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Ultrafilters, IP sets, Dynamics, and Combinatorial Number Theory

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Abstract. We survey the connection between ultrafilters, ergodic theory, and combinatorics.

1. Introduction

The main theme of this survey is the multifaceted and mutually perpetuating connection between ultrafilters, ergodic theory, and combinatorics. In this short introductory section we collect some general definitions and facts about ultrafilters. The reader will find more detail and discussion in [**Ber2**], [**Ber4**], and [**Ber6**]. For a comprehensive treatment of topological algebra in the Stone-Čech compactification the reader is referred to [**HS**].

Given a set S, a filter on S is a nonempty family $\mathcal{S} \subset \mathcal{P}(S)$ (where $\mathcal{P}(S)$ denotes the power set of S) with the properties

(i) $\emptyset \notin \mathcal{S}$,

(ii) $A \in \mathcal{S}$ and $A \subseteq B$ implies $B \in \mathcal{S}$,

(iii) $A \in \mathcal{S}$ and $B \in \mathcal{S}$ implies $A \cap B \in \mathcal{S}$.

A filter $\mathcal{S}$ is an *ultrafilter* if, in addition, it satisfies

(iv) For any $A \subseteq S$ either $A \in \mathcal{S}$ or $A^c \in \mathcal{S}$ (but not both, in view of (i) and (iii)).

It is not hard to see that property (iv) is equivalent to

(v) If, for some integer r, $S = A_1 \cup \ldots \cup A_r$, then for some i, $1 \leq i \leq r$, one has $A_i \in \mathcal{S}$.

Indeed, (v) immediately implies (iv) by considering $S = A \cup A^c$. On the other hand, (iv) implies (v) since if, say, $A_1 \notin \mathcal{S}$ then $A_1^c \in \mathcal{S}$ which implies, by (ii), that $S_1 = A_2 \cup \ldots \cup A_r \in \mathcal{S}$, and the rest of the argument is clear.

It is easy to come up with numerous examples of filters (just take your favorite family of sets with the finite intersection property). On the other hand, when S is infinite, the only explicit example of an ultrafilter is of a somewhat degenerate nature and can be described as follows. Fix an element $a \in S$ and let

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$$A = \{R \subseteq S : a \in R\}.$$

Ultrafilters of this form are in 1-1 correspondence with elements of S and are called *principal*. All the other ultrafilters on S are called nonprincipal. When S is infinite, the proof of existence of nonprincipal ultrafilters requires Zorn's lemma - see [**CoN**], pp. 161-162. So, we have a somewhat strange situation: on one hand, one can show that the cardinality of the set of ultrafilters on S is $2^{2^{|S|}}$ (where $|S|$ denotes the cardinality of S, see [**Pos**]) but, on the other hand, it is not clear how to envision them.

Still, recognizing some members of a given ultrafilter (or, at least, knowing something about properties of members of an ultrafilter) often provides useful insights and leads to interesting applications.

A typical statement in *partition* Ramsey theory has the following (admittedly, vague) form:

> (P) For any finite partition of an infinite "well organized" set S,
> one of the cells of the partition is also "well organized".[1]

For example, put $S \subseteq \mathbb{N} = \{1, 2, \ldots\}$ and interpret "well organized" as :
(i) containing arbitrarily long arithmetic progressions;
(ii) containing a *finite sums set*, that is a set of the form
$FS((x_i)_{i\in\mathbb{N}}) = \{x_{i_1} + x_{i_2} + \ldots + x_{i_k} : i_1 < i_2 < \ldots < i_k; k \in \mathbb{N}\}$[2],
where $(x_i)_{i\in\mathbb{N}}$ is (for convenience) an increasing sequence in $\mathbb{N}$.

Then the statement (P) holds for each of the two interpretations of "well organized" and corresponds to (i) van der Waerden's theorem ([**vdW**]), and (ii) Hindman's finite sums theorem ([**H1**]).

The following observation due to Hindman shows that there is a natural connection between (correct) statements as above and ultrafilters.

Theorem ([**H2**], Theorem 6.7). *Let $\mathcal{A}$ be a family of subsets of a set S. The following statements are equivalent.*

(i) For any finite partition $S = \bigcup_{i=1}^{r} C_i$, one of the C_i belongs to $\mathcal{A}$

(ii) There exists an ultrafilter $\mathcal{S}$ on S such that for every $A \in \mathcal{S}$ there exists $G \in \mathcal{A}$ with $G \subseteq A$.

Assume now that $(S, \cdot)$ is a countably infinite discrete semigroup. The semigroup operation on S allows one to naturally transform the set of ultrafilters on S into a left topological semigroup. This left topological semigroup is nothing else but βS, the Stone-Čech compactification of S. See [**HS**] for more information and discussion. To define said operation it will be convenient to modify our point of view on ultrafilters.

Namely, rather than viewing an ultrafilter as a *maximal filter* (that is, a family of sets satisfying the properties (i),(ii), (iii), and (iv) or (v) above), we will find it useful to interpret the ultrafilters as $\{0, 1\}$-valued finitely additive probability measures on $\mathcal{P}(S)$. Indeed, given an ultrafilter $\mathcal{S}$ on S, assign to a set $A \subseteq S$ measure 1 if and only if $A \in \mathcal{S}$. It is easy to see that this indeed defines a finitely additive probability measure on $\mathcal{P}(S)$ (the formula (v) being responsible for its

[1] Compare with statement (D) in Section 7.

[2] Such sets are also called *IP sets*, a term introduced in [**FW**].

finite additivity). In the other direction, it is easy to see that given any $\{0,1\}$-valued finitely additive probability measure μ on $\mathcal{P}(S)$, the set of μ-large sets (that is, sets having measure 1) forms an ultrafilter.

At this point we, in accordance with a well established tradition, will switch notation and denote ultrafilter measures by lower case letters p, q, r, s, t, etc.

The perspicacious reader will observe that the following definition of product of ultrafilters is nothing but convolution of measures.

Definition. Let p, q be ultrafilters on a semigroup $(S,\cdot)$. The ultrafilter $p\cdot q$ is defined by

$$A \in p\cdot q \Leftrightarrow \{x\in S : Ax^{-1}\in p\}\in q.$$

(The set Ax^{-1} is defined by the rule $y\in Ax^{-1}$ iff $yx\in A$).

It is a routine exercise to verify that this operation is well defined and associative (see, for example, [**H2**], Lemma 8.4). We also would like to remark that, when restricted to principal ultrafilters, the above operation coincides with the operation on $(S,\cdot)$ (we identify the principal ultrafilters with the elements of S).

Finally, we describe the topology on βA. Given $A\subseteq S$, let $\bar{A}=\{p\in\beta S : A\in p\}$. The collection $\{\bar{A} : A\subseteq S\}$ forms a basis for the open sets (and for closed sets as well) and makes βS into a (non-metrizable) compact Hausdorff space. One can also verify that for any fixed $p\in\beta S$, the function $\lambda_p(q)=p\cdot q$ is a continuous self-map of βS (see [**H2**] and [**HS**] for more information).

With the operation introduced above, $(\beta S,\cdot)$ becomes a compact *left topological* semigroup. By Ellis' Theorem, which we will presently formulate, $(\beta S,\cdot)$ has idempotents, that is, elements satisfying $x\cdot x=x$.

This result will be utilized numerous times during the course of this survey. When $S=\mathbb{N}$, one actually has two semigroup structures $(\mathbb{N},+)$ and $(\mathbb{N},\cdot)$ and, consequently, one has additive idempotents and multiplicative idempotents in $\beta\mathbb{N}$. The additive idempotents are elements of $(\beta\mathbb{N},+)$ and satisfy the equation $p+p=p$, where $+$ denotes the extension of the operation $+$ to $\beta\mathbb{N}$. Similarly, the multiplicative idempotents are elements of $(\beta\mathbb{N},\cdot)$ and satisfy the equation $q\cdot q=q$. It is the interplay between the two operations on $\beta\mathbb{N}$ which is behind many Ramsey-theoretical results that will be discussed below.

Theorem (Ellis, [**E**]). *If $(G,\star)$ is a compact left topological semigroup (i.e. for any $x\in G$ the function $\lambda_x(y)=x\star y$ is continuous), then G has an idempotent.*

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2. Hindman's Finite Sums Theorem

We start our discussion with Hindman's (by now classical) finite sums theorem.

Theorem 2.1 ([**H1**]). *For any finite partition $\bigcup_{i=1}^{r} C_i$, there exist an infinite sequence $(n_i)_{i\in\mathbb{N}}\subset\mathbb{N}$ and $j\in\{1,2,\ldots,r\}$ such that C_j contains the set*

$$FS((n_i)_{i\in\mathbb{N}})=\{n_{i_1}+n_{i_2}+\ldots+n_{i_k} : i_1<i_2<\ldots<i_k;\ k\in\mathbb{N}\}.$$

The original proof of Theorem 2.1 in [**H1**] was elementary but very complicated. The short proof via ultrafilters which we will present below was found somewhat later and is due to F. Galvin and S. Glazer (see the account of the history of the ultrafilter proof in [**H5**] and [**HS**], pp. 102-103). To stress the dynamical nature of the ultrafilter proof of Theorem 2.1, we will make first a short digression into some basic ergodic theory stemming from the classical work of Poincaré on celestial mechanics ([**Poi1**], [**Poi2**]). This digression will also allow us to motivate the introduction of certain notions of largeness which will be utilized in subsequent sections.

Let $(X, \mathcal{B}, \mu)$ be a probability space and $T : X \to X$ a measure preserving transformation[3], meaning that for any set $A \in \mathcal{B}$ one has $\mu(T^{-1}A) = \mu(A)$.

The classical Poincaré recurrence theorem[4] asserts that for any $A \in \mathcal{B}$ with $\mu(A) > 0$ almost every point $x \in A$ returns to A under some nonzero power of T. More formally, Poincaré's recurrence theorem states that $\forall A \in \mathcal{B}$ with $\mu(A) > 0$, one has

$$\mu(\{x \in A : \exists n \in \mathbb{N} \text{ such that } T^n x \in A\}) = \mu(A). \tag{2.1}$$

Poincaré's recurrence theorem immediately follows from the following statement.

Proposition 2.2. [5] *For any measure preserving system $(X, \mathcal{B}, \mu, T)$, and any $A \in \mathcal{B}$ with $\mu(A) > 0$, there exists $n \in \mathbb{N}$ such that*

$$\mu(A \cap T^{-n}A) > 0. \tag{2.2}$$

Proof. Let $(m_i)_{i\in\mathbb{N}} \subseteq \mathbb{N}$ be an arbitrary increasing sequence, and consider the sets $T^{-m_i}A$, $i \in \mathbb{N}$. Since T is measure preserving, one has $\mu(T^{-m_i}A) = \mu(A)\ \forall i \in \mathbb{N}$. If $k > \frac{1}{\mu(A)}$ then, due to the fact that μ is an additive function on $\mathcal{B}$ and $\mu(X) = 1$, there exist $1 \le i < j \le k$ such that
$\mu(T^{-m_i}A \cap T^{-m_j}A) = \mu(A \cap T^{-(m_j - m_i)}A) > 0$, and so $n = m_j - m_i$ satisfies (2.2). ☐

Remark 2.3. The above proof works for any finitely additive probability measure. This rather trivial observation will be utilized below in the ultrafilter proof of Hindman's finite sums theorem.

Remark 2.4. Given r integers $n_1 < n_2 < \ldots < n_r$, the set of differences $\{n_j - n_i : 1 \le i < j \le r\}$ is called a Δ_r set. A set $E \subseteq \mathbb{N}$ is called Δ_r^* if it has nontrivial intersection with any Δ_r set.[6] What was actually shown in the course of

[3]The quadruple $(X, \mathcal{B}, \mu, T)$, where $(X, \mathcal{B}, \mu)$ is a probability space and $T : X \to X$ is measure-preserving, is called a measure-preserving system.

[4]cf. [**Poi1**], § 8 and [**Poi2**], §§ 291-296.

[5]To derive (2.1) from Proposition 2.2, one argues as follows. Let A_0 be the (measurable!) set $\{x \in A : (\forall n \in \mathbb{N})\ T^n x \notin A\}$. If $\mu(A_0) > 0$ then for some $n \in \mathbb{N}$ one will have $\mu(A_0 \cap T^{-n}A_0) > 0$. But then for any $x \in A_0 \cap T^{-n}A_0$ one will have $T^n x \in A_0$, which gives a contradiction.

[6]We have here an instance of a natural way of introducing a notion of largeness. More generally, given a family $\mathcal{A}$ of subsets of a set A, one defines a dual family $\mathcal{A}^* = \{S \subseteq A : \forall B \in \mathcal{A}, S \cap B \ne \emptyset\}$. We will encounter many examples of important dual families, the most important of which is the family of *IP** sets, that is, the family of sets having nontrivial intersection with any IP set.

the proof of Proposition 2.2 is that the set $R_A = \{n : \mu(A \cap T^{-n}A) > 0\}$ is a Δ_r^* set for any $r > 1/\mu(A)$. This, in turn, implies that R_A is a *syndetic* set, that is, a set which has a nontrivial intersection with any long enough interval. Indeed, if this was not the case, R_A^c would contain arbitrarily long intervals, which leads to a contradiction since, as it is not hard to see, for any fixed r, any sufficiently long interval contains a Δ_r set.[7]

Let $(X, \mathcal{B}, \mu, T)$ be a measure preserving system. Let $A \in \mathcal{B}$ with $\mu(A) > 0$. By Poincaré's recurrence theorem, we can find $n_1 \in \mathbb{N}$ such that $\mu(A \cap T^{-n_1}A) > 0$. Applying Poincaré's recurrence theorem again to the set $A_1 = A \cap T^{-n_1}A$, we can find $n_2 > n_1$ such that

$$\begin{aligned} \mu(A_1 \cap T^{-n_2}A_1) &= \mu((A \cap T^{-n_1}A) \cap T^{-n_2}(A \cap T^{-n_1}A)) \\ &= \mu(A \cap T^{-n_1}A \cap T^{-n_2}A \cap T^{-(n_1+n_2)}A) > 0. \end{aligned}$$

Continuing in this manner we will obtain an infinite sequence $(n_i)_{i\in\mathbb{N}} \subseteq \mathbb{N}$ such that for each element $m \in FS((n_i)_{i\in\mathbb{N}})$ one has $\mu(A \cap T^{-m}A) > 0$. We see that the finite sums sets naturally appear in the process of repeated applications of Poincaré's recurrence theorem. Since ultrafilters on $(\mathbb{N}, +)$ are finitely additive probability measures, one could use this "iterative" approach to get a proof of Hindman's finite sums theorem if, in addition, the operation $n \mapsto n+1$ could be interpreted as "measure-preserving". We will momentarily see that, for the idempotent ultrafilters in $(\beta\mathbb{N}, +)$, something like this is the case.

Let $p \in (\beta\mathbb{N}, +)$ satisfy $p + p = p$. By the definition of the operation $+$ in $\beta\mathbb{N}$ (see Introduction), we have

$$A \in p \Leftrightarrow A \in p + p \Leftrightarrow \{n \in \mathbb{N} : (A - n) \in p\} \in p. \tag{2.3}$$

Formula (2.3) implies that if A is p-large, then, for p-many $n \in \mathbb{N}$, the set $A - n$ is also p-large. This is the translation-invariance we were looking for.

Proof of Theorem 2.1. Let a partition $\bigcup_{i=1}^r C_i$ be given. Let $p = p + p$ be an idempotent ultrafilter. Then one of the cells of the partition, call it C, is p-large. By (2.3), the set $\{n : (C - n) \in p\}$ is also p-large and hence one can find $n_1 \in C \cap \{n : (C - n) \in p\}$ such that $C_1 = C \cap (C - n_1) \in p$. Repeating this procedure, let $n_2 \in C_1 \cap \{n : (C_1 - n) \in p\}$ be such that $n_2 > n_1$ and $C_2 = C_1 \cap (C_1 - n_2) = C \cap (C - n_1) \cap (C - n_2) \cap (C - (n_1 + n_2)) \in p$. Note that $n_1, n_2, n_1 + n_2 \in C$. Choosing $n_3 \in C_2 \cap \{n : (C_2 - n) \in p\}$ will give us $FS((n_i)_{i=1}^3) \subseteq C$. Continuing in this way, we will obtain an increasing sequence $(n_i)_{i\in\mathbb{N}}$ such that, for any $k \in \mathbb{N}$, $FS((n_i)_{i=1}^k) \subseteq C$. We are done. $\square$

The (proof of) Theorem 2.1 tells us that if $p \in (\beta\mathbb{N}, +)$ is an idempotent, then any p-large set A contains an IP set $FS((n_i)_{i\in\mathbb{N}})$. The proof, however, does not guarantee that the set $FS((n_i)_{i\in\mathbb{N}})$ obtained in the course of the proof is itself p-large. Moreover, it is easy to see that there are IP sets in A which cannot be p-large. The following proposition (attributed in [**HS**] to F. Galwin) shows that nevertheless, for any IP set E there exists an idempotent $q \in (\beta\mathbb{N}, +)$ such that $E \in q$.

[7] The sets in $(\mathbb{N}, +)$ (or in $(\mathbb{Z}, +)$) which contain arbitrarily long intervals are called *thick*. If $\mathcal{T}$ denotes the family of thick sets and $\mathcal{S}$ denotes the family of syndetic sets, then, clearly, $\mathcal{T}^* = \mathcal{S}$ and $\mathcal{S}^* = \mathcal{T}$. We will see below that these notions can be meaningfully defined in any semigroup.

Theorem 2.5. *Given any sequence $(n_i)_{i\in\mathbb{N}} \subseteq \mathbb{N}$, there is an idempotent $p \in (\beta\mathbb{N}, +)$ such that, for any $m \in \mathbb{N}$, $FS((n_i)_{i=m}^{\infty}) \in p$.*

Proof. Let $\overline{FS((n_i)_{i=m}^{\infty})}$ denote the closure in $\beta\mathbb{N}$, and let

$$S = \bigcap_{m=1}^{\infty} \overline{FS((n_i)_{i=m}^{\infty})}.$$

S is an intersection of a decreasing sequence of compact sets and hence is compact and nonempty. We will show now that S is a semigroup. Let $p, q \in S$. To show that $p + q \in S$ one needs to verify that, for any $m \in \mathbb{N}$, $A = FS((n_i)_{i=m}^{\infty}) \in p + q$, which is equivalent to showing that $\{x \in \mathbb{N} : (A - x) \in p\} \in q$. Let $a \in A$. Then $a = n_{i_1} + n_{i_2} + \ldots + n_{i_l}$, where $m \le n_{i_1} < n_{i_2} < \ldots < n_{i_l}$. Let $k = l + 1$. Then $FS((n_i)_{i=k}^{\infty}) \subseteq A - a$. But $FS((n_i)_{i=k}^{\infty}) \in p$ which implies that $A - a \in p$. So

$$A \subseteq \{x \in \mathbb{N} : (A - x) \in p\} \in q,$$

and we are done. □

Theorem 2.5 gives an easy answer to another important question: which ultrafilters (besides the idempotent ones) have the property that their members contain IP sets?

Let Γ be the closure in $\beta\mathbb{N}$ of the (nonempty!) set of idempotents:

$$\Gamma = \mathrm{cl}\{p \in (\beta\mathbb{N}, +) : p + p = p\}.$$

Theorem 2.6. *An ultrafilter p belongs to Γ if and only if every p-large set contains an IP set.*

Proof. $\Rightarrow$: Let $p \in \Gamma$ and let $A \in p$. Then $\bar{A}$ is a neighborhood of p in $\beta\mathbb{N}$ so there is $q \in \beta\mathbb{N}$ such that $q = q + q$ and $q \in \bar{A}$, or, which is the same, $A \in q$. Then, by Theorem 2.1, A has to contain an IP set.

$\Leftarrow$: Let p be given and assume that every $A \in p$ contains an IP set. We have to show that $p \in \Gamma$. Fix $A \in p$ and let $E \subseteq A$ be an IP set. Then, by Theorem 2.5 there is an idempotent $q = q + q$ such that $E \in q$. This implies that $q \in \bar{E}$ and hence $q \in \bar{A}$. So we see that, for any $A \in p$, $\bar{A} \cap \{q \in (\beta\mathbb{N}, +) : q + q = q\} \neq \emptyset$. This implies that $p \in \Gamma$. □

3. Many Equivalent Forms of Hindman's Finite Sums Theorem

We start this section with the observation that from Theorem 2.1 one can easily derive its multiplicative analog.

Theorem 3.1. *For any finite partition $\mathbb{N} = \bigcup_{i=1}^{r} B_i$, one of the B_i contains a finite products set, namely a set of the form*

$$FP((n_i)_{i\in\mathbb{N}}) = \{n_{i_1} \cdot n_{i_2} \cdot \ldots \cdot n_{i_k} : i_1 < i_2 < \ldots < i_k; k \in \mathbb{N}\}.^{8}$$

Proof. Let $C_i = \{n \in \mathbb{N} : 2^n \in B_i\}$ and apply Theorem 2.1. □

Remark 3.2. Another approach to Theorem 3.1 is to invoke the existence of idempotents in $(\beta\mathbb{N}, \cdot)$ and to mimic the proof of Theorem 2.1. This approach leads to a stronger result since it shows that for any multiplicative idempotent $p \in (\beta\mathbb{N}, \cdot)$, any $A \in p$ contains a multiplicative IP set.

[8] We will also call such a set a *multiplicative IP set* .

The above remark can be applied to any semigroup $(S,\cdot)$, since, by Ellis' theorem, $(\beta S,\cdot)$ always has an idempotent. So we have the following general result.

THEOREM 3.3. *Let $(S,\cdot)$ be a semigroup. For any finite partition $S=\bigcup_{i=1}^{r} C_i$, one of the C_i contains a set of the form*

$$FP((x_i)_{i\in\mathbb{N}})=\{x_{i_k}\cdot x_{i_{k-1}}\cdot\ldots\cdot x_{i_1}:i_1<i_2<\ldots<i_k;k\in\mathbb{N}\}.[9]$$

We will introduce now one more, set-theoretical, version of Theorem 2.1, which is often utilized in various applications.

Let $\mathcal{F}$ denote the family of all finite nonempty subsets of $\mathbb{N}$.[10] $\mathcal{F}$ forms a natural semigroup with respect to the operation of taking unions. Applying Theorem 3.3 to $(\mathcal{F},\cup)$, one obtains the fact that for any finite partition $\mathcal{F}=\bigcup_{i=1}^{r} C_i$, one of the C_i has to contain a *finite unions set* of the form

$$FU((\alpha_i)_{i\in\mathbb{N}})=\{\alpha_{i_1}\cup\alpha_{i_2}\cup\ldots\cup\alpha_{i_k}:i_1<i_2<\ldots<i_k;k\in\mathbb{N}\}.$$

Unfortunately, this formulation, due to the idempotent nature of the operation $\cup$, is not strong enough to be useful. The following enhanced version is free of this flaw.

THEOREM 3.4 (cf. [**Ba**]). *For any partition $\mathcal{F}=\bigcup_{i=1}^{r} C_i$ there exist $j\in\{1,2,\ldots,r\}$ and a sequence $(\alpha_i)_{i\in\mathbb{N}}\subseteq\mathcal{F}$ such that*

(i) $\min\alpha_{k+1}>\max\alpha_k$ for each $k\in\mathbb{N}$.

(ii) $\forall\beta\in\mathcal{F}$, $\bigcup_{t\in\beta}\alpha_t\in C_j$.

An ostensibly stronger version of the finite sums theorem states that given an IP set $A=FS((n_i)_{i\in\mathbb{N}})\subseteq\mathbb{N}$ and a finite coloring $A=\bigcup_{i=1}^{r} C_i$, one of the C_i contains an IP set.

This fact, however, is just one more equivalent form of the finite sums theorem.

THEOREM 3.5 ([**BerHi3**], Lemma 2.1). *The following statements are equivalent.*

(i) Let $(S,\cdot)$ be a semigroup, let $r\in\mathbb{N}$ and let $(x_n)_{n\in\mathbb{N}}\subseteq S$. If $FP((x_n)_{n\in\mathbb{N}})=\bigcup_{i=1}^{r} C_i$, then there is $i\in\{1,2,\ldots,r\}$ and a sequence $(y_n)_{n\in\mathbb{N}}\subseteq S$, such that $FP((y_n)_{n\in\mathbb{N}})\subseteq C_i$.

(ii) Let $r\in\mathbb{N}$ and let $\mathbb{N}=\bigcup_{i=1}^{r} C_i$. There exist $i\in\{1,2,\ldots,r\}$ and a sequence $(x_n)_{n\in\mathbb{N}}\subseteq\mathbb{N}$ such that $FS((x_n)_{n\in\mathbb{N}})\in C_i$.

(iii) Let $r\in\mathbb{N}$ and let $\mathcal{F}=\bigcup_{i=1}^{r} C_i$. There exist $i\in\{1,2,\ldots,r\}$ and a sequence $(\alpha_n)_{n\in\mathbb{N}}\subseteq\mathcal{F}$ such that $\min\alpha_{n+1}>\max\alpha_n$ for each $n\in\mathbb{N}$ and $\cup_{n\in\beta}\alpha_n\in C_i$ whenever $\beta\in\mathcal{F}$.

PROOF. (i) $\Longrightarrow$ (ii): This implication immediately follows from the fact that $(\mathbb{N},+)$ is $\mathbb{N}=FS((2^{n-1})_{n\in\mathbb{N}})$.

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[9] Note that the finite products set $FP((x_i)_{i\in\mathbb{N}})$ is made of products $x_{i_k}\cdot x_{i_{k-1}}\cdot\ldots\cdot x_{i_1}$ in decreasing order of indices. By switching the operation in $(S,\cdot)$ from $x\cdot y$ to $y\cdot x$ (which affects the operation in βS as well), one can guarantee the products in the increasing order as well. Of course, when S is commutative, one does not have to care about such things.

[10] Note that the elements of $FS((n_i)_{i\in\mathbb{N}})$ are naturally indexed by the elements of $\mathcal{F}$: for any $\alpha=\{i_1,i_2,\ldots,i_k\}\in\mathcal{F}$, let $n_\alpha=\sum_{i\in\alpha}n_i=n_{i_1}+\ldots+n_{i_k}$.

This observation applies of course to multiplicative IP sets as well.

(ii) $\Longrightarrow$ (iii): Let $\mathcal{F} = \bigcup_{i=1}^{r} C_i$. Let, for $i \in \{1, 2, \ldots, r\}$, $A_i = \{\sum_{n\in\alpha} 2^n : \alpha \in C_i\}$, and let A_0 be the set of odd natural numbers. Pick $i \in \{1, 2, \ldots, r\}$ and a sequence $(x_n)_{n\in\mathbb{N}} \subseteq \mathbb{N}$ with $FS((x_n)_{n\in\mathbb{N}}) \subseteq A_i$.

Let $\gamma_1 = \{1\}$ and $y_1 = \gamma_1$. Inductively, given $\gamma_n \in \mathcal{F}$ and $y_n = \sum_{i\in\gamma_n} x_i$, pick $\alpha_n \in \mathcal{F}$ such that $y_n = \sum_{i\in\alpha_n} 2^i$, let $l = \max \alpha_n$ and $m = \max \gamma_n$. Let γ_{n+1} consist of 2^{l+1} members of $\{m+1, m+2, \ldots\}$ such that for any $t, s \in \gamma_{n+1}$ one has $x_t \equiv x_s \mod 2^{l+1}$. Then, letting $y_{n+1} = \sum_{i\in\gamma_{n+1}} x_i$, one has that 2^{n+1} divides y_{n+1}, so if $\alpha_{n+1} \in \mathcal{F}$ is chosen so that $y_{n+1} = \sum_{i\in\alpha_{n+1}} 2^i$, one will have $\min \alpha_{n+1} > \max \alpha_n$. Let now $\beta \in \mathcal{F}$ and let $\gamma = \bigcup_{n\in\beta} \gamma_n$, $\alpha = \bigcup_{n\in\beta} \alpha_n$. Then $\sum_{n\in\beta} y_n = \sum_{n\in\gamma} x_n \in A_i$ and $\sum_{n\in\beta} y_n = \sum_{i\in\alpha} 2^i$, so $\alpha \in C_i$.

(iii) $\Longrightarrow$ (i): Assume that $FP((x_n)_{n\in\mathbb{N}}) = \bigcup_{i=1}^{r} C_i$ and, for each $i \in \{1, 2, \ldots, r\}$ let $C_i = \{\alpha \in \mathcal{F} : \prod_{n\in\alpha} x_n \in C_i\}$ (note that $\prod_{n\in\alpha} x_n$ denotes the product taken in decreasing order of indices). Choose $i \in \{1, 2, \ldots, r\}$ and a sequence $(\alpha_n)_{n\in\mathbb{N}} \subseteq \mathcal{F}$ as guaranteed by (iii). For each $n \in \mathbb{N}$, let $y_n = \prod_{i\in\alpha_n} x_i$. Then, given $\beta \in \mathcal{F}$, $\prod_{n\in\beta} y_n = \prod_{i\in\alpha} x_i$, where $\alpha = \bigcup_{n\in\beta} \alpha_n$. Since $\alpha \in C_i$, we have $\prod_{n\in\beta} y_n \in C_i$. $\square$

4. Additive and Multiplicative IP Sets in One Cell of a Partition

As we have seen in the previous section, for any finite coloring $\mathbb{N} = \bigcup_{i=1}^{r} C_i$ there must exist $i, j \in \{1, 2, \ldots, r\}$ such that C_i contains an additive IP set $FS((n_i)_{i\in\mathbb{N}})$ and C_j contains a multiplicative IP set $FP((m_i)_{i\in\mathbb{N}})$. This leads to the natural question whether one can have $i = j$. We will see in this section that the answer to this question is YES.[11] Actually, we will present two proofs of this interesting fact. The first proof, due to Hindman [**H3**], utilizes the topological algebra in $\beta\mathbb{N}$. The second one, obtained in [**BerHi3**], utilizes the combinatorial richness of IP* sets.

Let

$$\Gamma = \{p \in (\beta\mathbb{N}, +) : \text{any } A \in p \text{ contains an IP set}\}.$$

We have seen already (see Theorem 2.6) that

$$\Gamma = \text{cl}\{p \in (\beta\mathbb{N}, +) : p + p = p\}.$$

The following lemma shows that Γ is a right ideal of $(\beta\mathbb{N}, \cdot)$ meaning that, for any $p \in \Gamma$, $p \cdot \beta\mathbb{N} \subseteq \Gamma$.

Lemma 4.1. *Γ is a right ideal in $(\beta\mathbb{N}, \cdot)$.*

Proof. Γ is certainly nonempty. Let $p \in \Gamma$ and $q \in \beta\mathbb{N}$, and let us show that $p \cdot q \in \Gamma$. Let $A \in p \cdot q$. By the definition of the operation in $(\beta\mathbb{N}, \cdot)$, we have $\{n \in \mathbb{N} : A/n \in p\} \in q$. Take any $m \in \mathbb{N}$ with $A/m \in p$ and let $FS((n_i)_{i\in\mathbb{N}})$ be an (additive) IP set contained in A/m. (The existence of such a set follows from the fact that $A/m \in p \in \Gamma$). This implies that A contains an IP set, and we are done. $\square$

Theorem 4.2. *For an arbitrary finite partition $\mathbb{N} = \bigcup_{i=1}^{r} C_i$ there exist $j \in \{1, 2, \ldots, r\}$ and two increasing sequences $(n_i)_{i\in\mathbb{N}}$, $(m_i)_{i\in\mathbb{N}}$ such that $FS((n_i)_{i\in\mathbb{N}}) \subseteq C_j$ and $FP((m_i)_{i\in\mathbb{N}} \subseteq C_j$.*

[11]Encouraged by this answer, one may ask if it is also always possible to have $n_i = m_i$, $i \in \mathbb{N}$. This time the answer is NO. See [**H4**].

Proof. We know that $\Gamma = \mathrm{cl}\{p \in (\beta\mathbb{N}, +) : p+p = p\}$ is a closed right ideal in $(\beta\mathbb{N}, \cdot)$ and hence, by Ellis' theorem, contains a multiplicative idempotent $q = q \cdot q$. Let $j \in \{1, 2, \ldots, r\}$ be such that $C_j \in q$. Then, since $q \in \Gamma$, C_j contains an additive IP set $FS((n_i)_{i\in\mathbb{N}})$. On the other hand, since $q = q \cdot q$, C_j has to contain a multiplicative IP set $FP((m_i)_{i\in\mathbb{N}})$ as well. □

We will now present an elementary proof of Theorem 4.2. Before doing so we will introduce and briefly discuss some important notions of largeness.

Definition 4.3. Let $(S, \cdot)$ be a semigroup. A set $A \subseteq S$ is called an IP* set if it has nontrivial intersection with any IP set contained in S.

We collect some useful facts about IP* sets in the following lemma.

Lemma 4.4. *(i) Let $(S, \cdot)$ be a semigroup and assume that $A \subseteq S$ is an IP* set. Then for any IP set $E \subseteq S$, $A \cap E$ contains an IP set.*

(ii) Let $(S, \cdot)$ be a semigroup. A set $A \subseteq S$ is an IP* set if and only if $A \in p$ for every idempotent $p = p \cdot p$ in $(\beta S, \cdot)$.

(iii) Let $(S, \cdot)$ be any semigroup, $k \in \mathbb{N}$, and let $A_1, A_2, \ldots, A_k$ be IP* sets in S. Then $\bigcap_{i=1}^{k} A_i$ is also an IP* set.

Proof. To prove (i), consider the partition $E = (A \cap E) \cup (A^c \cap E)$. If $A \cap E$ does not contain an IP set, then $A^c \cap E$ does, but this contradicts the fact that A is an IP* set.

To prove (ii), assume first that A is an IP* set in S. If, for some idempotent p, $A \notin p$, then $A^c \in p$ and hence there is an IP set $E \subseteq A^c$, which contradicts (i). In the other direction, let us assume that $A \in p$ for any $p = p \cdot p$. If A is not an IP* set, then there exists an IP set E such that $A \cap E = \emptyset$. But then A^c contains the IP set E, and by theorem 2.5 there exists an idempotent p such that $E \in p$. Hence $A^c \in p$ and $A \notin p$. Contradiction.

As for (iii), it immediately follows from (ii). □

Definition 4.5. Let $(S, \cdot)$ be a semigroup. A set $A \subseteq S$ is *thick* if it contains a translate of every finite set $F \subseteq S$. Formally, A is thick if, for every finite $F \subseteq S$, there exists $t \in S$ such that $tF \subseteq S$.[12]

It is easy to see that a set $A \subseteq (\mathbb{N}, +)$ is thick if and only if it contains arbitrarily long intervals. Equivalently, $A \subseteq (\mathbb{N}, +)$ is thick if and only if, for any $n \in \mathbb{N}$, one has $A \cap (A-1) \cap (A-2) \cap \ldots \cap (A-n) \neq \emptyset$. Similarly, $B \subseteq (\mathbb{N}, \cdot)$ is thick if and only if, for any $n \in \mathbb{N}$, $B \cap B/2 \cap B/3 \cap \ldots \cap B/n \neq \emptyset$.

Lemma 4.6. *Let $A \subseteq (\mathbb{N}, +)$ be an IP* set. Then A is multiplicatively thick.*

Proof. It is enough to verify that, for any $k \in \mathbb{N}$, the set A/k is also IP*. (The result in question will then follow from the fact that, $\forall n \in \mathbb{N}$, $A \cap A/2 \cap A/3 \cap \ldots \cap A/n$ is, by Lemma 4.4, an IP* set and hence is nonempty).

Let $E \subseteq (\mathbb{N}, +)$ be an IP set. Then kE is also an IP set and, by Lemma 4.4, there exists an IP set $E_0 \subseteq E$ such that $kE_0 \subseteq kE \cap A$. Then $E_0 \subseteq A/k$ and we are done. □

Lemma 4.7. *If $A \subseteq \mathbb{N}$ is multiplicatively thick, then A contains a multiplicative IP set $FP((n_i)_{i\in\mathbb{N}})$.*

[12] To be more precise such a set ought to be called left thick (the right thick sets being the sets which contain a right translate of any finite set).

Proof. The proof goes along the same lines as the proof of Theorem 2.1. Let $n_1 \in A$. Then $A \cap A/n_1 \neq \emptyset$. Let $n_2 \in A \cap A/n_1$. Clearly

$$A \cap A/n_1 \cap A/n_2 \cap A/n_1 n_2 \neq \emptyset.$$

And so on. □

Second Proof of Theorem 4.2 (Cf [**BerHi3**], Thm. 2.4.) Let $\mathbb{N} = \bigcup_{i=1}^{r} C_i$ and let $I = \{i \in \{1, 2, \dots, r\} : C_i \text{ contains an additive IP set}\}$. Let $A = \bigcup_{i \in I} C_i$. Clearly, A is an IP* set. By Lemma 4.6, A is multiplicatively thick and by Lemma 4.7 contains a multiplicative IP set E. Now, $E \subseteq \bigcup_{i \in I} C_i$, so by Theorem 3.5, one of the C_i, $i \in I$, has to contain a multiplicative IP set. Since for every $i \in I$, C_i contains an additive IP set, we are done. □

5. Additively and Multiplicatively Central Sets

In this section we will introduce the notion of centrality, one more useful notion of largeness. It was originally introduced by Furstenberg via the notions of proximality and uniform recurrence (see [**F2**], Def 8.3, p.161) and only somewhat later was shown to have an equivalent form in terms of ultrafilters (see [**BerHi1**]).

A topological dynamical system (with "time" $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$) is a pair (X, T) where X is a compact (not necessarily metrizable) space and
$T : X \to X$ a continuous map. The system (X, T) is *minimal* if for any $x \in X$ one has $\overline{\{T^n x : n \in \mathbb{N}_0\}} = X$. One can show by a simple application of Zorn's lemma that any topological dynamical system (X, T) has a minimal subsystem (Y, T), where Y denotes a T-invariant nonempty closed subset of X (and, by slight abuse of notation, the restriction of T to Y is denoted by the same symbol). Let $\sigma : \mathbb{N} \to \mathbb{N}$ denote the shift operation: $\sigma(x) = x + 1$, $x \in \mathbb{N}$. In Section 1.1 we have observed that Hindman's finite sums theorem can be viewed as an instance of application of Poincaré's recurrence theorem to the "measure-preserving system" $(\mathbb{N}, \mathcal{P}(\mathbb{N}), p, \sigma)$, where p is an arbitrary idempotent in $(\beta\mathbb{N}, +)$. As we will momentarily see, a certain subclass of idempotent ultrafilters allows one to make a useful connection between minimal right ideals in $(\beta\mathbb{N}, +)$ and minimal topological systems.

Extend the shift operation σ from $\mathbb{N}$ to $\beta\mathbb{N}$ by the rule $\sigma(q) = q + 1$, $q \in \beta\mathbb{N}$ (here 1 is identified with the principal ultrafilter of sets containing the integer 1). This makes the pair $(\beta\mathbb{N}, \sigma)$ a topological dynamical system.

Theorem 5.1. *The family of minimal closed σ-invariant subsets of $\beta\mathbb{N}$ coincides with the family of minimal right ideals of $(\beta\mathbb{N}, +)$.*

Proof. We first observe that closed σ-invariant sets in $\beta\mathbb{N}$ coincide with right ideals. Indeed, if I is a right ideal, i.e. satisfies $I + \beta\mathbb{N} \subseteq I$, then for any $p \in I$ one has $p + 1 \in I + \beta\mathbb{N} \subseteq I$, so that I is σ-invariant. On the other hand, if S is a closed σ-invariant set in $\beta\mathbb{N}$ and $p \in S$, then $p + \beta\mathbb{N} = p + \bar{\mathbb{N}} = \overline{p + \mathbb{N}} \subseteq \overline{S} = S$, which implies $S + \beta\mathbb{N} \subseteq S$.

Now the theorem follows from the simple general fact that any minimal right ideal in a compact left-topological semigroup $(G, \cdot)$ is closed. Indeed, if R is a right ideal in $(G, \cdot)$ and $x \in R$, then xG is compact as the continuous image of G and is an ideal. Hence the minimal ideal containing x is compact as well. (The fact that R contains a minimal ideal follows by a routine application of Zorn's lemma to the non-empty family $\{I : I \text{ is a closed right ideal of } G \text{ and } I \subseteq R\}$). □

Observe now that any minimal right ideal in $(\beta\mathbb{N},+)$, being a compact left-topological semigroup, contains, by Ellis' theorem, an idempotent.

Definition 5.2. An idempotent $p \in (\beta\mathbb{N},+)$ is called *minimal* if p belongs to a minimal right ideal.

It is not hard to show that any minimal right ideal R of $(\beta\mathbb{N},+)$ is of the form $q+\beta\mathbb{N}$ for some $q \in R$. Indeed, for any $q \in R$, $q+\beta\mathbb{N} \subseteq R+\beta\mathbb{N} = R$. Since R is minimal, we get $q+\beta\mathbb{N} = R$. Note that since $q+\beta\mathbb{N}$ is the continuous image of $\beta\mathbb{N}$ under the function $\lambda_q(p) = q+p$, minimal right ideals in $(\beta\mathbb{N},+)$ are compact. It follows that one can choose q to be an idempotent. This gives the following result.

Theorem 5.3. *Any minimal subsystem of $(\beta\mathbb{N},\sigma)$ is of the form $(p+\beta\mathbb{N},\sigma)$, where p is a minimal idempotent in $(\beta\mathbb{N},+)$.*

We are going to show that, if p is a minimal idempotent in $(\beta\mathbb{N},+)$ and $A \in p$, then A is a *piecewise syndetic* set, namely, a set of the form $S \cap T$, where T is a thick set and S is syndetic (i.e. has bounded gaps). A useful equivalent definition of piecewise syndeticity is given by the following lemma, the proof of which is left to the reader.

Lemma 5.4. *A set $A \subseteq (\mathbb{N},+)$ is piecewise syndetic if and only if there exists a finite set $F \subseteq \mathbb{N}$ such that the family*

$$\Big\{\bigcup_{t\in F}(A-t)-n : n \in \mathbb{N}\Big\}$$

has the finite intersection property.

Theorem 5.5. *Let p be a minimal idempotent in $(\beta\mathbb{N},+)$.*
(i) For any $A \in p$, the set $B = \{n \in \mathbb{N} : (A-n) \in p\}$ is syndetic.
(ii) Any $A \in p$ is piecewise syndetic.

Proof. Statement (i) follows immediately from the fact that $(p+\beta\mathbb{N},\sigma)$ is a minimal system. Indeed, note that the assumption $A \in p$ just means that $p \in \bar{A}$, i.e. $\bar{A}$ is a (clopen) neighborhood of p. Now, by minimality, every point in $(p+\beta\mathbb{N},\sigma)$ is *uniformly recurrent*, i.e. visits any of its neighborhoods V along a syndetic set. This implies that the set $\{n : p+n \in \bar{A}\} = \{n : A \in p+n\} = \{n : A-n \in p\}$ is syndetic.

(ii) Since the set $B = \{n : A-n \in p\}$ is syndetic, the union of finitely many shifts of B covers $\mathbb{N}$, i.e. for some finite set $F \subseteq \mathbb{N}$ one has $\bigcup_{t\in F}(B-t) = \mathbb{N}$. So, for any $n \in \mathbb{N}$ there exists $t \in F$ such that $n \in B-t$, or $n+t \in B$. By the definition of B this implies that $(A-(n+t)) \in p$. It follows that for any n the set $(\bigcup_{t\in F}(A-t))-n$ belongs to p, and consequently the family $\{(\bigcup_{t\in F}(A-t))-n : n \in \mathbb{N}\}$ has the finite intersection property. By Lemma 5.4, this is equivalent to piecewise syndeticity of A, and we are done. □

At this point we want to make a simple but important observation. Namely, all the definitions, results and proofs in this section which pertain to $(\mathbb{N},+)$ can be transferred (usually verbatim) to the more general situation where the semigroup $(\mathbb{N},+)$ is replaced by a (discrete) semigroup $(S,\cdot)$. In particular, this remark applies to the semigroup $(\mathbb{N},\cdot)$.

We collect for the reader's convenience some definitions and results related to $(\beta\mathbb{N},\cdot)$.

Definition 5.6. (i) An idempotent $p \in (\beta\mathbb{N}, \cdot)$ is *minimal* if it belongs to a minimal right ideal of $(\beta\mathbb{N}, \cdot)$.

(ii) A set $A \subseteq (\beta\mathbb{N}, \cdot)$ is *syndetic* if there exists a finite set $F \subseteq \mathbb{N}$ such that $\bigcup_{n\in F} A/n = \mathbb{N}$.

(iii) A set $A \subseteq (\beta\mathbb{N}, \cdot)$ is *piecewise syndetic* if A is of the form $A = S \cap T$, where A is multiplicatively syndetic and T is multiplicatively thick.

Theorem 5.7. *Let p be a minimal idempotent in $(\beta\mathbb{N}, \cdot)$.*

(i) For any $A \in p$, the set $B = \{n \in \mathbb{N} : (A/n \in p\}$ is (multiplicatively) syndetic.

(ii) Any $A \in p$ is (multiplicatively) piecewise syndetic.

Definition 5.8. (i) A set $A \subseteq (\mathbb{N}, +)$ is *additively central* if it is a member of a minimal idempotent $p \in (\beta\mathbb{N}, +)$.

(ii) A set $A \subseteq (\mathbb{N}, \cdot)$ is *multiplicatively central* if it is a member of a minimal idempotent $p \in (\beta\mathbb{N}, \cdot)$.

(iii) A set $A \subseteq (\mathbb{N}, +)$ is *additively central** (or AC*) if for any central set $S \subseteq (\mathbb{N}, +)$, $A \cap S \neq \emptyset$.

(iv) A set $A \subseteq (\mathbb{N}, \cdot)$ is *multiplicatively central** (or MC*) if for any central set $S \subseteq (\mathbb{N}, \cdot)$, $A \cap S \neq \emptyset$.

Remark 5.9. (i) One can show (see for example the proof of Theorem 5.4 in [**BerHi1**]) that if p is a minimal idempotent in $(\beta\mathbb{N}, +)$, then so is np for any $n \in \mathbb{N}$. This implies that if A is a central set in $(\mathbb{N}, +)$, then, for any $n \in \mathbb{N}$, A/n is also central.

(ii) It is easy to see that a set $A \subseteq \mathbb{N}$ is additively (multiplicatively) central* if and only if A is a member of any minimal additive (multiplicative) idempotent.

The usefulness of minimal idempotents in Ramsey theory stems from the fact that their members, central sets, are both large (in particular, are piecewise syndetic) and combinatorially rich. For example, one can show that any central set in $(\mathbb{N}, +)$ not only contains an IP set, but also contains arbitrarily long arithmetic progressions, and, more generally, contains a solution of any partition regular system of linear equations. (See [**F2**], Ch. 8). Similarly, any central set in $(\mathbb{N}, \cdot)$ contains a multiplicative IP set, as well as, for any $k \in \mathbb{N}$, *geoarithmetic* configurations of the form $\{b(a + id)^j : 0 \leq i, j \leq k\}$, where $a, b, d \in \mathbb{N}$. (See [**Ber5**], [**BeiBerHS**], [**Bei1**], [**M**]).

Lemma 5.10. *Any additively thick set in $\mathbb{N}$ is additively central and any multiplicatively thick set is multiplicatively central.*

Proof. We will deal with the multiplicative case, the other being practically identical. Let $A \subseteq \mathbb{N}$ be a multiplicatively thick set. Since this is equivalent to the fact that for any $n \in \mathbb{N}$, $A \cap A/2 \cap \ldots \cap A/n \neq \emptyset$, which, in turn, implies that there is $x_n \in A$ such that $\{x_n, 2x_n, \ldots, nx_n\} \subseteq A$, we will assume that for some infinite sequence $(x_n)_{n\in\mathbb{N}} \subseteq \mathbb{N}$, $A = \bigcup_{n\in\mathbb{N}}\{x_n, 2x_n, \ldots, nx_n\}$. Now, any infinite subset of $\mathbb{N}$ is a member of *some* nonprincipal ultrafilter, so let $p \in \beta\mathbb{N} \setminus \mathbb{N}$ be such that $\{x_n : \mathbb{N} \in \mathbb{N}\} \in p$.

We claim that $p \cdot \beta\mathbb{N} \subseteq \bar{A}$. Indeed, since, for any $n \in \mathbb{N}$,
$\{x_m : m \geq n\} \subseteq A/n$, we have $A/n \in p$, and so $\{n : A/n \in p\} = \mathbb{N} \in q$ for any idempotent $q \in \beta\mathbb{N}$. Note that $p \cdot \beta\mathbb{N}$ is a right ideal of $(\beta\mathbb{N}, \cdot)$ so by Zorn's lemma it contains a minimal ideal R which, as was already remarked above, is necessarily

closed. So, by Ellis' theorem, R contains an idempotent q. Then $A \in q$ which implies that A is multiplicatively central. $\square$

LEMMA 5.11. *Any AC* set in* $\mathbb{N}$ *is multiplicatively thick.*

PROOF. The proof is similar to that of Lemma 4.6. Let $A \subseteq \mathbb{N}$ be an AC* set. It is enough to check that, for any $k \in \mathbb{N}$, A/k is also an AC* set. (Indeed, it will follow that $A \cap A/2 \cap \ldots \cap A/n$ is AC* and hence nonempty). Now, to see that A/k is an AC* set, one argues as follows. By Remark 5.9, if p is a minimal idempotent, then so is kp for any $k \in \mathbb{N}$ and since A is an AC* set, it is a member of kp which implies $A/k \in p$. So $A \cap A/2 \cap \ldots \cap A/n \in p$ for any minimal idempotent $p \in (\beta\mathbb{N}, +)$. We are done. $\square$

COROLLARY 5.12. *Any AC* set in* $\mathbb{N}$ *is multiplicatively central.*

PROOF. Follows immediately from Lemma 5.11. $\square$

In view of Theorem 2.5, Theorem 4.2 says that for any finite partition $\mathbb{N} = \bigcup_{i=1}^{r} C_i$, there exist an additive idempotent in $(\beta\mathbb{N}, +)$, a multiplicative idempotent $q \in (\beta\mathbb{N}, \cdot)$ and $j \in \{1, 2, \ldots, r\}$ such that $C_j \in p$ and $C_j \in q$. The following theorem is a strengthening of this fact.

THEOREM 5.13 (cf [**BerHi1**], Corollary 5.5). *For any finite partition* $\mathbb{N} = \bigcup_{i=1}^{r} C_i$*, one of the* C_i *is both additively and multiplicatively central.*

PROOF. Similar to the proof of Theorem 4.2, there are (at least) two possible approaches. The first one utilizes the fact that the set

$$M = \mathrm{cl}\{p \in \beta\mathbb{N} : p \text{ is a minimal idempotent in } (\beta\mathbb{N}, +)\}$$

is a right ideal in $(\beta\mathbb{N}, \cdot)$. (See [**BerHi1**], Theorem 5.4). By Zorn's lemma M contains a minimal right ideal R which contains a minimal idempotent $q = q \cdot q$. So, for some $j \in \{1, 2, \ldots, r\}$, $C_j \in q$. But then, by the definition of M, C_j is also a member of some additive minimal idempotent. So, C_j is both additively and multiplicatively central.

The other approach follows the lines of our second proof of Theorem 4.2. Namely, let $I = \{i \in \{1, 2, \ldots, r\} : C_i \text{ is additively central}\}$. Then $A = \bigcup_{i \in I} C_i$ is an AC* set. Now, by Lemma 5.11, A is multiplicatively thick and hence, by Lemma 5.10, is multiplicatively central. But then one of the C_i, $i \in I$ is multiplicatively central and since every C_i, $i \in I$ is additively central, we are done. $\square$

Now that we know that, for any finite partition of $\mathbb{N}$, one of the cells of the partition is both additively and multiplicatively central, it is natural to ask whether all additively central sets must contain rich multiplicative structure and similarly whether all multiplicatively central sets must contain rich additive structure. The following two results show that the answers turn out to be NO (Proposition 5.14) and YES (Theorem 5.15) respectively.

PROPOSITION 5.14 ([**BerHi2**], Theorem 3.4). *There is an additively central set* $A \subseteq \mathbb{N}$ *such that for no* $x, y \in \mathbb{N}$ *is* $\{x, y, x \cdot y\} \subseteq A$.

PROOF. One can actually construct an additively thick set $A = \bigcup_{n=1}^{\infty} \{x_n, x_n + 1, \ldots, x_n + y_n\}$ which satisfies the requirements. To make it work one has just to choose $x_1 \geq 2$, to make sure that x_n grows fast enough so that for no $i, j < n$ will one have $x_i x_j \in \{x_n, x_n + 1, \ldots, x_n + y_n\}$ and to pick (increasing) $y_n < x_2 x_n$. $\square$

Theorem 5.15 ([**BerHi2**], Theorem 3.5). *Let $A \subseteq \mathbb{N}$ be a multiplicatively central set. Then for each m there exists an m-element sequence $(y_n)_{n=1}^m$ such that $FS((y_n)_{n=1}^m) \subseteq A$.*

Proof. Let $T = \{p \in \beta\mathbb{N} :$ for all $B \in p$ and all $m \in \mathbb{N}$ there exists $(y_n)_{n=1}^m$ with $FS((y_n)_{n=1}^m) \subseteq B\}$. Now all additive idempotents are in T so $T \neq \emptyset$. We claim that T is a two sided ideal of $(\beta\mathbb{N}, \cdot)$. To this end let $p \in T$ and let $q \in \beta\mathbb{N}$. To see that $p \cdot q \in T$, let $B \in p \cdot q$ and $m \in \mathbb{N}$ be given. Then $\{n \in \mathbb{N} : B/n \in p\} \in q$ so pick $n \in \mathbb{N}$ with $B/n \in p$. Pick $(y_t)_{t=1}^m$ with $FS((y_t)_{t=1}^m) \subseteq B/n$. Then $FS((n \cdot y_t)_{t=1}^m) \subseteq B$.

To see that $q \cdot p \in T$, let $B \in q \cdot p$ and $m \in \mathbb{N}$ be given. Then $\{n \in \mathbb{N} : B/n \in q\} \in p$, so pick $(y_t)_{t=1}^m$ with

$$FS((y_t)_{t=1}^m) \subseteq \{n \in \mathbb{N} : B/n \in q\}.$$

Since $FS((y_t)_{t=1}^m)$ is finite we have $\bigcap\{B/n : n \in FS((y_t)_{t=1}^m)\} \in q$ so pick $a \in \bigcap\{B/n : n \in FS((y_t)_{t=1}^m)\}$. Then $FS((a \cdot y_t)_{t=1}^m) \subseteq B$.

Now A is multiplicatively central so pick a minimal idempotent $p \in (\beta\mathbb{N}, \cdot)$ with $A \in p$. Pick a minimal right ideal R of $(\beta\mathbb{N}, \cdot)$ with $p \in R$. Since T is a two sided ideal, $R \subseteq T$. (Since T is a left ideal $T \cap R \neq \emptyset$ and hence $T \cap R$ is a right ideal so $T \cap R = R$.) Then $p \in T$. Since $A \in p$, we are done. □

Remark 5.16. It is natural to ask whether any multiplicatively central set in $\mathbb{N}$ contains an infinite additive IP set. The answer is NO: one can construct a multiplicatively central $A \subseteq \mathbb{N}$ such that for no $(y_n)_{n\in\mathbb{N}} \subseteq \mathbb{N}$ is $FS((y_n)_{n\in\mathbb{N}}) \subseteq A$. See [**BerHi2**], Theorem 3.6.

The following result will be needed in the next section.

Theorem 5.17 (cf. [**BerHi1**], Theorem 5.6). *There is a minimal idempotent $q \in (\beta\mathbb{N}, \cdot)$ such that every member of q is additively central.*

Proof. Let $M = \mathrm{cl}\{p \in \beta\mathbb{N} : p$ is a minimal idempotent in $(\beta\mathbb{N}, +)\}$. As was already mentioned in the proof of Theorem 5.13, M is a right ideal in $(\beta\mathbb{N}, \cdot)$ and hence contains a minimal right ideal R. Let $q \in R$ be a minimal multiplicative idempotent. Let $A \in q$. Then $q \in \bar{A} \cap M$, which implies that A is additively central. □

6. An Application: Partition Regularity of the Equation a+b = cd.

In this short section we will utilize Theorem 5.17 to show the partition regularity of the equation $a + b = cd$, thereby providing an affirmative answer to a question posed in [**CsGSa**]. For another solution to this question see [**H6**].

Theorem 6.1. *For any finite coloring $\mathbb{N} = \bigcup_{i=1}^r C_i$, one of the C_i contains arbitrarily large and distinct a, b, c, d such that $a + b = cd$.*

Proof. Let $p \in \beta\mathbb{N}$ be a minimal multiplicative idempotent with the property that any member of p is additively central (see Theorem 5.17). Let a partition $\mathbb{N} = \bigcup_{i=1}^r C_i$ be given and let $i \in \{1, 2, \ldots, r\}$ be such that $C_i \in p$. For convenience of notation we will denote this C_i by C. Since $C \in p = p \cdot p$, we have $\{n : C/n \in p\} \in p$. So there exists $d \in C$ such that $C/d \in p$ and hence $C \cap C/d \in p$. (Note that there are "many" such d's).

Now since any member of p is additively central, there exists an additive idempotent q such that $C \cap C/d \in q$. This implies that

$$\{n : (C \cap C/d) - n \in q\} \in q.$$

Let $b' \in C \cap C/d$ be such that $(C \cap C/d) - b' \in q$. Then

$$(C \cap C/d) \cap ((C \cap C/d) - b') \in q$$

and hence is nonempty. Note now that it follows from the choice of b' that $b = b'd \in C$. Now, since $(C \cap C/d) \bigcap ((C \cap C/d) - b') \neq \emptyset$, we obtain $E = (dC \cap C) \cap ((dC \cap C) - b) \neq \emptyset$. Choose $a \in E$. Then in particular $a \in C$, $b \in C$, and $a + b \in dC$, so that for some $c \in C$ we get $a + b = cd$.

It is clear from the proof that a, b, c, d can be chosen arbitrarily large and distinct. We are done. $\square$

7. Ultrafilters and Diophantine Approximation

Let X be a topological space, and let $p \in (\beta\mathbb{N}, +)$. Given a sequence $(x_n)_{n\in\mathbb{N}}$ in X, we shall write $p\text{-}\lim_{n\in\mathbb{N}} x_n = y$ if, for every neighborhood U of y one has

$$\{n \in \mathbb{N} : x_n \in U\} \in p.$$

It is easy to see that $p\text{-}\lim_{n\in\mathbb{N}} x_n$ exists and is unique in any compact Hausdorff space.

THEOREM 7.1. *Let X be a compact Hausdorff space, let $p, q \in \beta\mathbb{N}$ and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X. Then*

$$(q + p)\text{-}\lim_{r\in\mathbb{N}} x_r = p\text{-}\lim_{t\in\mathbb{N}} q\text{-}\lim_{s\in\mathbb{N}} x_{s+t}.$$

In particular, if p is an idempotent and $p = q$ one has

$$p\text{-}\lim_{r\in\mathbb{N}} x_r = p\text{-}\lim_{s\in\mathbb{N}} p\text{-}\lim_{t\in\mathbb{N}} x_{s+t}.$$

PROOF. Recall that

$$q + p = \{A \subseteq \mathbb{N} : \{n \in \mathbb{N} : (A - n) \in q\} \in p\}.$$

Let $x = (q + p)\text{-}\lim_{r\in\mathbb{N}} x_r$. It will suffice for us to show that for any neighborhood U of x, we have that for p-many t, $q\text{-}\lim_{s\in\mathbb{N}} x_{s+t} \in U$. Fix such a U. We have $\{r : x_r \in U\} \in q + p$, so that

$$\{t : \{x : x_s \in U\} - t \in q\} = \{t : \{x : x_{s+t} \in U\} \in q\} \in p.$$

This implies, in particular, that for p-many t, $q\text{-}\lim_{s\in\mathbb{N}} x_{s+t} \in U$. $\square$

As an immediate application of Theorem 7.1, let X be the one dimensional torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ and let, for some $a \in \mathbb{T}$, $x_n = na$. (It is convenient to interpret $\mathbb{T}$ as the unit interval $[0, 1]$ with the ends glued up and $x_n = na$ as corresponding to the real sequence $na \mod 1 \in [0, 1)$).

We claim that for any idempotent $p \in (\beta\mathbb{N}, +)$ one has $p\text{-}\lim_{n\in\mathbb{N}} na = 0$. To see this, let $c = p\text{-}\lim_{n\in\mathbb{N}} na$. Then

$$\begin{aligned} c = (p+p)\text{-}\lim_{n\in\mathbb{N}} na &= p\text{-}\lim_{n\in\mathbb{N}} p\text{-}\lim_{m\in\mathbb{N}}(n+m)a \\ &= p\text{-}\lim_{n\in\mathbb{N}}(c+na) \\ &= 2c, \end{aligned}$$

and it follows that $c = 0$.

It is now easy to inductively extend this observation to polynomial sequences of the form $x_n = a_1 n + \ldots + a_k n^k$, where $a_i \in \mathbb{T}$, $i = 1, 2, \ldots, k$. For example, if $x_n = n^2 a$ and $p = p + p$, one has

$$\begin{aligned} p\text{-}\lim_{n\in\mathbb{N}} n^2 a &= p\text{-}\lim_{n\in\mathbb{N}} p\text{-}\lim_{m\in\mathbb{N}}(n^2+2nm+m^2)a \\ &= p\text{-}\lim_{n\in\mathbb{N}}(n^2 a + 2n(p\text{-}\lim_{m\in\mathbb{N}} ma) + p\text{-}\lim_{m\in\mathbb{N}} m^2 a) \\ &= p\text{-}\lim_{n\in\mathbb{N}} n^2 a + p\text{-}\lim_{m\in\mathbb{N}} m^2 a \\ &= 2(p\text{-}\lim_{n\in\mathbb{N}} n^2 a). \end{aligned}$$

which implies $p\text{-}\lim_{n\in\mathbb{N}} n^2 a = 0$. (Note that we used the "linear" fact that for any fixed m and $a \in \mathbb{T}$, $p\text{-}\lim_{n\in\mathbb{N}} 2nma = 0$). So, modulo the completely trivial details of a routine inductive proof, we have established the following result.

THEOREM 7.2. *For any $k \in \mathbb{N}$, $a_i \in \mathbb{T}$, $i = 1, \ldots, k$, and $p \in \beta\mathbb{N}$ with $p = p+p$, one has $p\text{-}\lim_{n\in\mathbb{N}}(a_1 n + \ldots + a_k n^k) = 0$.*

Let $\|\cdot\|$ denote the distance to a closest integer in $\mathbb{R}$. Utilizing the characterization of IP* given by Lemma 4.4 (ii), we have the following immediate corollary.

COROLLARY 7.3 (cf. [**F2**], Theorem 2.19). *For any $\epsilon > 0$, $l \in \mathbb{N}$, and any real polynomials g_i satisfying $g_i(0) = 0$, $i = 1, \ldots, l$, the set*

$$R_\epsilon = \{n \in \mathbb{N} : \|g_i(n)\| < \epsilon,\ i = 1, \ldots, l\}$$

is IP*.

We will discuss now a strengthening of Corollary 7.3 which involves multiplicatively central sets. First, we need a definition.

DEFINITION 7.4. Given any r integers $n_1, \ldots, n_r \in \mathbb{N}$, call the finite sums set $FS((n_i)_{i=1}^r)$ an IP$_r$ set. A set $A \subseteq \mathbb{N}$ is IP$_r^*$ set if for any IP$_r$ set E one has $A \cap E \neq \emptyset$.

The following theorem is an immediate consequence of Theorem 5.15.

THEOREM 7.5. *If $r \in \mathbb{N}$ and $A \subseteq \mathbb{N}$ is an IP$_r^*$ set then A is an MC* set.*

One can show that the set R_ϵ appearing in the formulation of Corollary 7.3 is an IP$_r^*$ set for some r (which depends only on ϵ, on k, and on the maximal degree of the polynomials g_i, $i = 1, 2, \ldots, k$). To give a flavour of the reasoning leading to this statement, let us show for example that for any real number x and any $\epsilon > 0$ there exists r such that the set $\{n \in \mathbb{N} : \|n^2 x\| \leq \epsilon\}$ is an IP$_r^*$ set.

We will use the following special case of the Hales-Jewett theorem (see the discussion of various equivalent forms of the Hales-Jewett thoerem in [**BerL2**] and [**Ber2**], Section 4). Given a finite set F, let $\mathcal{P}(F)$ denote the set of all subsets of F.

THEOREM 7.6. *For any $t \in \mathbb{N}$ there exists $r = r(t)$ such that for any t-coloring*

$$\mathcal{P}(\{1,2,\dots,r\}) \times \mathcal{P}(\{1,2,\dots,r\}) = \bigcup_{i=1}^{t} C_i,$$

one of the C_i contains a configuration of the form

$$\{(\alpha_1,\alpha_2),(\alpha_1\cup\gamma,\alpha_2),(\alpha_1,\alpha_2\cup\gamma),(\alpha_1\cup\gamma,\alpha_2\cup\gamma)\},$$

where $\gamma \subseteq \{1,2,\dots,r\}$ is nonempty and disjoint from α_1 and α_2.

To show that the set $\{n \in \mathbb{N} : \|n^2x\| < \epsilon\}$ is an IP_r^* set for some r, one argues as follows. First, assume for convenience and without loss of generality that $\epsilon = 1/t$ for some $t \in \mathbb{N}$ and that $x \notin \mathbb{Q}$. Let now $r = r(t)$, as in Theorem 7.6, and let an r-element subset $\{n_1,\dots,n_r\} \subseteq \mathbb{N}$ be given. For any nonempty $\alpha \subseteq \{1,2,\dots,r\}$ we will write $n_\alpha = \sum_{i\in\alpha} n_i$. Also set $n_\emptyset = 0$. Let us take the partition of $[0,1)$ into t semiopen intervals $I_1, I_2, \dots, I_t$ (of length $1/t$ each) and correspond to each pair

$$(\alpha,\beta) \subseteq \{1,2,\dots,r\} \times \{1,2,\dots,r\}$$

the unique subinterval I_j for which $n_\alpha n_\beta x \mod 1 \in I_j$. This induces a t-coloring of

$$\mathcal{P}(\{1,2,\dots,r\}) \times \mathcal{P}(\{1,2,\dots,r\})$$

and by Theorem 7.6 we have that for some $\alpha_1,\alpha_2,\gamma \subseteq \{1,2,\dots,r\}$ where γ is nonempty and disjoint from α_1 and α_2, and some $j \in \{1,2,\dots,t\}$, the four numbers $n_{\alpha_1}n_{\alpha_2}x \mod 1$, $(n_{\alpha_1}+n_\gamma)n_{\alpha_2}x \mod 1$, $n_{\alpha_1}(n_{\alpha_2}+n_\gamma)x \mod 1$, and $(n_{\alpha_1}+n_\gamma)(n_{\alpha_2}+n_\gamma)x \mod 1$ are all in I_j. Applying the identity

$$n_{\alpha_1}n_{\alpha_2} - (n_{\alpha_1}+n_\gamma)n_{\alpha_2} - n_{\alpha_1}(n_{\alpha_2}+n_\gamma) + (n_{\alpha_1}+n_\gamma)(n_{\alpha_2}+n_\gamma) = n_\gamma^2$$

and taking into account that the length of I_j is $1/t$ and that $x \notin \mathbb{Q}$, we get $\|n_\gamma^2 x\| < 1/t$. Since $\gamma \subseteq \{1,2,\dots,r\}$, we established the fact that $\{n \in \mathbb{N} : \|n^2x\| < \epsilon\}$ is an IP_r^* set.

A similar argument shows that for any $k \in \mathbb{N}$, any $x \in \mathbb{R}$, and any $\epsilon > 0$, the set $\{n \in \mathbb{N} : \|n^kx\| < \epsilon\}$ is an IP_r^* set for some r. Now, one can show that for any $\mathrm{IP}_{r_i}^*$ sets A_i, $i = 1,2,\dots,k$, there exists $r \in \mathbb{N}$ such that the set $A_1 \cap A_2 \cap \dots \cap A_k$ is an IP_r. This implies that for any $\epsilon > 0$, $k \in \mathbb{N}$, and $x_1,x_2,\dots,x_k \in \mathbb{R}$, the set $\{n \in \mathbb{N} : \|nx_1 + n^2x_2 + \dots + n^kx_k\| < \epsilon\}$ is an IP_r set for some r. Moreover, and for the same reason, this is also true for any finite set of polynomials. We summarize this in the following theorem. (Note that the last claim of this theorem follows from Theorem 7.5).

THEOREM 7.7. *For any $\epsilon > 0$, $l \in \mathbb{N}$, and any real polynomials g_i satisfying $g_i(0) = 0$, $i = 1,2,\dots,l$, the set*

$$R_\epsilon = \{n \in \mathbb{N} : \|g_i(n)\| < \epsilon,\ i = 1,2,\dots,l\}$$

is an IP_r^* set for some r. Moreover, R_ϵ is a multiplicatively central* set.

8. Ultrafilters and Measure Preserving Systems

As we have seen in previous sections, the usefulness of ultrafilters in *partition* Ramsey theory stems from the fact that for any ultrafilter $p \in (\beta S, \cdot)$ and any finite partition $S = \bigcup_{i=1}^{r} C_i$ one (and only one) of the C_i is a member of p. If it is known that members of p always posses a certain property, then one cell of the partition will have this property as well.

For example, one can show that if p is a minimal idempotent in $(\beta\mathbb{N}, +)$ then every $A \in p$ is *AP-rich*, that is, contains arbitrarily long arithmetic progressions (see [**BerHi1**] Section 3, [**BerFHiK**], and [**Ber4**], Theorem 2.10). This immediately implies the classical van der Waerden theorem stating that for any finite partition $\mathbb{N} = \bigcup_{i=1}^{r} C_i$, one of the C_i is AP-rich.

This leads to the question whether there is any way to tell which cell of a given partition has the property of being AP-rich. Questions of this kind are dealt with by *density* Ramsey theory, which "upgrades" the results of the form (P) (see Introduction) to the following:

> (D) Any "large" subset of an infinite "well organized" set S is "well organized".

For example, if $S = \mathbb{N}$, "well organized" means being AP-rich, and "large" is interpreted as the property of a set $E \subseteq \mathbb{N}$ to have positive upper Banach density, $d^*(E) = \limsup_{N-M\to\infty} \frac{|E\cap\{M+1,\ldots,N\}|}{N-M}$, then (D) is the celebrated Szemerédi theorem on arithmetic progressions ([**Sz**]).[13]

In [**F**] Furstenberg proved Szemerédi's theorem by deriving it as a corollary of the following beautiful and far reaching extension of Poincaré's recurrence theorem.

THEOREM 8.1. *For any probability measure preserving system $(X, \mathcal{B}, \mu, T)$, any $A \in \mathcal{B}$ with $\mu(A) > 0$, and any $k \in \mathbb{N}$, there exists $n > 0$ such that*

$$\mu(A \cap T^{-n}A \cap T^{-2n}A \cap \ldots \cap T^{-kn}A) > 0. \tag{8.1}$$

Note that if for $E \subseteq \mathbb{N}$ one has $d^*(E) > 0$ then the fact that E contains a $(k+1)$-element arithmetic progression just means that for some $n > 0$

$$E \cap (E-n) \cap (E-2n) \cap \ldots \cap (E-kn) \neq \emptyset. \tag{8.2}$$

It is not hard to see that, under the hypotheses of Theorem 8.1, the validity of (8.2) for some $n > 0$ implies the ostensibly stronger conclusion

$$d^*(E \cap (E-n) \cap (E-2n) \cap \ldots \cap (E-kn)) > 0. \tag{8.3}$$

Note that for any $n \in \mathbb{N}$ one has $d^*(E-n) = d^*(E)$. So we see that both Szemerédi's theorem and Furstenberg's ergodic Szemerédi theorem are about iterations of a "size" preserving transformation. To derive Szemerédi's theorem from Theorem 8.1 one can use the following form of Furstenberg's correspondence principle (see for example [**Ber1**] or [**Ber3**]).

THEOREM 8.2 (Furstenberg's correspondence principle). *Given a set $E \subseteq \mathbb{Z}$ with $d^*(E) > 0$ there is a probability measure preserving system $(X, \mathcal{B}\mu, T)$ and a*

[13] For more discussion and examples see [**Ber1**], Section 1, and [**Ber6**].

set $A \in \mathcal{B}$ with $\mu(A) = d^*(E)$ such that for any $k \in \mathbb{N}$ and any $n_1, n_2, \ldots, n_k \in \mathbb{Z}$ one has:

$$d^*(E \cap (E-n_1) \cap (E-n_2) \cap \ldots \cap (E-n_k)) \geq \mu(A \cap T^{-n_1}A \cap T^{-n_2}A \cap \ldots \cap T^{-n_k}A).$$

Fix a probability measure preserving system $(X, \mathcal{B}, \mu, T)$, a set $A \in \mathcal{B}$ with $\mu(A) > 0$, an integer $k \in \mathbb{N}$, and consider the following set:

$$R_{k,A} = \{n \in \mathbb{N} : \mu(A \cap T^{-n}A \cap T^{-2n}A \cap \ldots \cap T^{-kn}A) > 0\}.$$

When $k = 1$ this is just the set R_A which we encountered in Section 1. As we have seen, R_A is large in quite a strong sense, namely, R_A is a Δ_r^* set for any $r > \frac{1}{\mu(A)}$, and, in particular, is syndetic.[14]

It is natural to inquire to what extent the largeness properties of R_A generalize to $R_{k,A}$. The fact that $R_{k,A}$ is syndetic was already contained in Furstenberg's original paper [**F**]. Indeed, Furstenberg actually proved in [**F**] that

$$\liminf_{N-M\to\infty} \frac{1}{N-M} \sum_{n=M}^{N-1} \mu(A \cap T^{-n}A \cap T^{-2n}A \cap \ldots \cap T^{-kn}A) > 0,$$

which implies that for any $A \in \mathcal{B}$ with $\mu(A) > 0$ there is a constant $a > 0$ such that

$$\left\{n \in \mathbb{N} : \mu(A \cap T^{-n}A \cap T^{-2n}A \cap \ldots \cap T^{-kn}A) > a\right\} \text{ is syndetic.} \tag{8.4}$$

We remark in passing that neither the original combinatorial proof in [**Sz**], nor the more recent harmonic analysis proof by Gowers [**G**], leads to the syndeticity of the set $R_{k,A}$.

A much stronger result in this direction was obtained by H. Furstenberg and Y. Katznelson in [**FK1**] where they showed that $R_{k,A}$ is an IP* set. As a matter of fact they established an even stronger fact. Recall that a set $E \subseteq \mathbb{N}$ is called IP$_r^*$ if for any r-element set $\{n_1, n_2, \ldots, n_r\} \subseteq \mathbb{N}$, E has nontrivial intersection with the set $FS((n_i)_{i=1}^r)$.

Theorem 8.3 ([**FK1**], [**FK2**]). *Let $k \in \mathbb{N}$ and let $T_1, T_2, \ldots, T_k$ be commuting measure preserving transformations of a probability space $(X, \mathcal{B}, \mu)$. Then for any $A \in \mathcal{B}$ with $\mu(A) > 0$, there exist $c > 0$ and $r \in \mathbb{N}$ such that*

$$\left\{n \in \mathbb{N} : \mu(A \cap T_1^{-n}A \cap T_2^{-n}A \cap \ldots \cap T_k^{-n}A) > c\right\}$$

is an IP$_r^*$ set.

In view of Theorem 7.5, we have the following corollary.

Corollary 8.4. *Under the conditions and notation of Theorem 8.3, the set*

$$\left\{n \in \mathbb{N} : \mu(A \cap T_1^{-n}A \cap T_2^{-n}A \cap \ldots \cap T_k^{-n}A) > c\right\}$$

is a multiplicatively central* set.

We will formulate now (a special case of) the IP polynomial Szemerédi theorem obtained in [**BerM2**]. It is an open question whether the set R appearing in the formulation is an IP$_r^*$ set.

[14]Note that in view of Furstenberg's correspondence principle this fact implies that for any set $E \subseteq \mathbb{N}$ with $d^*(E) > 0$, the set of differences $E - E = \{x - y : x, y \in E\}$ is syndetic.

Theorem 8.5. *For any $k \in \mathbb{N}$, let $T_1, T_2, \ldots, T_k$ be commuting measure preserving transformations of a probability space $(X, \mathcal{B}, \mu)$ and let g_i be polynomials satisfying $g_i(\mathbb{Z}) \subseteq \mathbb{Z}$ and $g_i(0) = 0$, $i = 1, 2, \ldots, k$. Then, for any $A \in \mathcal{B}$ with $\mu(A) > 0$, there exists $c > 0$ such that the set*

$$R = \left\{ n \in \mathbb{N} : \mu(A \cap T_1^{g_1(n)} A \cap T_2^{g_2(n)} A \cap \ldots \cap T_k^{g_k(n)} A) > c \right\}$$

is an IP* set.

The fact that the set R appearing above is IP* is useful (via Furstenberg's correspondence principle) in various combinatorial applications. To formulate one such application, we need to define the notion of *multiplicatively large* sets.

Definition 8.6. A set $A \subseteq \mathbb{N}$ is *multiplicatively large* if for some sequence of positive integers $(a_n)_{n \in \mathbb{N}}$ one has

$$\limsup_{n \to \infty} \frac{|A \cap a_n F_n|}{|F_n|} > 0,$$

where $F_n = \{p_1^{i_1} \cdot p_2^{i_2} \cdot \ldots \cdot p_n^{i_n} : 0 \leq i_j \leq n, 1 \leq j \leq n\}$ and where the sequence $(p_i)_{i \in \mathbb{N}}$ consists of the primes in some arbitrary order.

Remark 8.7. (i) The notion of mulitplicatively large is a natural analog of the corresponding additive property of having positive upper Banach density.

(ii) It is not hard to see that multiplicatively syndetic, and more generally, multiplicatively piecewise syndetic, sets are multiplicatively large.

(iii) The notions of largeness based on additive and multiplicative densities do not overlap. For example, the set $2\mathbb{N} - 1$ of odd natural numbers has density $1/2$ along any sequence of intervals $[a_n, b_n]$ with $b_n - a_n \to \infty$. On the other hand, it is not hard to see that this set has density zero with respect to any averaging scheme in $(\mathbb{N}, \cdot)$. In the other direction, consider the set $S = \bigcup_{n=1}^{\infty} a_n F_n$, where F_n are defined above and the integers a_n satisfy $a_n > |F_n|$, $n = 1, 2, \ldots$. Then it is not had to check that S has zero upper Banach density. At the same time, S has multiplicative density one with respect to the sequence $(a_n F_n)_{n \in \mathbb{N}}$.

It turns out that multiplicatively large sets are much richer than the sets having positive density in $(\mathbb{N}, +)$. In particular, any multiplicatively large set contains not only arbitrarily long geometric progressions (as could be expected by mere analogy), but also arbitrarily long arithmetic progressions and more general configurations of mixed type such as
$\{q^i(a + jd) : 0 \leq i, j \leq k\}$ (see [**Ber5**], Theorem 1.5).

The following theorem (in the proof of which Theorem 8.5 plays a decisive role) is yet another manifestation of the combinatorial richness of multiplicatively large sets.

Theorem 8.8 ([**Ber5**], Theorem 3.15). *Let $E \subseteq \mathbb{N}$ be a multiplicatively large set. For any $k \in \mathbb{N}$, there exist $a, b, d \in \mathbb{N}$ such that*

$$\left\{ b(a + id)^j : 0 \leq i, j \leq k \right\} \subseteq E.$$

As was already mentioned in Section 6, for any sequence $(x_n)_{n \in \mathbb{N}} \subseteq X$ in a compact Hausdorff space, $p\text{-}\lim_{n \in \mathbb{N}} x_n$ exists for any $p \in \beta\mathbb{N}$. Since the unit ball in a separable Hilbert space is compact in the weak topology, and since the unit ball is preserved under the action of unitary operators, this opens interesting possibilities of applications of p-limits to measure preserving dynamics (and hence, via Furstenberg's correspondence principle, to combinatorics). We will describe now some examples of such applications.

THEOREM 8.9 ([**Ber2**], Theorem 3.12. See also [**BerFM**] and [**BerHåM**]). *Let $q(t) \in \mathbb{Q}[t]$ with $q(\mathbb{Z}) \subseteq \mathbb{Z}$ and $q(0) = 0$. Let U be a unitary operator on a Hilbert space $\mathcal{H}$ and let $p \in (\beta\mathbb{N}, +)$ be an idempotent. Then, letting $p\text{-}\lim_{n\in\mathbb{N}} U^{q(n)} f = P_p(f)$, where the limit is in the weak topology, P_p is an orthogonal projection onto a subspace of $\mathcal{H}$.*

COROLLARY 8.10. *Let $E \subseteq \mathbb{N}$ satisfy $d^*(E) > 0$. Then, for any $\epsilon > 0$, for any polynomial $q(t) \in \mathbb{Q}[t]$ with $q(\mathbb{Z}) \subseteq \mathbb{Z}$ and $q(0) = 0$, the set*

$$\left\{n \in \mathbb{N} : d^*(E \cap (E - q(n))) > (d^*(E))^2 - \epsilon\right\}$$

is an IP* set.

PROOF. We will show that it follows from Theorem 8.10 that for any invertible measure preserving system $(X, \mathcal{B}, \mu, T)$ and any $A \in \mathcal{B}$ with $\mu(A) > 0$,

$$p\text{-}\lim \mu(A \cap T^{q(n)} A) \geq \mu(A)^2.$$

The result in question will follow then from Furstenberg's correspondence principle.

Take $\mathcal{H} = L^2(X, \mathcal{B}, \mu)$, and take U to be the unitary operator induced by T, that is, $Ug(x) = g(Tx)$, and let $f = \mathbf{1}_A$. We have:

$$\begin{aligned} p\text{-}\lim_{n\in\mathbb{N}} \mu(A \cap T^{q(n)} A) &= p\text{-}\lim_{n\in\mathbb{N}} \langle U^{q(n)} f, f\rangle = \langle P_p f, f\rangle \\ &= \langle P_p f, P_p f\rangle \langle 1, 1\rangle \geq (\langle P_p f, 1\rangle)^2 = (\langle f, 1\rangle)^2 \\ &= (\langle \mathbf{1}_A, 1\rangle)^2 = (\mu(A))^2. \end{aligned}$$

□

In recent years, the class of the so-called *essential* (see [**BerD**]) idempotent ultrafilters in $(\beta\mathbb{N}, +)$, which is broader than that of minimal ones, has started to gain importance. The defining property of essential idempotents is that all their members have positive upper Banach density. For example, one can show that members of essential idempotents, called $\mathcal{D}$ sets, share much in the way of combinatorial richness with central sets (see [**BeiBerDF**]).

In [**BerM4**], convergence along essential idempotents was employed to obtain an extension of the polynomial Szemerédi theorem (see [**BerL1**] and [**BerM1**]). Before formulating it, we have to introduce the notion of generalized polynomials. A function $f : \mathbb{Z} \to \mathbb{Z}$ is a *generalized polynomial* if it can be obtained regular polynomials with the help of the greatest integer function $[\cdot]$ and the usual arithmetic operators. Thus the functions given by expressions like $[n^2\alpha][n^5\beta] - [n^3\delta]$ are generalized polynomials.

Note that, unlike conventional polynomials, generalized polynomials need not be eventually monotone (consider $[[n\alpha]n\beta] - [n^2\alpha\beta]$), may take only finitely many values (for example, $[(n+1)\alpha] - [n\alpha] - [\alpha]$ takes only the values zero and one), and may vanish on sets of positive density while growing without bound on other such sets (multiply the previous example by n).

Despite such oddities, new evidence has begun to emerge that generalized polynomials do possess certain strong regularities. In particular, it was shown in [**BL3**] that any bounded generalized polynomial g can be expressed as $g(n) = f(T^n x)$, where T is a translation on a nilmanifold X (that is, $X = N/\Gamma$, where N is a nilpotent group and Γ is a cocompact lattice) and f is a Riemann integrable function on X.[15]

Here now is the promised formulation of the extension of the polynomial Szemerédi theorem.

THEOREM 8.11 ([**BerM4**]). *Let $k \in \mathbb{N}$, let $q_i(x)$ be generalized polynomials, $1 \le i \le k$, and let $p \in (\beta\mathbb{N}, +)$ be an essential idempotent. Then there exist constants c_i, $1 \le i \le k$, such that if $E \subseteq \mathbb{N}$ satisfies $d^*(E) > 0$, then the set*

$$\Big\{ n \in \mathbb{N} : \exists a \in \mathbb{N} : \{a, a + q_1(n) - c_1, \dots, a + q_k(n) - c_k\} \subseteq E \Big\}$$

belongs to p.

The reader will find additional interesting applications of ultrafilters in [**BerM3**].

9. Beiglböck's Proof of Jin's Sumsets Theorem

Let $A, B \subseteq \mathbb{R}$ satisfy $\lambda(A) > 0$, $\lambda(B) > 0$, where λ denotes Lebesgue measure, and consider the sumset $A + B = \{x + y : x \in A, y \in B\}$. The classical lemma of Steinhaus states that $A + B$ has to contain an open interval. This result is an instance of the sumset phenomenon, which manifests itself in results where the sum of two "large" sets is "very large".

Another example of the sumset phenomenon is provided by the following very interesting theorem of R. Jin proved in [**J1**]. (See also [**J2**], [**BerFW**], and [**BeiBerF**]). For a set $C \subseteq \mathbb{Z}$, the upper Banach density, $d^*(C)$, is defined by

$$d^*(C) = \limsup_{N - M \to \infty} \frac{|C \cap \{M+1, \dots, N\}|}{N - M}.$$

THEOREM 9.1 ([**J1**]). *For any $A, B \subseteq \mathbb{Z}$ satisfying $d^*(A) > 0$, $d^*(B) > 0$, the sumset $A + B$ is piecewise syndetic.*

The original proof of Theorem 9.1 in [**J1**] utilized nonstandard analysis. This proof was converted to a standard one in [**J2**]. Later on, additional approaches were found, which allow one to strengthen Jin's result and to extend it to general amenable groups (see [**JK**], [**BerFW**], [**BeiBerF**]). We will present now a most recent proof of Jin's theorem due to Beiglböck [**Bei2**]. This proof is short and sweet and makes a nice use of ultrafilters.

For a set $A \subseteq \mathbb{Z}$ and an ultrafilter $p \in \beta\mathbb{Z}$, let

$$A - p = \{k \in \mathbb{Z} : A - k \in p\}.$$

(Note that when p is a principal ultrafilter, this reduces to the usual definition of a shifted set).

LEMMA 9.2. *For any $A, B \subseteq \mathbb{Z}$ there exists $p \in \beta\mathbb{Z}$ such that $d^*(A \cap (B - p)) \ge d^*(A) d^*(B)$.*

To see that Theorem 9.1 follows from Lemma 9.2, assume that $d^*(A), d^*(B) > 0$. Then by the Lemma, there exists $p \in \beta\mathbb{Z}$ such that the set $C = (-A) \cap (B - p)$ has $d^*(C) > 0$. By Footnote 14, $C - C$ is syndetic and so is $S := A + (B - p) \supseteq C - C$.

[15] One can show that if T is a translation on a nilmanifold X then for any idempotent $p \in (\beta\mathbb{N}, +)$ and any $x \in X$, one has $p\text{-}\lim_{n \in \mathbb{N}} T^n x = x$. This leads to interesting Diophantine applications. See, for example, Theorem D in [**BL3**].

Note that if $s \in A+(B-p)$ then $A+B-s \in p$. (Indeed, if $s \in A+\{k \in \mathbb{Z} : B-k \in p\}$ then, for some $a \in A$, $B-(s+a) \in p$, which implies $A+B-s \in p$).

So, for every finite set $\{s_1, s_2, \ldots, s_n\} \subseteq A+(B-p)$ we have $\bigcap_{i=1}^{n}(A+B-s_i) \in p$, and hence this intersection is nonempty. This, in turn, implies that for some $t \in \mathbb{Z}$, $t+\{s_1, s_2, \ldots, s_n\} \subseteq A+B$. So we see that $A+B$ contains shifts of all finite subsets of the syndetic set $A+(B-p)$ which implies that $A+B$ is piecewise syndetic.

It remains to prove Lemma 9.2. Before doing so, we summarize some facts which will be used in the proof.

First, we note that, given a set $A \subseteq \mathbb{Z}$, one can always find an *invariant mean*, i.e. a shift-invariant finitely additive probability measure m on $\mathbb{Z}, \mathcal{P}(\mathbb{Z}))$, such that $m(A) = d^*(A)$. To see this, let finite intervals $I_n \subseteq \mathbb{Z}$ be such that $d^*(A) = \lim_{n\to\infty} m_n(A)$, where for $B \in \mathcal{P}(\mathbb{Z})$, $m_n(B) := \frac{|B\cap I_n|}{|I_n|}$. Now take m to be a cluster point of the set $\{m_n : n \in \mathbb{N}\}$ in the (compact) set $[0,1]^{\mathcal{P}(\mathbb{Z})}$.

Since $B(\mathbb{Z})$, the space of bounded functions on $\mathbb{Z}$, is isomorphic to $C(\beta\mathbb{Z})$, it follows from the Riesz representation theorem that there exists a regular Borel probability measure μ on $\beta\mathbb{Z}$ such that $m(A) = \mu(\bar{A})$ for all $A \subseteq \mathbb{Z}$ (here $\bar{A} = \{p \in \beta\mathbb{Z} : A \in p\}$).

To prove the lemma, pick a sequence of intervals $I_n \subseteq \mathbb{Z}$, $n \in \mathbb{N}$, with $|I_n| \to \infty$ and $d^*(B) = \lim_{n\to\infty} \frac{|B\cap I_n|}{|I_n|}$, and pick an invariant mean m such that $m(A) = d^*(A)$. Finally, define $f_n : \beta\mathbb{Z} \to [0,1]$ by

$$f_n(p) := \frac{|I_n \cap B \cap \{k \in \mathbb{Z} : (A-k) \in p\}|}{|I_n|} = \frac{1}{|I_n|} \sum_{k \in I_n \cap B} \mathbf{1}_{\overline{A-k}}(p),$$

and let $f(p) = \limsup_{n\to\infty} f_n(p) \leq d^*\big(B \cap \{k \in \mathbb{Z} : (A-k) \in p\}\big)$. By Fatou's lemma,

$$\begin{aligned} \int f d\mu \geq \limsup_{n\to\infty} \int \frac{1}{|I_n|} \sum_{k \in I_n \cap B} \mathbf{1}_{\overline{A-k}} d\mu &= \limsup_{n\to\infty} \frac{1}{|I_n|} \sum_{k \in I_n \cap B} m(A-k) \\ &= d^*(A) \cdot d^*(B) \end{aligned}$$

This implies that for some $p \in \beta\mathbb{Z}$, $d^*(A) \cdot d^*(B) \leq f(p)$, and we are done.

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