

$$\langle f, 1_B \rangle = \langle P1_B, P1_B \rangle \langle 1, 1 \rangle \geq \langle P1_B, 1 \rangle^2 = \mu(B)^2$$

as desired. ∎

We obtain as a corollary the following result from [MW14], which uses the following terminology. Let again G be an abelian group. An ultrafilter p on G is **essential** if it is idempotent and $\mathrm{d}^*(A) > 0$ for all $A \in \mathsf{p}$. A **D set** in G is any subset of G that belongs to an essential ultrafilter on G, and a subset of G is D^* if its intersection with any D is non-empty.

Corollary 3.7 ([MW14, Corollary 5]): *Let F be a countable field of finite characteristic and let $p : F \to F^n$ be a polynomial mapping. For any action T of F^n on a probability space $(X, \mathscr{B}, \mu)$, any B in $\mathscr{B}$ and any $\varepsilon > 0$, the set*

$$\{x \in F : \mu(B \cap T^{p(x)} B) \geq \mu(B)^2 - \varepsilon\} \tag{3.8}$$

is D^*.

Proof. It follows from the proof of Theorem 1.2 that (3.8) is of the form $A \setminus B$ where A is IP_r^* for some $r \in \mathbb{N}$ and B has zero upper Banach density. Any IP_r^* subset of G is IP^* and therefore belongs to every idempotent ultrafilter on G, so A certainly belongs to every essential ultrafilter on G. By the filter property, removing from A a set of zero upper Banach density does not change this fact, because every set in an essential idempotent has positive upper Banach density. ∎

It has recently been shown [MZ14] that there are D^* subsets of $\mathbb{Z}$ that are not AIP^*. This is also the case in countable fields of finite characteristic [McC14]. Thus our result constitutes a genuine strengthening of Corollary 3.7.

4. Multiplicative structure

According to Theorem 1.2 the set (1.3) is large in terms of the additive structure of F^n. In this section we connect the largeness of (1.3) when $n = 1$ to the multiplicative structure of F by showing that (1.3) is almost an MC^* subset of F. Here MC stands for **multiplicatively central** and a set is MC^* if its intersection with every multiplicatively central set is non-empty.

To define what a multiplicatively central set is, recall that, given a commutative ring R, we can extend the multiplication on R to a binary operation $*$ on the set βR of all ultrafilters on R by

$$\mathsf{p} * \mathsf{q} = \{A \subset R : \{u \in R : Au^{-1} \in \mathsf{p}\} \in \mathsf{q}\}$$

for all $\mathsf{p}, \mathsf{q} \in \beta R$. One can check that this makes βR a semigroup. It is also possible to equip βR with a compact, Hausdorff topology with respect to which the binary operation is right continuous. See [Ber03] or [HS12] for the details of these constructions. A subset A of R is then called **multiplicatively central** or **MC** if it belongs to an ultrafilter that is both idempotent and contained in a minimal right ideal of βR. The following version of [BH94, Theorem 3.5] relates IP_r sets in R to multiplicatively central sets.

Proposition 4.1: *Let R be a commutative ring and let $A \subset R$ be a multiplicatively central set. For every $r \in \mathbb{N}$ one can find $x_1, \ldots, x_r$ in R such that $\mathrm{FS}(x_1, \ldots, x_r) \subset A$.*

Proof. Consider the family T of ultrafilters p on R having the property that every set in p contains an IP_r set for every $r \in \mathbb{N}$. We claim that T is a two-sided ideal in βR. Indeed fix $\mathsf{p} \in T$ and $\mathsf{q} \in \beta R$. We need to prove that $\mathsf{p} * \mathsf{q}$ and $\mathsf{q} * \mathsf{p}$ belong to T.

For the former, fix $B \in \mathsf{p} * \mathsf{q}$ and $r \in \mathbb{N}$. We can find $u \in R$ such that $Bu^{-1} \in \mathsf{p}$ so Bu^{-1} contains $\mathrm{FS}(x_1, \ldots, x_r)$ for some $x_1, \ldots, x_r$ in R. This immediately implies that $\mathrm{FS}(x_1 u, \ldots, x_r u) \subset B$ as desired.

For the latter, fix $B \in \mathsf{q} * \mathsf{p}$ and $r \in \mathbb{N}$. We can find $x_1, \ldots, x_r$ in R such that $\mathrm{FS}(x_1, \ldots, x_r) \subset \{u \in R : Bu^{-1} \in \mathsf{q}\}$. But by the filter property

$$\cap\{Bu^{-1} : u \in \mathrm{FS}(x_1, \ldots, x_r)\} \in \mathsf{q} \tag{4.2}$$

and choosing a from this intersection gives $\mathrm{FS}(ax_1, \ldots, ax_r) \subset B$.

Our set A is multiplicatively central so it is contained in some idempotent ultrafilter p that belongs to a minimal right ideal S. Since T is also a right ideal $S \subset T$ and $\mathsf{p} \in T$ as desired. ∎

Note that it is not possible to prove this way that multiplicatively central sets contain IP sets, as that would require an infinite intersection in (4.2). In fact, as shown in [BH94, Theorem 3.6], there are multiplicatively central sets in $\mathbb{N}$ that do not contain IP sets.

We say that a subset of R is MC* if its intersection with every multiplicatively central set is non-empty. As noted in [Ber10], the preceding result implies that every IP$_r^*$ set is MC*. Call a set AMC* (with A again standing for "almost") if it is of the form $A \setminus B$ where A is MC* and B has zero upper Banach density in $(R, +)$. The following result is then an immediate consequence of Theorem 1.2.

Theorem 4.3: *Let F be a countable field and let T be an action of the additive group of F on a probability space $(X, \mathscr{B}, \mu)$. For any polynomial $\phi \in F[x]$, any $B \in \mathscr{B}$ and any $\varepsilon > 0$, the set*

$$\{u \in F : \mu(B \cap T^{\phi(u)} B) > \mu(B)^2 - \varepsilon\} \tag{4.4}$$

is AMC*.

We conclude by mentioning that all AMC* sets have positive upper Banach density in $(F, +)$. This follows from the fact that every MC* set belongs to every minimal multiplicative idempotent, and a straightforward generalization of [BH90, Theorem 5.6], which guarantees the existence of a minimal idempotent for $*$ all of whose members have positive upper Banach density in $(F, +)$.

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