Recurrence for Semigroup Actions and a non-Commutative Schur Theorem

Vitaly Bergelson and Randall McCutcheon

1. Introduction

This note is concerned with the topological, measure theoretical and combinatorial aspects of *recurrence* and their interplay. The classical Poincaré recurrence theorem asserts that for any finite measure preserving system $(X, \mathcal{B}, \mu, T)$ and any set $A \in \mathcal{B}$ with $\mu(A) > 0$, μ-almost every point $x \in A$ returns to A under a power of the transformation T, that is $\mu(\{x \in A : \text{ there exists } n \in \mathbf{N} \text{ with } T^n x \in A\}) = \mu(A)$. This result, which deals with recurrence of points, can easily be obtained from a result dealing with recurrence of sets: if $\mu(A) > 0$ then for some $n \in \mathbf{N}$ one has $\mu(A \cap T^{-n}A) > 0$. It is the recurrence of sets in measure preserving, topological and combinatorial set-ups which will be the focus of our attention in this paper. (A standing exercise, which we offer to the reader, is to detect the recurrence of points lying in the background.)

A far-reaching refinement of the Poincaré recurrence theorem was obtained by Furstenberg in [F1]: for any finite measure preserving system $(X, \mathcal{B}, \mu, T)$, any $A \in \mathcal{B}$ with $\mu(A) > 0$, and any $k \in \mathbf{N}$ there exists $n \in \mathbf{N}$ such that $\mu(A \cap T^{-n}A \cap \cdots \cap T^{-kn}A) > 0$. In the same paper Furstenberg showed that this *multiple recurrence* theorem implies Szemerédi's celebrated theorem on arithmetic progressions ([Sz]), thereby establishing a fruitful and mutually perpetuating link between ergodic theory and *density* Ramsey theory.

In [FW] Furstenberg and Weiss established a similar link between topological dynamics and *partition* Ramsey theory, which deals with theorems like van der Waerden's, Hindman's, etc. (see [GRS] for background; see also [F2] and [B2]). For example, one of the results in [FW] asserts that for any continuous self-mapping T of a compact metric space X, any $k \in \mathbf{N}$, and any $\epsilon > 0$, there exists $x \in X$ and $n \in \mathbf{N}$ such that the diameter of the set $\{x, T^n x, \cdots, T^{kn}x\}$ is smaller than ϵ. This result implies van der Waerden's theorem on arithmetic progressions, which is the natural counterpart in partition Ramsey theory to Szemerédi's theorem. Again, this result about recurrence of points follows from a corresponding result about recurrence of sets: if (X, T) is a minimal dynamical system, then for any $k \in \mathbf{N}$ and any non-empty open set U one has $U \cap T^{-n}U \cap \cdots \cap T^{-kn}U \neq \emptyset$ for some $n \in \mathbf{N}$. Alternatively: if (X, T) is a (not necessarily minimal) dynamical system and $\{U_1, \cdots, U_t\}$ is an open cover of X then for some i, $1 \leq i \leq t$, and some $n \in \mathbf{N}$ one has $U_i \cap T^{-n}U_i \cap \cdots \cap T^{-kn}U_i \neq \emptyset$.

While the measurable and topological multiple recurrence results alluded to provide one with strong Ramsey-theoretical results, already there are some interesting questions and results concerning the relationship between these two modes of recurrence for the single recurrence case. A good illustration is given by a result due to Kriz ([K]), which we now describe. In an attempt to better understand the connection between results of density Ramsey theory and their counterparts in partition Ramsey theory, the first author asked whether it is true that if $\mathcal{R}$ is a shift-invariant family of finite sets in $\mathbf{N}$ having the property that some member of $\mathcal{R}$ can always be found in one cell of an arbitrary finite partition of $\mathbf{N}$, then any set of positive upper density in $\mathbf{N}$ contains a member of $\mathcal{R}$. Kriz demonstrated the answer to be no.

To more fully elucidate what he proved, we will introduce two notions. First, let a subset R of $\mathbf{N}$ be called *density intersective* provided that for every $A \subset \mathbf{N}$ with $\overline{d}(A) > 0$ there exists $n \in R$ such that $A \cap (A - n) \neq \emptyset$. R will be called *chromatically intersective* if for every partition of $\mathbf{N}$ into r cells, one of the cells, call it C, satisfies $C \cap (C - n) \neq \emptyset$ for some $n \in R$. What Kriz did was construct a chromatically intersective set R which is not density intersective. Letting then $\mathcal{R}$ be the set of configurations $\{a, a+n\}$, where $n \in R$ and $a \in \mathbf{N}$, $\mathcal{R}$ is easily seen to indeed be a shift-invariant class having the property that for every finite partition of $\mathbf{N}$, some cell contains a member of $\mathcal{R}$, yet there exists a set A with $\overline{d}(A) > 0$ which does not contain a member of $\mathcal{R}$.

Due to the aforementioned connection between ergodic theory and density combinatorics (see also Theorem 2.1), one may show that a set $R \subset \mathbf{N}$ is density intersective if and only if for every measure preserving transformation T of a probability space $(X, \mathcal{B}, \mu)$ and every $A \in \mathcal{B}$ with $\mu(A) > 0$ there exists $n \in R$ such that $\mu(A \cap T^{-n}A) > 0$. For this reason, density intersective sets are also called *sets of measurable recurrence.* Similarly, the connection between Ramsey theory and topological dynamics provides a dynamically formulated condition equivalent to that of being a set of chromatic intersectivity. There are several equivalent forms, however the one most commonly used ([Fo], [M]) has been the following: R is chromatically intersective if and only if for every invertible, minimal dynamical system (X, T), where X is a compact metric space, and every non-empty open set U, there exists $n \in R$ such that $U \cap T^{-n}U \neq \emptyset$. For this reason, chromatically intersective sets in $\mathbf{N}$ are also called *sets of topological recurrence.* Thus, the set constructed by Kriz is a set of topological recurrence which is not a set of measurable recurrence.

One of our purposes is to extend these notions, defined here in $\mathbf{N}$, to arbitrary countable semigroups S, exploring their interrelationships both with each other and with combinatorics. We remark that most of the definitions and results of this paper make sense for actions of uncountable discrete semigroups S (in other words, actions with no continuity restrictions), however we do not pursue this notion, in part because we have no combinatorial results for uncountable semigroups which are not already obtainable as corollaries of results for countable semigroups.

Moreover, usually when one considers actions of uncountable semigroups, these actions are taken to be continuous with respect to some given topology, and for this reason there may be some cause for confusion in what should be considered a set of recurrence in a group such as $\mathbf{R}$. Indeed, by [BBB, p.43] for any $\alpha > 0$, any continuous measure preserving flow $\{T_t : t \in \mathbf{R}\}$ on a probability space $(X, \mathcal{B}, \mu)$ and any $A \in \mathcal{B}$ with $\mu(A) > 0$ there exists $n \in \mathbf{N}$ such that $\mu(A \cap T_{n^\alpha} A) > 0$. On the other hand, for all but countably many α, $\{n^\alpha : n \in \mathbf{N}\}$ is linearly independent over $\mathbf{Q}$ (see [BBB, Lemma 2.9 and Remark on p. 38]). For such an $\alpha > 1$, [BBB, Theorem D] allows one to find a measurable set $E \subset \mathbf{R}$ such that

$$d(E) = \lim_{t \to \infty} \frac{m(E \cap [0, t])}{t} = \frac{1}{2}$$

and $E \cap (E - n^\alpha) = \emptyset$ for all $n \in \mathbf{N}$. Using the methods of Section 2 (namely the proof of Theorem 2.1, which works for uncountable *discrete* semigroups) one may use this example to construct a (non-continuous!) measure preserving $\mathbf{R}$-action $\{T_g\}_{g\in\mathbf{R}}$ on a probability space $(X, \mathcal{B}, \mu)$ such that for some $A \in \mathcal{B}$ with $\mu(A) > 0$ we have $\mu(A \cap T_{n^\alpha} A) = 0$ for all $n \in \mathbf{N}$. Therefore, whether or not one should call $\{n^\alpha : n \in \mathbf{N}\}$ a set of measurable recurrence appears to be a matter of perspective. This is another reason we choose not to pursue this issue here.

Definition 1.1 Let S be a countable semigroup and suppose $R \subset S$.

(a) R is called a *set of measurable recurrence* if for every measure preserving S-action $\{T_g\}_{g\in S}$ on a probability space $(X, \mathcal{B}, \mu)$ and every $A \in \mathcal{B}$ satisfying $\mu(A) > 0$ we have $\mu(A \cap T_g^{-1}A) > 0$ for some $g \in R$.

(b) R is called a *set of topological recurrence* if for every minimal S-action $\{T_g\}_{g\in S}$ by continuous self-mappings of a compact metric space X and every open set U we have $U \cap T_g^{-1}U \neq \emptyset$ for some $g \in R$.

Unfortunately, for non-abelian semigroups S the property of Definition 1.1 (b) is not what is needed for the Ramsey theoretic connection. The correct notion, which for abelian semigroups (but not in general) is easily shown (see Theorem 2.6 (b)) to be equivalent, is the following.

Definition 1.2 A subset R of a countable semigroup S is called a *set of chromatic recurrence* if for every S-action $\{T_g\}_{g\in S}$ by continuous self-mappings of a compact metric space X and every open covering $\{U_1, \cdots, U_r\}$ of X there exists i with $1 \leq i \leq r$ such that $U_i \cap T_g^{-1}U_i \neq \emptyset$ for some $g \in R$.

Remark. Our definitions of sets of recurrence are such that if S contains an identity e then $\{e\}$ is automatically a set of recurrence. This is a departure from the definition of sets of recurrence for $S = \mathbf{Z}$ given in [F2], in which R is a set of recurrence if and only if $R \setminus \{0\}$ is a set of recurrence. We are changing things a little for the sake of convenience. The difference, of course, is more philosophical than actual.

Equivalent combinatorial formulations of each of these three types of sets of recurrence are given in Section 2. For sets of topological recurrence and sets of chromatic recurrence, these equivalences hold for all countable semigroups S, however for sets of measurable recurrence the equivalence is only valid in the event that S is countable and left amenable. We then show that sets of topological recurrence are always sets of chromatic recurrence, and that for countable left amenable semigroups sets of measurable recurrence are always sets of topological recurrence. Meanwhile, since for abelian semigroups chromatic and topological recurrence are equivalent, [K] provides us with a set of chromatic recurrence which is not a set of measurable recurrence.

The reason sets of measurable recurrence in countable left amenable semigroups must also be sets of topological recurrence is that for any action of such a semigroup S by continuous self-mappings of a compact metric space X, there exists an invariant measure. For non-amenable (semi)groups, no such measure is guaranteed, removing any a priori reason why a set of measurable recurrence should be a set of topological recurrence. Indeed, one may almost effortlessly find sets of measurable recurrence in free groups which are not sets of topological recurrence (see Theorem 2.6 (c)). A more interesting question is whether one may find sets of measurable recurrence which are not sets of *chromatic* recurrence.

There is some evidence (Theorem 4.5) which suggests that there may be such examples, but as of now we haven't found any. Section 4 is a short section devoted to expounding upon this and various other open questions, a few of which pertain to *multiple* recurrence.

Section 3 contains a non-commutative version of Schur's theorem. Recall that given a finite partition of a semigroup S, $S = \bigcup_{i=1}^{r} C_i$, one of the cells C_i contains a configuration of the form $\{x, y, xy\}$. This result, which in the case $S = (\mathbf{N}, +)$ is due to Schur (see [S], [GRS]), has interesting consequences in number theory (cf. [Si], [B1]).

Schur's theorem is extended considerably by the following marvelous theorem of Hindman ([H]): for any finite partition of a semigroup S there is a sequence $\{x_i\}_{i\in\mathbf{N}}$ with the property that all its elements together with all finite products of the form $x_{i_1}x_{i_2}\cdots x_{i_k}$, where $k \in \mathbf{N}$ is arbitrary and $i_1 < \cdots < i_k$, belong to the same cell of the partition.

We use a result about *double recurrence* recently obtained in [BMZ] to advance Schur's theorem in a non-commutative direction. Namely, we address the following question inspired by Hindman's theorem and stemming from the general philosophy of Ramsey theory: Is it true that for any finite partition of any "sufficiently non-commutative" group and any natural number k, one of the cells of the partition contains pairwise non-commuting elements $x_1, \cdots, x_k$ together with all possible products $x_{i_1}x_{i_2}\cdots x_{i_m}$ (where $i_r \neq i_t$ for $r \neq t$)? The vague formulation ("sufficiently non-commutative") we have given is intentionally open-ended, however at this stage its meaning for us is that, letting A be the subgroup (to see that A is in fact a subgroup, see the beginning of the proof of Theorem 3.4 in Section 3) of G consisting of all elements x such that $[G : C(x)] < \infty$, where $C(x)$ is the centralizer of x, we have $[G : A] = \infty$. In other words, there are neglectably few elements which commute with many elements of G. For amenable groups at least, this seems to be appropriate. Plenty of countable amenable groups satisfy this condition, for example the Heisenberg group and the group S_∞ of finite permutations of $\mathbf{N}$. Indeed, we know of no example of a countable amenable group G having no abelian subgroups of finite index which fails the condition. At any rate, our modest contribution to the question introduced above is the following theorem.

Theorem 3.4 Suppose that G is a countable amenable group and that $G = \bigcup_{i=1}^{r} C_i$ is a finite partition. Let $A = \{g \in G : [G : C(g)] < \infty\}$. If $[G : A] = \infty$ then there exist $x, y \in G$ and i, $1 \leq i \leq r$, with $xy \neq yx$ and such that $\{x, y, xy, yx\} \subset C_i$.

Although on the face of it this theorem belongs strictly to partition Ramsey theory, it belongs to a sub-class of results in this field for which we have at present no proof which avoids ergodic theory (see [BMZ, Corollary 7.2] for another). In Section 5 we offer a very short philosophical discussion which hints at some of the reasons for this phenomenon (at least in non-abelian groups) by indicating where the available topological methods fail.

2. Combinatorial formulations of and relationships between the various types.

In this section we will formulate equivalent combinatorial conditions for the three kinds of sets of recurrence defined in Section 1. Then, we will point out a few of the inclusive and exclusive relationships these concepts have with each other. First we handle sets of measurable recurrence. To this end, we give a general form of a correspondence principle due to Furstenberg which is valid for countable left amenable semigroups. Recall that a semigroup S is left amenable if there exists a *left invariant mean* m on $l^\infty(S)$. Namely, m is a member of $l^\infty(S)^*$ with $m(\mathbf{1})=1$, $m(g)\geq 0$ if $g(s)\geq 0$ for all $s\in S$, and $m({}_sg)=m(g)$ for all $s\in S$ (where ${}_sg(t)=g(st)$). Given a left invariant mean $m\in l^\infty(S)^*$ and $E\subset S$, we will follow convention and write $m(E)$ for $m(1_E)$. Written this way, m becomes a finitely additive, shift invariant (i.e. $m(s^{-1}E)=m(E)$ for all $s\in S$ and $E\subset S$, where $s^{-1}E=\{g\in S: sg\in E\}$) probability measure on $\mathcal{P}(S)$, the power set of S.

Theorem 2.1 Let S be a countable left amenable semigroup, let m be a left invariant mean and suppose $A\subset S$. There exists a probability space $(X,\mathcal{B},\mu)$ (X is in fact a compact metric space and $\mathcal{B}$ is the σ-algebra of Borel sets), a set $U\in\mathcal{B}$ (which is open and closed), and a measure preserving S-action $\{T_g\}_{g\in S}$ on X (the transformations are continuous) such that for every $h_1,\cdots,h_k\in S$ we have

$$\mu(T_{h_1}^{-1}U\cap T_{h_2}^{-1}U\cap\cdots\cap T_{h_k}^{-1}U)=m(h_1^{-1}A\cap h_2^{-1}A\cap\cdots\cap h_k^{-1}A). \tag{2.1}$$

Proof. By adding it if necessary we will assume that S contains an identity e. Let $X=\{0,1\}^S$. With the product topology, X is compact and metric. Let $\mathcal{C}$ be the collection of *cylinder sets*, that is sets of the form

$$C=\{\gamma\in X:\gamma(h_1)=\epsilon_1,\gamma(h_2)=\epsilon_2,\cdots,\gamma(h_k)=\epsilon_k\}, \tag{2.2}$$

where $k\in\mathbf{N}$, $h_i\in S$ are distinct and $\epsilon_i\in\{0,1\}$, $1\leq i\leq k$. Note that $\mathcal{C}$ is closed under finite intersections. Let $\mathcal{A}$ be the algebra generated by $\mathcal{C}$, that is, the set of finite unions of members of $\mathcal{C}$. For $C\in\mathcal{C}$ given by (2.2), let $\lambda(C)=m(h_1^{-1}A_1\cap h_2^{-1}A_2\cap\cdots\cap h_k^{-1}A_k)$, where $A_i=A$ if $\epsilon_i=1$ and $A_i=A^c$ if $\epsilon_i=0$, $1\leq i\leq k$. One easily checks (due to additivity of m) that if C_1 and C_2 are disjoint members of $\mathcal{C}$ and $(C_1\cup C_2)\in\mathcal{C}$ then $\lambda(C_1\cup C_2)=\lambda(C_1)+\lambda(C_2)$. It follows that λ may be extended to a finitely additive set function on $\mathcal{A}$. Furthermore, every member of $\mathcal{A}$ is both open and closed in X. Therefore if $(B_i)_{i=1}^\infty$ is a sequence of pairwise disjoint members of $\mathcal{A}$ whose union lies in $\mathcal{A}$ then by compactness all but finitely many of the sets B_i must be empty. In particular, $\lambda(\bigcup B_i)=\sum\lambda(B_i)$. It follows that λ is a *premeasure* on $\mathcal{A}$ and hence extends to a measure μ on the σ-algebra $\mathcal{B}$ of Borel sets. One easily checks that, due to invariance of m, μ is invariant under the shift action $\{T_g\}_{g\in S}$, where $T_g\gamma(h)=\gamma(hg)$, $\gamma\in X$, $h,g\in S$ (it suffices to check this on $\mathcal{A}$). Finally we let $U=\{\gamma\in X:\gamma(e)=1\}$. Then

$$\begin{aligned}\mu(T_{h_1}^{-1}U\cap T_{h_2}^{-1}U\cap\cdots\cap T_{h_k}^{-1}U)&=\mu(\{\gamma\in X:\gamma(h_1)=1,\gamma(h_2)=1,\cdots,\gamma(h_k)=1\})\\&=m(h_1^{-1}A\cap h_2^{-1}A\cap\cdots\cap h_k^{-1}A).\end{aligned}$$

□

This in hand, we are able to give combinatorial necessary and sufficient conditions for a set R to be a set of measurable recurrence. These conditions involve the return of a "large" set $A\subset S$ to itself, in the sense that $A\cap g^{-1}A\neq\emptyset$ for some $g\in R$. (See also Theorem 3.2 and the discussion following Definition 4.1.)

Theorem 2.2 Suppose that S is a countable left amenable semigroup. Then $R \subset S$ is a set of measurable recurrence if and only if for every left invariant mean m and every $A \subset S$ with $m(A) > 0$ we have $A \cap g^{-1}A \neq \emptyset$ for some $g \in R$.

Proof. ($\Rightarrow$) Suppose m is a left invariant mean and $A \subset S$ with $m(A) > 0$. Let $(X, \mathcal{B}, \mu)$ and $U \in \mathcal{B}$ be as guaranteed by Theorem 2.1. In particular, (2.1) holds. Since R is a set of measurable recurrence there exists $g \in R$ such that $\mu(U \cap T_g^{-1}U) > 0$. Therefore by (2.1) $m(A \cap g^{-1}A) > 0$.

($\Leftarrow$) Let $\{T_g\}_{g \in S}$ be a measure preserving S-action on a probability space $(X, \mathcal{B}, \mu)$ and suppose that $U \in \mathcal{B}$ with $\mu(U) > 0$. Delete from X every set of the form $U \cap T_g^{-1}U$ which is of measure 0, $g \in S$. The reason we are doing this is that in the space which remains, $U \cap T_g^{-1}U \neq \emptyset$ implies $\mu(U \cap T_g^{-1}U) > 0$. For every $x \in X$, let $E_x = \{g \in S : x \in T_g^{-1}U\}$. We claim that $m(E_x) > 0$ for some $x \in X$ and some left invariant mean m. Supposing the claim to be valid, by hypothesis there exists $g \in R$ such that $E_x \cap g^{-1}E_x \neq \emptyset$. Letting $k \in E_x \cap g^{-1}E_x$, we have $T_k x \in U$ and $T_{gk}x \in U$, so that $T_k x \in U \cap T_g^{-1}U$. According to our earlier stipulation, $\mu(U \cap T_g^{-1}U) > 0$. Therefore, all we must do is establish the claim.

We take $l^1(S)$ to consist of functions $f : S \to \mathbf{R}$ satisfying

$$||f|| = \sum_{s \in S} |f(s)| < \infty.$$

Members f of $l^1(S)$ induce elements $\tilde{f}$ of $l^\infty(S)^*$ by the formula

$$\tilde{f}(g) = \sum_{s \in S} f(s)g(s), \quad g \in l^\infty(S).$$

Let $\Phi \subset l^1(S)$ consist of all functions $f : S \to [0,1]$ such that $\{s : f(s) > 0\}$ is finite and $\sum_{s \in S} f(s) = 1$. For all $f \in \Phi$ one has $\tilde{f}(\mathbf{1}) = 1$ and $\tilde{f}(g) \geq 0$ if $g \geq 0$, hence members of Φ are called *finite means*. For $s \in S$ and $f \in \Phi$, define $s * f \in \Phi$ by $s * f(t) = \sum_{\{h : sh = t\}} f(h)$. A theorem of Day ([D]) states that left amenability of S is equivalent to existence of a net $(f_\gamma) \subset \Phi$ satisfying $\lim_\gamma ||s * f_\gamma - f_\gamma|| = 0$. Using the fact that S is countable, one can assume this net to be a *sequence* $(f_i)_{i=1}^\infty$.

For $E \subset S$, let $\overline{m}(E) = \limsup_{n \to \infty} \tilde{f}_n(E)$ (by $\tilde{f}_n(E)$ we mean $\tilde{f}_n(1_E) = \sum_{s \in E} f_n(s)$). Suppose for some $x \in X$ we were to have $\overline{m}(E_x) > 0$. Then for some increasing sequence $(n_k) \subset \mathbf{N}$, $\lim_k \tilde{f}_{n_k}(E) > 0$ and letting m be any weak-* limit point of $\{\tilde{f}_{n_k}\}_{k=1}^\infty$ in $l^\infty(S)^*$, m will be a left invariant mean satisfying $m(E_x) > 0$, completing the proof. We have therefore reduced the problem to finding x with $\overline{m}(E_x) > 0$.

Suppose then that $\overline{m}(E_x) = 0$ for all $x \in X$ (we will arrive at a contradiction). For every x there exists some least number $n_x \in \mathbf{N}$ having the property that for all $n \geq n_x$ we have $\tilde{f}_n(E_x) < \frac{\mu(U)}{2}$. That n_x is a measurable function of x follows from the fact that $x \to \tilde{f}_n(E_x) = \sum_{x \in T_s^{-1}U} f_n(s)$ is measurable for each n. Let $n_0 \in \mathbf{N}$ have the property that $\mu(\{x : n_x > n_0\}) < \frac{\mu(U)}{2}$. For every $n \geq n_0$, we have

$$\int \tilde{f}_n(E_x)\, d\mu(x) < \frac{\mu(U)}{2} + \frac{\mu(U)}{2} = \mu(U).$$

On the other hand,

$$
\begin{aligned}
\int \tilde{f}_n(E_x)\, d\mu(x) &= \int \sum_{s\in E_x} f_n(s)\, d\mu(x) \\
&= \int \sum_{s\in S} f_n(s) 1_{E_x}(s)\, d\mu(x) \\
&= \int \sum_{s\in S} f_n(s) 1_{T_s^{-1}U}(x)\, d\mu(x) \\
&= \sum_{s\in S} f_n(s)\mu(T_s^{-1}U) = \mu(U).
\end{aligned}
$$

This contradiction completes the proof.

□

Thus we see that sets of positive measure return to themselves under R if and only if sets of positive density return to themselves in the sense that $A \cap g^{-1}A \neq \emptyset$ for some $g \in R$. (Of course for $\mathbf{Z}$-actions this is already well known and Theorem 2.2 is a routine extension of this fact.) For topological recurrence we need a different notion of largeness for subsets of S, namely that of *syndeticity*. A subset of E of $\mathbf{N}$ is said to be syndetic if it has bounded gaps. Equivalently, $E \subset \mathbf{N}$ is syndetic if $\mathbf{N}$ is the union of finitely many shifts of E. For non-abelian semigroups, there are two versions of syndeticity, namely left and right, depending on which direction one shifts from. An idea related to syndeticity is that of *thickness*. In an abelian semigroup, a set is thick if it contains a shifted copy of every finite configuration (hence in $\mathbf{Z}$ a set is thick if and only if it contains arbitrarily long intervals). For non-abelian semigroups there are again two versions. Since we deal with left semigroup actions, left syndeticity and left thickness are the right notions for us (irony unintended).

Definition 2.3 Let S be a countable semigroup and suppose $E \subset S$.

(a) E is said to be *left* (respectively *right*) *syndetic* if there exists a finite set $H \subset S$ such that $S = \bigcup_{t\in H} t^{-1}E$ (respectively $S = \bigcup_{t\in H} Et^{-1}$).

(b) E is said to be *left* (respectively *right*) *thick* if for every finite set $H \subset S$ there exists $x \in S$ such that $Hx \subset E$ (respectively $xH \subset E$).

It is easy to show that E is left (respectively right) syndetic if and only if E^c fails to be left (respectively right) thick. This observation is used in our next proof, as is the well-known fact that if $\{T_g\}_{g\in S}$ is an S-action by continuous self-mappings of a compact space Y then there exists a closed set $X \subset Y$ which is invariant (i.e. $T_g X \subset X$ for all $g \in S$) and with respect to which $\{T_g\}_{g\in S}$ is minimal. That is, if $W \subset X$ is a closed invariant set then $W = X$ or $W = \emptyset$.

Theorem 2.4 In a countable semigroup S, a set R is a set of topological recurrence if and only if for every left syndetic set $A \subset S$ there exists $g \in R$ such that $A \cap g^{-1}A \neq \emptyset$.

Proof. ($\Rightarrow$) Let $A \subset S$ be left syndetic. By adjoining one if necessary, we will assume that S has an identity e. Put $Z = \{0,1\}^S$. Endowed with the product topology, Z is a compact metric space. A left S-action $\{T_g\}_{g\in S}$ by continuous self-mappings of Z may be given by $T_g\gamma(h) = \gamma(hg)$, $\gamma \in Z$. Let $Y = \overline{\{T_g 1_A : g \in S\}}$. One easily checks that Y is S-invariant. Choose a closed invariant set $X \subset Y$ with respect to which the action $\{T_g\}_{g\in S}$ is minimal. Let $U = \{\gamma \in X : \gamma(e) = 1\}$. U is clearly open in X. We claim it is also non-empty. Indeed, for any $\gamma \in X$, $T_g\gamma \in U$ for some $g \in S$. Equivalently, $\gamma(g) = 1$ for some $g \in S$, that is, the function which sends every element of S to 0 does not lie in X. Otherwise, for every finite set $H \subset S$ there would exist $k \in G$ such that $T_k 1_A(s) = 0$ for all $s \in H$, in other words $Hk \subset A^c$, which since H is arbitrary is to say that A^c is left thick, a contradiction. This establishes the claim. Since U is non-empty, and R is a set of topological recurrence, there exists $g \in R$ such that $U \cap T_g^{-1}U \neq \emptyset$. In particular, there exists $\gamma \in Y$ and $g \in R$ such that $\gamma(e) = \gamma(g) = 1$, which implies that for some $h \in S$ we have $T_h 1_A(e) = T_h 1_A(g) = 1$, meaning that h and gh lie in A. That is, $h \in A \cap g^{-1}A$.

($\Leftarrow$) Let $\{T_g\}_{g\in S}$ be a minimal S-action on a compact metric space X and let $U \subset X$ be open. There exists a finite set $H \subset S$ such that $X = \bigcup_{h\in H} T_h^{-1}U$. Let $x \in X$ and put $E = \{g \in S : T_g x \in U\}$. Notice that for every $g \in S$ we have $T_g x \in T_h^{-1}U$ for some $h \in H$, that is $T_{hg}x \in U$, which is to say $g \in h^{-1}E$. Hence $S = \bigcup_{h\in H} h^{-1}E$, so that in particular E is left syndetic and by hypothesis there exists $g \in R$ such that $E \cap T_g^{-1}E \neq \emptyset$. Let $a \in E \cap T_g^{-1}E$ and put $z = T_a x$. Then $z \in U$ and $T_g z \in U$. In other words, $z \in U \cap T_g^{-1}U$.

Finally, here is the equivalence result for sets of chromatic recurrence.

Theorem 2.5 In a countable semigroup S, a set R is a set of chromatic recurrence if and only if for every partition of S into finitely many cells $A_1, \cdots, A_r$ there exists i, $1 \leq i \leq r$ such that $A_i \cap g^{-1}A_i \neq \emptyset$ for some $g \in R$.

Proof. ($\Rightarrow$) Suppose a partition $S = A_1 \cup \cdots \cup A_r$ is given. If S contains an identity, call it e and let $A_{r+1} = \emptyset$. Otherwise, adjoin an identity e to S and put $A_{r+1} = \{e\}$. Let $Z = \{1, 2, \cdots, r+1\}^S$ and define $\gamma \in Z$ by $\gamma(g) = i$ if and only if $g \in A_i$. Let $X = \overline{\{T_g\gamma : g \in S\}}$, where $T_g\alpha(h) = \alpha(hg)$, $g \in S$, $\alpha \in Z$. Then X is S-invariant. Let $U_i = \{\alpha \in X : \alpha(e) = i\}$, $1 \leq i \leq r+1$. Then $\{U_1, \cdots, U_{r+1}\}$ is an open covering of X. Since R is a set of chromatic recurrence, there exists i, $1 \leq i \leq r+1$, and $g \in R$ such that $U_i \cap T_g^{-1}U_i \neq \emptyset$. This implies that $T_k\gamma \in U_i \cap T_g^{-1}U_i$ for some $k \in S$. Hence $k \in A_i$ and $gk \in A_i$, that is, $k \in A_i \cap g^{-1}A_i$. If $i \neq r+1$ we are done, but if $i = r+1$ then $k = g = e$, a contradiction since $g \in R \subset \bigcup_{i=1}^r A_i$.

($\Leftarrow$) Let $\{T_g\}_{g\in S}$ be an S-action by continuous self-mappings of a compact metric space X and suppose that $\{U_1, \cdots, U_r\}$ is an open covering. Let $x \in X$ and for every $g \in S$ let i_g be the minimal i such that $T_g x \in U_i$. Let $A_i = \{g \in S : i_g = i\}$. Then $S = A_1 \cup \cdots \cup A_r$ is a finite partition. By hypothesis, there exists i, $1 \leq i \leq r$, and $g \in R$ such that $A_i \cap g^{-1}A_i \neq \emptyset$. Let $k \in A_i \cap g^{-1}A_i$. Then $T_k x \in U_i$ and $T_{gk}x \in U_i$. It follows that $T_k x \in U_i \cap T_g^{-1}U_i$.

□

The following theorem indicates in part the relative strength of the three notions of sets of recurrence we have defined.

Theorem 2.6 Suppose that S is a countable semigroup and $R \subset S$.

(a) If R is a set of topological recurrence then R is a set of chromatic recurrence.

(b) If R is a set of chromatic recurrence and $R \subset Z(S) = \{s \in S : st = ts \text{ for all } t \in S\}$ (in particular, if S is abelian) then R is a set of topological recurrence.

(c) There exists an example where S is a group and R is both a set of measurable recurrence and a set of chromatic recurrence, but R is not a set of topological recurrence.

(d) If R is a set of measurable recurrence and S is left amenable then R is both a set of topological recurrence and a set of chromatic recurrence.

(e) There exists an example in which $S = \mathbf{N}$, R is both a set of topological recurrence and a set of chromatic recurrence, but R is not a set of measurable recurrence.

Proof. (a) Suppose that $\{T_g\}_{g \in S}$ is an S-action by continuous self-mappings of a compact metric space X and suppose that $\{U_1, \cdots, U_r\}$ is an open covering of X. Let $Y \subset X$ be a minimal invariant set. For some i we have $U = Y \cap U_i \neq \emptyset$. Since R is a set of topological recurrence and U is a non-empty open subset of Y, $U \cap T_g^{-1} U \neq \emptyset$ for some $g \in R$. Clearly we have $U_i \cap T_g^{-1} U_i \neq \emptyset$ as well. (The reader is urged to supply an alternative proof of (a) using the equivalent combinatorial characterizations of Theorems 2.4 and 2.5.)

(b) Suppose that R is a set of chromatic recurrence contained in $Z(S)$ and that $L \subset S$ is left syndetic. Then there exists a finite set $H \subset S$ such that $S = \bigcup_{h \in H} h^{-1} L$. Since R is a set of chromatic recurrence there exist $h \in H$ and $g \in R$ such that $(h^{-1}L) \cap g^{-1}(h^{-1}L) \neq \emptyset$. Let $k \in (h^{-1}L) \cap g^{-1}(h^{-1}L)$. Then $hk \in L$ and $hgk \in L$. But g commutes with h, so $hk \in L \cap g^{-1}L$. By Theorem 2.4, R is a set of topological recurrence.

(c) Let S be the free group on the letters a and b and let R be the set $\{a^k : k \in \mathbf{N}\}$. Let $S = A_1 \cup \cdots \cup A_r$ be any partition of S and choose $m < n$ such that a^m and a^n lie in the same cell A_i. Then $a^m \in A_i \cap (a^{n-m})^{-1} A_i$. This shows that R is a set of chromatic recurrence. Let $\{T_g\}_{g \in S}$ be a measure preserving S-action of a probability space $(X, \mathcal{B}, \mu)$ and suppose $U \in \mathcal{B}$ with $\mu(U) > 0$. Choose $m < n$ with $\mu(T_{a^m}^{-1} U \cap T_{a^n}^{-1} U) > 0$. Then $\mu(U \cap T_{a^{n-m}}^{-1} U) > 0$. This shows that R is a set of measurable recurrence. To see that R is not a set of topological recurrence, consider the left syndetic set B consisting of all words that begin with either b or b^{-1}. Clearly $B \cap (a^k)^{-1} B = \emptyset$ for all $k \in \mathbf{N}$. By Theorem 2.4 R cannot be a set of topological recurrence.

(d) Suppose that R is a set of measurable recurrence and S is left amenable. Let m be a left invariant mean and suppose $L \subset S$ is left syndetic. For some finite set $\{h_1, \cdots, h_n\} \subset S$, $S = \bigcup_{i=1}^n h_i^{-1} L$. We must have $m(L) \geq \frac{1}{n}$. Since R is a set of measurable recurrence, by Theorem 2.2 there exists $g \in R$ such that $L \cap g^{-1} L \neq \emptyset$. Therefore by Theorem 2.4 R is a set of topological recurrence and by (a) a set of chromatic recurrence as well.

(e) This non-trivial result, mentioned in the introduction, is due to Kriz ([K], see also [Fo] and [M]).

□

We now see that, while sets of measurable recurrence and sets of chromatic recurrence are partition regular, sets of topological recurrence are not.

Theorem 2.7 Let S be a countable semigroup. Suppose $R_1, R_2, \cdots, R_k \subset S$ and let $R = R_1 \cup R_2 \cup \cdots \cup R_k$.

(a) If R is a set of measurable recurrence then R_i is a set of measurable recurrence for some i, $1 \le i \le k$.

(b) If R is a set of chromatic recurrence then R_i is a set of chromatic recurrence for some i, $1 \le i \le k$.

(c) There exists an example in which S is a group, R is a set of topological recurrence and no R_i is a set of topological recurrence, $1 \le i \le k$.

Proof. (a) Suppose not. Then there exist probability spaces $(X_i, \mathcal{B}_i, \mu_i)$, positive measure sets $A_i \in \mathcal{B}_i$, and measure preserving S-actions $\{T_g^{(i)}\}_{g \in S}$ such that

$$\mu_i(A_i \cap (T_g^{(i)})^{-1} A_i) = 0, \;\; g \in R_i, \; 1 \le i \le k.$$

Let $X = \prod_{i=1}^k X_i$, $\mu = \prod_{i=1}^k \mu_i$, $T_g = \prod_{i=1}^k T_g^{(i)}$ and put $A = \prod_{i=1}^k A_i$. Then $\mu(A) > 0$ and yet $\mu(A \cap T_g^{-1} A) = 0$ for all $g \in R$, a contradiction.

(b) Suppose not. Then there exist S-actions $\{T_g^{(i)}\}_{g \in S}$ by continuous self-mappings of compact metric spaces X_i, and open covers $\{U_1^{(i)}, \cdots, U_{t_i}^{(i)}\}$ of X_i such that

$$U_j^{(i)} \cap (T_g^{(i)})^{-1} U_j^{(i)} = \emptyset, \;\; 1 \le i \le k, \; 1 \le j \le t_i, \; g \in R_i.$$

Let $X = \prod_{i=1}^k X_i$, $T_g = \prod_{i=1}^k T_g^{(i)}$ and let $\mathcal{U}$ be the open cover of X consisting of sets of the form $\prod_{i=1}^k U_{j_i}^{(i)}$, where $1 \le j_i \le t_i$, $1 \le i \le k$. Then for every $U \in \mathcal{U}$ and every $g \in R$ we have $U \cap T_g^{-1} U = \emptyset$, a contradiction.

(c) Let S be the free group on the letters $\{g, h, k\}$. $R = S \setminus \{e\}$ is left thick and hence a set of topological recurrence (see Example 4.4 (b)). If $w_1, w_2 \in R$, let $w_1 \sim w_2$ if the first letter of w_1 in its reduced form is the same as or the inverse of the first letter of w_2 in its reduced form, and if the last letter of w_1 in its reduced form is the same as or the inverse of the last letter of w_2 in its reduced form. Let $R_1, \cdots, R_9$ be the set of equivalence classes under $\sim$. Given i, $1 \le i \le 9$, one easily finds a left syndetic set L such that $L \cap g^{-1} L = \emptyset$ for all $g \in R_i$. For example, suppose that R_i is the set of words beginning with g or g^{-1} and ending with h or h^{-1}. Let L be the set of words beginning with k or k^{-1}. L is left syndetic, but $L \cap g^{-1} L = \emptyset$ for $g \in R_i$. By Theorem 2.5 none of the sets R_i is a set of topological recurrence.

□

Remark. At this stage the alert reader may wonder why we even care about sets of "topological recurrence". Insofar as they fail to provide a connection to partition Ramsey theory, and insofar as they fail to possess many of the good properties of their counterparts, sets of measurable recurrence and sets of chromatic recurrence, they may appear in many ways to be a useless and inappropriate idea. Nevertheless, we have our reasons, which will be very briefly expounded upon in Section 5.

We now move to discussion which will serve as a preparation for the non-commutative Schur theorem we aim to prove in the next section. Schur's theorem, which states that for any finite coloring of a semigroup S a configuration of the type $\{x, y, xy\}$ can be found in one color, has been given many different proofs. Schur's original method ([S], see also [Si], [H]) employed counting techniques. Another way is to use *Ramsey's theorem*. A proof along these lines may be found in [GRS]. A third idea is to derive Schur's theorem as a consequence of Hindman's theorem or its proof, for example by using the existence of an idempotent ultrafilter in βS.

We do not see any way to generalize any of these methods to get $\{x, y, xy, yx\}$ in one color with $xy \neq yx$. Consider, however, the following proof of Schur's theorem which uses partition regularity of sets of recurrence and is valid for countable amenable semigroups (this proof is an adaptation of the proof of a *density Schur theorem* in [B1]): Suppose $S = \bigcup_{i=1}^{r} C_i$ is a finite partition of a countable, left amenable semigroup S and let m be a left invariant mean. Renumbering if necessary we may assume that there exists r_0 such that $m(C_i) > 0$ when $1 \leq i \leq r_0$, and $m(C_i) = 0$ when $r_0 < i \leq r$. Then $m(\bigcup_{i=1}^{r} C_i) = 1$, so that in particular $\bigcup_{i=1}^{r} C_i$ is left thick ([P, Proposition 1.21]) and hence a set of measurable recurrence (Example 4.4 (a) gives something even stronger). By Theorem 2.7 (a), therefore, there exists i with $1 \leq i \leq r_0$ such that C_i is a set of measurable recurrence. In particular, by Theorem 2.2 we have $C_i \cap x^{-1}C_i \neq \emptyset$ for some $x \in C_i$. Letting $y \in C_i \cap x^{-1}C_i$, one obtains $\{x, y, xy\} \subset C_i$.

The preceding proof uses a recurrence fact, namely the fact that left thick sets are sets of measurable recurrence. The proof in the next section substitutes for this assertion the following stronger result from [BMZ]: in amenable groups, left thick sets are sets of measurable *double recurrence*. What exactly this means will be explained presently.

3. Double recurrence and a non-commutative Schur theorem.

A special case of Szemerédi's theorem due to K. Roth ([R]) asserts that if $E \subset \mathbf{N}$ has positive upper density then E contains an arithmetic progression of length 3. Roth's theorem is easily shown via Furstenberg correspondence to be a consequence of the following *double recurrence* strengthening of the Poincaré recurrence theorem: for any finite measure preserving system $(X, \mathcal{B}, \mu, T)$ and any $A \in \mathcal{B}$ with $\mu(A) > 0$ there exists $n \in \mathbf{N}$ such that $\mu(A \cap T^{-n}A \cap T^{-2n}A) > 0$. This of course is a special case of Furstenberg's multiple recurrence theorem, however it may also be obtained (see [F1], [F2]) as a consequence of what Furstenberg calls the *ergodic Roth theorem*, which asserts that $\frac{1}{N}\sum_{n=1}^{N} T^n f T^{2n} g$ converges in $L^2(X, \mathcal{B}, \mu)$ for all $f, g \in L^\infty(X, \mathcal{B}, \mu)$ and identifies the limit.

The phenomenon of double recurrence prompts the question of which sets it occurs along. In the following definition and elsewhere, we take two S-actions $\{T_g\}_{g \in S}$ and $\{S_g\}_{g \in S}$ to be commuting if $T_g S_h = S_h T_g$ for all $g, h \in S$.

Definition 3.1 Let R be a subset of a countable semigroup S. R is called a *set of measurable 2-recurrence* if for any pair of commuting measure preserving S-actions $\{T_g\}_{g \in S}$ and $\{S_g\}_{g \in S}$ on a probability space $(X, \mathcal{B}, \mu)$ and every $A \in \mathcal{B}$ with $\mu(A) > 0$ there exists $g \in R$ such that

$$\mu(A \cap T_g^{-1}A \cap (T_g S_g)^{-1}A) > 0. \tag{3.1}$$

Like sets of single measurable recurrence, this notion has an equivalent, combinatorial formulation which may be proved by methods similar to those used for Theorem 2.2.

Theorem 3.2 Let $R \subset S$, where S is a countable left amenable semigroup. R is a set of measurable 2-recurrence if and only if for every left invariant mean m on $l^\infty(S \times S)$ and every $A \subset S \times S$ with $m(A) > 0$, A contains a configuration of the form $\{(a,b),(ga,b),(ga,gb)\}$, where $g \in R$.

We would now like to say a few words about inequality (3.1). If S is abelian and (3.1) were to be replaced by the alternate expression $\mu(A \cap T_g^{-1}A \cap S_g^{-1}A) > 0$, the resulting definition for sets of 2-recurrence would be equivalent to the given one. However, if S is non-abelian, the corresponding alternative definition is neither thought to be equivalent to Definition 3.1, nor has this alternate version been conducive to study. Indeed, it is unknown whether or not for every countable amenable group S, given commuting measure preserving S-actions $\{T_g\}_{g\in S}$ and $\{S_g\}_{g\in S}$ on a probability space $(X,\mathcal{B},\mu)$, and $A \in \mathcal{B}$ with $\mu(A) > 0$ there exists $g \neq e$ with $\mu(A \cap T_g^{-1}A \cap S_g^{-1}A) > 0$. On the other hand, one has the following.

Theorem 3.3 ([BMZ]) If G is a countable amenable group, $\{T_g\}_{g\in G}$ and $\{S_g\}_{g\in G}$ are commuting measure preserving G-actions on a probability space $(X,\mathcal{B},\mu)$, and $A \in \mathcal{B}$ with $\mu(A) > 0$ then the set $\{g \in G : \mu(A \cap T_g^{-1}A \cap (T_gS_g)^{-1}A) > 0\}$ is both left and right syndetic. In particular, any left or right thick subset of G is a set of measurable 2-recurrence.

As an application of Theorem 3.3 we present the main original result of this paper, namely a non-commutative Schur theorem for amenable groups. Recall that a sequence of finite subsets $\{\Phi_n\}_{n=1}^\infty$ in a countable amenable group G is called a *left (respectively right) Følner sequence* if for every $s \in G$ we have $\frac{|\Phi_n \cap s\Phi_n|}{|\Phi_n|} \to 1$ (respectively $\frac{|\Phi_n \cap \Phi_n s|}{|\Phi_n|} \to 1$) as $n \to \infty$. $\{\Phi_n\}_{n=1}^\infty$ will be called a *two-sided Følner sequence* if it is both a left and a right Følner sequence. It is well known that every countable amenable group admits a two-sided Følner sequence. If $\{\Phi_n\}_{n=1}^\infty$ is a (left, right or two-sided) Følner sequence and $E \subset G$, the *upper density* of E with respect to $\{\Phi_n\}_{n=1}^\infty$ is defined to be the number

$$\overline{d}(E) = \limsup_{n\to\infty} \frac{|E \cap \Phi_n|}{|\Phi_n|}.$$

One easily checks that if $\{\Phi_n\}_{n=1}^\infty$ is a left (respectively right) Følner sequence then $\overline{d}(sE) = \overline{d}(E)$ (respectively $\overline{d}(Es) = \overline{d}(E)$) for all $s \in G$ and $E \subset G$. If $\lim_{n\to\infty} \frac{|E\cap\Phi_n|}{|\Phi_n|}$ exists, we call the limit $d(E)$ and say E has *density* $d(E)$. Working with d has an advantage over working with $\overline{d}$ in that d is additive in the following sense: if $E \cap F = \emptyset$ and $d(E)$ and $d(F)$ both exist then $d(E \cup F) = d(E) + d(F)$.

The following well known facts will be used in the proof of Theorem 3.4:

(a) Suppose $E_i \subset G$, $i \in \mathbf{N}$, and $\{\Phi_n\}_{n=1}^\infty$ is a left Følner sequence for G. There exists an increasing sequence $(n_k)_{k\in\mathbf{N}} \subset \mathbf{N}$ such that for all $i \in \mathbf{N}$, $d(E_i)$ exists with respect to the left Følner sequence $\{\Phi_{n_k}\}_{k\in\mathbf{N}}$.

(b) If $\{\Phi_n\}_{n=1}^\infty$ is a left Følner sequence then there exists a left invariant mean m having the property that for every $E \subset G$ for which $d(E)$ (taken with respect to $\{\Phi_n\}_{n=1}^\infty$) exists one has $m(E) = d(E)$.

(c) If G is a countable amenable group then $G \times G$ is as well. Furthermore, if $\{\Phi_n\}_{n=1}^{\infty}$ and $\{\Psi_n\}_{n\in\mathbf{N}}$ are any two left Følner sequences for G then $\{\Phi_n \times \Psi_n\}_{n\in\mathbf{N}}$ is a left Følner sequence for $G \times G$.

(d) If G is a countable amenable group, $A \subset G$ is a subgroup, and $\{\Phi_n\}_{n=1}^{\infty}$ is any left Følner sequence for G then $d(A)$ exists and is equal to $\frac{1}{[G:A]}$ if $[G : A] < \infty$ and 0 if $[G : A] = \infty$. (This is a consequence of shift invariance of $\overline{d}$ and the fact that G is the disjoint union of the left cosets of A.)

(e) If G is a group having two subgroups of finite index A and B then $A \cap B$ is of finite index as well. (If $[G : A] = m$ and $[G : B] = n$, consider the homomorphism $g \to (\phi_1(g), \phi_2(g))$ of G into $S_m \times S_n$ (S_m being the group of permutations of the left cosets of A and S_n being the group of permutations of the left cosets of B), where $\phi_1(g)(hA) = (gh)A$ and $\phi_2(g)(hB) = (gh)B$. The kernel lies in $A \cap B$.)

Theorem 3.4 Suppose that G is a countable amenable group and that $G = \bigcup_{i=1}^{r} C_i$ is a finite partition. Let $A = \{g \in G : [G : C(g)] < \infty\}$. If $[G : A] = \infty$ then there exist $x, y \in G$ and i, $1 \le i \le r$, with $xy \ne yx$ and such that $\{x, y, xy, yx\} \subset C_i$.

Proof. That A is a subgroup of G is a consequence of (e), since $C(g) \cap C(h)$ is contained in $C(gh)$. Let $\{\Phi_n\}_{n=1}^{\infty}$ be a *two-sided* Følner sequence for G. Passing if necessary to a sub-sequence of $\{\Phi_n\}_{n=1}^{\infty}$ we may by (a) assume that $d(C_i)$ exists, $1 \le i \le r$. Renumbering if necessary, we may also assume that $d(C_i) > 0$, $1 \le i \le r_0$, and $d(C_i) = 0$, $r_0 < i \le r$. Then $d(\bigcup_{i=1}^{r_0} C_i) = 1$. Let $E = G \setminus A$ and replace C_i by $C_i \cap E$, $1 \le i \le r_0$. By (d) we have $d(E) = 1$, so we still have $d(\bigcup_{i=1}^{r_0} C_i) = 1$, and moreover $\bigcup_{i=1}^{r_0} C_i \subset E$. By (b) there exists some left (or right, since $\{\Phi_n\}_{n=1}^{\infty}$ is two-sided) invariant mean l such that $l(\bigcup_{i=1}^{r_0} C_i) = 1$. In particular by [P, Theorem 1.21] $\bigcup_{i=1}^{r_0} C_i$ is both left and right thick. Since a subset F of G is left (respectively right) thick if and only if F^{-1} is right (respectively left) thick, it follows that $\bigcup_{i=1}^{r_0} C_i^{-1}$ is also both left and right thick.

Let $(k_n)_{n\in\mathbf{N}}$ be a sequence in $\mathbf{N}$ having the property that

$$\frac{|\Phi_{k_n} \cap \Phi_{k_n} g^{-1}|}{|\Phi_{k_n}|} > 1 - \frac{1}{n} \tag{3.2}$$

for all $n \in \mathbf{N}$ and all $g \in \Phi_n$ (this is possible since $\{\Phi_n\}_{n=1}^{\infty}$ is a right Følner sequence). By (c), $\{\Phi_{k_n} \times \Phi_n\}_{n\in\mathbf{N}}$ is a left Følner seqence for $G \times G$. For $C \subset G$ let $\tilde{C} = \{(a, b) \in G \times G : ab^{-1} \in C\}$. We claim that if $d(C)$ exists then $d(\tilde{C})$ (measured with respect to $\{\Phi_{k_n} \times \Phi_n\}_{n\in\mathbf{N}}$) exists and equals $d(C)$. To see this, consider that

$$\begin{aligned}
\lim_{n\to\infty} \frac{|\tilde{C} \cap (\Phi_{k_n} \times \Phi_n)|}{|\Phi_{k_n} \times \Phi_n|} &= \lim_{n\to\infty} \frac{1}{|\Phi_{k_n}||\Phi_n|} \sum_{g\in\Phi_n} |\tilde{C} \cap (\Phi_{k_n} \times \{g\})| \\
&= \frac{1}{|\Phi_{k_n}||\Phi_n|} \sum_{g\in\Phi_n} |\Phi_{k_n} \cap Cg| \\
&= \lim_{n\to\infty} \frac{1}{|\Phi_n|} \sum_{g\in\Phi_n} \frac{|\Phi_{k_n} g^{-1} \cap C|}{|\Phi_{k_n}|} \\
&= \lim_{n\to\infty} \frac{1}{|\Phi_n|} \sum_{g\in\Phi_n} \frac{|\Phi_{k_n} \cap C|}{|\Phi_{k_n}|} = d(C).
\end{aligned}$$

(We have used (3.2) in the final equality.) In particular for every i, $1 \le i \le r_0$, $d(\tilde{C}_i)$ exists and is positive. By (a), we may by passing to a sub-sequence of the Følner sequence $\{\Phi_{k_n} \times \Phi_n\}_{n\in\mathbf{N}}$ assume that for all $(g_1,h_1),(g_2,h_2) \in G\times G$,

$$d(\tilde{C}_i \cap (g_1,h_1)^{-1}\tilde{C}_i \cap (g_2,h_2)^{-1}\tilde{C}_i)$$

exists, $1 \le i \le r_0$. Using (b), let m be a left-invariant mean on $G\times G$ having the property that $m(E) = d(E)$ for all $E \subset G\times G$ for which $d(E)$ exists. By Theorem 2.1, for $1 \le i \le r_0$ there exist probability spaces $(X_i, \mathcal{B}_i, \mu_i)$, sets $A_i \in \mathcal{B}_i$ with $\mu_i(A_i) > 0$, and measure preserving $G\times G$-actions $\{T^{(i)}_{(g,h)}\}_{(g,h)\in G\times G}$ such that for any (g_1,h_1) and (g_2,h_2) in $G\times G$ we have

$$\begin{aligned} d\big(\tilde{C}_i \cap (g_1,h_1)^{-1}\tilde{C}_i \cap (g_2,h_2)^{-1}\tilde{C}_i\big) &= m\big(\tilde{C}_i \cap (g_1,h_1)^{-1}\tilde{C}_i \cap (g_2,h_2)^{-1}\tilde{C}_i\big) \\ &= \mu\big(A_i \cap (T^{(i)}_{(g_1,h_1)})^{-1}A_i \cap (T^{(i)}_{(g_2,h_2)})^{-1}A_i\big), \end{aligned}$$

$1 \le i \le r_0$. Let $X = \prod_{i=1}^{r_0} X_i$, $\mu = \prod_{i=1}^{r_0}\mu_i$, $T_{(g,h)} = \prod_{i=1}^{r_0} T^{(i)}_{(g,h)}$, and $A = \prod_{i=1}^{r_0} A_i$. Then $\mu(A) > 0$, so by Theorem 3.3 the set

$$R = \{g \in G : \mu(A \cap T^{-1}_{(g,e)}A \cap T^{-1}_{(g,g)}A) > 0\}$$

is both left and right syndetic in G. We have

$$d(\tilde{C}_i \cap (g^{-1},e)\tilde{C}_i \cap (g^{-1},g^{-1})\tilde{C}_i) > 0,\ 1 \le i \le r_0,\ g \in R.$$

Since R is left and right syndetic and $\bigcup_{i=1}^{r_0} C_i^{-1}$ is left and right thick there exists $g \in R \cap \bigcup_{i=1}^{r_0} C_i^{-1}$. Hence for some i, $1 \le i \le r_0$, we have $g \in R \cap C_i^{-1}$. Letting $C_0 = G \setminus C(g)$ we have by (d) that $d(C_0) = 1$ with respect to $\{\Phi_n\}_{n=1}^{\infty}$ (since $g^{-1} \in \bigcup_{i=1}^{r_0} C_i \subset E$), so that $d(\tilde{C}_0) = 1$ with respect to $\{\Phi_{k_n} \times \Phi_n\}_{n\in\mathbf{N}}$. Since

$$d(\tilde{C}_i \cap (g^{-1},e)\tilde{C}_i \cap (g^{-1},g^{-1})\tilde{C}_i) > 0,$$

there exists

$$(a,b) \in (\tilde{C}_0 \cap \tilde{C}_i \cap (g^{-1},e)\tilde{C}_i \cap (g^{-1},g^{-1})\tilde{C}_i).$$

Then ab^{-1} doesn't commute with g and $\{g^{-1}, ab^{-1}, gab^{-1}, gab^{-1}g^{-1}\} \subset C_i$. Letting $x = g^{-1}$ and $y = gab^{-1}$ we have $xy \ne yx$ and $\{x,y,xy,yx\} \subset C_i$.

□

4. k-recurrence, questions, and examples.

As an extension of the ideas brought forth in the previous two sections, we now discuss those classes of sets along which one has higher orders of recurrence. The reader will notice that the case $k = 1$ in what is to follow gives the notions of Section 2, while the case $k = 2$ in part (a) corresponds to Definition 3.1.

Definition 4.1 Let $k \in \mathbf{N}$ and suppose that R is a subset of a countable semigroup S.

(a) R is called a *set of measurable k-recurrence* if for every k commuting measure preserving S-actions $\{T_g^{(i)}\}_{g\in S}$, $1 \le i \le k$, on a probability space $(X, \mathcal{B}, \mu)$ and every $A \in \mathcal{B}$ with $\mu(A) > 0$ we have

$$\mu(A \cap (T_g^{(1)})^{-1}A \cap (T_g^{(1)}T_g^{(2)})^{-1}A \cap \cdots \cap (T_g^{(1)} \cdots T_g^{(k)})^{-1}A) > 0$$

for some $g \in R$.

(b) R is called a *set of topological k-recurrence* if for every k commuting S-actions $\{T_g^{(i)}\}_{g\in S}$, $1 \le i \le k$, by continuous self-mappings of a compact metric space X, which is minimal with respect to the resulting S^k-action, and every open set $U \subset X$ we have

$$U \cap (T_g^{(1)})^{-1}U \cap (T_g^{(1)}T_g^{(2)})^{-1}U \cap \cdots \cap (T_g^{(1)} \cdots T_g^{(k)})^{-1}U \neq \emptyset$$

for some $g \in R$.

(c) R is called a *set of chromatic k-recurrence* if for every k commuting S-actions $\{T_g^{(i)}\}_{g\in S}$, $1 \le i \le k$, by continuous self-mappings of a compact metric space X, and every open covering $\{U_1, \cdots, U_r\}$ of X there exists i with $1 \le i \le r$ such that

$$U_i \cap (T_g^{(1)})^{-1}U_i \cap (T_g^{(1)}T_g^{(2)})^{-1}U_i \cap \cdots \cap (T_g^{(1)} \cdots T_g^{(k)})^{-1}U_i \neq \emptyset$$

for some $g \in R$.

If a set R is a set of measurable (respectively topological, chromatic) k-recurrence for every $k \in \mathbf{N}$, then R is said to be a set of measurable (respectively topological, chromatic) *multiple* recurrence. The following remarks are given as motivation for the questions to follow.

Remarks 4.2

A. In $S = \mathbf{N}$ there exist sets of measurable 1-recurrence which are known to fail 2-recurrence ([F2, p. 177]), or for which it is unknown whether or not they are sets of 2-recurrence, eg. $\{p - 1 : p \text{ prime}\}$ (for a proof that this set is density intersective and hence a set of measurable recurrence see [Sa]). However, every set $R \subset \mathbf{N}$ which is known to be a set of 2-recurrence is also known to be a set of multiple recurrence.

B. In $S = \mathbf{N}$ there exist sets, eg. the example of Kriz, which are sets of chromatic 1-recurrence but fail to be sets of measurable 1-recurrence.

C. According to Theorem 2.6 (d) every set of measurable recurrence in a countable left amenable semigroup S is a set of chromatic recurrence.

D. According to Theorem 2.6 (b), for countable abelian groups any set of chromatic recurrence is a set of topological recurrence, but there exist sets of chromatic recurrence in free groups which are not sets of topological recurrence.

Questions 4.3

A Does there exist a set $R \subset \mathbf{N}$ which is a set of measurable 2-recurrence but not a set of measurable multiple recurrence?

B. Does there exist a set $R \subset \mathbf{N}$ which is a set of chromatic 2-recurrence but not a set of measurable 1-recurrence?

C. Does there exist an example of a countable semigroup S and a set $R \subset S$ which is a set of measurable recurrence but is not a set of chromatic recurrence?

D. For which classes of countable semigroups, if any, intermediate to abelian and free, are all sets of chromatic recurrence sets of topological recurrence?

Remark. Our guess is that the answers to Questions 4.3 A, B, and C should all be *yes*.

For the purposes of describing the current state of knowledge, we offer a few examples of types of sets that are known to be sets of k-recurrence for various k. The two most commonly encountered, non-trivial classes of sets known to have strong recurrence properties in semigroups are left thick sets (see Definition 2.3 (b)), and *IP sets*. Recall that in a semigroup S, the (left) *IP set* generated by a sequence $\langle y_n \rangle_{n\in\mathbf{N}} \subset S$ is the set of products of finite subsets of the sequence taken in order of increasing indices, namely the set $FP(\langle y_n \rangle_{n\in\mathbf{N}}) = \{y_{i_1} \cdots y_{i_k} : i_1 < \cdots < i_k\}$. Every left thick set L contains an IP set. Indeed, let $y_1 \in L$, and having chosen $y_1, y_2, \cdots, y_n \in S$ such that $FP(\langle y_k \rangle_{1\le k\le n}) \subset L$, choose y_{n+1} with $FP(\langle y_k \rangle_{1\le k\le n})y_{n+1} \subset L$. It follows that in the examples to follow any assertion made about IP sets is true for left thick sets as well.

Examples 4.4

(a) Every IP set in a countable semigroup is both a set of measurable recurrence and a set of chromatic recurrence.

(b) Every left thick set in a countable semigroup is a set of topological recurrence.

(c) Every IP set in a countable group is a set of chromatic 2-recurrence ([BH, Corollary 2.7]). IP sets in countable groups need not, however, be sets of chromatic 3-recurrence ([BH, Corollary 4.6]), nor are IP sets in semigroups necessarily sets of chromatic 2-recurrence (see Theorem 4.5 below).

(d) Every IP set in a countable abelian group is a set of measurable multiple recurrence ([FK]) and a set of topological (or chromatic, which in the abelian case is the same) multiple recurrence ([FW]).

(e) ([BMZ]) Every left thick set in a countable amenable group is a set of measurable 2-recurrence, a set of topological 2-recurrence, and a set of chromatic 3-recurrence.

Although it does not provide an answer to Question 4.3 C, consider that if R is a set of topological recurrence in a countable semigroup S, $(X, \mathcal{B}, \mu)$ is a finite probability space, $A \in \mathcal{A}$ with $\mu(A) > 0$ and $\{R_g\}_{g\in S}$, $\{T_g\}_{g\in S}$ are commuting measure preserving S-actions on X, for some $g \in R$ we have $\mu(R_g^{-1}A \cap (R_gT_g)^{-1}A) = \mu(A \cap T_g^{-1}A) > 0$. The analogous "chromatic" statement is false:

Theorem 4.5 There exist a countable semigroup S, a set $R \subset S$ which is a set of chromatic (and measurable) recurrence, a compact space X, two commuting S-actions by continuous self-mappings of X $\{T_g\}_{g\in S}$ and $\{R_g\}_{g\in S}$, and an open cover $\{U_1, \cdots, U_r\}$ of X such that $R_g^{-1}U_i \cap (R_gT_g)^{-1}U_i = \emptyset$ for all $g \in R$ and $1 \le i \le r$.

Proof. Let S be the free semigroup on the countable family of letters $\langle y_n \rangle_{n\in\mathbf{N}}$ with identity e, and let R be the IP set $FP(\langle y_n \rangle_{n\in\mathbf{N}})$. R is a set of chromatic and measurable recurrence. According to [BH, Theorem 4.3], there exists a finite partition $\{A_1, \cdots, A_r\}$ of $S \times S$ such that no cell contains a configuration of the form $\{(ga, b), (ga, gb)\}$ for any $g \in FP(\langle y_n \rangle_{n\in\mathbf{N}})$ (any reader who looks up this reference is advised that we are performing a left-right switch in the semigroup multiplication due to the fact that IP sets are taken with multiplication in *decreasing* order of indices there).

Let $Z = \{1, \cdots, r\}^{S\times S}$. Let ξ be the element of Z defined by $\xi(s,t) = i$ if and only if $(s,t) \in A_i$, $1 \le i \le r$, $s,t \in S$. Define commuting S-actions $\{R_g\}_{g\in S}$ and $\{T_g\}_{g\in S}$ on X by $R_g\gamma(h,k) = \gamma(hg,k)$ and $T_g\gamma(h,k) = (h,kg)$. Put $X = \overline{\{T_gR_h\xi : (g,h) \in S \times S\}}$. X is invariant under both actions $\{R_g\}_{g\in S}$ amd $\{T_g\}_{g\in S}$. Let $U_i = \{\gamma \in X : \gamma(e,e) = i\}$, $1 \le i \le r$. Then $\{U_1, \cdots, U_r\}$ is an open cover of X.

Suppose now for some i, $1 \le i \le r$, and some $g \in R$, we have $R_g^{-1}U_i \cap (R_gT_g)^{-1}U_i \ne \emptyset$. Then for some $(a,b) \in S \times S$ we have $R_aT_b\xi \in R_g^{-1}U_i \cap (R_gT_g)^{-1}U_i$. Then $R_{ga}T_b\xi$ and $R_{ga}T_{gb}$ both lie in U_i, that is $\{(ga,b),(ga,gb)\} \subset A_i$, a contradiction.

□

5. Topological recurrence vs. chromatic recurrence.

We would like to conclude with a few words concerning the relationship of topological k-recurrence to chromatic k-recurrence. For abelian groups these notions are the same, and a close inspection reveals that it is this equivalence which lies at the heart of the inductive proofs of multiple recurrence results in topological dynamics, e.g. the topological proof of van der Waerden's theorem. One is able to show that if a class of sets (such as thick sets or IP sets) having certain translation properties (for example, that any set in the class may be shifted by many of its own members and yield another set in the class) are sets of topological k-recurrence then they are also sets of topological $k+1$-recurrence. In the non-abelian case, this induction breaks down, but in an interesting way. One is still able to show that if members of the class are sets of topological k-recurrence then they are sets of chromatic $k+1$-recurrence. An argument of this nature was carried out in [BMZ, Theorem 7.1], where it was shown that left thick sets are sets of chromatic 3-recurrence for countable amenable groups. (This example, incidentally, marks one of our reasons for even bothering about "topological recurrence", which owing to results such as 2.6 (c) and 2.7 (c) seems otherwise to be a rather ineffectual notion.)

The difficulty is that one may not automatically conclude (as one does in the abelian case) that members of the class in question are sets of *topological* $k+1$-recurrence, which seems necessary in order for the induction to proceed. Indeed, the only way we have at present to circumvent this snag, and this method is valid only in the presence of amenability, is to establish *measurable* $k+1$-recurrence. For example, a consequence of Theorem 3.3 is that any left or right thick subset of a countable amenable group is a set of topological 2-recurrence. Oddly, we have no proof of this topological fact which doesn't use (measurable) ergodic theory. It would be interesting to see whether there are legitimate reasons for this being so or whether there exists a more elementary method, perhaps one which would yield arbitrarily high orders of chromatic recurrence, without any appeal to measurable methodology.

REFERENCES

[B1] V. Bergelson, A density statement generalizing Schur's theorem, *J. Combinatorial Theory (A)* **43** (1986), 338-343.

[B2] V. Bergelson, Ergodic Ramsey theory–an update, *Ergodic Theory of* $\mathbf{Z}^d$*-actions* (edited by M. Pollicott and K. Schmidt), *London Math. Soc. Lecture Note Series* **228**, Cambridge University Press, 1996, 1-61.

[BBB] V. Bergelson, M. Boshernitzan and J. Bourgain, Some results on non-linear recurrence, *J. d'Analyse Math.*, **62** (1994), 29-46.

[BH] V. Bergelson and N. Hindman, Some topological semicommutative van der Waerden type theorems and their combinatorial consequences, *J. London Math. Soc.* (2) **45** (1992), 385-403.

[BMZ] V. Bergelson, R. McCutcheon, and Q. Zhang, A Roth theorem for amenable groups, *American Journal of Mathematics*, to appear.

[D] M. Day, Amenable semigroups, *Illinois J. Math* **1** (1957), 509-544.

[Fo] A. Forrest, Ph.D. thesis, The Ohio State University, 1990.

[F1] H. Furstenberg, Ergodic behavior of diagonal measures and a theorem of Szemerédi on arithmetic progressions, *J. d'Analyse Math.* **31** (1977), 204-256.

[F2] H. Furstenberg, *Recurrence in Ergodic Theory and Combinatorial Number Theory*, Princeton Univ. Press, Princeton, New Jersey, 1981.

[FK] H. Furstenberg and Y. Katznelson, An ergodic Szemerédi theorem for IP-systems and combinatorial theory, *J. d'Analyse Math.* **45** (1985), 117-168.

[FW] H. Furstenberg and B. Weiss, Topological dynamics and combinatorial number theory, *J. d'Analyse Math.* **34** (1978), 61-85.

[GRS] R. Graham, B. Rothschild and J. Spenser, *Ramsey Theory*, Wiley, New York, 1980.

[H] N. Hindman, Ultrafilters and combinatorial number theory, *Number Theory Carbondale*, M. Nathanson, ed., *Lecture Notes in Math.* **751** (1979), 119-184.

[K] I. Kriz, Large independent sets in shift-invariant graphs. Solution of Bergelson's problem, *Graphs and Combinatorics* **3** (1987), 145-158.

[M] R. McCutcheon, Three results in recurrence, *Proceedings of Alexandria Conference on Harmonic Analysis and its Applications to Ergodic Theory* (edited by K. Petersen I. Salama), *London Math. Soc. Lecture Note Series* **205** (1995) 349-358.

[P] A. Paterson, *Amenability*, American Mathematical Society, Providence, 1988.

[R] K. Roth, Sur quelques ensembles d'entiers, *C.R. Acad. Sci. Paris* **234** (1952), 388-390.

[Sa] A. Sárközy, On difference sets of sequences of numbers III, *Acta. Acad. Sci. Hung.* **31** (1978), 355-386.

[S] I. Schur, Uber die Kongruenz $x^m + y^m = z^m \pmod{p}$, *Jber. Deutsch. Math.-Verein.* **25** (1916), 114-116.

[Si] W. Sierpiński, *Elementary Theory of Numbers*, Warszawa, 1959.

[Sz] E. Szemerédi, On sets of integers containing no k elements in arithmetic progression, *Acta. Arith.* **27** (1975), 199-245.

Vitaly Bergelson, Department of Mathematics, 231 West 18th Avenue, The Ohio State University, Columbus, OH 43210; *email:* vitaly@math.ohio-state.edu

Randall McCutcheon, Department of Mathematics, Case Western Reserve University, Cleveland, OH 44106; *email:* rxm16@po.cwru.edu