MATH. 4552, SUMMER '20

ADDITIONAL HOMEWORK

June 12, 2020

1. Find all complex numbers z such that

$$8z^3 - 12\sqrt{3}z^2 + 2z + 13\sqrt{3} = 0\,.$$

2. Find all complex solutions z to the cubic equation

$$z^3 - 3z + \sqrt{2} = 0\,.$$

MATH. 4552, SUMMER '20

ADDITIONAL HOMEWORK

June 15, 2020

1. Find all complex solutions z to the quartic equation

$$2z^4 - 6z + 3 = 0 .$$

Additional homework, June 12: Solutions

1. We need to rewrite the equation as $w^3 + pw + q = 0$ with a new unknown w and use the identity $(u+v)^3 - 3uv(u+v) - (u^3+v^3) = 0$ to obtain a solution $w = u+v$, finding the complex numbers u, v from a quadratic equation for which their cubes u^3, v^3 are the roots. Specifically, for

$$w = z - \frac{\sqrt{3}}{2}$$

the equation becomes

$$w^3 - 2w + \sqrt{3} = 0 .$$

As $(u+v)^3 - 3uv(u+v) - (u^3+v^3) = 0$, equation $w^3 - 3uvw - (u^3+v^3) = 0$ is satisfied by $w = u+v$, and so we can find a solution w to our problem by choosing u, v with

$$uv = \frac{2}{3}, \qquad u^3 + v^3 = -\sqrt{3} .$$

For such u and v, the cubes u^3, v^3 must be the roots of the equation

$$\xi^2 + \sqrt{3}\,\xi + \frac{8}{27} = 0$$

with the unknown ξ. Solving it for ξ, we obtain

$$\xi = -\frac{\sqrt{3}}{18}\,(9 \pm 7), \quad \text{i.e.,} \quad \xi = -\frac{8\sqrt{3}}{9} \quad \text{or} \quad \xi = -\frac{\sqrt{3}}{9} .$$

Choosing the cubic roots of these solutions to be

$$u = -\frac{2\sqrt{3}}{3}, \qquad v = -\frac{\sqrt{3}}{3},$$

we obtain u, v with the required properties, leading to the solution $w = -\sqrt{3}$. We can now easily factorize

$$w^3 - 2w + \sqrt{3} = (w + \sqrt{3})(w^2 - \sqrt{3}\,w + 1),$$

so that we get the solutions

$$w_1 = -\sqrt{3}, \qquad w_2 = \frac{\sqrt{3}}{2} + \frac{i}{2}, \qquad w_3 = \frac{\sqrt{3}}{2} - \frac{i}{2},$$

and, for the original equation

$$z_1 = -\frac{\sqrt{3}}{2}, \qquad z_2 = \sqrt{3} + \frac{i}{2}, \qquad z_3 = \sqrt{3} - \frac{i}{2} .$$

2. Again, the identity $(u + v)^3 - 3uv(u + v) - (u^3 + v^3) = 0$ allows us to obtain a solution $z = u + v$ provided that we can find complex numbers u, v such that $-3uv$ and $-(u^3 + v^3)$ are our coefficients. To find u, v we write a quadratic equation for which u^3, v^3 are the roots. To be specific, equation $z^3 - 3uvz - (u^3 + v^3) = 0$ holds for $z = u + v$, and so we can find a solution z to our original problem by choosing u, v with $uv = 1$ and $u^3 + v^3 = -\sqrt{2}$. For such u and v, the cubes u^3, v^3 must be the roots of the equation $\xi^2 + \sqrt{2}\,\xi + 1 = 0$. Solving that for the unknown ξ, we obtain

$$\xi = -\frac{\sqrt{2}}{2}(1 \pm i) = e^{\mp 3\pi i/4}.$$

Choosing the cubic roots of these two solutions ξ to be

$$u = e^{\pi i/4} = \frac{\sqrt{2}}{2}(1+i), \qquad v = e^{-\pi i/4} = \frac{\sqrt{2}}{2}(1-i),$$

we obtain u, v with the required properties, leading to the solution $z = u + v = \sqrt{2}$. We now use this root to factorize

$$z^3 - 3z + \sqrt{2} = (z - \sqrt{2})(z^2 + \sqrt{2}\,z - 1),$$

so that we get the following solutions to the original equation:

$$z_1 = \sqrt{2}, \qquad z_2 = \frac{\sqrt{6} - \sqrt{2}}{2}, \qquad z_3 = -\frac{\sqrt{6} + \sqrt{2}}{2}.$$

Additional homework, June 15: Solution

1. The coefficient of z^3 already is zero, so we may proceed with searching for a factorization

$$2z^4 - 6z + 3 = 2(z^2 - \alpha z + \beta)(z^2 + \alpha z + \gamma),$$

which amounts to finding α, β, γ that satisfy the system (Q.5), (Q.6) with

$$p = 0, \qquad q = -3, \qquad r = \frac{3}{2},$$

that is,

$$(*) \qquad \beta + \gamma = \alpha^2, \qquad \beta\gamma = \frac{3}{2}, \qquad \alpha(\beta - \gamma) = -3.$$

The first two relations mean that our β, γ are the roots of a quadratic equation depending on α, and the third relation then is a "consistency condition" which may be used to determine α. Specifically, $\alpha \neq 0$ (by the third relation) and

$$\frac{9}{\alpha^2} = (\beta - \gamma)^2 = (\beta + \gamma)^2 - 4\beta\gamma = \alpha^4 - 6,$$

i.e.,

$$\alpha^6 - 6\alpha^2 - 9 = 0.$$

This is a cubic equation in α^2, which we can solve using Cardano's formulae, setting $\alpha^2 = u + v$, with the unknown u, v subject to conditions (C.6) and (C.7) (where p, q now stand for -6 and -9), that is,

$$uv = 2, \qquad u^3 + v^3 = 9.$$

An easy guess is that we may use

$$u = 1, \qquad v = 2.$$

(Proceeding systematically, we note that, as $(\xi - u^3)(\xi - v^3) = \xi^2 - 9\xi + 8$, the cubes u^3, v^3 of u and v must be the roots of the quadratic equation $\xi^2 - 9\xi + 8 = 0$. Solving it we obtain

$$\xi = \frac{9 \pm 7}{2},$$

and we may choose the cubic roots of these solutions to be $u = 1$, $v = 2$.)

The corresponding solution α with $\alpha^2 = u + v = 3$ is $\alpha = \pm\sqrt{3}$. We may choose the sign of α to be positive, and then solve (for β and γ) our equations $(*)$, which now read

$$\beta + \gamma = 3, \qquad \beta\gamma = \frac{3}{2}, \qquad \sqrt{3}\,(\beta - \gamma) = -3.$$

Thus, β and γ are the roots ζ of the quadratic equation $0 = (\zeta-\beta)(\zeta-\gamma) = \zeta^2-3\zeta+3/2$, i.e., they are the numbers $(3 \pm \sqrt{3})/2$, and ordering them so that $\beta - \gamma < 0$ we have

$$\alpha = \sqrt{3}\,, \qquad \beta = \frac{3-\sqrt{3}}{2}\,, \qquad \gamma = \frac{3+\sqrt{3}}{2}\,.$$

The required the decomposition of the original quartic polynomial now is

$$2z^4 - 6z + 3 = 2\,[z^2 - \sqrt{3}\,z + (3-\sqrt{3})/2] \cdot [z^2 + \sqrt{3}\,z + (3+\sqrt{3})/2]\,.$$

Solving the two separate quadratic equations

$$z^2 - \sqrt{3}\,z + (3-\sqrt{3})/2 = 0\,, \qquad z^2 + \sqrt{3}\,z + (3+\sqrt{3})/2 = 0\,,$$

we now find the solutions

$$\frac{1}{2}\left[\sqrt{3} \pm \sqrt{2\sqrt{3} - 3}\right], \qquad \frac{1}{2}\left[-\sqrt{3} \pm i\sqrt{2\sqrt{3} + 3}\right]$$

to the original problem.