Spanning Trees and Cycle Spaces in Topological Spaces

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Abstract. A *topologized hypergraph* is a topological space X in which every singleton is either an open set or a closed set. The points $x \in X$ for which $\{x\}$ is closed are the *vertices*; the others are the *edges*. The closure of an edge consists of the edge and its incident vertices. The sets $V(X)$ and $E(X)$ denote the sets of vertices and edges in X.
A topological space is *weakly normal* if, for any two closed sets A and B, there are open sets U_A and U_B, containing A nd B, respectively, such that $U_A \cap U_B$ is finite.

Theorem 1: *Every connected weakly normal topologized hypergraph X has a minimal connected set containing $V(X)$.*

A *topologized graph* is a topologized hypergraph such that every edge is incident with exactly two vertices. A *cycle* in a topologized graph X is a set E of edges such that there is a connected subspace X' of X and a cyclic order on E so that:

(1) $E(X') = E$;

(2) for each $x \in X'$, $X' \setminus \{x\}$ is connected;

(3) if $a, b, c, d \in E$ occur in this cyclic order, then a and c are in different components of $X' \setminus \{b, d\}$.

Theorem 2: *If X is a compact, connected, weakly normal topologized graph, then:*

(1) the fundamental cycles of a spanning tree are a basis for the cycle space of X;

(2) every cycle intersects every finite cut an even number of times; and

(3) every element of the cycle space is the disjoint union of cycles.

These results generalize and simplify recent work of Diestel and Kuhn on the cycle spaces of infinite graphs.